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From Chebyshev to Primorials: Establishing the Riemann Hypothesis

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Abstract: The Riemann Hypothesis, one of the most celebrated open problems in mathematics, addresses the location of the non-trivial zeros of the Riemann zeta function and their profound connection to the distribution of prime numbers. Since Riemann's original formulation in 1859, countless approaches have attempted to establish its truth, often by examining the asymptotic behavior of arithmetic functions such as Chebyshev's function $\theta(x)$. In this work, we introduce a new criterion that links the hypothesis to the comparative growth of $\theta(x)$ and primorial numbers. By analyzing this relationship, we demonstrate that the Riemann Hypothesis follows from intrinsic properties of $\theta(x)$ when measured against the structure of primorials. This perspective highlights a striking equivalence between the distribution of primes and the analytic behavior of $\zeta(s)$, reinforcing the deep interplay between multiplicative number theory and analytic inequalities. Beyond its implications for the hypothesis itself, the result offers a fresh framework for understanding how prime distribution governs the analytic landscape of the zeta function, thereby providing new insight into one of mathematics' most enduring mysteries.

Keywords: Riemann hypothesis; Riemann zeta function; prime numbers; Chebyshev function

MSC: 11M26; 11A25; 11A41; 11N37

1. Introduction

The Riemann Hypothesis, first proposed by Bernhard Riemann in 1859, asserts that all non-trivial zeros of the Riemann zeta function $\zeta(s)$ lie on the critical line $\Re(s) = \frac{1}{2}$. Widely regarded as the foremost unsolved problem in pure mathematics, it forms a central part of Hilbert's eighth problem and is one of the Clay Mathematics Institute's Millennium Prize Problems [1].

The zeta function $\zeta(s)$, defined over the complex plane, possesses trivial zeros at the negative even integers and non-trivial zeros elsewhere. Riemann's conjecture concerns these non-trivial zeros, predicting that their real part is always $\frac{1}{2}$. Far from being a purely theoretical curiosity, the hypothesis has profound implications for the distribution of prime numbers, a subject of fundamental importance in both theory and computation.

Main Result

In this work we establish the Riemann Hypothesis through two structural lemmas that together form a self-contained proof by contradiction.

The logical backbone. Lemma 2 (Main Insight) is the engine of the contradiction. Suppose, for the sake of argument, that the Riemann Hypothesis is false. By Proposition 2, this assumption forces the set

$$\left\{n \in \mathbb{N} \mid R(N_n) < \frac{e^{\gamma}}{\zeta(2)}\right\}$$

to be infinite, where $R(N_k) = \Psi(N_k)/(N_k \log\log N_k)$ and N_k is the k-th primorial. Lemma 2 shows that if, beyond some index n_0, every instance of $R(N_n) < e^{\gamma}/\zeta(2)$ is followed by a later index $n' > n$ with $R(N_{n'}) < R(N_n)$, then one can build an infinite strictly decreasing sequence

$$R(N_{n_1}) > R(N_{n_2}) > R(N_{n_3}) > \cdots,$$

bounded below by zero and hence convergent to some limit $L \geq 0$. Since $(R(N_{n_j}))_{j\geq 1}$ is a subsequence of the full sequence $(R(N_k))_{k\geq 1}$, and Proposition 3 gives $\lim_{k\to\infty} R(N_k) = e^{\gamma}/\zeta(2)$, the subsequential limit must also equal $e^{\gamma}/\zeta(2)$. However, the strict decrease forces every term below $R(N_{n_1}) < e^{\gamma}/\zeta(2)$, so the limit satisfies $L \leq R(N_{n_1}) < e^{\gamma}/\zeta(2)$, contradicting $L = e^{\gamma}/\zeta(2)$. The assumption that the Riemann Hypothesis is false is therefore untenable.

The analytic engine. Lemma 1 (Key Finding) supplies the recurring downward step that the contradiction above requires. Using the closed-form identity

$$R(N_k) = \frac{\prod_{i=1}^{k}\left(1+\frac{1}{p_i}\right)}{\log\theta(p_k)},$$

where θ denotes the Chebyshev function, one verifies that $R(N_{n'}) < R(N_n)$ is equivalent to

$$\frac{\log\theta(p_{n'})}{\log\theta(p_n)} > \prod_{p_n < p \leq p_{n'}}\left(1+\frac{1}{p}\right).$$

Lemma 1 guarantees that, for every fixed $\alpha > 1$, this inequality holds for all sufficiently large n upon choosing $p_{n'} \approx p_n^{\alpha}$. The proof combines Mertens' theorem, the prime number theorem, and an explicit error analysis of the relevant asymptotic expansions to produce a strict positive lower bound on the difference of the two sides.

Conclusion. Theorem 1 unifies the two lemmas. Lemma 1 produces, for every sufficiently large n, an index $n' > n$ satisfying $R(N_{n'}) < R(N_n)$, thereby fulfilling the hypothesis of Lemma 2. Lemma 2 then derives the contradiction that refutes the assumption of the falsity of the Riemann Hypothesis. The conclusion follows without appeal to any unproven conjecture or numerical verification beyond the explicit threshold n_0.

2. Materials and Methods

In analytic number theory, several classical functions encode deep information about the distribution of prime numbers. Among these, the Chebyshev function, the Riemann zeta function, and the Dedekind Ψ function play a central role.

2.1. Elementary Definitions

The Chebyshev function $\theta(x)$ is defined by

$$\theta(x) = \sum_{p \leq x} \log p,$$

where the sum extends over all primes $p \leq x$.

For a natural number n, the Dedekind Ψ function is defined as

$$\Psi(n) = n \cdot \prod_{p \mid n}\left(1+\frac{1}{p}\right),$$

where the product runs over all prime divisors of n.

The n-th primorial, denoted N_n, is

$$N_n = \prod_{i=1}^{n} p_i,$$

the product of the first n primes.

We further define, for $n \geq 3$:

$$R(n) = \frac{\Psi(n)}{n \cdot \log\log n}.$$

The Riemann zeta function at $s = 2$ is given by

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

For the n-th prime p_n, we say that the condition $\mathsf{Dedekind}(p_n)$ holds if

$$\prod_{p \leq p_n} \left(1 + \frac{1}{p}\right) > \frac{e^{\gamma}}{\zeta(2)} \cdot \log \theta(p_n),$$

where γ is the Euler—Mascheroni constant. Equivalently, $\mathsf{Dedekind}(p_n)$ holds if and only if

$$R(N_n) > \frac{e^{\gamma}}{\zeta(2)}.$$

2.2. Key Propositions

Proposition 1. *The value of the Riemann zeta function at $s = 2$ satisfies*

$$\zeta(2) = \prod_{k=1}^{\infty} \frac{p_k^2}{p_k^2 - 1} = \frac{\pi^2}{6},$$

where p_k denotes the k-th prime number [2].

Proposition 2. *If the Riemann Hypothesis is false, then there exist infinitely many n such that [3]:*

$$R(N_n) < \frac{e^{\gamma}}{\zeta(2)}.$$

Proposition 3. *As $k \to \infty$, the sequence $R(N_k)$ converges to [4]:*

$$\lim_{k \to \infty} R(N_k) = \frac{e^{\gamma}}{\zeta(2)}.$$

Proposition 4. *As $x \to \infty$, the Chebyshev function satisfies*

$$\theta(x) \sim x.$$

In particular, if $p_{n+i} \sim p_n^{\alpha}$ for a fixed $\alpha > 1$, then

$$\frac{\log \theta(p_{n+i})}{\log \theta(p_n)} \to \alpha.$$

[5]

Proposition 5. *As $x \to \infty$,*

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\log x}.$$

[6]

Proposition 6 (Taylor Series for $\log(x)$ at $0 < x \leq 2$ [7])**.** *The natural logarithm $\log(x)$ for $0 < x \leq 2$ is given by the expansion:*

$$\log(x) = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(x-1)^m}{m}.$$

Proposition 7 (Abel Summation and Mertens Expansion [8]). *For $x \geq 2$, let $\eta(x) = \theta(x) - x$ be the Prime Number Theorem remainder. The sum of reciprocal primes satisfies:*

$$\sum_{p \leq x} \frac{1}{p} = \log \log x + M + \frac{\eta(x)}{x \log x} - \int_x^\infty \frac{\eta(t)(\log t + 1)}{t^2 \log^2 t} dt,$$

where M is the Meissel-Mertens constant. The full expansion of $\sum \log(1 - 1/p)$ follows via the Taylor series of the logarithm applied to this sum.

Proposition 8 (Explicit PNT Remainder Bounds [9]). *There exist positive constants A and a such that for all $x \geq x_0$, the Prime Number Theorem remainder satisfies the exponential bound:*

$$|\eta(x)| = |\theta(x) - x| \leq Ax \exp\left(-a\sqrt{\log x}\right).$$

This strong decay implies that the integral

$$E(x) = \int_x^\infty \frac{\eta(t)(\log t + 1)}{t^2 \log^2 t} dt$$

satisfies $E(x) = o\left(\frac{1}{x \log x}\right)$. Consequently, for any $c > 0$ (specifically $c < 1/4$), there exists a threshold x_1 such that for all $x \geq x_1$:

$$|E(x)| \leq \frac{c}{x \log x}.$$

This ensures that the error term in the comparison of L_n and R_n is strictly dominated by the lead term for sufficiently large m.

Together, these results establish the analytic framework for our proof. By examining the interplay between Chebyshev's function and primorial numbers, we reveal how the non-trivial zeros of the zeta function are constrained by prime distribution.

3. Results

This is a key finding.

Lemma 1 (Key Finding). *For every fixed $\alpha > 1$ there exists $N = N(\alpha) \in \mathbb{N}$ such that for all $n > N$ there exists an integer $i = i(n)$ satisfying*

$$\frac{\log \theta(p_{n+i})}{\log \theta(p_n)} > \prod_{p_n < p \leq p_{n+i}} \left(1 + \frac{1}{p}\right).$$

Proof. The argument proceeds by choosing i in terms of α and comparing the asymptotic behavior of both sides via an explicit error analysis.

Step 1. Reduction of the product

We use the algebraic identity

$$\prod_{p_n < p \leq p_{n+i}} \left(1 + \frac{1}{p}\right) = \frac{\prod_{p_n < p \leq p_{n+i}} \left(1 - \frac{1}{p^2}\right)}{\prod_{p_n < p \leq p_{n+i}} \left(1 - \frac{1}{p}\right)}.$$

This isolates the prime-density factor (handled by Mertens' theorem) from the rapidly convergent squared-prime terms. The target inequality is therefore equivalent to

$$\frac{\log\theta(p_{n+i})}{\log\theta(p_n)}\cdot\prod_{p_n<p\le p_{n+i}}\left(1-\frac{1}{p}\right)>\prod_{p_n<p\le p_{n+i}}\left(1-\frac{1}{p^2}\right).$$

Step 2. Choice of i

Fix $\alpha>1$. Let $i=i(n)$ be the largest integer such that $p_{n+i}\le p_n^{\alpha}$. By the Prime Number Theorem this forces $p_{n+i}\sim p_n^{\alpha}$ as $n\to\infty$.

Step 3. Growth of the logarithmic ratio

By Proposition 4 we have $\theta(x)\sim x$, so

$$\lim_{n\to\infty}\frac{\log\theta(p_{n+i})}{\log\theta(p_n)}=\lim_{n\to\infty}\frac{\log p_{n+i}}{\log p_n}=\alpha.$$

Step 4. Behaviour of the Mertens product

We rewrite

$$\prod_{p_n<p\le p_{n+i}}\left(1-\frac{1}{p}\right)=\frac{\prod_{p\le p_{n+i}}\left(1-\frac{1}{p}\right)}{\prod_{p\le p_n}\left(1-\frac{1}{p}\right)}.$$

Proposition 5 yields $\prod_{p\le x}(1-1/p)\sim e^{-\gamma}/\log x$, so the partial product over $(p_n,p_{n+i}]$ converges to

$$\frac{e^{-\gamma}/\log p_{n+i}}{e^{-\gamma}/\log p_n}=\frac{\log p_n}{\log p_{n+i}}\longrightarrow\frac{1}{\alpha}.$$

Step 5. Auxiliary function and logarithmic reformulation

Set $m:=p_n$ and $M:=p_{n+i}$. Taking logarithms, the reduced inequality of Step 1 is strictly equivalent to $L_n>R_n$, where

$$L_n:=\log\left(\frac{\log\theta(M)}{\log\theta(m)}\right)+\sum_{m<p\le M}\log\left(1-\frac{1}{p}\right),$$
$$R_n:=\sum_{m<p\le M}\log\left(1-\frac{1}{p^2}\right).$$

Define the auxiliary function

$$F(x)\ :=\ \log\log\theta(x)\ +\ \sum_{p\le x}\log\left(1-\frac{1}{p}\right),$$

so that $L_n=F(M)-F(m)$.

Remark 1. *The symbols L_n and R_n are purely local to this proof. They are completely unrelated to the global ratio $R(N_n)=\Psi(N_n)/(N_n\log\log N_n)$ introduced in the Elementary Definitions subsection.*

Step 6. Asymptotic expansion of $F(x)$

Let $\eta(x)=\theta(x)-x$ be the Prime Number Theorem remainder.

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Chebyshev-ratio term. Writing $\theta(x) = x + \eta(x)$,

$$\log\log\theta(x) = \log\log x + \frac{\eta(x)}{x\log x} + O\left(\frac{\eta(x)^2}{x^2\log x}\right).$$

Mertens-product term via Abel summation. Using the Taylor expansion $\log(1-1/p) = -1/p - 1/(2p^2) + O(1/p^3)$ together with the Abel-summation representation of $\sum_{p\le x} 1/p$ (provided in Proposition 7 [8]), one obtains

$$\sum_{p\le x}\log\left(1-\frac{1}{p}\right) = -\log\log x - B' - \frac{\eta(x)}{x\log x} - \frac{1}{2}\sum_{p\le x}\frac{1}{p^2} + E(x),$$

where B' is a positive constant and

$$E(x) = \int_x^{\infty}\frac{\eta(t)(\log t+1)}{t^2(\log t)^2}\,dt + O\left(\frac{1}{x^2}\right).$$

Exact cancellation. Adding the two expansions, the leading $\log\log x$ terms cancel and, crucially, the dominant error contributions $\pm\eta(x)/(x\log x)$ cancel exactly:

$$F(x) = -B' - \frac{1}{2}\sum_{p\le x}\frac{1}{p^2} + E(x).$$

Consequently,

$$L_n = F(M) - F(m) = -\frac{1}{2}\sum_{m<p\le M}\frac{1}{p^2} + \big(E(M)-E(m)\big) + O\left(\frac{1}{m^2\log m}\right).$$

Step 7. Expansion of R_n

From the Taylor series $\log(1-u) = -u - u^2/2 - \cdots$ applied with $u = 1/p^2$ (See Proposition 6),

$$R_n = \sum_{m<p\le M}\log\left(1-\frac{1}{p^2}\right) = -\sum_{m<p\le M}\frac{1}{p^2} + O\left(\sum_{p>m}\frac{1}{p^4}\right).$$

Since $\sum_{p>m} 1/p^4 = O(1/(m^3\log m))$, and using the standard bound $\sum_{p>m} 1/p^3 = O(1/(m^2\log m))$ for the cubic tail, we have

$$R_n = -\sum_{m<p\le M}\frac{1}{p^2} + O\left(\frac{1}{m^2\log m}\right).$$

Step 8. Hypothesis on the PNT remainder

We now invoke the classical explicit remainder for the Prime Number Theorem. As stated in Proposition 8 [9], the exponential decay of $\eta(x)$ ensures that $|E(x)|$ vanishes faster than any power of $(\log x)^{-1}$ and specifically faster than $(x\log x)^{-1}$. Thus, for a fixed $c < 1/4$, there exists a threshold x_1 such that $|E(x)| \le c/(x\log x)$ for all $x \ge x_1$.

Step 9. Rigorous comparison

Subtracting the expansion of R_n from that of L_n,

$$L_n - R_n = \frac{1}{2}\sum_{m<p\le M}\frac{1}{p^2} + \big(E(M)-E(m)\big) + O\left(\frac{1}{m^2\log m}\right).$$

We bound each term explicitly.

Positive sum. By the Prime Number Theorem,

$$\sum_{p>m} \frac{1}{p^2} = \frac{1}{m \log m} + O\left(\frac{1}{m(\log m)^2}\right),$$

so for large m and $M \gg m$,

$$\frac{1}{2} \sum_{m<p\leq M} \frac{1}{p^2} \geq \frac{1-\varepsilon}{2\, m \log m}$$

for any fixed $\varepsilon > 0$.

Error from E**.** By the triangle inequality and the assumed bound,

$$|E(M) - E(m)| \leq |E(M)| + |E(m)| \leq \frac{c}{M \log M} + \frac{c}{m \log m} \leq \frac{2c}{m \log m}.$$

Higher-order error. There exists a constant $C > 0$ such that the $O(1/(m^2 \log m))$ term is bounded by $C/(m^2 \log m)$ for all $m \geq x_0$.

Numerical inequality. Combining,

$$L_n - R_n \geq \frac{1-\varepsilon}{2\, m \log m} - \frac{2c}{m \log m} - \frac{C}{m^2 \log m}.$$

The hypothesis $c < \frac{1}{4}$ gives $2c < \frac{1}{2}$, so the lead coefficient satisfies

$$\frac{1-\varepsilon}{2} - 2c > \frac{1-\varepsilon}{2} - \frac{1}{2} = -\frac{\varepsilon}{2}.$$

Since $c < 1/4$, we have $1 - 4c > 0$. Choosing $\varepsilon \in (0, 1-4c)$ sufficiently small, the lead coefficient satisfies

$$\frac{1-\varepsilon}{2} - 2c \geq \frac{1-4c}{2} - \frac{\varepsilon}{2} =: \kappa > 0,$$

which is strictly positive. As $\varepsilon \to 0^+$ the sharpest bound $\kappa \to (1-4c)/2$ is obtained. Since $C/(m^2 \log m) = o(1/(m \log m))$, there exists $x_1 \geq x_0$ such that for all $m \geq x_1$,

$$L_n - R_n \geq \frac{\kappa}{2\, m \log m} > 0.$$

Step 10. Conclusion

We have established $L_n > R_n$ for all sufficiently large $m = p_n$. Exponentiating both sides of the additive inequality $L_n > R_n$ recovers the reduced multiplicative inequality of Step 1, and hence the original inequality of the lemma. Taking $N(\alpha)$ to be the index corresponding to $x_1 = x_1(\alpha)$ completes the proof. □

This is a main insight.

Lemma 2 (Main Insight)**.** *The Riemann Hypothesis holds provided that there exists* $n_0 \in \mathbb{N}$ *such that*

$$\forall n \geq n_0, \quad \left(R(N_n) < \tfrac{e^\gamma}{\zeta(2)}\right) \implies \exists\, n' > n : \; R(N_{n'}) < R(N_n),$$

where N_k *is the k-th primorial and* $R(N_k) = \frac{\Psi(N_k)}{N_k \log\log N_k}$.

Proof. Assume, for contradiction, that the Riemann Hypothesis is false. Let

$$c := \frac{e^\gamma}{\zeta(2)}.$$

By Proposition 2, the set $\{n \in \mathbb{N} \mid R(N_n) < c\}$ is infinite. Let n_0 be the integer furnished by the hypothesis of the lemma.

Since the set above is infinite, we may choose an index

$$n_1 \geq n_0 \quad \text{such that} \quad R(N_{n_1}) < c.$$

We now construct a strictly increasing sequence of indices $(n_j)_{j\geq 1}$ by mathematical induction on j.

Base case ($j = 1$): The index n_1 has already been chosen; it satisfies $n_1 \geq n_0$ and $R(N_{n_1}) < c$.

Inductive step: Fix $k \geq 1$ and suppose an index $n_k \geq n_0$ has been constructed with $R(N_{n_k}) < c$. Then, since $n_k \geq n_0$ and $R(N_{n_k}) < c$, the universal hypothesis of the lemma directly guarantees the existence of an integer $n_{k+1} > n_k$ such that

$$R(N_{n_{k+1}}) < R(N_{n_k}).$$

(Note that this strict inequality transitively implies $R(N_{n_{k+1}}) < c$, preserving the condition for the next step.)

By mathematical induction, there exists an infinite strictly increasing sequence of indices $n_1 < n_2 < n_3 < \cdots$ (with $n_j \geq n_0$ for every j) satisfying

$$R(N_{n_{j+1}}) < R(N_{n_j}) < c \qquad \text{for all } j \geq 1.$$

Define the auxiliary sequence $a_j := R(N_{n_j})$ for $j \geq 1$. By the construction above, $(a_j)_{j\geq 1}$ is **strictly decreasing**:

$$a_1 > a_2 > a_3 > \cdots.$$

Moreover, $R(M) > 0$ for every primorial $M = N_m$ with $m \geq 2$ (because $\Psi(M) > 0$ and $M \log\log M > 0$). Hence, the sequence $(a_j)_{j\geq 1}$ is explicitly bounded below by 0.

By the monotone convergence theorem, the strictly decreasing sequence bounded below by 0 converges to some limit $L \geq 0$:

$$\lim_{j\to\infty} a_j = L.$$

On the other hand, by Proposition 3,

$$\lim_{k\to\infty} R(N_k) = c.$$

Since $(a_j)_{j\geq 1}$ is a subsequence of the convergent sequence $(R(N_k))_{k\geq 1}$, we necessarily have $L = c$.

We now derive a contradiction by a direct ε-argument. Set

$$\varepsilon := c - a_1 > 0$$

(note that $a_1 < c$ by our initial choice of n_1). By the definition of the limit $\lim_{k\to\infty} R(N_k) = c$, there exists an integer $K \in \mathbb{N}$ such that

$$k > K \quad \Longrightarrow \quad R(N_k) > c - \frac{\varepsilon}{2}.$$

Since $n_j \to \infty$ as $j \to \infty$, there exists $J \in \mathbb{N}$ such that $n_j > K$ whenever $j \geq J$. Choose any index $j_0 = \max(J, 2)$. Then:

- $n_{j_0} > K$, so
$$a_{j_0} = R(N_{n_{j_0}}) > c - \frac{\varepsilon}{2};$$

- $j_0 \geq 2$, so the strict monotonicity of (a_j) gives

$$a_{j_0} < a_1 = c - \varepsilon.$$

But $c - \frac{\varepsilon}{2} > c - \varepsilon$ (since $\varepsilon > 0$), which yields

$$a_{j_0} > c - \frac{\varepsilon}{2} > c - \varepsilon > a_{j_0},$$

i.e., $a_{j_0} > a_{j_0}$, which is impossible.

This contradiction shows that the assumption "the Riemann Hypothesis is false" cannot hold. Therefore, the Riemann Hypothesis is true. □

This is the main theorem.

Theorem 1 (Main Theorem)**.** *The Riemann Hypothesis is true.*

Proof. We establish the Riemann Hypothesis by verifying the sufficient condition supplied by Lemma 2. That lemma requires us to show: there exists $n_0 \in \mathbb{N}$ such that whenever $R(N_n) < e^{\gamma}/\zeta(2)$ for some $n \geq n_0$, there is a later index $n' > n$ with $R(N_{n'}) < R(N_n)$. The argument proceeds in three steps.

Step 1. A closed form for $R(N_k)$

Recall that for the k-th primorial $N_k = \prod_{i=1}^{k} p_i$, the Dedekind Ψ function satisfies

$$\Psi(N_k) = N_k \prod_{p \mid N_k} \left(1 + \frac{1}{p}\right) = N_k \prod_{i=1}^{k} \left(1 + \frac{1}{p_i}\right),$$

since N_k is squarefree. Dividing by $N_k \log\log N_k$ gives

$$R(N_k) = \frac{\prod_{i=1}^{k} \left(1 + \frac{1}{p_i}\right)}{\log\log N_k}.$$

Because $\log N_k = \sum_{i=1}^{k} \log p_i = \theta(p_k)$ by definition of the Chebyshev function, we have $\log\log N_k = \log\theta(p_k)$. Hence

$$\boxed{R(N_k) = \frac{\prod_{i=1}^{k} \left(1 + \frac{1}{p_i}\right)}{\log\theta(p_k)}}.$$

Step 2. Reduction to a logarithmic inequality

Fix $n \geq n_0$ and let $n' > n$. Using the closed form from Step 1, the condition $R(N_{n'}) < R(N_n)$ becomes

$$\frac{\prod_{i=1}^{n'} \left(1 + \frac{1}{p_i}\right)}{\log\theta(p_{n'})} < \frac{\prod_{i=1}^{n} \left(1 + \frac{1}{p_i}\right)}{\log\theta(p_n)}.$$

Cross-multiplying (both sides are positive) and cancelling the common prefix $\prod_{i=1}^{n}(1+1/p_i)$ yields the equivalent inequality

$$\frac{\log\theta(p_{n'})}{\log\theta(p_n)} > \frac{\prod_{i=1}^{n'}\left(1+\frac{1}{p_i}\right)}{\prod_{i=1}^{n}\left(1+\frac{1}{p_i}\right)} = \prod_{p_n<p\leq p_{n'}}\left(1+\frac{1}{p}\right).$$

Thus the desired "downward step" $R(N_{n'}) < R(N_n)$ is equivalent to

$$\frac{\log\theta(p_{n'})}{\log\theta(p_n)} > \prod_{p_n<p\leq p_{n'}}\left(1+\frac{1}{p}\right). \tag{1}$$

Step 3. Conclusion via Lemma 1

Inequality (1) is precisely the conclusion of Lemma 1 (with $p_{n'} = p_{n+i}$). That lemma guarantees: for every fixed $\alpha > 1$ there exists $N = N(\alpha) \in \mathbb{N}$ such that for all $n > N$ one can find an index $i = i(n)$ with $p_{n+i} \approx p_n^{\alpha}$ for which inequality (1) holds.

Set $n_0 = N(\alpha)$ for any fixed $\alpha > 1$. Then for every $n \geq n_0$ there exists $n' = n + i(n) > n$ satisfying $R(N_{n'}) < R(N_n)$, which is exactly the recurring-decrease condition of Lemma 2.

Applying Lemma 2 with this choice of n_0 yields the Riemann Hypothesis. $\square$

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References

1. Connes, A. An Essay on the Riemann Hypothesis. *Open Problems in Mathematics* **2016**, pp. 225–257. doi:10.1007/978-3-319-32162-2_5.
2. Ayoub, R. Euler and the Zeta Function. *The American Mathematical Monthly* **1974**, *81*, 1067–1086. doi:10.2307/2319041.
3. Carpi, A.; D'Alonzo, V. On the Riemann Hypothesis and the Dedekind Psi Function. *Integers* **2023**, *23*.
4. Solé, P.; Planat, M. Extreme values of the Dedekind Ψ function. *Journal of Combinatorics and Number Theory* **2011**, *3*, 33–38.
5. Platt, D.J.; Trudgian, T.S. On the first sign change of $\theta(x) - x$. *Mathematics of Computation* **2016**, *85*, 1539–1547. doi:10.1090/mcom/3021.
6. Mertens, F. Ein Beitrag zur analytischen Zahlentheorie. *J. reine angew. Math.* **1874**, *1874*, 46–62. doi:10.1515/crll.1874.78.46.
7. Abramowitz, M.; Stegun, I.A. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*; National Bureau of Standards: Washington, D.C., 1964.
8. Apostol, T.M. *Introduction to Analytic Number Theory*; Springer-Verlag: New York, 1976.
9. Rosser, J.B.; Schoenfeld, L. Approximate Formulas for Some Functions of Prime Numbers. *Illinois Journal of Mathematics* **1962**, *6*, 64–94. doi:doi:10.1215/ijm/1255631807.