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Optimal Investment of Merton Model for Multi-Investors with Frictions

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Abstract: In this paper, we extend the Merton model of investment in discrete time to the cases when there is a finite number of investors and the market is with frictions represented by convex penalty functions defined for each investor. In the main result of this paper, we proved the existence of optimal strategy of investment by using a new approach based on the formulation of an equivalent general equilibrium economy model.

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1. Introduction

The optimal investment of Merton model introduced in [12,13] has been investigated by researchers and extended in different contexts since it appears. One important extension in continuous time is due to Magill and Constantinides [11], where a linear transaction costs function is used in the context of Merton problem. In discrete time, the study of Merton model with linear transaction costs was developed by Jouini and Kallal in [8]. We can also cite the papers of Shreve and Soner [14] which extended Merton problem by including viscosity theory and Cetin, Jarrow and Protter [3] which studied the Merton model for illiquid markets.

Recently, Chebbi and Soner in [1] extend the Merton model in discrete time and finite horizon to the case of market with frictions represented by a convex penalty function defined for one investor. They proved the existence of an optimal strategy by solving a dynamic optimization problem. Then Ounaies, Bonnisseau, Chebbi and Soner in [15] extend this model to the infinite horizon and and they proved the existence of optimal strategy by an argument of fixed points.

In this paper, we will take this direction of extension in order to prove the existence of optimal strategy in Merton model for market frictions in infinte horizon when there are finite number of investors. our approach is very different and based on constructing an equivalent general equilibrium model with multiple agents. The idea to use the general equilibrium theory is inspired by the paper of Le Van and Dana [10].

Sections of this paper are organized as follow: In section 2, We give a description of Merton model of investment problem in infinite horizon and with market frictions modeled by convex) penalty functions defined for each investor and constraints conditions about liquidation value is defined consequently.

In section 3, we construct an equivalent general equilibrium economy model to Merton model of investment.

In Section 4, we prove the the existence of an equilibrium for the model of general equilibrium economy and the optimal strategy of Merton problem of invested will be this obtained equilibrium.

2. The Model

Let $(\Omega, \mathcal{F}, P)$ be the probability space where $\Omega = (\mathbb{R}^N)^\infty$ is the space of events $(\omega_t)_{t\geq 1}$. For $t \in \mathbb{N}^*$, let $\mathcal{F}_t = \sigma(B_s; s \in \{1, 2, ..., t\})$ be the σ-field generated by the canonical mapping process

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$B_t(\omega) = \omega_t$, $t \geq 1$, $\omega \in \Omega$. We denote by $\mathcal{F}_\infty = \sigma(\bigcup_{t\in\mathbb{N}} \mathcal{F}_t)$, where $\mathcal{F}_0 = \{\emptyset, \Omega\}$ is the trivial σ-algebra and by $P : \mathcal{F} \to [0,1]$, the probability measure.

In the discrete time model of this paper, we suppose that the market is with a money market account paying a return $r > 0$ and N risky assets that provide a random return of $R = (R_t)_{t\geq 1}$ with values in $[-1,\infty)^N$ that are supposed to be identically and independently distributed over time. We denote by $(p^j)_{1\leq j\leq N}$, the strictly positive asset price process that will is supposed to satisfy the following condition:

$$p_t^j = p_0^j \prod_{k\geq 1}[1 + R_k^j] \Longleftrightarrow R_t^j = \frac{p_t^j - p_{t-1}^j}{p_{t-1}^j}, \quad j = 1, \cdots, N. \tag{2.1}$$

where p_0^j is the initial stock value. The return vector at time t is given by $R_t(\omega) = B_t(\omega) = \omega_t$, $t \in \mathbb{N}^*$, $j = 1, \cdots, N$. Then R_t's are $\mathcal{F}_t$-measurable and consequently, $R = (R_t)_{t\geq 1}$ is an $(\mathbb{R})^N$-valued, $\mathcal{F}$-adapted process. The process p is an $(\mathbb{R}^+)^N$-valued $\mathcal{F}$-adapted process.

In our multi-investors model, we suppose that there is a finite number m of investors labeled i, $(i = 1, 2, \cdots, m)$. Each investor has to choose a portfolio of assets j, $(j = 0, 1, 2, \cdots, N)$. We denote by $y = (y_{i,t}^j)_{t\geq 1}$, the individual i's process of money invested in the j-th stock at any time t prior to the portfolio adjustment. The riskless asset $x = (x_{i,t})_{t\geq 1}$ will be the process of money invested in the money market account at any time t. Shares are traded at determined price vector $p_t = (p_t^1, \cdots, p_t^N)$. For $t \geq 1$, the process $z_{i,t}$ will denote the number of shares held by the i-th investor at time t with values in $\mathbb{R}^N$ and we have:

$$y_{i,t}^j = z_{i,t}^j p_t^j, \qquad j = 1, \cdots, N, \quad i = 1, \cdots, m, \quad t \geq 1.$$

In our model of markets with frictions, we assume that there is a penalty function $g_i : \mathbb{R}^N \to \mathbb{R}_+^N$ for each investor i due to transaction costs. The dynamics of the riskless asset will be as follow:

$$x_{i,t+1} = \left(x_{i,t} - \alpha_{i,t} \cdot 1 - p_t g_i\big((z_{i,t+1} - z_{i,t})\big) \cdot 1 - c_{i,t}\right)(1 + r), \qquad t \geq 1, \tag{2.2}$$

where the $\mathcal{F}$-adapted process c_i denotes the *consumption* of the i-th investor, and α_i is the portfolio adjustment process given by:

$$\alpha_{i,t}^j := p_t^j \Delta_t z_i^j = p_t^j\,(z_{i,t+1}^j - z_{i,t}^j), \quad j = 1, \ldots, N, \ t \geq 1. \tag{2.3}$$

Note taht rebalancing of portfolio will occur between time t and time $t+1$ and it is easy to see that:

$$y_{i,t+1}^j = \left(y_{i,t}^j + \alpha_{i,t}^j\right)\,(1 + R_{t+1}^j), \tag{2.4}$$

and the mark-to-market value is given by:

$$\omega_{i,t} := x_{i,t} + y_{i,t} \cdot 1 = x_{i,t} + \sum_{j=1}^N y_{i,t}^j$$

3. General equilibrium model of Merton investment problem

Given a portfolio position $(x, y) \in \mathbb{R} \times (\mathbb{R}^+)^N$, the after-liquidation value will be defined as follows:

$$\begin{aligned} L(x_{i,t}, y_{i,t}) &= a_{i,t} + b_{i,t} \cdot 1 - p_t g_i\big((z_{i,t+1} - z_{i,t})\big) \cdot 1 \\ &= x_{i,t} + p_t z_{i,t} \cdot 1 - p_t g_i\big((z_{i,t+1} - z_{i,t})\big) \cdot 1 \end{aligned} \tag{3.1}$$

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and the solvency condition is given by the requirement that $L(x_{i,t}, y_{i,t}) \geq 0$ for all $t \geq 1$, P-almost surely. Hence, our optimal investment problem will be formulated by the following optimization problem:

$$\begin{array}{rcl} Q_i(x,y) & : & \sup_{(c_{i,t}, z_{i,t})} E\left[\sum_{t=0}^{\infty} \rho_i^t u_i(c_{i,t})\right] \\ \text{subject to} & : & x_{i,t} + p_t z_{i,t} \cdot 1 - p_t g_i\big((z_{i,t+1} - z_{i,t})\big) \cdot 1 \geq 0 \quad a.e. \end{array}$$

where for each investor i, u_i is the utility function and ρ_i^t is the impatience parameter.

The infinite-horizon sequence of prices and quantities are given by:

$$(p, (c_i, z_i)_{i=1}^m)$$

where, for each $i = 1, \cdots, m$,

$$(p, c_i, z_i) = \big((p_t)_{t=0}^{+\infty}, (c_{i,t})_{t=0}^{+\infty}, (z_{i,t})_{t=0}^{+\infty}\big) \in (\mathbb{R}_+^{+\infty})^N \times \mathbb{R}_+^{+\infty} \times (\mathbb{R}_+^{+\infty})^N,$$

Now let $\mathcal{E}$ be the economy characterized by:

$$\mathcal{E} = (\mathbb{R}^N, (u_i, \rho_i, z_{i,-1})_{i=1}^m)$$

Equilibrium of this economy is determined by the set of consumption policies and price processes for which each agent maximizes his/her expected utility. More precisely:

Definition 1. *The process* $\big(\bar{p}_t, (\bar{c}_{i,t}, \bar{z}_{i,t})_{i=1}^m\big)_{t=0}^{\infty}$ *is an equilibrium of the economy* $\mathcal{E}$ *if the following conditions are satisfied:*

1. *Price positivity:* $\bar{p}_t > 0$ *for* $t \geq 0$.
2. *Market clearing: at each* $t \geq 0$,

$$\begin{array}{ll} \sum_{i=1}^m & \bar{c}_{i,t} + p_t g_i\big((z_{i,t+1} - z_{i,t}\big) \cdot 1 = \omega_t, \qquad a.e. \\ \sum_{j=1}^N & z_{i,t}^j = 1 \quad a.e., \quad \forall i \in \{1, \cdots, m\}, \\ \sum_{i=1}^m & z_{i,t}^0 = 0 \qquad a.e. \end{array}$$

3. *Optimal consumption plans: for each* i, $\big((\bar{c}_{i,t}, \bar{z}_{i,t})_{i=1}^m\big)_{t=0}^{\infty}$ *is a solution of the problem* $Q_i(x,y)$.

4. Existence of Equilibrium

We will use the following standard assumptions in order to prove the existence of equilibrium:
- Assumption (H1): For each $i = 1, \cdots, m$, u_i is continuously differentiable, strictly increasing and concave function satisfying $u_i(0) = 0$, $u_i'(0) = \infty$.
- Assumption (H2): At initial period 0, $z_{i,-1} \geq 0$, and $z_{i,-1} \neq 0$ for $i = 1, \cdots, m$ with $\sum_{i=1}^m z_{i,-1} = 1_m$.
- Assumption (H3): $g_i : \mathbb{R}^N \to \mathbb{R}_+^N$, is convex with $g_i(0) = 0$ and $g_i \geq 0$ for $i = 1, \cdots, m$.
- Assumption (H4): The utility of each agent i is finite:

$$\sum_{t=0}^{\infty} \rho_i^t u_i(c_{i,t}) < \infty.$$

We now constructed the T-truncated economy $\mathcal{E}^T$as $\mathcal{E}$ in which we suppose that there are no activities from period $T+1$ to the infinity and by using a classical argument, we compactify this economy by using the bounded economy $\mathcal{E}_b^T$ as $\mathcal{E}^T$ in which all random variables are bounded. Consider a finite-horizon bounded economy which goes on for $T+1$ periods: $t = 0, \cdots, T$ with B_c, B_z defined by:

$$\begin{aligned} C_i &:= \left\{(c_{i,0}, \cdots, c_{i,T}) : 0 \le c_{i,t} \le B_c, \quad \forall t \in \{1, \cdots, T\}\right\} = [0, B_c]^{T+1}; \\ Z_i &:= \left\{(z^j_{i,1}, \cdots, z^j_{i,T}) : 0 \le z^j_{i,t} \le B_z, \quad \forall t \in \{1, \cdots, T\}\right\} = [0, B_z]^T. \end{aligned}$$

The solvency set is given by:

$$\mathbb{U}_i^T(x,y) := \left\{(c_i, z_i) \in C_i \times Z_i \ : \ x_{i,t} + p_t z_{i,t} \cdot 1 - p_t g_i(z_{i,t+1} - z_{i,t}) \cdot 1 \ge 0, \ P - a.s\right\}.$$

Now, we define the economy $\mathcal{E}_b^{T,\epsilon}$, for each $\epsilon > 0$ such that $m\epsilon < 1$, by adding ϵ units for each agent at date 0. This condition assure the non-emptiness of the solvency set. Thus, the feasible set of each agent i will be:

$$\begin{aligned} \mathbb{U}_i^{T,\epsilon}(x,y) &:= \Big\{(c_i, z_i) \in \mathbb{R}_+^{T+1} \times (\mathbb{R}_+^{T+1})^N \ : \\ &\quad (x_{i,0} + p_0(z_{i,0} + \epsilon) \cdot 1 - p_0 g_i(-(z_{i,0} + \epsilon)) \cdot 1 - c_{i,0})(1+r) \ge 0, \\ &\quad \text{for each } 1 \le t \le T : x_{i,t} + p_t z_{i,t} \cdot 1 - p_t g_i(z_{i,t+1} - z_{i,t}) \cdot 1 \ge 0, \ P - a.s.\Big\} \end{aligned}$$

$$\begin{aligned} \mathbb{L}_i^{T,\epsilon}(x,y) &:= \Big\{(c_i, z_i) \in \mathbb{R}_+^{T+1} \times (\mathbb{R}_+^{T+1})^N \ : \\ &\quad (x_{i,0} + p_0(z_{i,0} + \epsilon) \cdot 1 - p_0 g_i(-(z_{i,0} + \epsilon)) \cdot 1 - c_{i,0})(1+r) > 0, \\ &\quad \text{for each } 1 \le t \le T : x_{i,t} + p_t z_{i,t} \cdot 1 - p_t g_i(z_{i,t+1} - z_{i,t}) \cdot 1 > 0, \ P - a.s.\Big\} \end{aligned}$$

Lemma 4.1. *The set $\mathbb{L}_i^{T,\epsilon}(x,y)$ is non empty, for $t = 0, \cdots, T$.*

Proof. Indeed,

$$\begin{aligned} & L \quad (x_{i,1}, y_{i,1}) \\ &= \ L\left((x_{i,0} + p_0(z_{i,0} + \epsilon) \cdot 1 - p_0 g_i(-(z_{i,0} + \epsilon)) \cdot 1 - c_{i,0})(1+r), (y^{j\epsilon}_{i,0} + \alpha^{j\epsilon}_{i,0})(1 + R^j_1)\right) \\ &= \ L((x_{i,0} + p_0(z_{i,0} + \epsilon) \cdot 1 - p_0 g_i(-(z_{i,0} + \epsilon)) \cdot 1 - c_{i,0})(1+r), 0) \\ &= \ (x_{i,0} + p_0(z_{i,0} + \epsilon) \cdot 1 - p_0 g_i(-(z_{i,0} + \epsilon)) \cdot 1 - c_{i,0})\,(1+r) \ge 0 \end{aligned}$$

Now, since $\epsilon, (z_{i,0} + \epsilon) > 0$, we can select $c_{i,0} \in (0, B_c)$ and $z_{i,0} \in (0, B_z)$ such that

$$(x_{i,0} + p_0(z_{i,0} + \epsilon) \cdot 1 - p_0 g_i(-(z_{i,0} + \epsilon)) \cdot 1 - c_{i,0})\,(1+r) > 0$$

□

Lemma 4.2. *The set $\mathbb{U}_i^T(x,y)$ has convex values.*

Proof. Now we want to show that $\mathbb{U}(x,y)$ is convex. Take $(c^k_{i,t}, \alpha^k_{i,t}) \in \mathbb{U}(x^k, y^k)$, for $k = 1, 2$ and $t \ge 1$. For $\lambda \in [0,1]$, we note by $\bar{c}_{i,t} = \lambda c^1_{i,t} + (1-\lambda) c^2_{i,t}$ and similarly $\bar{x}_{i,t}$, $\bar{z}_{i,t}$. We have:

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$$\begin{aligned} L(\bar{x}_{i,t}, \bar{y}_{i,t}) &= \bar{x}_{i,t} + \bar{p}_t \bar{z}_{i,t} \cdot 1 - \bar{p}_t g(\bar{z}_{i,t+1} - \bar{z}_{i,t}) \cdot 1 \\ &= \lambda(x^1_{i,t} + \bar{p}_t z^1_{i,t} \cdot 1) + (1-\lambda)(x^2_{i,t} + \bar{p}_t z^2_{i,t} \cdot 1) - \bar{p}_t g(\bar{z}_{i,t+1} - \bar{z}_{i,t}) \cdot 1 \\ &\geq \bar{p}_t[\lambda g(z^1_{i,t+1} - z^1_{i,t}) \cdot 1 + (1-\lambda) g(z^2_{i,t+1} - z^2_{i,t}) \cdot 1 - g(\bar{z}_{i,t+1} - \bar{z}_{i,t}) \cdot 1] \\ &\geq 0 \end{aligned}$$

since g is convex and $(x^k_{i,t}, y^k_{i,t}) \in \mathbb{L}$ for $k = 1, 2$ and $t \geq 1$.
□

For simplicity, we denote $U_i = C_i \times Z_i$.

Lemma 4.3. *$\mathbb{L}^{T,\epsilon}_i(x,y)$ is lower semi-continuous correspondence on U_i and $\mathbb{U}^{T,\epsilon}_i(x,y)$ is upper semi-continuous with compact convex values.*

Proof. Since $\mathbb{L}^{T,\epsilon}_i(x,y)$ is non-empty and has open graph, then it is lower semi-continuous correspondence. Since U_i is compact and the correspondence $\mathbb{U}^{T,\epsilon}_i(x,y)$ has a closed graph, then $\mathbb{U}^{T,\epsilon}_i(x,y)$ is upper semi-continuous with compact values.
□

Definition 2. *The stochastic process $\left(\bar{p}_t, (\bar{c}_{i,t}, \bar{z}_{i,t})^m_{i=1}\right)^T_{t=0}$ is an equilibrium of the economy $\mathcal{E}^T_b$ if it satisfies the following conditions:*

1. *Price positivity: $\bar{p}_t > 0$ for $t = 0, 1, \cdots, T$*
2. *Market clearing:*

$$\begin{aligned} \sum_{i=1}^m \bar{c}_{i,0} + p_0 g_i(-(\bar{z}_{i,0} + \epsilon)) &= \sum_{i=1}^m x_{i,0} + p_0(\bar{z}^j_{i,0} + \epsilon) \cdot \vec{1}, \qquad a.e. \\ \sum_{i=1}^m \bar{c}_{i,t} + p_t g_i(\bar{z}_{i,t+1} - \bar{z}_{i,t}) &= \sum_{i=1}^m x_{i,t} + \bar{p}_t \bar{z}_{i,t} \cdot 1, \qquad a.e. \end{aligned}$$

3. *Optimal consumption plans: for each i, $\left(\bar{c}_{i,t}, \bar{z}_{i,t}\right)^T_{t=1}$ is a solution of the maximization problem of agent i with the feasible set $\mathbb{U}^{T,\epsilon}_i(x,y)$ such that:*

$$Q^{T,\epsilon}_i(x,y) : \sup_{(c_{i,t}, z_{i,t})} \mathbb{E}\left[\sum_{t=0}^T \rho^t u_i(c_{i,t})\right].$$

For $i = 0, \cdots, m$, consider an element $h = (h_i)$ defined on $X := B \times \prod_{i=1}^m U_i$ by

$$h_i = \begin{cases} p & \text{for } i = 0 \\ (c_i, z_i) & \text{for } i = 1, \cdots, m \end{cases}$$

where $B = \{p \in \mathbb{R}^N | \|p\| \leq 1\}$.
Now let φ_0 be the correspondence defined by:

$$\varphi_0 : \prod_{i=1}^m U_i \to 2^B$$

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$$\begin{aligned}\varphi_0((h_i)_{i=0}^m) \quad &:= \quad \arg\max_{p\in B}\Big\{(\sum_{i=1}^m c_{i,0} + p_0 g_i(-(z_{i,0}+\epsilon))\cdot 1 - x_{i,0} - p_0(z_{i,0}+\epsilon)\cdot 1 \\ &+ \quad \sum_{t=1}^T\sum_{i=1}^m c_{i,t} + p_t g_i(z_{i,t+1}-z_{i,t})\cdot 1 - x_{i,t} - p_t z_{i,t}^j \cdot 1\Big\}.\end{aligned}$$

and for each $i = 1, \cdots, m$, consider:

$$\varphi_i : B \to 2^{\mathcal{U}_i}$$

$$\varphi_i(p) := \arg\max_{(c_i,z_i)\in\mathbb{U}(x,y)} \mathbb{E}\left[\sum_{t=0}^T \rho_i^t u_i(c_{i,t})\right].$$

Lemma 4.4. *The correspondence φ_i is upper semi-continuous with non-empty, convex, compact valued for each $i = 1, \cdots, m$.*

Proof. This is a direct consequence of the Maximum Theorem. □

According to the Kakutani Theorem, there exists $\big(\bar{p}, (\bar{c}_{i,t}, \bar{z}_{i,t})\big)$ such that

$$\bar{p} \quad \in \quad \varphi_0\big((\bar{c}_i, \bar{z}_i)_{i=1}^m\big) \tag{4.1}$$
$$(\bar{c}_i, \bar{z}_i) \quad \in \quad \varphi_i(\bar{p}). \tag{4.2}$$

For simplicity, we denote by

$$\begin{aligned}\bar{E}_t &= \sum_{i=1}^m \bar{c}_{i,t} - x_{i,t}, \quad t \geq 0 \\ \bar{F}_0 &= \sum_{i=1}^m g_i(-(\bar{z}_{i,0}^j + \epsilon)) - (\bar{z}_{i,0}^j + \epsilon)\cdot 1 \\ \bar{F}_t &= \sum_{i=1}^m g_i(\bar{z}_{i,t+1} - \bar{z}_{i,t}) - \bar{z}_{i,t}^j \cdot 1, \quad t \geq 1\end{aligned}$$

Lemma 4.5. *Under Assumptions (H1), (H2) and (H3), there exists an equilibrium for the finite-horizon bounded ϵ-economy $\mathcal{E}_b^{T,\epsilon}$.*

Proof. We start proving that $\bar{E}_t + \bar{p}_t\bar{F}_t = 0$ and $\bar{p}_t > 0$ for $t = 0, \cdots, T$. Indeed, From (4.1), one can easily check that for every $p \in B$, we have:

$$\sum_{t=0}^T (p_t - \bar{p}_t)\bar{F}_t \leq 0. \tag{4.3}$$

We recall the solvency constraint,

$$x_{i,t} + \bar{p}_t z_{i,t}\cdot 1 - \bar{p}_t g_i(z_{i,t+1} - z_{i,t})\cdot 1 \geq 0$$

Moreover, the value of an agent's consumption cannot exceed the value of his wealth and the following inequality will be satisfied:

$$x_{i,t} + \bar{p}_t \bar{z}_{i,t} \cdot 1 - \bar{p}_t g_i(\bar{z}_{i,t+1} - \bar{z}_{i,t}) \cdot 1 \geq \bar{c}_{i,t}$$
$$x_{i,t} - \bar{c}_{i,t} + \bar{p}_t \bar{z}_{i,t} \cdot 1 - \bar{p}_t g_i(\bar{z}_{i,t+1} - \bar{z}_{i,t}) \cdot 1 \geq 0 \quad (4.4)$$

By summing the inequality (4.4) over i, we obtain that, for each t:

$$\sum_{i=1}^{m} x_{i,t} - \bar{c}_{i,t} + \bar{p}_t \left[\sum_{i=1}^{m} \bar{z}^j_{i,t} \cdot 1 - g_i(\bar{z}_{i,t+1} - \bar{z}_{i,t}) \cdot 1 \right] \geq 0$$
$$\bar{E}_t + \bar{p}_t \bar{F}_t \leq 0 \quad (4.5)$$

If $\bar{p}_t = 0$, we deduce that $\bar{c}_{i,t} = B_c > \omega_{i,t}$. Therefore for all t, $\sum_{i=1}^{m} \bar{c}_{i,t} > \sum_{i=1}^{m} x_{i,t}$, which contradicts (4.5). Hence, we obtain as a result, $\bar{p}_t > 0$.

Since prices are strictly positive and the utility functions are strictly increasing, all budget constraints are binding. By summing over i) at date t, we obtain:

$$\bar{E}_t + \bar{p}_t \bar{F}_t = 0.$$

Hence, the optimality of $(\bar{c}_i, \bar{z}_i)$ is from (4.2). □

Lemma 4.6. *Suppose Assumptions (H1), (H2) and (H3) are satisfied, then there exists an equilibrium for the finite-horizon bounded economy $\mathcal{E}_b^T$.*

Proof. We have proved that for each $\epsilon = \frac{1}{n} > 0$, where n is an integer and large enough, there exists an equilibrium denoted:

$$equi(n) := \left(\bar{p}(n), (\bar{c}_{i,t}(n), \bar{z}_{i,t}(n))_{i=1}^{m}\right)_{t=0}^{T};$$

for the economy, $\mathcal{E}_b^{T,\epsilon_n}$. Since prices and allocations are bounded, there exists a sub-sequence $(n_1, n_2, \cdots,)$ such that $equi(n_s)$ converges. without loss of generality, we can assume that

$$\left(\bar{p}(n), (\bar{c}_i(n), \bar{z}_i(n))_{i=1}^{m}\right) \to \left(\bar{p}, (\bar{c}_i, \bar{z}_i)_{i=1}^{m}\right)$$

when n tends to infinity. Moreover, by taking limit of market clearing conditions of the $\mathcal{E}_b^{T,\epsilon_n}$, we obtain the corresponding conditions of the bounded truncated economy $\mathcal{E}_b^T$. □

Remark 1. *It should be noticed that at equilibrium, we have $\bar{p}_0 > 0$ according to (2.1).*

Lemma 4.7. *For each i, $(\bar{c}_i, \bar{z}_i)$ is optimal.*

Proof. Since $\sum_{i=1}^{m} z^j_{i,-1} = 1$, for all $j \in \{1, \cdots, N\}$, there exists an agent i such that $z_{i,-1} > 0$. According to Remark 1, we have $\mathbb{L}_i^T(x, y) \neq \emptyset$. We now prove the optimality of $(\bar{c}_i, \bar{z}_i)$.
Let (c_i, z_i) be a feasible allocation of the maximization problem of agent i with the feasible set $\mathbb{U}_i^T(\bar{x}, \bar{y})$.
We should prove that $\mathbb{E}\left[\sum_{t=0}^{T} \rho_i^t u_i(c_{i,t})\right] \leq \mathbb{E}\left[\sum_{t=0}^{T} \rho_i^t u_i(\bar{c}_{i,t})\right]$.

Since $\mathbb{L}_i^T(\bar{x}, \bar{y}) \neq \emptyset$, there exists $(h)_{h \geq 0}$ and $(c_i^h, z_i^h) \in \mathbb{L}_i^T(\bar{x}, \bar{y})$ such that (c_i^h, z_i^h) converges to (c_i, z_i). Then, for each i, we have

$$x_{i,t} + \bar{p}_t z^h_{i,t} \cdot \vec{1} - \bar{p}_t g_i(z^h_{i,t+1} - z^h_{i,t}) > 0.$$

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Fix h. Let n_0 be high enough such that for every $n \geq n_0$, $(c_i^h, z_i^h) \in \mathbb{U}_i^{T,\frac{1}{n}}(\bar{x}(n), \bar{y}(n))$. Then

$$\mathbb{E}\left[\sum_{t=0}^{T} \rho_i^t u_i(c_{i,t}^h)\right] \leq \mathbb{E}\left[\sum_{t=0}^{T} \rho_i^t u_i(\bar{c}_{i,t}(n))\right].$$

Tend n tend to infinity, we obtain

$$\mathbb{E}\left[\sum_{t=0}^{T} \rho_i^t u_i(c_{i,t}^h)\right] \leq \mathbb{E}\left[\sum_{t=0}^{T} \rho_i^t u_i(\bar{c}_{i,t})\right].$$

Let tends h to infinity, we obtain $\mathbb{E}\left[\sum_{t=0}^{T} \rho_i^t u_i(c_{i,t})\right] \leq \mathbb{E}\left[\sum_{t=0}^{T} \rho_i^t u_i(\bar{c}_{i,t})\right]$.

We have just showed that $(\bar{c}_i, \bar{z}_i)$ is an optimal solution. We now prove that $\bar{p}_t > 0$ for every t. Indeed, if $\bar{p}_t = 0$, the optimality of $(\bar{c}_i, \bar{z}_i)$ implies that $\bar{c}_{i,t} = B_c > x_{i,t}$, contradiction.
□

After proving the existence of the equilibrium when ϵ tends to 0, we deduce that this equilibrium holds for the truncated unbounded economy.

Lemma 4.8. *An equilibrium for $\mathcal{E}_b^T$ is an equilibrium for $\mathcal{E}^T$.*

Proof. Let $\left(\bar{p}_t, (\bar{c}_{i,t}, \bar{z}_{i,t})_{i=1}^m\right)_{t=0}^T$ be an equilibrium of $\mathcal{E}_b^T$. Note that $z_{i,T+1} = 0$ for every $i = 1, \cdots T$. We can see that conditions (i) and (ii) in Definition (2) are satisfied. We will show that condition (iii) is also verified. Let $a_i := (\bar{c}_{i,t}, \bar{z}_{i,t})_{t=0}^T$ be a feasible plan of agent i. Suppose that $\sum_{t=0}^T \rho_i^t u_i(c_{i,t}) > \sum_{t=0}^T \rho_i^t u_i(\bar{c}_{i,t})$. For each $\gamma \in (0,1)$, we define $a_i(\gamma) := \gamma a_i + (1-\gamma)\bar{a}_i$. By definition of $\mathcal{E}_b^T$,we can choose γ sufficiently close to 0 such that $a_i(\gamma) \in C_i \times Z_i$. It is clear that $a_i(\gamma)$ is a feasible allocation. By the concavity of the utility function, we have

$$\begin{aligned}\sum_{t=0}^{T} \rho_i^t u_i(c_{i,t}(\gamma)) &\geq \gamma \sum_{t=0}^{T} \rho_i^t u_i(c_{i,t}) + (1-\gamma) \sum_{t=0}^{\infty} \rho_i^t u_i(\bar{c}_{i,t}) \\ &> \sum_{t=0}^{T} \rho_i^t u_i(\bar{c}_{i,t})\end{aligned}$$

We deduce that:

$$\mathbb{E}\left[\sum_{t=0}^{T} \rho_i^t u_i(c_{i,t}(\gamma))\right] > \mathbb{E}\left[\sum_{t=0}^{T} \rho_i^t u_i(\bar{c}_{i,t})\right]$$

which contradicts the optimality of $\bar{a}_i$.

We denote by $(\bar{p}^T, (\bar{c}_i^T, \bar{z}_i^T)_{i=1}^m)$ an equilibrium of the T-truncated economy $\mathcal{E}^T$. Since $\|\bar{p}_t\| \leq 1$, for every $t \leq T$, $\bar{c}_i^T \leq B_c$ and$\sum_{i=1}^m \bar{z}_i^T = 1$.Thus, we can assume that:

$$(\bar{p}^T, (\bar{c}_i^T, \bar{z}_i^T)_{i=1}^m) \longrightarrow (\bar{p}, (\bar{c}_i, \bar{z}_i)_{i=1}^m)$$

when T goes to infinity.
One can easily check that all markets clear.

Now we can give the main result of this paper:

Theorem 4.1. *If hypothesis (H1), (H2),(H3) and (H4) are satisfied, then there exists an equilibrium of the infinite horizon economy $\mathcal{E}$.*

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Proof. We have proved previously that for each $T \geq 1$, there exists an equilibrium for the economy $\mathcal{E}^T$. Let (c_i, z_i) be a feasible allocation of the problem $Q_i(\bar{p}, \bar{z})$. We will prove that $\mathbb{E}\left[\sum_{t=0}^{\infty} \rho_i^t u_i(c_{i,t})\right] \leq \mathbb{E}\left[\sum_{t=0}^{\infty} \rho_i^t u_i(\bar{c}_{i,t})\right]$.

We define $(c_i', z_i')_{t=0}^T$ as follows:

$$\begin{aligned} z'_{i,t} &= z_{i,t} \quad \text{if } t \leq T-1, \\ c'_{i,t} &= c_{i,t} \quad \text{if } t \leq T-1, \\ c_{i,t} &= z_{i,t} = 0 \quad \text{if } t > T \\ x_{i,T} + \bar{p}_T z'_{i,T} - \bar{p}_T g_i(-z'_{i,T}) &= x_{i,T} + \bar{p}_T z_{i,T} - \bar{p}_T g_i(-z_{i,T}) \end{aligned}$$

We can see that $(c_i', z_i')_{t=0}^T \in \mathbb{U}_i^T(\bar{x}, \bar{y})$.
Since $\mathbb{L}_i^T(\bar{x}, \bar{y}) \neq \emptyset$, there exists a sequence $((c_i^n, z_i^n)_{t=0}^T)_{n=0}^{\infty} \in \mathbb{L}_i^T(\bar{x}, \bar{y})$ with $z_{i,T+1}^n = 0$ and this sequence converges to $(c_i', z_i')_{t=0}^T$ when n tends to infinity . We have

$$x_{i,t}^n + \bar{p}_t z_{i,t}^n - \bar{p}_t g_i(z_{i,t+1}^n - z_{i,t}^n) > 0.$$

We can choose s_0 high enough such that $s_0 > T$ and for every $s \geq s_0$, we have

$$x_{i,t}^n + \bar{p}_t^s z_{i,t}^n - \bar{p}_t^s g_i(z_{i,t+1}^n - z_{i,t}^n) > 0.$$

Consequently $(c_i^n, z_i^n)_{t=0}^T \in \mathbb{U}_i^T(\bar{x}^s, \bar{y}^s)$. Therefore, we get

$$\sum_{t=0}^{T} \rho_i^t u_i(c_{i,t}^n) \leq \sum_{t=0}^{s} \rho_i^t u_i(\bar{c}_{i,t}^s)$$

. Tends s to infinity, we obtain $\sum_{t=0}^{T} \rho_i^t u_i(c_{i,t}^n) \leq \sum_{t=0}^{\infty} \rho_i^t u_i(\bar{c}_{i,t})$. Now, if we tend n tend to infinity, we obtain $\sum_{t=0}^{T} \rho_i^t u_i(c_{i,t}) \leq \sum_{t=0}^{\infty} \rho_i^t u_i(\bar{c}_{i,t})$ for every T. Consequently:

$$\sum_{t=0}^{T-1} \rho_i^t u_i(c_{i,t}) \leq \sum_{t=0}^{\infty} \rho_i^t u_i(\bar{c}_{i,t}).$$

Let T tend to infinity, we obtain

$$\sum_{t=0}^{\infty} \rho_i^t u_i(c_{i,t}) \leq \sum_{t=0}^{\infty} \rho_i^t u_i(\bar{c}_{i,t}).$$

Then,

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \rho_i^t u_i(c_{i,t})\right] \leq \mathbb{E}\left[\sum_{t=0}^{\infty} \rho_i^t u_i(\bar{c}_{i,t})\right].$$

Hence, we have proved the optimality of $(\bar{c}_i, \bar{z}_i)$. Note that prices $\bar{p}_t$ are strictly positive since the utility function of agent i is strictly increasing.
□

The obtained equilibrium is the expected optimal strategy of Merton investment problem in the case of multi-investors and in markets with frictions formulated by a penalty functions for every investor due to loss in trading. Our model and main result extends models and results obtained by Chebbi and Soner in [1] and by Ounaies, Bonnisseau, Chebbi and Soner in [15]

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