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Not peer-reviewed version

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Posted Date: 3 November 2025

doi: 10.20944/preprints202510.2532.v1

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Article

MultiRough, MultiGrey, MultiGranular, MultiInterval, and MultiFunctorial Numbers

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Abstract

A Multi-Structure replaces classical operations with mappings from tuples to finite multisets, allowing multiple outputs for a single input tuple simultaneously and flexibly. An Iterative Multi-Structure extends these multiset operations across hierarchical levels, combining multisets of multisets through k iterative stages of layered aggregation. In this paper, we define the MultiRough, MultiGrey, MultiGranular, MultiInterval, and MultiFunctorial Numbers by extending the Rough, Grey, Granular, Interval, and Functorial Numbers using the frameworks of Multi-Structure and Iterative Multi-Structure. Furthermore, we introduce their corresponding Iterative versions—Iterative MultiRough, Iterative MultiGrey, Iterative MultiGranular, Iterative MultiInterval, and Iterative MultiFunctorial Numbers—and briefly discuss their fundamental properties.

Keywords: multi-structure; iterative multi-structure; number

1. Preliminaries

This section introduces the fundamental concepts and definitions that underpin the discussions in this paper. Throughout, all sets are assumed to be finite.

1.1. Multi-Structure

A Multi-Structure replaces classical operations with maps from tuples to finite multisets, enabling multiple outputs per input tuple flexibly simultaneously [1–4].

Definition 1 (Finite Multiset). (cf.[5–8]) Let H be a nonempty set. A finite multiset on H is a function

$$m : H \longrightarrow \mathbb{N}_0$$

with finite support $\{x \in H \mid m(x) > 0\}$. We denote by $\mathcal{M}(H)$ the collection of all such finite multisets on H . Equivalently, an element of $\mathcal{M}(H)$ can be written as $\{x_1^{k_1}, x_2^{k_2}, \dots, x_r^{k_r}\}$, where each $x_i \in H$ and $k_i = m(x_i) \in \mathbb{N}$.

Definition 2 (MultiOperation). Let H be a nonempty set and fix an integer $m \geq 1$. A multi-operation of arity m on H is a map

$$\begin{aligned} \#^{(m)} : H^m &\longrightarrow \mathcal{M}(H), \\ (x_1, \dots, x_m) &\mapsto \#^{(m)}(x_1, \dots, x_m) \in \mathcal{M}(H). \end{aligned}$$

Thus, instead of producing a single element of H , a multi-operation assigns a finite multiset of elements of H .

Definition 3 (MultiStructure). A MultiStructure is a pair

$$\mathcal{MS} = (H, \{\#^{(m)} : H^m \rightarrow \mathcal{M}(H)\}_{m \in \mathbb{I}}),$$

where H is a nonempty carrier set and $\mathcal{I} \subseteq \mathbb{Z}_{>0}$ indexes a family of multi-operations of various arities. No further axioms are imposed unless specified.

Example 1 (Real-life Multi-Structure: Two-hop urban transit with route multiplicities). Let H be the set of stations $H = \{A, B, C, D, E\}$. Suppose there are labeled directed edges (station-to-station services) with possibly multiple lines per edge:

$$A \xrightarrow{\{R,B\}} B, \quad A \xrightarrow{\{G\}} C, \quad B \xrightarrow{\{R,B\}} D, \quad C \xrightarrow{\{G\}} D, \quad B \xrightarrow{\{R\}} E, \quad E \xrightarrow{\{B\}} D.$$

Define a binary multi-operation $\#^{(2)} : H^2 \rightarrow \mathcal{M}(H)$ by sending an origin–destination pair (x, y) to the multiset of feasible transfer stations v that admit a 2-hop trip $x \rightarrow v \rightarrow y$, with multiplicity equal to the number of distinct line-pairs:

$$\#^{(2)}(x, y) := \{v^{m(v)} : v \in H, (x \rightarrow v), (v \rightarrow y)\}, \quad m(v) := |\text{Lines}(x \rightarrow v)| \cdot |\text{Lines}(v \rightarrow y)|.$$

Concretely,

$$\#^{(2)}(A, D) = \{B^4, C^1\} \quad \text{since } |\text{Lines}(A \rightarrow B)| = 2, |\text{Lines}(B \rightarrow D)| = 2 \Rightarrow m(B) = 4,$$

and $|\text{Lines}(A \rightarrow C)| = |\text{Lines}(C \rightarrow D)| = 1 \Rightarrow m(C) = 1$. Likewise,

$$\#^{(2)}(A, E) = \{B^2\}, \quad \#^{(2)}(B, D) = \{(no\ transfer)\} = \emptyset \quad (\text{when transfers are required}).$$

Thus $\mathcal{MS} = (H, \{\#^{(2)}\})$ is a Multi-Structure whose output multiset counts route alternatives via multiplicities, reflecting a concrete, operational transit planning task.

1.2. Iterative Multi-Structure

An Iterative Multi-Structure extends multiset operations across levels, combining multisets of multisets iteratively through k hierarchical stages in layered aggregation [9,10].

Definition 4 (Iterative Multi-Structure of Order k). Let H be a nonempty set and fix an integer $k \geq 1$. Define iteratively the multiset powersets

$$\mathcal{M}^0(H) = H, \quad \mathcal{M}^{i+1}(H) = \mathcal{M}(\mathcal{M}^i(H)), \quad i = 0, 1, \dots, k-1,$$

where $\mathcal{M}(X)$ denotes the collection of finite multisets on X (Definition 1). Let $\mathcal{I} \subseteq \mathbb{Z}_{>0}$ index a family of arities. An Iterative Multi-Structure of order k is a tuple

$$\mathcal{IMS}^{(k)} = (H, \{\#^{(m,i)} : (\mathcal{M}^i(H))^m \longrightarrow \mathcal{M}^{i+1}(H)\}_{m \in \mathcal{I}, 0 \leq i < k}),$$

where for each $i = 0, \dots, k-1$ and each $m \in \mathcal{I}$,

$$\#^{(m,i)}(x_1, \dots, x_m) \in \mathcal{M}^{i+1}(H), \quad x_j \in \mathcal{M}^i(H).$$

Thus $\#^{(m,0)}$ is an ordinary Multi-Structure operation on H , $\#^{(m,1)}$ combines multisets of multisets, and so on, up to level k .

Example 2 (Real-life Iterative Multi-Structure: From items to baskets to pallet candidates). Let $H = \{M, C, E, P, B\}$ denote SKUs: Milk (M), Cereal (C), Eggs (E), IcePack (P), BoxSmall (B). Level-0 elements are items ($\mathcal{M}^0(H) = H$), level-1 elements are baskets (finite multisets of items, $\mathcal{M}^1(H)$), and level-2 elements are batches of baskets ($\mathcal{M}^2(H)$).

Level-0 operation $\#^{(2,0)} : H^2 \rightarrow \mathcal{M}^1(H)$.

Given two items, return a recommended basket fragment (items plus required packaging/cooling):

$$\#^{(2,0)}(M, C) = \{M^1, C^1, B^1\}, \quad \#^{(2,0)}(E, M) = \{E^1, M^1, P^1, B^1\},$$

and by default $\#^{(2,0)}(x, y) = \{x^1, y^1\}$ when no special handling is required.

Level-1 operation $\#^{(2,1)} : (\mathcal{M}^1(H))^2 \rightarrow \mathcal{M}^2(H)$.

Given two baskets $X, Y \in \mathcal{M}^1(H)$, propose a pallet candidate set containing each basket and their multiset union (to reflect possible consolidation):

$$\#^{(2,1)}(X, Y) := \{X, Y, X \uplus Y\} \in \mathcal{M}^2(H),$$

where $\uplus$ denotes multiset sum (adding multiplicities).

Concrete computation.

Let

$$X = \#^{(2,0)}(M, C) = \{M^1, C^1, B^1\}, \quad Y = \#^{(2,0)}(E, M) = \{E^1, M^1, P^1, B^1\}.$$

Then

$$X \uplus Y = \{M^2, C^1, E^1, P^1, B^2\},$$

and

$$\#^{(2,1)}(X, Y) = \{\{M^1, C^1, B^1\}, \{E^1, M^1, P^1, B^1\}, \{M^2, C^1, E^1, P^1, B^2\}\} \in \mathcal{M}^2(H).$$

Hence $\mathcal{IMS}^{(2)} = (H, \{\#^{(2,0)}, \#^{(2,1)}\})$ is an Iterative Multi-Structure: level-0 combines items into baskets, while level-1 combines baskets into pallet candidates—both as finite multisets that preserve concrete counts.

1.3. Rough Number

A rough number is an interval of lower and upper mean attribute values over rough approximations under an equivalence relation [11–13]. Extensions of the rough number, such as the Fuzzy Rough Number and Neutrosophic Rough Number, have also been studied [14–19].

Definition 5 (Universe and Equivalence Classes). (cf.[11]) Let U be a nonempty finite set (the universe). Let

$$R \subseteq U \times U$$

be an equivalence relation on U . For each $x \in U$, denote its equivalence class by

$$[x]_R = \{y \in U \mid (x, y) \in R\}.$$

Definition 6 (Lower and Upper Approximations). (cf.[11]) For any $G \subseteq U$, define its lower approximation and upper approximation with respect to R by

$$\underline{G} = \{x \in U \mid [x]_R \subseteq G\}, \quad \overline{G} = \{x \in U \mid [x]_R \cap G \neq \emptyset\}.$$

Here $\underline{G}$ collects all elements that definitely belong to G , while $\overline{G}$ collects those that possibly belong.

Definition 7 (Rough Number). (cf.[11,20,21]) Let $R = \{G_1, G_2, \dots, G_t\}$ be an ordered partition of U with $G_1 < \dots < G_t$. For each class G_q , define:

$$\text{Lim}^L(G_q) = \frac{1}{|\underline{G}_q|} \sum_{y \in \underline{G}_q} R(y), \quad \text{Lim}^U(G_q) = \frac{1}{|\overline{G}_q|} \sum_{y \in \overline{G}_q} R(y).$$

Then the rough number of G_q is the interval

$$\text{RN}(G_q) = [\text{Lim}^L(G_q), \text{Lim}^U(G_q)].$$

Example 3 (Rough Number: Product quality under coarse inspection groups). Let the universe be six tomatoes $U = \{u_1, \dots, u_6\}$. Define an equivalence relation “same inspection batch” with classes

$$C_1 = \{u_2, u_3\}, \quad C_2 = \{u_1\}, \quad C_3 = \{u_4, u_5\}, \quad C_4 = \{u_6\}.$$

Suppose we stratify quality into an ordered partition $\{G_1 < G_2 < G_3\}$ and focus on the mid band $G_2 = \{u_2, u_3, u_4\}$. Let the numeric attribute (e.g. Brix) be

$$R(u_1) = 6.0, \quad R(u_2) = 6.5, \quad R(u_3) = 7.0, \quad R(u_4) = 7.2, \quad R(u_5) = 8.0, \quad R(u_6) = 5.5.$$

Lower/upper approximations of G_2 (w.r.t. the batch classes) are

$$\underline{G}_2 = \{x \in U \mid [x] \subseteq G_2\} = \{u_2, u_3\} \quad (\text{since } C_1 \subseteq G_2),$$

$$\overline{G}_2 = \{x \in U \mid [x] \cap G_2 \neq \emptyset\} = \{u_2, u_3, u_4, u_5\} \quad (\text{since } C_1, C_3 \text{ intersect } G_2).$$

Thus the lower/upper mean limits are

$$\text{Lim}^L(G_2) = \frac{R(u_2) + R(u_3)}{2} = \frac{6.5 + 7.0}{2} = 6.75, \quad \text{Lim}^U(G_2) = \frac{6.5 + 7.0 + 7.2 + 8.0}{4} = \frac{28.7}{4} = 7.175.$$

Hence the rough number of G_2 is

$$\text{RN}(G_2) = [6.75, 7.175],$$

capturing the imprecision induced by batch-level (coarse) observations.

1.4. Grey Number

A grey number denotes an unknown real value constrained between known bounds, represented as interval $[g^-, g^+]$, modeling incomplete information (cf.[22–27]).

Definition 8 (Grey Number). [28,29] A grey number is a real quantity constrained by known lower and upper bounds $g^- \leq x \leq g^+$, while the exact position x within $[g^-, g^+]$ is unknown; we denote it by $g^\pm := [g^-, g^+]$.

Example 4 (Grey Number: Contracted electricity tariff without a fixed rate). A commercial customer has negotiated next month’s unit price but it is yet to be fixed; the contract guarantees

$$g^\pm = [g^-, g^+] = [24.0, 26.5] \text{ JPY/kWh}.$$

The exact x is unknown except $24.0 \leq x \leq 26.5$. Useful summaries are midpoint and half-width:

$$m(g^\pm) = \frac{24.0 + 26.5}{2} = 25.25, \quad w(g^\pm) = \frac{26.5 - 24.0}{2} = 1.25.$$

This grey number succinctly encodes incomplete pricing information for budgeting and risk buffers.

1.5. Granular Numbers

A granular number encodes a connected set of possible values, typically as center and radius, optionally weighted, capturing structured imprecision [30–33].

Definition 9 (Granular Number (G-number)). A granular number is an extension of an ordinary number: instead of a single value, it denotes a connected set of possible values. A unified representation uses a center and radius, $X = G(c_X, r_X)$, optionally augmented by a granule weight $A(X)$ as $X = G_{A(X)}(c_X, r_X)$.

Example 5 (Granular Number: Thermostat comfort range with a weight). *An office occupant reports a comfortable temperature band encoded as a granular number*

$$X = G_{A(X)}(c_X, r_X) = G_{0.8}(22.0, 1.5),$$

meaning the acceptable set is $\{t \in \mathbb{R} : |t - 22.0| \leq 1.5\} = [20.5, 23.5]$ with a confidence/priority weight $A(X) = 0.8$. This compactly models a connected tolerance set (center-radius) together with how strongly to respect it in HVAC control.

1.6. Interval Number

An interval number is a closed real interval $[a, b]$, representing all possible values between both lower and upper bounds, inclusively [34,35].

Definition 10 (Interval Number). *An interval number is a nonempty closed interval of the real line,*

$$A = [a_L, a_R] = \{a \in \mathbb{R} \mid a_L \leq a \leq a_R\},$$

with $a_L \leq a_R$. The degenerate case $a_L = a_R$ identifies $A = [a, a]$ with the real number a . Equivalently, one writes $A = \langle m(A), w(A) \rangle$ with midpoint and half-width

$$m(A) = \frac{a_L + a_R}{2}, \quad w(A) = \frac{a_R - a_L}{2}.$$

Example 6 (Interval Number: Mechanical tolerance on a shaft diameter). *A machining specification requires the diameter D of a shaft to lie in the closed interval*

$$D \in [24.95, 25.05] \text{ mm.}$$

As an interval number $A = [a_L, a_R] = [24.95, 25.05]$, its midpoint and half-width are

$$m(A) = \frac{24.95 + 25.05}{2} = 25.00, \quad w(A) = \frac{25.05 - 24.95}{2} = 0.05,$$

which drive gauge selection, process capability indices, and pass/fail inspection.

1.7. Functorial Numbers

A functorial number assigns each object a commutative semiring naturally, with operations preserved along morphisms[36].

Definition 11 (Functorial Number). *Let $\mathcal{C}$ be a category with finite products. A Functorial Number is a tuple*

$$(N, \oplus, \otimes, 0, 1)$$

where $N : \mathcal{C} \rightarrow \mathbf{Set}$ is a functor, $\oplus, \otimes : N \times N \Rightarrow N$ and $0, 1 : \Delta 1 \Rightarrow N$ are natural transformations, such that for each object X ,

$$(N(X), \oplus_X, \otimes_X, 0_X, 1_X)$$

is a commutative semiring, and for every morphism $f : X \rightarrow Y$ and $a, b \in N(X)$,

$$N(f)(a \oplus_X b) = N(f)a \oplus_Y N(f)b, \quad N(f)(a \otimes_X b) = N(f)a \otimes_Y N(f)b, \quad N(f)(0_X) = 0_Y, \quad N(f)(1_X) = 1_Y.$$

Equivalently, a Functorial Number is a functor $\tilde{N} : \mathcal{C} \rightarrow \mathbf{CRig}$ (commutative semirings and homomorphisms), composed with the forgetful functor to $\mathbf{Set}$.

Example 7 (Functorial Number: Aggregating SKU variants via a semiring functor). Let $\mathcal{C} = \mathbf{FinSet}$ and for each finite set X define the commutative semiring

$$N(X) := \mathbb{N}_0[X] \quad (\text{polynomials with variables indexed by } X).$$

For a function $f : X \rightarrow Y$, let $N(f) : \mathbb{N}_0[X] \rightarrow \mathbb{N}_0[Y]$ be the semiring homomorphism sending each variable $x \in X$ to the variable $f(x) \in Y$ and extending by linearity and multiplicativity.

Real-life reading. Let $X = \{\text{redMug}, \text{blueMug}, \text{plate}\}$ (SKU variants) and $Y = \{\text{MUG}, \text{PLATE}\}$ (base classes), with $f(\text{redMug}) = f(\text{blueMug}) = \text{MUG}$, $f(\text{plate}) = \text{PLATE}$. A basket polynomial

$$p = 3 \text{redMug} + 2 \text{blueMug} + (\text{plate})^2 \in \mathbb{N}_0[X]$$

is aggregated by

$$N(f)(p) = 3 \text{MUG} + 2 \text{MUG} + (\text{PLATE})^2 = 5 \text{MUG} + (\text{PLATE})^2 \in \mathbb{N}_0[Y].$$

Moreover, for $q = \text{redMug} \cdot \text{blueMug}$ we have

$$N(f)(p + q) = N(f)(p) + N(f)(q) = (5 \text{MUG} + (\text{PLATE})^2) + (\text{MUG})^2,$$

$$N(f)(p \cdot q) = N(f)(p) \cdot N(f)(q) = (5 \text{MUG} + (\text{PLATE})^2) \cdot (\text{MUG})^2,$$

showing that N preserves $+$ and $\cdot$ (semiring homomorphism). Thus $(N, \oplus = +, \otimes = \cdot, 0, 1)$ is a Functorial Number that models SKU-variant aggregation consistently across remappings f , as required in retail analytics.

2. Main Results

2.1. MultiRough Number

We generalize the classical rough number by allowing *multiple attributes / scenarios* to contribute simultaneously and by recording their intervals as a *finite multiset* of intervals.

Notation 1 (Interval universe). Let $\mathbb{I} := \{[a, b] \subseteq \mathbb{R} \mid a \leq b\}$ denote the set of all closed real intervals (cf. the notion of Interval Number).

Definition 12 (MultiRough Number (MRN)). Let U be a nonempty finite universe, let $\sim$ be an equivalence relation on U , and let $\underline{G}, \overline{G}$ be the lower/upper approximations of a subset $G \subseteq U$ with respect to $\sim$. Fix a nonempty finite index set A (attributes / scenarios) and for each $a \in A$ a numeric attribute $v_a : U \rightarrow \mathbb{R}$. For each $G \subseteq U$ define

$$L_a(G) := \frac{1}{|\underline{G}|} \sum_{x \in \underline{G}} v_a(x), \quad U_a(G) := \frac{1}{|\overline{G}|} \sum_{x \in \overline{G}} v_a(x),$$

and the associated rough interval $I_a(G) := [L_a(G), U_a(G)] \in \mathbb{I}$. The MultiRough Number of G (with respect to A and $\{v_a\}$) is the finite multiset

$$\text{MRN}_A(G) \in \mathcal{M}(\mathbb{I}), \quad \text{MRN}_A(G) := \{I_a(G) : a \in A\},$$

where multiplicities may optionally encode weights/frequencies of scenarios.

Example 8 (MultiRough Number: Produce quality across crates and attributes). *Setting.* Let the universe be six oranges $U = \{o_1, \dots, o_6\}$ packed in three crates

$$B_1 = \{o_1, o_2\}, \quad B_2 = \{o_3, o_4\}, \quad B_3 = \{o_5, o_6\},$$

and let $\sim$ be the equivalence relation “in the same crate” (so $[o_i]_{\sim} = B_j$). Consider the target set $G := B_1 \cup \{o_3\} = \{o_1, o_2, o_3\}$. Then

$$\underline{G} = \{x \in U : [x]_{\sim} \subseteq G\} = B_1 = \{o_1, o_2\}, \quad \overline{G} = \{x \in U : [x]_{\sim} \cap G \neq \emptyset\} = B_1 \cup B_2.$$

Attributes. Take two numeric attributes $A = \{\text{sugar}, \text{firm}\}$ with readings

$$v_{\text{sugar}}(o_1, \dots, o_6) = (11.0, 12.0, 13.5, 14.0, 10.0, 10.5),$$

$$v_{\text{firm}}(o_1, \dots, o_6) = (6.0, 6.5, 7.0, 7.5, 5.5, 5.0).$$

MRN computation (Def. 12). For each $a \in A$,

$$L_a(G) = \frac{1}{|\underline{G}|} \sum_{x \in \underline{G}} v_a(x), \quad U_a(G) = \frac{1}{|\overline{G}|} \sum_{x \in \overline{G}} v_a(x), \quad I_a(G) = [L_a(G), U_a(G)].$$

Sugar:

$$L_{\text{sugar}}(G) = \frac{11.0+12.0}{2} = 11.5, \quad U_{\text{sugar}}(G) = \frac{11.0+12.0+13.5+14.0}{4} = \frac{50.5}{4} = 12.625,$$

so $I_{\text{sugar}}(G) = [11.5, 12.625]$. Firmness:

$$L_{\text{firm}}(G) = \frac{6.0+6.5}{2} = 6.25, \quad U_{\text{firm}}(G) = \frac{6.0+6.5+7.0+7.5}{4} = \frac{27}{4} = 6.75,$$

so $I_{\text{firm}}(G) = [6.25, 6.75]$.

MultiRough Number. The MultiRough Number of G under the two attributes is the finite multiset

$$\text{MRN}_A(G) = \{I_{\text{sugar}}(G), I_{\text{firm}}(G)\} = \{[11.5, 12.625], [6.25, 6.75]\} \in \mathcal{M}(\mathbb{I}),$$

which jointly summarizes crate-induced imprecision for both quality dimensions.

Example 9 (Customer Credit Assessment as a MultiRough Number). A bank evaluates a small business customer G using three partially observed indicators: monthly turnover (T), inventory value (I), and delayed payments (D). Because data come from different reporting systems, each indicator can only be approximated by a rough interval, i.e. by lower/upper bounds induced by an indiscernibility relation (same filing period, same accountant, same tax form).

$$\underline{T}(G) = [1.8, 2.0] \times 10^4 \text{ USD}, \quad \overline{T}(G) = [1.8, 2.5] \times 10^4 \text{ USD},$$

$$\underline{I}(G) = [4.5, 5.0] \times 10^3 \text{ USD}, \quad \overline{I}(G) = [4.5, 6.2] \times 10^3 \text{ USD},$$

$$\underline{D}(G) = [0, 2] \text{ days}, \quad \overline{D}(G) = [0, 7] \text{ days}.$$

Define the (single-level) MultiRough Number of G as

$$\text{MRN}(G) = \{(\text{Turnover}, [\underline{T}(G), \overline{T}(G)]), (\text{Inventory}, [\underline{I}(G), \overline{I}(G)]), (\text{Delay}, [\underline{D}(G), \overline{D}(G)])\} \in \mathcal{M}^1(\mathbb{I}).$$

It collects several rough, approximation-based numerical descriptors of the same real entity G , each coming from incomplete/indiscernible information.

Theorem 1 (MRN is representable in a MultiStructure). There exists a MultiStructure $\mathcal{MS} = (H, \{\#^{(m)}\})$ (Definition 3) that represents MRN_A in the sense that for every $G \subseteq U$,

$$\exists \#_G^{(1)} : H \rightarrow \mathcal{M}(H) \text{ with } \#_G^{(1)}(\star) = \text{MRN}_A(G) \text{ (as a multiset of intervals)}.$$

Proof. Let the carrier be $H := \mathbb{I} \cup \{\star\}$, where $\star$ is a distinguished token. For each $G \subseteq U$, define a unary multi-operation

$$\#_G^{(1)} : H \rightarrow \mathcal{M}(H), \quad \#_G^{(1)}(\star) := \text{MRN}_A(G), \quad \#_G^{(1)}(h) := \emptyset \quad (h \in \mathbb{I}).$$

By construction $\#_G^{(1)}(h) \in \mathcal{M}(H)$ for all $h \in H$ and each element of $\#_G^{(1)}(\star)$ lies in $\mathbb{I} \subseteq H$, so types match. Hence $\mathcal{MS} = (H, \{\#_G^{(1)}\}_{G \subseteq U})$ is a valid MultiStructure that realizes $\text{MRN}_A(G)$ as the image of $\star$ under $\#_G^{(1)}$. $\square$

Theorem 2 (MRN generalizes the classical Rough Number). *Let $A = \{a_0\}$ be a singleton and let $v_{a_0} = v$ be the (classical) attribute used to define the rough number $\text{RN}_v(G) := [L_v(G), U_v(G)]$. Then*

$$\text{MRN}_{\{a_0\}}(G) = \{ [L_v(G), U_v(G)] \} \in \mathcal{M}(\mathbb{I}),$$

and the selection map $\text{Sel} : \mathcal{M}(\mathbb{I}) \rightarrow \mathbb{I}$ that returns the unique element of a singleton multiset satisfies

$$\text{Sel}(\text{MRN}_{\{a_0\}}(G)) = \text{RN}_v(G).$$

Proof. For $A = \{a_0\}$ we have, by Definition 12,

$$\text{MRN}_{\{a_0\}}(G) = \{I_{a_0}(G)\} = \{ [L_{a_0}(G), U_{a_0}(G)] \} = \{ [L_v(G), U_v(G)] \}.$$

Applying Sel to this singleton multiset yields the unique element $[L_v(G), U_v(G)]$, which is exactly $\text{RN}_v(G)$. $\square$

2.2. Iterative MultiRough Number

We now lift MultiRough Numbers through multiple multiset layers, obtaining elements of $\mathcal{M}^k(\mathbb{I})$ that capture hierarchical scenario groupings.

Definition 13 (Iterative MultiRough Number (IMRN)). *Within the setting of Definition 12, fix an integer $k \geq 1$. For each level $i = 1, \dots, k$, let A_i be a nonempty finite index set and let*

$$F_1(G, a_1) \in \mathbb{I} \quad (a_1 \in A_1), \quad F_i(G, a_i) \in \mathcal{M}^{i-1}(\mathbb{I}) \quad (i \geq 2, a_i \in A_i)$$

be given level- i building blocks. Define inductively the level- i iterative MultiRough Numbers by

$$\text{IMRN}^{(1)}(G) := \{ F_1(G, a_1) : a_1 \in A_1 \} \in \mathcal{M}^1(\mathbb{I}),$$

$$\text{IMRN}^{(i)}(G) := \{ F_i(G, a_i) : a_i \in A_i \} \in \mathcal{M}^i(\mathbb{I}) \quad (i = 2, \dots, k).$$

We call $\text{IMRN}^{(k)}(G) \in \mathcal{M}^k(\mathbb{I})$ the (order- k) Iterative MultiRough Number of G .

Example 10 (Iterative MultiRough Number: Month-over-month portfolios of attribute intervals). *Level structure.* Keep the same crates, relation $\sim$, and target set G as in Example 8. Form two time scenarios $T = \{M1, M2\}$ (two months). Level 1 elements are interval multisets (MultiRough Numbers per month); level 2 elements are multisets of those multisets (a portfolio across months).

Month M1 (repeat of Example 8).

$$\text{IMRN}_{M1}^{(1)}(G) = \{ [11.5, 12.625], [6.25, 6.75] \} \in \mathcal{M}^1(\mathbb{I}).$$

Month M2 (updated readings). New attribute values:

$$\begin{aligned} v_{sugar}^{M2}(o_1, \dots, o_6) &= (12.0, 12.5, 13.0, 13.5, 10.5, 11.0), \\ v_{firm}^{M2}(o_1, \dots, o_6) &= (6.2, 6.4, 6.8, 7.2, 5.6, 5.4). \end{aligned}$$

Then

$$\begin{aligned} I_{sugar}^{M2}(G) &= \frac{12.0+12.5}{2} = 12.25, & U_{sugar}^{M2}(G) &= \frac{12.0+12.5+13.0+13.5}{4} = \frac{51.0}{4} = 12.75, \\ I_{firm}^{M2}(G) &= \frac{6.2+6.4}{2} = 6.30, & U_{firm}^{M2}(G) &= \frac{6.2+6.4+6.8+7.2}{4} = \frac{26.6}{4} = 6.65, \end{aligned}$$

hence

$$\text{IMRN}_{M2}^{(1)}(G) = \{ [12.25, 12.75], [6.30, 6.65] \} \in \mathcal{M}^1(\mathbb{I}).$$

Order-2 Iterative MultiRough Number (Def. 13). Collecting the two months as a multiset of level-1 elements yields

$$\text{IMRN}^{(2)}(G) = \{ \text{IMRN}_{M1}^{(1)}(G), \text{IMRN}_{M2}^{(1)}(G) \} \in \mathcal{M}^2(\mathbb{I}),$$

i.e., a portfolio of attribute-interval summaries across months, suitable for planning under seasonal variation.

Example 11 (Multi-Agency Poverty Scoring as an Iterative MultiRough Number). Suppose a municipality wants to estimate a poverty-support score for a household G , but data are stored in three agencies: social services (A_1), health insurance (A_2), and education support (A_3). Each agency provides only rough numerical evidence because of missing or coarsened attributes (e.g. income bracket instead of exact income, last-year hospitalization instead of current health, school-fee waiver instead of exact fee).

Level 1 (agency level): for each agency $a_1 \in A_1$ define

$$F_1(G, a_1) \in \mathbb{I}$$

as the agency's rough interval for the household's support-need score, obtained from its own indiscernibility classes (same district, same family size).

Level 2 (municipal consolidation): the municipality groups the level-1 rough scores from all agencies into a set

$$F_2(G, a_2) = \{ F_1(G, a_1) : a_1 \in A_1 \} \in \mathcal{M}^1(\mathbb{I}) \quad (a_2 \in A_2),$$

where each a_2 may correspond to a municipal department (welfare, budget, audit) that needs to see the whole family of agency-rough scores.

Level 3 (regional or national roll-up): the central authority collects all municipal consolidations for auditing or prioritization,

$$F_3(G, a_3) = \{ F_2(G, a_2) : a_2 \in A_2 \} \in \mathcal{M}^2(\mathbb{I}) \quad (a_3 \in A_3).$$

By Definition 13,

$$\text{IMRN}^{(1)}(G) = \{ F_1(G, a_1) : a_1 \in A_1 \}, \quad \text{IMRN}^{(2)}(G) = \{ F_2(G, a_2) : a_2 \in A_2 \}, \quad \text{IMRN}^{(3)}(G) = \{ F_3(G, a_3) : a_3 \in A_3 \}.$$

Thus the order-3 Iterative MultiRough Number $\text{IMRN}^{(3)}(G) \in \mathcal{M}^3(\mathbb{I})$ stores, in a single nested object, (i) rough numeric evaluations from heterogeneous sources, (ii) their municipal-level groupings, and (iii) their upper-level aggregations, while preserving the rough (lower/upper) nature of every estimate.

Theorem 3 (IMRN is representable in an Iterative Multi-Structure). There exists an Iterative Multi-Structure of order k (Definition 4)

$$\mathcal{IMS}^{(k)} = \left(H, \{ \#^{(1,i)} : (\mathcal{M}^i(H)) \rightarrow \mathcal{M}^{i+1}(H) \}_{i=0}^{k-1} \right),$$

with carrier $H = \mathbb{I} \cup \{\star\}$, such that for each $G \subseteq U$ there are level-wise unary multi-operations $\#_G^{(1,i)}$ satisfying

$$\#_G^{(1,0)}(\star) = \text{IMRN}^{(1)}(G), \quad \#_G^{(1,1)}(\{\star\}) = \text{IMRN}^{(2)}(G), \quad \dots, \quad \#_G^{(1,k-1)}(\underbrace{\{\dots \{\star\} \dots\}}_{k \text{ times}}) = \text{IMRN}^{(k)}(G).$$

Proof. Let $H := \mathbb{I} \cup \{\star\}$. Define a seed at each level by $s^{(0)} := \star \in H$ and $s^{(i+1)} := \{s^{(i)}\} \in \mathcal{M}^{i+1}(H)$ (a singleton multiset). For each $G \subseteq U$ and level $i = 0, \dots, k-1$, define the unary multi-operation

$$\#_G^{(1,i)} : \mathcal{M}^i(H) \longrightarrow \mathcal{M}^{i+1}(H), \quad \#_G^{(1,i)}(s^{(i)}) := \text{IMRN}^{(i+1)}(G), \quad \#_G^{(1,i)}(x) := \emptyset \quad (x \neq s^{(i)}).$$

By Definition 13, $\text{IMRN}^{(i+1)}(G) \in \mathcal{M}^{i+1}(\mathbb{I}) \subseteq \mathcal{M}^{i+1}(H)$, so the codomains are correct. Therefore $\mathcal{IMS}^{(k)}$ is an Iterative Multi-Structure realizing the stated identities at each level. $\square$

Theorem 4 (IMRN generalizes MRN). *For $k = 1$ and any choice of A_1 and $F_1(G, a_1) = [L_{a_1}(G), U_{a_1}(G)]$ as in Definition 12, one has*

$$\text{IMRN}^{(1)}(G) = \text{MRN}_{A_1}(G) \in \mathcal{M}^1(\mathbb{I}).$$

More generally, for any $k \geq 2$, if each $F_i(G, a_i)$ is a singleton multiset lift of $\text{IMRN}^{(i-1)}(G)$, then the levelwise unnesting map $\text{Unnest}_{k-1} : \mathcal{M}^k(\mathbb{I}) \rightarrow \mathcal{M}^1(\mathbb{I})$ recovers $\text{MRN}_{A_1}(G)$:

$$\text{Unnest}^{k-1}(\text{IMRN}^{(k)}(G)) = \text{IMRN}^{(1)}(G) = \text{MRN}_{A_1}(G).$$

Proof. The case $k = 1$ is immediate from the definitions:

$$\text{IMRN}^{(1)}(G) = \{F_1(G, a_1) : a_1 \in A_1\} = \{[L_{a_1}(G), U_{a_1}(G)] : a_1 \in A_1\} = \text{MRN}_{A_1}(G).$$

For $k \geq 2$, assume each $F_i(G, a_i)$ is a singleton multiset containing an element of $\mathcal{M}^{i-1}(\mathbb{I})$; then $\text{IMRN}^{(i)}(G)$ is a multiset of singletons. Applying Unnest removes one level of singleton braces at a time. After $k-1$ steps we obtain $\text{IMRN}^{(1)}(G)$, which equals $\text{MRN}_{A_1}(G)$ by the $k = 1$ case. $\square$

2.3. MultiGrey Number

We generalize the classical Grey Number (a single interval $[g^-, g^+]$) by allowing *multiple* scenario/attribute intervals to be recorded simultaneously as a *finite multiset* of intervals.

Definition 14 (MultiGrey Number (MGN)). *Let $\mathbb{I} := \{[a, b] \subseteq \mathbb{R} \mid a \leq b\}$ be the set of closed real intervals. Let A be a nonempty finite index set (scenarios/attributes). For each $a \in A$, let the associated Grey Number be*

$$g_a^\pm := [g_a^-, g_a^+] \in \mathbb{I} \quad \text{with } g_a^- \leq g_a^+.$$

Optionally, let $w : A \rightarrow \mathbb{N}$ encode multiplicities (frequencies/weights). The MultiGrey Number of A is the finite multiset

$$\text{MGN}(A) \in \mathcal{M}(\mathbb{I}), \quad \text{MGN}(A) := \{(g_a^\pm)^{w(a)} : a \in A\}.$$

If $w \equiv 1$, then $\text{MGN}(A) = \{g_a^\pm : a \in A\}$ (all multiplicities 1).

Example 12 (MultiGrey Number: Competing electricity quotes for next month). *Let $\mathbb{I} = \{[a, b] \subseteq \mathbb{R} \mid a \leq b\}$ be the set of closed real intervals. A company solicits per-kWh unit-price quotes for next month from three suppliers. Each quote is a grey interval capturing the contractually guaranteed bounds:*

$$q_1 = [24.0, 26.5], \quad q_2 = [23.8, 26.2], \quad q_3 = [25.0, 27.0] \quad (\text{JPY/kWh}).$$

The MultiGrey Number (MGN) collecting all scenarios is the finite multiset

$$\text{MGN} = \{q_1, q_2, q_3\} \in \mathcal{M}(\mathbb{I}).$$

For budgeting, one may summarize each interval by midpoint and half-width:

$$\begin{aligned} m(q_1) &= \frac{24.0+26.5}{2} = 25.25, & w(q_1) &= \frac{26.5-24.0}{2} = 1.25, \\ m(q_2) &= \frac{23.8+26.2}{2} = 25.00, & w(q_2) &= \frac{26.2-23.8}{2} = 1.20, \\ m(q_3) &= \frac{25.0+27.0}{2} = 26.00, & w(q_3) &= \frac{27.0-25.0}{2} = 1.00. \end{aligned}$$

Thus MGN simultaneously records all supplier uncertainties as a multiset of grey numbers.

Example 13 (Maintenance Cost Planning as a MultiGrey Number). A factory wants to estimate the quarterly maintenance cost of a critical machine, but only incomplete historical data are available. For each month $m \in \{\text{Jan}, \text{Feb}, \text{Mar}\}$ the technician can only bound the cost by an interval, i.e. a grey number in the interval space $\mathbb{I}$:

$$C_{\text{Jan}} = [480, 520] \text{ USD}, \quad C_{\text{Feb}} = [450, 560] \text{ USD}, \quad C_{\text{Mar}} = [500, 600] \text{ USD}.$$

Define the multi-grey number

$$G = \{(\text{Jan}, C_{\text{Jan}}), (\text{Feb}, C_{\text{Feb}}), (\text{Mar}, C_{\text{Mar}})\} \in \mathcal{M}^1(\mathbb{I}).$$

Here each component is a grey number (interval) reflecting data incompleteness (missing bills, inflation, unrecorded parts). Decision makers can aggregate G to obtain a quarterly grey cost

$$C_{Q1} = C_{\text{Jan}} \oplus C_{\text{Feb}} \oplus C_{\text{Mar}} = [1430, 1680] \text{ USD},$$

where $\oplus$ is interval addition. Thus G is a real-life MultiGrey Number: one object collecting several time-stamped interval estimates.

Theorem 5 (MGN is representable in a MultiStructure). There exists a MultiStructure $\mathcal{MS} = (H, \{\#^{(m)}\})$ that represents $\text{MGN}(A)$: taking $H := \mathbb{I} \cup \{\star\}$, there is a unary multi-operation

$$\#_A^{(1)} : H \longrightarrow \mathcal{M}(H) \quad \text{such that} \quad \#_A^{(1)}(\star) = \text{MGN}(A) \quad (\text{as a multiset of intervals}).$$

Proof. Define $\#_A^{(1)}$ by

$$\#_A^{(1)}(\star) := \text{MGN}(A) \in \mathcal{M}(\mathbb{I}) \subseteq \mathcal{M}(H), \quad \#_A^{(1)}(h) := \emptyset \quad (h \in \mathbb{I}).$$

Then $\#_A^{(1)}$ maps H into $\mathcal{M}(H)$, and its value at $\star$ is exactly the desired multiset of intervals. Hence $\mathcal{MS}$ represents $\text{MGN}(A)$. $\square$

Theorem 6 (MGN generalizes the Grey Number). If $A = \{a_0\}$ is a singleton and $w(a_0) = 1$, then

$$\text{MGN}(A) = \{g_{a_0}^\pm\} \in \mathcal{M}(\mathbb{I}).$$

Let $\text{Sel} : \mathcal{M}(\mathbb{I}) \rightarrow \mathbb{I}$ be the selection map that returns the unique element of a singleton multiset. Then

$$\text{Sel}(\text{MGN}(\{a_0\})) = g_{a_0}^\pm,$$

i.e. the classical Grey Number is recovered.

Proof. For $A = \{a_0\}$ and $w(a_0) = 1$, Definition 14 gives $\text{MGN}(A) = \{g_{a_0}^\pm\}$. By definition of Sel on singletons, $\text{Sel}(\{g_{a_0}^\pm\}) = g_{a_0}^\pm$. $\square$

2.4. Iterative MultiGrey Number

We now lift MultiGrey Numbers through multiple *multiset levels*, yielding elements of $\mathcal{M}^k(\mathbb{I})$ that encode hierarchical groupings (e.g., portfolios of scenario sets).

Definition 15 (Iterative MultiGrey Number (IMGN)). *Fix an integer $k \geq 1$. For each level $i = 1, \dots, k$ let A_i be a nonempty finite index set and let*

$$F_1(a_1) \in \mathbb{I} \quad (a_1 \in A_1), \quad F_i(a_i) \in \mathcal{M}^{i-1}(\mathbb{I}) \quad (i \geq 2, a_i \in A_i)$$

be level- i building blocks. Define inductively

$$\text{IMGN}^{(1)} := \{ F_1(a_1) : a_1 \in A_1 \} \in \mathcal{M}^1(\mathbb{I}),$$

$$\text{IMGN}^{(i)} := \{ F_i(a_i) : a_i \in A_i \} \in \mathcal{M}^i(\mathbb{I}) \quad (i = 2, \dots, k).$$

We call $\text{IMGN}^{(k)} \in \mathcal{M}^k(\mathbb{I})$ the (order- k) Iterative MultiGrey Number.

Example 14 (Iterative MultiGrey Number: Week-by-week portfolios of supplier intervals). *Using the setting of Example 12, suppose quotes are refreshed weekly. For Week 1 we have*

$$\text{MGN}_{W1} = \{ [24.0, 26.5], [23.8, 26.2], [25.0, 27.0] \} \in \mathcal{M}^1(\mathbb{I}).$$

In Week 2, market conditions shift, giving

$$\text{MGN}_{W2} = \{ [24.5, 26.8], [24.2, 26.4], [25.5, 27.3] \} \in \mathcal{M}^1(\mathbb{I}).$$

Midpoints and half-widths for Week 2 are

$$[24.5, 26.8] : m = 25.65, w = 1.15; \quad [24.2, 26.4] : m = 25.30, w = 1.10; \quad [25.5, 27.3] : m = 26.40, w = 0.90.$$

The order-2 Iterative MultiGrey Number aggregates weeks as a multiset of multisets:

$$\text{IMGN}^{(2)} = \{ \text{MGN}_{W1}, \text{MGN}_{W2} \} \in \mathcal{M}^2(\mathbb{I}),$$

i.e., a portfolio of week-specific supplier uncertainty sets suitable for planning across short-term horizons.

Example 15 (Citywide Water-Supply Uncertainty as an Iterative MultiGrey Number). *Consider a city with three administrative levels:*

- *Level 1 (households): each household $h \in A_1$ reports its daily water consumption as an interval $F_1(h) \in \mathbb{I}$ because meters are read only roughly, e.g. $F_1(h) = [180, 210]$ liters.*
- *Level 2 (districts): each district $d \in A_2$ stores the set of its households' grey consumptions,*

$$F_2(d) = \{ F_1(h) : h \text{ is in district } d \} \in \mathcal{M}^1(\mathbb{I}).$$

- *Level 3 (city): the central office stores the collection of all district-level grey sets,*

$$F_3(c) = \{ F_2(d) : d \text{ is a district in the city } c \} \in \mathcal{M}^2(\mathbb{I}).$$

By Definition 15,

$$\text{IMGN}^{(1)} = \{ F_1(h) : h \in A_1 \}, \quad \text{IMGN}^{(2)} = \{ F_2(d) : d \in A_2 \}, \quad \text{IMGN}^{(3)} = \{ F_3(c) : c \in A_3 \}.$$

The order-3 object $\text{IMGN}^{(3)}$ is an Iterative MultiGrey Number: level 1 captures household-level imprecision (coarse readings), level 2 groups those grey numbers by district, and level 3 groups district-level grey sets for

the whole city. Planners can roll up to obtain a citywide grey demand interval for a day, or drill down to locate districts with the largest uncertainty, all within the same nested grey structure.

Remark 1. Taking $F_1(a_1) = g_{a_1}^\pm$ and $F_i(a_i)$ as chosen multisets of level- $(i-1)$ objects (e.g. singleton lifts of $\text{IMGN}^{(i-1)}$) recovers the standard “nesting by singletons” used in hierarchical multiset constructions.

Theorem 7 (IMGN is representable in an Iterative Multi-Structure). *There exists an Iterative Multi-Structure of order k (with carrier $H := \mathbb{I} \cup \{\star\}$)*

$$\mathcal{IMS}^{(k)} = \left(H, \{ \#^{(1,i)} : \mathcal{M}^i(H) \rightarrow \mathcal{M}^{i+1}(H) \}_{i=0}^{k-1} \right)$$

and seeds $s^{(0)} := \star, s^{(i+1)} := \{s^{(i)}\}$ such that

$$\#^{(1,0)}(s^{(0)}) = \text{IMGN}^{(1)}, \quad \#^{(1,1)}(s^{(1)}) = \text{IMGN}^{(2)}, \quad \dots, \quad \#^{(1,k-1)}(s^{(k-1)}) = \text{IMGN}^{(k)}.$$

Proof. For each level $i = 0, \dots, k-1$, define the unary multi-operation

$$\#^{(1,i)} : \mathcal{M}^i(H) \rightarrow \mathcal{M}^{i+1}(H), \quad \#^{(1,i)}(s^{(i)}) := \text{IMGN}^{(i+1)}, \quad \#^{(1,i)}(x) := \emptyset \quad (x \neq s^{(i)}).$$

By Definition 15, $\text{IMGN}^{(i+1)} \in \mathcal{M}^{i+1}(\mathbb{I}) \subseteq \mathcal{M}^{i+1}(H)$, so types match and the identities hold. $\square$

Theorem 8 (IMGN generalizes MGN). *If $k = 1$ and $F_1(a_1) = g_{a_1}^\pm$ for $a_1 \in A_1$, then*

$$\text{IMGN}^{(1)} = \{ g_{a_1}^\pm : a_1 \in A_1 \} = \text{MGN}(A_1).$$

More generally, if for $i \geq 2$ each $F_i(a_i)$ is a singleton lift containing an element of $\mathcal{M}^{i-1}(\mathbb{I})$, then the levelwise unnesting map $\text{Unnest}^{k-1} : \mathcal{M}^k(\mathbb{I}) \rightarrow \mathcal{M}^1(\mathbb{I})$ satisfies

$$\text{Unnest}^{k-1}(\text{IMGN}^{(k)}) = \text{IMGN}^{(1)} = \text{MGN}(A_1).$$

Proof. The case $k = 1$ follows immediately from the definitions. For the general case, if each $F_i(a_i)$ is a singleton multiset, then $\text{IMGN}^{(i)}$ is a multiset of singletons at every level $i \geq 2$. Applying Unnest removes one singleton layer at a time; after $k-1$ steps we obtain $\text{IMGN}^{(1)}$, which equals $\text{MGN}(A_1)$. $\square$

2.5. MultiGranular Numbers

We generalize the classical *Granular Number* $G(c, r)$ (center $c \in \mathbb{R}$, radius $r \in \mathbb{R}_{\geq 0}$) by allowing multiple granules to be recorded simultaneously as a *finite multiset* of granular numbers.

Notation 2 (Space of granular numbers). Let $\mathbb{G} := \{ G(c, r) \mid c \in \mathbb{R}, r \in \mathbb{R}_{\geq 0} \}$ denote the set of all granular numbers (cf. the preliminaries on Granular Numbers). When needed, we identify $G(c, r)$ with the closed ball $[c - r, c + r] \subset \mathbb{R}$.

Definition 16 (MultiGranular Number (MGN)). Let A be a nonempty finite index set (scenarios/attributes). For each $a \in A$, let $X_a := G(c_a, r_a) \in \mathbb{G}$ be a granular number and let $w : A \rightarrow \mathbb{N}$ encode multiplicities (frequencies/weights). The MultiGranular Number determined by $(A, \{X_a\}, w)$ is the finite multiset

$$\text{MGN}(A) \in \mathcal{M}(\mathbb{G}), \quad \text{MGN}(A) := \{ (X_a)^{w(a)} : a \in A \}.$$

If $w \equiv 1$, then $\text{MGN}(A) = \{ X_a : a \in A \}$.

Example 16 (MultiGranular Number: Shared office HVAC comfort from multiple occupants). Consider three occupants sharing an office. Each provides a granular number (i.e., a connected comfort band with weight)

of the form $X_i = G_{A_i}(c_i, r_i)$ where c_i is the preferred center temperature ($^{\circ}\text{C}$), r_i the half-width (tolerance), and $A_i \in [0, 1]$ the priority/confidence weight:

$$X_1 = G_{0.9}(22.0, 1.0), \quad X_2 = G_{0.7}(21.5, 1.2), \quad X_3 = G_{0.6}(23.0, 0.8).$$

Their acceptable intervals are

$$I_1 = [21.0, 23.0], \quad I_2 = [20.3, 22.7], \quad I_3 = [22.2, 23.8].$$

The MultiGranular Number is the finite multiset

$$\text{MGN} = \{X_1, X_2, X_3\} \quad (\text{a multiset of granular numbers}).$$

Two useful summaries.

- (a) **Consensus intersection (hard agreement).** $I_{\cap} = \bigcap_{i=1}^3 I_i = [\max\{21.0, 20.3, 22.2\}, \min\{23.0, 22.7, 23.8\}] = [22.2, 22.7]$. Thus a strict common band is $[22.2, 22.7]$, with midpoint 22.45 and half-width 0.25.
- (b) **Weighted consensus granule (soft aggregation).** Let $W = \sum_i A_i = 0.9 + 0.7 + 0.6 = 2.2$. Define

$$c^* = \frac{\sum_i A_i c_i}{W} = \frac{0.9 \cdot 22.0 + 0.7 \cdot 21.5 + 0.6 \cdot 23.0}{2.2} = \frac{48.65}{2.2} \approx 22.1136,$$

$$r^* = \frac{\sum_i A_i r_i}{W} = \frac{0.9 \cdot 1.0 + 0.7 \cdot 1.2 + 0.6 \cdot 0.8}{2.2} = \frac{2.22}{2.2} \approx 1.0091.$$

A soft (weighted) consensus is then $G_1(c^*, r^*) \approx G_1(22.1136, 1.0091)$.

Both summaries are derived from the same MGN, which records all occupant bands simultaneously.

Example 17 (Utility Billing as a MultiGranular Number). Consider a household that receives resource-consumption data at three natural granularities:

- Hourly electricity usage (in kWh),
- Daily water usage (in liters),
- Monthly gas usage (in m^3).

Let the base granular domain be

$$\mathbb{G} = \{\text{kWh, liter, m}^3\}.$$

Define a (single-step) multigranular number

$$M = \{(\text{hour, 1.8 kWh}), (\text{day, 420 liters}), (\text{month, 18 m}^3)\}.$$

Here, each component carries a value together with its own granularity tag (hour/day/month). The consumer, the billing system, or a decision-support tool can select or aggregate values at the granularity that matches the current task (short-term control uses the hourly electricity, anomaly detection uses the daily water, budgeting uses the monthly gas). Thus M is a real-life instance of a MultiGranular Number: one object, several coexisting, non-conflicting granular values.

Theorem 9 (MGN is representable in a MultiStructure). There exists a MultiStructure $\mathcal{MS} = (H, \{\#^{(m)}\})$ (Definition 3) that represents $\text{MGN}(A)$: taking $H := \mathbb{G} \cup \{\star\}$, there is a unary multi-operation

$$\#_A^{(1)} : H \longrightarrow \mathcal{M}(H) \quad \text{such that} \quad \#_A^{(1)}(\star) = \text{MGN}(A) \quad (\text{as a multiset of granular numbers}).$$

Proof. Define $\#_A^{(1)}$ by

$$\#_A^{(1)}(\star) := \text{MGN}(A) \in \mathcal{M}(\mathbb{G}) \subseteq \mathcal{M}(H), \quad \#_A^{(1)}(h) := \emptyset \quad (h \in \mathbb{G}).$$

Then $\#_A^{(1)} : H \rightarrow \mathcal{M}(H)$ is a valid unary multi-operation and its value at $\star$ is exactly $\text{MGN}(A)$. Hence $\mathcal{MS}$ represents $\text{MGN}(A)$. $\square$

Theorem 10 (MGN generalizes the classical Granular Number). *If $A = \{a_0\}$ and $w(a_0) = 1$, then*

$$\text{MGN}(A) = \{X_{a_0}\} \in \mathcal{M}(\mathbb{G}).$$

Let $\text{Sel} : \mathcal{M}(\mathbb{G}) \rightarrow \mathbb{G}$ return the unique element of a singleton multiset. Then

$$\text{Sel}(\text{MGN}(\{a_0\})) = X_{a_0} = G(c_{a_0}, r_{a_0}),$$

i.e., the classical Granular Number is recovered.

Proof. Immediate from Definition 16 and the definition of Sel on singletons. $\square$

2.6. Iterative MultiGranular Numbers

We now lift MultiGranular Numbers through multiple *multiset levels*, obtaining elements of $\mathcal{M}^k(\mathbb{G})$ that encode hierarchical groupings (e.g., portfolios of granule sets across organizations, time windows, or experts).

Definition 17 (Iterative MultiGranular Number (IMGN)). *Fix an integer $k \geq 1$. For each level $i = 1, \dots, k$ let A_i be a nonempty finite index set and let*

$$F_1(a_1) \in \mathbb{G} \quad (a_1 \in A_1), \quad F_i(a_i) \in \mathcal{M}^{i-1}(\mathbb{G}) \quad (i \geq 2, a_i \in A_i)$$

be level- i building blocks. Define inductively

$$\text{IMGN}^{(1)} := \{F_1(a_1) : a_1 \in A_1\} \in \mathcal{M}^1(\mathbb{G}),$$

$$\text{IMGN}^{(i)} := \{F_i(a_i) : a_i \in A_i\} \in \mathcal{M}^i(\mathbb{G}) \quad (i = 2, \dots, k).$$

We call $\text{IMGN}^{(k)} \in \mathcal{M}^k(\mathbb{G})$ the (order- k) Iterative MultiGranular Number.

Example 18 (Iterative MultiGranular Number: Zone portfolios in a building). *A building has two zones, EAST and WEST, each with its own MultiGranular set of occupant comfort bands.*

EAST zone.

$$\text{MGN}_E = \{G_{0.9}(22.0, 1.0), G_{0.7}(21.5, 1.2), G_{0.6}(23.0, 0.8)\}.$$

Hard intersection: $I_\cap^E = [22.2, 22.7]$ (as in Example 16). Weighted consensus:

$$c_E^* = \frac{48.65}{2.2} \approx 22.1136, \quad r_E^* = \frac{2.22}{2.2} \approx 1.0091.$$

WEST zone.

$$\text{MGN}_W = \{G_{0.8}(24.0, 0.7), G_{0.6}(23.5, 1.0), G_{0.5}(22.8, 0.9)\}.$$

Intervals: $[23.3, 24.7], [22.5, 24.5], [21.9, 23.7] \Rightarrow I_\cap^W = [\max\{23.3, 22.5, 21.9\}, \min\{24.7, 24.5, 23.7\}] = [23.3, 23.7]$. Weighted consensus with $W = 0.8 + 0.6 + 0.5 = 1.9$:

$$c_W^* = \frac{0.8 \cdot 24.0 + 0.6 \cdot 23.5 + 0.5 \cdot 22.8}{1.9} = \frac{44.7}{1.9} \approx 23.5263,$$

$$r_W^* = \frac{0.8 \cdot 0.7 + 0.6 \cdot 1.0 + 0.5 \cdot 0.9}{1.9} = \frac{1.61}{1.9} \approx 0.8474.$$

Order-2 Iterative MultiGranular Number. Collect zone-level MultiGranular sets into a multiset of multisets:

$$\text{IMGN}^{(2)} = \{ \text{MGN}_E, \text{MGN}_W \},$$

which lives at level 2 (a portfolio across zones). Zone-specific hard/soft summaries $(I_\cap^E, G_1(c_E^*, r_E^*))$ and $(I_\cap^W, G_1(c_W^*, r_W^*))$ can be used by the building controller, while $\text{IMGN}^{(2)}$ preserves the full, hierarchical granularity of occupant preferences.

Example 19 (Smart-City Traffic Monitoring as an Iterative MultiGranular Number). Let a city collect traffic information in three iterative layers.

- Level 1 (sensor level): for each sensor s on a road segment, store the instantaneous vehicle count per minute, e.g. $F_1(s) \in \mathbb{G} = \{\text{veh/min}\}$.
- Level 2 (segment level): for each road segment r , store the set of its sensor reports from level 1 during the last 15 minutes,

$$F_2(r) = \{ F_1(s) : s \text{ belongs to segment } r \} \in \mathcal{M}^1(\mathbb{G}).$$

- Level 3 (district level): for each district d , store the collection of all segment-level objects inside that district,

$$F_3(d) = \{ F_2(r) : r \text{ is a segment in district } d \} \in \mathcal{M}^2(\mathbb{G}).$$

With A_1 the set of sensors, A_2 the set of road segments, and A_3 the set of districts, the construction in Definition 17 yields

$$\text{IMGN}^{(1)} = \{ F_1(s) : s \in A_1 \}, \quad \text{IMGN}^{(2)} = \{ F_2(r) : r \in A_2 \}, \quad \text{IMGN}^{(3)} = \{ F_3(d) : d \in A_3 \}.$$

The order-3 object $\text{IMGN}^{(3)}$ is an Iterative MultiGranular Number that nests minute-level counts (level 1) inside segment-level aggregations (level 2), which are themselves grouped into district-level traffic states (level 3). A traffic controller can drill down (district $\rightarrow$ segment $\rightarrow$ sensor) or roll up (sensor $\rightarrow$ segment $\rightarrow$ district) without changing the underlying data model.

Theorem 11 (IMGN is representable in an Iterative Multi-Structure). There exists an Iterative Multi-Structure of order k (Definition 4) with carrier $H := \mathbb{G} \cup \{\star\}$,

$$\mathcal{IMS}^{(k)} = \left(H, \{ \#^{(1,i)} : \mathcal{M}^i(H) \rightarrow \mathcal{M}^{i+1}(H) \}_{i=0}^{k-1} \right),$$

and seeds $s^{(0)} := \star, s^{(i+1)} := \{s^{(i)}\}$, such that

$$\#^{(1,0)}(s^{(0)}) = \text{IMGN}^{(1)}, \quad \#^{(1,1)}(s^{(1)}) = \text{IMGN}^{(2)}, \quad \dots, \quad \#^{(1,k-1)}(s^{(k-1)}) = \text{IMGN}^{(k)}.$$

Proof. For each $i = 0, \dots, k-1$, define

$$\#^{(1,i)} : \mathcal{M}^i(H) \rightarrow \mathcal{M}^{i+1}(H), \quad \#^{(1,i)}(s^{(i)}) := \text{IMGN}^{(i+1)}, \quad \#^{(1,i)}(x) := \emptyset \quad (x \neq s^{(i)}).$$

By Definition 17, $\text{IMGN}^{(i+1)} \in \mathcal{M}^{i+1}(\mathbb{G}) \subseteq \mathcal{M}^{i+1}(H)$, so each $\#^{(1,i)}$ is well typed and realizes the stated identities. $\square$

Theorem 12 (IMGN generalizes MGN). For $k = 1$ and $F_1(a_1) = X_{a_1} = G(c_{a_1}, r_{a_1})$, one has

$$\text{IMGN}^{(1)} = \{ X_{a_1} : a_1 \in A_1 \} = \text{MGN}(A_1).$$

More generally, if for $i \geq 2$ each $F_i(a_i)$ is a singleton lift containing an element of $\mathcal{M}^{i-1}(\mathbb{G})$, then the levelwise unnesting map $\text{Unnest}^{k-1} : \mathcal{M}^k(\mathbb{G}) \rightarrow \mathcal{M}^1(\mathbb{G})$ satisfies

$$\text{Unnest}^{k-1}(\text{IMGN}^{(k)}) = \text{IMGN}^{(1)} = \text{MGN}(A_1).$$

Proof. The case $k = 1$ is immediate from the definitions. If each $F_i(a_i)$ is a singleton multiset, then $\text{IMGN}^{(i)}$ is a multiset of singletons for all $i \geq 2$. Each application of Unnest removes one singleton layer, so after $k - 1$ steps we recover $\text{IMGN}^{(1)} = \text{MGN}(A_1)$. $\square$

2.7. MultiInterval Number

We generalize the classical *Interval Number* (a single closed interval $[a_L, a_R] \subseteq \mathbb{R}$) by allowing multiple intervals to be recorded simultaneously as a *finite multiset* of intervals.

Notation 3 (Interval universe). Let $\mathbb{I} := \{[a, b] \subseteq \mathbb{R} \mid a \leq b\}$ be the set of all closed real intervals.

Definition 18 (MultiInterval Number (MIN)). Let A be a nonempty finite index set (scenarios/attributes). For each $a \in A$, let $I_a = [\ell_a, r_a] \in \mathbb{I}$ be an interval and let $w : A \rightarrow \mathbb{N}$ encode multiplicities (frequencies/weights). The MultiInterval Number determined by $(A, \{I_a\}, w)$ is the finite multiset

$$\text{MIN}(A) \in \mathcal{M}(\mathbb{I}), \quad \text{MIN}(A) := \{ (I_a)^{w(a)} : a \in A \}.$$

If $w \equiv 1$, then $\text{MIN}(A) = \{ I_a : a \in A \}$.

Example 20 (MultiInterval Number: Lead-time windows from multiple suppliers). Let $\mathbb{I} = \{[a, b] \subseteq \mathbb{R} \mid a \leq b\}$ be the set of closed intervals. A manufacturer sources a part from three qualified suppliers; each provides a delivery lead-time window (days):

$$I_1 = [7, 10], \quad I_2 = [8, 12], \quad I_3 = [9, 11].$$

The MultiInterval Number (MIN) collecting all scenarios is the finite multiset

$$\text{MIN} = \{ I_1, I_2, I_3 \} \in \mathcal{M}(\mathbb{I}).$$

Useful summaries include:

$$\text{Intersection (common promise)} = I_1 \cap I_2 \cap I_3 = [\max\{7, 8, 9\}, \min\{10, 12, 11\}] = [9, 10],$$

$$\text{Union (overall envelope)} = I_1 \cup I_2 \cup I_3 = [7, 12],$$

and midpoints/half-widths

$$m(I_1) = \frac{7+10}{2} = 8.5, \quad w(I_1) = \frac{10-7}{2} = 1.5; \quad m(I_2) = 10, \quad w(I_2) = 2; \quad m(I_3) = 10, \quad w(I_3) = 1.$$

Thus MIN stores all supplier intervals simultaneously, while $[9, 10]$ gives a conservative shared window.

Example 21 (MultiInterval Number: Room rental price windows from multiple agencies). A tenant is looking for a 1LDK apartment in Tokyo. Three real-estate agencies provide rental price windows (all charges included, in JPY/month) for essentially the same set of properties:

$$I_1 = [125,000, 135,000], \quad I_2 = [128,000, 140,000], \quad I_3 = [130,000, 138,000].$$

The MultiInterval Number collecting all these agency-specific intervals is the finite multiset

$$\text{MIN} = \{ I_1, I_2, I_3 \} \in \mathcal{M}(\mathbb{I}).$$

From this we can derive, for instance, a conservative “guaranteed feasible” rent window as the intersection

$$I_{\cap} = I_1 \cap I_2 \cap I_3 = [\max\{125,000, 128,000, 130,000\}, \min\{135,000, 140,000, 138,000\}] = [130,000, 135,000],$$

and an overall envelope as the union

$$I_{\cup} = I_1 \cup I_2 \cup I_3 = [125,000, 140,000].$$

Thus the tenant keeps, in one object, every agency’s possible rent interval and can pick either a safe band $[130,000, 135,000]$ or the widest band $[125,000, 140,000]$ depending on budget.

Theorem 13 (MIN is representable in a MultiStructure). *There exists a MultiStructure $\mathcal{MS} = (H, \{\#^{(m)}\})$ (Definition 3) that represents $\text{MIN}(A)$: taking $H := \mathbb{I} \cup \{\star\}$, there is a unary multi-operation*

$$\#_A^{(1)} : H \longrightarrow \mathcal{M}(H) \quad \text{such that} \quad \#_A^{(1)}(\star) = \text{MIN}(A) \quad (\text{as a multiset of intervals}).$$

Proof. Define $\#_A^{(1)}$ by

$$\#_A^{(1)}(\star) := \text{MIN}(A) \in \mathcal{M}(\mathbb{I}) \subseteq \mathcal{M}(H), \quad \#_A^{(1)}(h) := \emptyset \quad (h \in \mathbb{I}).$$

Then $\#_A^{(1)} : H \rightarrow \mathcal{M}(H)$ is a valid unary multi-operation and its value at $\star$ is exactly $\text{MIN}(A)$. Hence $\mathcal{MS}$ represents $\text{MIN}(A)$. $\square$

Theorem 14 (MIN generalizes the classical Interval Number). *If $A = \{a_0\}$ and $w(a_0) = 1$, then*

$$\text{MIN}(A) = \{I_{a_0}\} \in \mathcal{M}(\mathbb{I}).$$

Let $\text{Sel} : \mathcal{M}(\mathbb{I}) \rightarrow \mathbb{I}$ return the unique element of a singleton multiset. Then

$$\text{Sel}(\text{MIN}(\{a_0\})) = I_{a_0} = [\ell_{a_0}, r_{a_0}],$$

i.e., the classical Interval Number is recovered.

Proof. Immediate from Definition 18 and the definition of Sel on singletons. $\square$

2.8. Iterative MultiInterval Number

We now lift MultiInterval Numbers through multiple *multiset levels*, obtaining elements of $\mathcal{M}^k(\mathbb{I})$ that encode hierarchical groupings (e.g., portfolios of interval sets across time windows, experts, or subsystems).

Definition 19 (Iterative MultiInterval Number (IMIN)). *Fix an integer $k \geq 1$. For each level $i = 1, \dots, k$ let A_i be a nonempty finite index set and let*

$$F_1(a_1) \in \mathbb{I} \quad (a_1 \in A_1), \quad F_i(a_i) \in \mathcal{M}^{i-1}(\mathbb{I}) \quad (i \geq 2, a_i \in A_i)$$

be level- i building blocks. Define inductively

$$\text{IMIN}^{(1)} := \{F_1(a_1) : a_1 \in A_1\} \in \mathcal{M}^1(\mathbb{I}),$$

$$\text{IMIN}^{(i)} := \{F_i(a_i) : a_i \in A_i\} \in \mathcal{M}^i(\mathbb{I}) \quad (i = 2, \dots, k).$$

We call $\text{IMIN}^{(k)} \in \mathcal{M}^k(\mathbb{I})$ the (order- k) Iterative MultiInterval Number.

Example 22 (Iterative MultiInterval Number: Quarter-by-quarter portfolios of lead times). *Using the same part and suppliers, suppose lead-time windows are updated quarterly due to seasonality.*

Quarter 1.

$$\text{MIN}_{Q1} = \{ [7, 10], [8, 12], [9, 11] \} \in \mathcal{M}^1(\mathbb{I}),$$

$$\text{with } \cap \text{MIN}_{Q1} = [9, 10], \quad \cup \text{MIN}_{Q1} = [7, 12].$$

Quarter 2.

$$\text{MIN}_{Q2} = \{ [6, 9], [8, 13], [10, 12] \} \in \mathcal{M}^1(\mathbb{I}),$$

for which

$$\cap \text{MIN}_{Q2} = [\max\{6, 8, 10\}, \min\{9, 13, 12\}] = [10, 9] = \emptyset \quad (\text{no common delivery window}),$$

$$\cup \text{MIN}_{Q2} = [6, 13] \quad (\text{connected via overlaps}).$$

Order-2 Iterative MultiInterval Number. *Aggregating quarters as a multiset of multisets yields*

$$\text{IMIN}^{(2)} = \{ \text{MIN}_{Q1}, \text{MIN}_{Q2} \} \in \mathcal{M}^2(\mathbb{I}),$$

a hierarchical portfolio capturing both within-quarter supplier variability and across-quarter drift.

Example 23 (Iterative MultiInterval Number: Weekly appointment slots aggregated by month). *A clinic manages doctor appointments in weekly time windows. For a given doctor and a given month, each week w offers several alternative daily appointment intervals in minutes (from opening time), e.g.*

$$\text{MIN}_{w_1} = \{ [60, 90], [120, 150], [210, 240] \}, \quad \text{MIN}_{w_2} = \{ [60, 80], [150, 180] \},$$

$$\text{MIN}_{w_3} = \{ [90, 120], [180, 210], [240, 270] \}, \quad \text{MIN}_{w_4} = \{ [60, 100], [130, 160] \}.$$

Each MIN_{w_i} is a level-1 MultiInterval Number (a multiset of daily slots). To represent the whole month hierarchically, the clinic forms the Iterative MultiInterval Number of order 2,

$$\text{IMIN}^{(2)} = \{ \text{MIN}_{w_1}, \text{MIN}_{w_2}, \text{MIN}_{w_3}, \text{MIN}_{w_4} \} \in \mathcal{M}^2(\mathbb{I}),$$

i.e. a multiset of weekly multisets. Scheduling software can then: (i) drill down to a particular week to pick one of its concrete intervals; or (ii) apply an “unnesting”/aggregation to list all feasible daily slots for the whole month. This captures, in a single object, both within-week variability and across-week drift of available appointment times.

Theorem 15 (IMIN is representable in an Iterative Multi-Structure). *There exists an Iterative Multi-Structure of order k (Definition 4) with carrier $H := \mathbb{I} \cup \{\star\}$,*

$$\mathcal{IMS}^{(k)} = \left(H, \{ \#^{(1,i)} : \mathcal{M}^i(H) \rightarrow \mathcal{M}^{i+1}(H) \}_{i=0}^{k-1} \right),$$

and seeds $s^{(0)} := \star, s^{(i+1)} := \{s^{(i)}\}$, such that

$$\#^{(1,0)}(s^{(0)}) = \text{IMIN}^{(1)}, \quad \#^{(1,1)}(s^{(1)}) = \text{IMIN}^{(2)}, \quad \dots, \quad \#^{(1,k-1)}(s^{(k-1)}) = \text{IMIN}^{(k)}.$$

Proof. For each $i = 0, \dots, k-1$, define

$$\#^{(1,i)} : \mathcal{M}^i(H) \rightarrow \mathcal{M}^{i+1}(H), \quad \#^{(1,i)}(s^{(i)}) := \text{IMIN}^{(i+1)}, \quad \#^{(1,i)}(x) := \emptyset \quad (x \neq s^{(i)}).$$

By Definition 19, $\text{IMIN}^{(i+1)} \in \mathcal{M}^{i+1}(\mathbb{I}) \subseteq \mathcal{M}^{i+1}(H)$, so each $\#^{(1,i)}$ is well typed and realizes the stated identities. $\square$

Theorem 16 (IMIN generalizes MIN). *For $k = 1$ and $F_1(a_1) = I_{a_1} \in \mathbb{I}$, one has*

$$\text{IMIN}^{(1)} = \{ I_{a_1} : a_1 \in A_1 \} = \text{MIN}(A_1).$$

More generally, if for $i \geq 2$ each $F_i(a_i)$ is a singleton lift containing an element of $\mathcal{M}^{i-1}(\mathbb{I})$, then the levelwise unnesting map $\text{Unnest}^{k-1} : \mathcal{M}^k(\mathbb{I}) \rightarrow \mathcal{M}^1(\mathbb{I})$ satisfies

$$\text{Unnest}^{k-1}(\text{IMIN}^{(k)}) = \text{IMIN}^{(1)} = \text{MIN}(A_1).$$

Proof. The case $k = 1$ is immediate from the definitions. If each $F_i(a_i)$ is a singleton multiset, then $\text{IMIN}^{(i)}$ is a multiset of singletons for all $i \geq 2$. Each application of Unnest removes one singleton layer, so after $k - 1$ steps we recover $\text{IMIN}^{(1)} = \text{MIN}(A_1)$. $\square$

2.9. MultiFunctorial Numbers

We extend a *Functorial Number* (a semiring-valued functor on a base category) to a *finite multiset* of such functorial numbers, so that multiple scenarios/instantiations can be recorded simultaneously.

Notation 4 (Space of functorial numbers). *Fix a category $\mathcal{C}$ with finite products. Write $\mathbf{CRig}$ for the category of commutative semirings and homomorphisms. A Functorial Number on $\mathcal{C}$ is (equivalently) a functor $\tilde{N} : \mathcal{C} \rightarrow \mathbf{CRig}$; composing with the forgetful functor $U : \mathbf{CRig} \rightarrow \mathbf{Set}$ gives $N := U \circ \tilde{N} : \mathcal{C} \rightarrow \mathbf{Set}$ together with natural transformations $(\oplus, \otimes, 0, 1)$ so that each $(N(X), \oplus_X, \otimes_X, 0_X, 1_X)$ is a commutative semiring and $N(f)$ preserves the semiring structure for every $f : X \rightarrow Y$. Let*

$$\mathbb{FN} := \left\{ (N, \oplus, \otimes, 0, 1) \text{ as above} \right\}$$

be the (ambient) set of all functorial numbers over $\mathcal{C}$.

Definition 20 (MultiFunctorial Number (MFN)). *Let A be a nonempty finite index set (scenarios/instances). For each $a \in A$, choose a functorial number $\mathcal{N}_a \in \mathbb{FN}$ and optionally a multiplicity $w(a) \in \mathbb{N}$ (frequency/weight). The MultiFunctorial Number determined by $(A, \{\mathcal{N}_a\}, w)$ is the finite multiset*

$$\text{MFN}(A) \in \mathcal{M}(\mathbb{FN}), \quad \text{MFN}(A) := \{ (\mathcal{N}_a)^{w(a)} : a \in A \}.$$

If $w \equiv 1$, then $\text{MFN}(A) = \{ \mathcal{N}_a : a \in A \}$.

Example 24 (MultiFunctorial Number: Meal-kit alternatives and SKU $\rightarrow$ Category aggregation). *Let $\mathcal{C} = \mathbf{FinSet}$ and define a (commutative-semiring-valued) functor*

$$N(X) := \mathbb{N}_0[X] \quad (\text{polynomials with variables indexed by } X),$$

and for $f : X \rightarrow Y$, let $N(f) : \mathbb{N}_0[X] \rightarrow \mathbb{N}_0[Y]$ be the unique semiring homomorphism sending each variable $x \in X$ to the variable $f(x) \in Y$ and extending linearly/multiplicatively.

Real-life setting. *A meal kit can be assembled via two SKU alternatives:*

$$X = \{\text{pastaA}, \text{pastaB}, \text{sauceR}, \text{sauceM}, \text{cheese}\}.$$

Two feasible kits (baskets) are encoded as polynomials (counts as coefficients/exponents):

$$p_1 = \text{pastaA} + \text{sauceR} + \text{cheese}, \quad p_2 = \text{pastaB} + \text{sauceM} + \text{cheese}.$$

The MultiFunctorial Number (multi-output across alternatives) is the finite multiset

$$\text{MFN} = \{ p_1, p_2 \} \in \mathcal{M}(\mathbb{N}_0[X]).$$

Aggregation map. Let $Y = \{\text{PASTA}, \text{SAUCE}, \text{CHEESE}\}$ and define $f : X \rightarrow Y$ by $f(\text{pastaA}) = f(\text{pastaB}) = \text{PASTA}$, $f(\text{sauceR}) = f(\text{sauceM}) = \text{SAUCE}$, $f(\text{cheese}) = \text{CHEESE}$. Then, functorially,

$$N(f)(p_1) = \text{PASTA} + \text{SAUCE} + \text{CHEESE}, \quad N(f)(p_2) = \text{PASTA} + \text{SAUCE} + \text{CHEESE}.$$

Applying $N(f)$ pointwise to the multiset of alternatives yields the aggregated multiset

$$N(f)_\#(\text{MFN}) = \{ N(f)(p_1), N(f)(p_2) \} = \{ \text{PASTA} + \text{SAUCE} + \text{CHEESE}, \text{PASTA} + \text{SAUCE} + \text{CHEESE} \},$$

which shows how a functorial number (polynomial counts) is combined with a multi-output choice set. The homomorphism property ensures $N(f)(p + q) = N(f)(p) + N(f)(q)$ and $N(f)(pq) = N(f)(p) N(f)(q)$, so aggregation is consistent with basket arithmetic.

Example 25 (MultiFunctorial Number: Cloud service costings across regions). Let $\mathcal{C} = \mathbf{FinSet}$ and define a functor

$$N(X) := \mathbb{N}_0[X] \quad (\text{nonnegative-integer polynomials in variables indexed by } X),$$

and for $f : X \rightarrow Y$ let $N(f) : \mathbb{N}_0[X] \rightarrow \mathbb{N}_0[Y]$ be the unique semiring homomorphism sending $x \mapsto f(x)$ and extending additively/multiplicatively.

Consider a cloud provider with a base SKU set

$$X = \{\text{vm-small}, \text{vm-medium}, \text{storage}, \text{backup}\}.$$

A functorial number for the US region is

$$\mathcal{N}_{US} : \mathcal{C} \rightarrow \mathbf{CRig}, \quad \mathcal{N}_{US}(X) = N(X),$$

interpreted as “how many units of each SKU a US customer consumes” (encoded as a polynomial). Similarly define $\mathcal{N}_{EU}$ and $\mathcal{N}_{APAC}$ using the same N but with region-specific pricing/discount structure in the target semiring.

Let

$$A = \{\text{US}, \text{EU}, \text{APAC}\}, \quad w \equiv 1.$$

Then the MultiFunctorial Number is the finite multiset

$$\text{MFN}(A) = \{ \mathcal{N}_{US}, \mathcal{N}_{EU}, \mathcal{N}_{APAC} \} \in \mathcal{M}(\mathbb{FN}),$$

which simultaneously records three regional functorial cost/count models on the same SKU category. If the provider later defines an aggregation functor $f : X \rightarrow Y$ that collapses $\{\text{vm-small}, \text{vm-medium}\} \mapsto \text{COMPUTE}$, then $N(f)$ acts functorially on each element of $\text{MFN}(A)$ and the multiset structure is preserved.

Theorem 17 (MFN is representable in a MultiStructure). There exists a MultiStructure $\mathcal{MS} = (H, \{\#^{(m)}\})$ (Definition of MultiStructure in the preliminaries) that represents $\text{MFN}(A)$: taking $H := \mathbb{FN} \cup \{\star\}$, there is a unary multi-operation

$$\#_A^{(1)} : H \longrightarrow \mathcal{M}(H) \quad \text{such that} \quad \#_A^{(1)}(\star) = \text{MFN}(A) \quad (\text{as a multiset of functorial numbers}).$$

Proof. Define $\#_A^{(1)}$ by

$$\#_A^{(1)}(\star) := \text{MFN}(A) \in \mathcal{M}(\mathbb{FN}) \subseteq \mathcal{M}(H), \quad \#_A^{(1)}(h) := \emptyset \quad (h \in \mathbb{FN}).$$

Then $\#_A^{(1)} : H \rightarrow \mathcal{M}(H)$ is a valid unary multi-operation and $\#_A^{(1)}(\star)$ is exactly $\text{MFN}(A)$. Hence $\mathcal{MS}$ represents $\text{MFN}(A)$. $\square$

Theorem 18 (MFN generalizes the Functorial Number). *If $A = \{a_0\}$ and $w(a_0) = 1$, then*

$$\text{MFN}(A) = \{\mathcal{N}_{a_0}\} \in \mathcal{M}(\mathbb{FN}).$$

Let $\text{Sel} : \mathcal{M}(\mathbb{FN}) \rightarrow \mathbb{FN}$ return the unique element of a singleton multiset. Then

$$\text{Sel}(\text{MFN}(\{a_0\})) = \mathcal{N}_{a_0},$$

i.e., the classical Functorial Number is recovered.

Proof. Immediate from Definition 20 and the definition of Sel on singletons. $\square$

2.10. Iterative MultiFunctorial Numbers

We now lift MultiFunctorial Numbers through multiple *multiset levels*, obtaining elements of $\mathcal{M}^k(\mathbb{FN})$ that encode hierarchical groupings (e.g., portfolios of scenario sets across teams, stages, or time).

Definition 21 (Iterative MultiFunctorial Number (IMFN)). *Fix an integer $k \geq 1$. For each level $i = 1, \dots, k$ let A_i be a nonempty finite index set and choose level- i building blocks*

$$F_1(a_1) \in \mathbb{FN} \quad (a_1 \in A_1), \quad F_i(a_i) \in \mathcal{M}^{i-1}(\mathbb{FN}) \quad (i \geq 2, a_i \in A_i).$$

Define inductively

$$\begin{aligned} \text{IMFN}^{(1)} &:= \{F_1(a_1) : a_1 \in A_1\} \in \mathcal{M}^1(\mathbb{FN}), \\ \text{IMFN}^{(i)} &:= \{F_i(a_i) : a_i \in A_i\} \in \mathcal{M}^i(\mathbb{FN}) \quad (i = 2, \dots, k). \end{aligned}$$

We call $\text{IMFN}^{(k)} \in \mathcal{M}^k(\mathbb{FN})$ the (order- k) Iterative MultiFunctorial Number.

Example 26 (Iterative MultiFunctorial Number: Week-by-week portfolios of meal-kit alternatives). *Using N as above, suppose alternatives change weekly due to supply.*

Week 1 (level 1).

$$p_1^{(1)} = \text{pastaA} + \text{sauceR} + \text{cheese}, \quad p_2^{(1)} = \text{pastaB} + \text{sauceM} + \text{cheese},$$

$$\text{MFN}_{W1} = \{p_1^{(1)}, p_2^{(1)}\} \in \mathcal{M}^1(\mathbb{N}_0[X]).$$

Week 2 (level 1). *A special bundle uses extra cheese:*

$$p_1^{(2)} = \text{pastaA} + \text{sauceR} + (\text{cheese})^2, \quad p_2^{(2)} = \text{pastaB} + \text{sauceM} + \text{cheese},$$

$$\text{MFN}_{W2} = \{p_1^{(2)}, p_2^{(2)}\} \in \mathcal{M}^1(\mathbb{N}_0[X]).$$

Order-2 Iterative MultiFunctorial Number (portfolio across weeks).

$$\text{IMFN}^{(2)} = \{\text{MFN}_{W1}, \text{MFN}_{W2}\} \in \mathcal{M}^2(\mathbb{N}_0[X]).$$

Applying the same $\text{SKU} \rightarrow \text{Category}$ map $f : X \rightarrow Y$ functorially at each level gives

$$N(f)_\#(\text{MFN}_{W1}) = \{\text{PASTA} + \text{SAUCE} + \text{CHEESE}, \text{PASTA} + \text{SAUCE} + \text{CHEESE}\},$$

$$N(f)_\#(\text{MFN}_{W2}) = \{ \text{PASTA} + \text{SAUCE} + (\text{CHEESE})^2, \text{PASTA} + \text{SAUCE} + \text{CHEESE} \},$$

and hence

$$N(f)_\#(\text{IMFN}^{(2)}) = \{ N(f)_\#(\text{MFN}_{W1}), N(f)_\#(\text{MFN}_{W2}) \} \in \mathcal{M}^2(\mathbb{N}_0[Y]).$$

This demonstrates an iterative (hierarchical) multi-output portfolio of functorial numbers, with aggregation along f respecting semiring laws and acting consistently across levels.

Example 27 (Iterative MultiFunctorial Number: Yearly portfolio of regional service models). Suppose the provider above stores, for each quarter $q \in A_2 = \{Q1, Q2, Q3, Q4\}$, the set of regional functorial numbers that were active in that quarter. At level 1 (regions) we have, for every $r \in A_1 = \{US, EU, APAC\}$, a functorial number

$$F_1(r) = \mathcal{N}_r \in \mathbb{FN},$$

so that

$$\text{IMFN}^{(1)} = \{ F_1(r) : r \in A_1 \} = \{ \mathcal{N}_{US}, \mathcal{N}_{EU}, \mathcal{N}_{APAC} \} \in \mathcal{M}^1(\mathbb{FN}).$$

For each quarter $q \in A_2$, define the quarter's multi-scenario object by

$$F_2(q) := \{ \text{the regional functorial numbers actually active in } q \} \subseteq \mathcal{M}^1(\mathbb{FN}),$$

e.g.

$$F_2(Q1) = \{ \mathcal{N}_{US}, \mathcal{N}_{EU} \}, \quad F_2(Q2) = \{ \mathcal{N}_{US}, \mathcal{N}_{EU}, \mathcal{N}_{APAC} \}, \quad F_2(Q3) = \{ \mathcal{N}_{US}, \mathcal{N}_{APAC} \}, \quad F_2(Q4) = \{ \mathcal{N}_{EU}, \mathcal{N}_{APAC} \}.$$

Then level 2 is

$$\text{IMFN}^{(2)} = \{ F_2(q) : q \in A_2 \} \in \mathcal{M}^2(\mathbb{FN}),$$

i.e. a multiset of multisets of functorial numbers, one for each quarter of the year.

If the provider defines a natural transformation “Convert to USD” that takes every regional semiring to a common USD-valued semiring (a functor $\mathbb{FN} \rightarrow \mathbb{FN}$ preserving semiring structure), it can be applied pointwise to every element inside every quarter in $\text{IMFN}^{(2)}$, and the nested multiset structure is kept unchanged. Thus $\text{IMFN}^{(2)}$ is an order-2 Iterative MultiFunctorial Number that captures a hierarchy: (region-level service models) inside (quarter-level business snapshots).

Theorem 19 (IMFN is representable in an Iterative Multi-Structure). *There exists an Iterative Multi-Structure of order k (see the preliminaries on Iterative Multi-Structure) with carrier $H := \mathbb{FN} \cup \{\star\}$,*

$$\mathcal{IMS}^{(k)} = \left(H, \{ \#^{(1,i)} : \mathcal{M}^i(H) \rightarrow \mathcal{M}^{i+1}(H) \}_{i=0}^{k-1} \right),$$

and seeds $s^{(0)} := \star, s^{(i+1)} := \{s^{(i)}\}$, such that

$$\#^{(1,0)}(s^{(0)}) = \text{IMFN}^{(1)}, \quad \#^{(1,1)}(s^{(1)}) = \text{IMFN}^{(2)}, \quad \dots, \quad \#^{(1,k-1)}(s^{(k-1)}) = \text{IMFN}^{(k)}.$$

Proof. For each $i = 0, \dots, k-1$, define

$$\#^{(1,i)} : \mathcal{M}^i(H) \rightarrow \mathcal{M}^{i+1}(H), \quad \#^{(1,i)}(s^{(i)}) := \text{IMFN}^{(i+1)}, \quad \#^{(1,i)}(x) := \emptyset \quad (x \neq s^{(i)}).$$

By Definition 21, $\text{IMFN}^{(i+1)} \in \mathcal{M}^{i+1}(\mathbb{FN}) \subseteq \mathcal{M}^{i+1}(H)$, so each $\#^{(1,i)}$ is well typed and realizes the stated identities. $\square$

Theorem 20 (IMFN generalizes MFN). *For $k = 1$ and $F_1(a_1) = \mathcal{N}_{a_1} \in \mathbb{FN}$, one has*

$$\text{IMFN}^{(1)} = \{ \mathcal{N}_{a_1} : a_1 \in A_1 \} = \text{MFN}(A_1).$$

More generally, if for $i \geq 2$ each $F_i(a_i)$ is a singleton lift containing an element of $\mathcal{M}^{i-1}(\mathbb{FN})$, then the levelwise unnesting map $Unnest^{k-1} : \mathcal{M}^k(\mathbb{FN}) \rightarrow \mathcal{M}^1(\mathbb{FN})$ satisfies

$$Unnest^{k-1}(\text{IMFN}^{(k)}) = \text{IMFN}^{(1)} = \text{MFN}(A_1).$$

Proof. The case $k = 1$ is immediate from the definitions. If each $F_i(a_i)$ is a singleton multiset, then $\text{IMFN}^{(i)}$ is a multiset of singletons for all $i \geq 2$. Each application of *Unnest* removes one singleton layer; after $k - 1$ steps we recover $\text{IMFN}^{(1)} = \text{MFN}(A_1)$. $\square$

3. Conclusions

In this paper, we defined the MultiRough, MultiGrey, MultiGranular, MultiInterval, and Multi-Functorial Numbers by extending the Rough, Grey, Granular, Interval, and Functorial Numbers using the frameworks of Multi-Structure and Iterative Multi-Structure. These extensions make it possible to represent complex real-world structures characterized by uncertainty in a more comprehensible and systematic manner.

In the future, we expect further developments in the extension of uncertainty modeling using Fuzzy Sets [37–39], Bipolar Fuzzy Sets [40,41], SuperHyperFuzzy Sets [42–44] Neutrosophic Sets [45,46,46–48], Double-Valued Neutrosophic Sets [49–53], Shadowed Sets [54–56], and Plithogenic Sets [57–60], as well as research on structural extensions based on Graphs [61], HyperGraphs [62–64], and SuperHyperGraphs [65–68]. Furthermore, it is hoped that future studies will advance the concepts discussed in this paper through research involving computational experiments.

This section is not mandatory, but may be added if there are patents resulting from the work reported in this manuscript.

Funding: No external funding was received for this work.

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Data Availability Statement: This paper is theoretical and did not generate or analyze any empirical data. We welcome future studies that apply and test these concepts in practical settings.

Research Integrity

The author confirms that this manuscript is original, has not been published elsewhere, and is not under consideration by any other journal.

Use of Computational Tools

All proofs and derivations were performed manually; no computational software (e.g., Mathematica, SageMath, Coq) was used.

Code Availability

No code or software was developed for this study.

Ethical Approval

This research did not involve human participants or animals, and therefore did not require ethical approval.

Use of Generative AI and AI-Assisted Tools

We use generative AI and AI-assisted tools for tasks such as English grammar checking, and We do not employ them in any way that violates ethical standards.

Disclaimer

The ideas presented here are theoretical and have not yet been validated through empirical testing. While we have strived for accuracy and proper citation, inadvertent errors may remain. Readers should verify any referenced material independently. The opinions expressed are those of the authors and do not necessarily reflect the views of their institutions.

Acknowledgments: We thank all colleagues, reviewers, and readers whose comments and questions have greatly improved this manuscript. We are also grateful to the authors of the works cited herein for providing the theoretical foundations that underpin our study. Finally, we appreciate the institutional and technical support that enabled this research.

Conflicts of Interest: The authors declare no conflicts of interest regarding the publication of this work.

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