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Article

Existence of Solutions to Fractional Differential Inclusions with Non-Separated and Multipoint Boundary Constraints

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Abstract

A novel and generalized class of fractional differential inclusions has been investigated under a combination of nonlocal, non-separated, multipoint, and integral boundary conditions. Fractional derivatives have been employed to capture memory effects and enhance the modeling accuracy of real-world phenomena. Uncertainty has been addressed through the formulation of set-valued differential inclusions. The proposed boundary conditions have been designed to encompass a wide spectrum of both local and nonlocal types previously reported in the literature. The existence of solutions has been established via the Bohnenblust–Karlin fixed point theorem. The applicability and effectiveness of the main results have been demonstrated through concrete examples.

Keywords: fractional calculus; non-separated boundary condition; fractional BVP; existence theorems

MSC: 26A33, 34A08, 34A60, 34B15

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1. Introduction

In recent years, the study of fractional differential inclusions has attracted significant attention due to their capacity to model uncertain and memory-dependent phenomena across various scientific fields. In particular, Shahid et al. [25] investigated nonlinear fractional differential inclusions involving fuzzy set-valued mappings of q-rung orthopair (q-ROF), establishing new existence results through fixed point theory. Recent research has expanded the analytical foundation of differential and integro-differential inclusions, particularly in the context of fractional and delayed systems. For example, Sasikumar and Vijayakumar [24] investigated approximate controllability results for second-order differential inclusions with state-dependent delay, highlighting the impact of damping in dynamic systems. Alshehri et al. [4] tackled delay integro-differential inclusions with practical applications, extending the solution framework for real-world systems involving memory effects. Bensalem et al. [5] studied infinite delay in second-order integro-differential inclusions, offering results on the topological structure of solution sets. In a coupled framework, Ma et al. [17] addressed existence results for nonlinear Hilfer fractional differential systems, combining differential equations and inclusions. A comprehensive survey on boundary value problems involving Hilfer-type fractional inclusions was presented by Ntouyas [19], offering a detailed classification of various solution methods. Khan et al. [13] proposed an abstract formulation for delayed fractional integro-differential systems using the α-resolvent operator, thereby making a notable contribution to the theory of fractional abstract Cauchy problems. In a related development, Jing et al. [12] examined hemivariational inequalities governed by Hilfer fractional differential evolution equations with history-dependent operators, which significantly enriched the theory of inclusion problems involving nonsmooth analysis and memory-based dynamics.

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With these advances, fractional differential equations (FDEs) have gained increasing relevance in modeling systems that exhibit memory and long-range dependence. Unlike classical models relying on standard derivatives, FDEs employ fractional (non-integer order) derivatives, enabling more accurate descriptions of phenomena such as viscoelasticity, anomalous diffusion, and biological processes [14,18]. Their wide applicability spans disciplines such as physics, engineering, biology, and economics. When coupled with differential inclusion theory, fractional calculus offers a more robust analytical framework, facilitating deeper investigation into existence, stability, and control of solutions under broad settings.

A general form of a fractional differential equation with boundary conditions is given by:

$$ {}^{c}D^{m}u(\xi) = \mathcal{F}(\xi, u(\xi)), \quad \text{where } 1 < m \leq 2, $$

where ${}^{c}D^{m}$ denotes the Caputo fractional derivative and $\mathcal{F}(\xi, u(\xi))$ is a given function. The boundary conditions may incorporate both the function and its fractional derivatives, for instance:

$$ au(0) + bu(1) = c, \quad a_1{}^{c}D^{p}u(0) + b_1{}^{c}D^{p}u(1) = c_1, \quad p \in (0,1]. $$

These boundary conditions impose essential constraints within a prescribed domain. The study of boundary value problems (BVPs) for FDEs represents an important field of mathematical analysis, with both theoretical and practical implications. Solving such problems often necessitates sophisticated mathematical tools since the inclusion of fractional-order derivatives at the boundaries introduces additional complexity [22**?**].

A powerful tool in this context is the Bohnenblust-Karlin fixed point theorem, which is widely used to prove existence results in set-valued analysis. This theorem guarantees the existence of fixed points for certain multivalued mappings defined on convex subsets of Banach spaces. Its application becomes especially valuable in the framework of differential inclusions, where the derivative of an unknown function is allowed to take values in a set. Such inclusions frequently appear in models involving uncertainties—examples include problems in economics, biological systems, and control theory [1,8].

We generalize the analysis by considering the nonlinear fractional differential inclusion:

$$ {}^{c}D^{m}u(\xi) \in \mathcal{F}(\xi, u(\xi)), \quad \text{where } 1 < m \leq 2, \tag{1} $$

subject to boundary conditions

$$ a_1u(0) + a_2u(1) = a_3\sum_{i=1}^{m-2}\psi_i u(\sigma_i) + a_4\sum_{j=1}^{p-2}\lambda_j\int_{\alpha}^{\beta}u(\tau)\,d\tau, \tag{2} $$

$$ b_1{}^{c}D^{p}u(0) + b_2{}^{c}D^{p}u(1) = b_3\sum_{i=1}^{m-2}\rho_i{}^{c}D^{p}u(\sigma_i) + b_4\sum_{j=1}^{p-2}\gamma_j\int_{\alpha}^{\beta}{}^{c}D^{p}u(\tau)\,d\tau, \quad 0 < p \leq 1. \tag{3} $$

Here, $\mathcal{F}(\xi, u(\xi))$ denotes a multivalued (set-valued) mapping that characterizes the possible values of the derivative. This generalization permits the modeling of systems whose evolution may not be uniquely determined and reflects multiple potential outcomes depending on system states.

The analysis of BVPs involving multivalued maps and fractional derivatives is intricate and often requires fixed-point techniques adapted to the multivalued setting [10,20]. The presence of integral and multipoint conditions involving both the function and its fractional derivatives further increases the analytical challenges. Yet, this formulation enhances the flexibility and realism of the models, making them more applicable to real-world phenomena.

The growing interest in these models is largely motivated by the limitations of classical integer-order differential equations when applied to systems with memory. Many real-world problems, especially in engineering and physics, inherently involve memory effects, which are more naturally

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described using fractional operators [26]. Moreover, the use of multivalued maps enables modeling of systems with uncertainty or nondeterminism, which frequently arise in areas such as decision theory, neural dynamics, and material science [7,9].

This paper focuses on proving the existence of solutions to a class of fractional differential inclusions governed by complex boundary conditions incorporating both fractional derivatives and multivalued terms. The Bohnenblust-Karlin fixed point theorem will be employed to derive sufficient conditions ensuring the existence of solutions. In doing so, this work extends existing results on fractional differential equations and broadens the analytical scope for solving practical problems in applied sciences [20].

2. Main Results

Before we proceed to the existence result for the nonlinear inclusion, we first consider the solution of the associated linear problem. For a given function $\Phi \in C[0,1]$, the solution of a linear fractional differential equation.

The following lemma is crucial for the existence of solutions to the proposed differential inclusion.

Lemma 1. *For $\Phi \in C[0,1]$, the solution of the following linear problem*

$$ {}^{c}D^{m}u(\jmath) = \Phi(\jmath), \text{ for } 0 < \jmath < 1, $$

with boundary conditions (2.2) *is*

$$
\begin{aligned}
u(\jmath) &= \int_0^{\jmath} \frac{(\jmath-\ell)^{q-1}}{\Gamma(q)}\Phi(\ell)d\ell \\
&\quad -\frac{1}{\hbar_3}\left[\frac{\hbar_2}{\hbar_1}\right]\left[\begin{array}{c} -b_2\int_0^1 \frac{(1-\ell)^{p-q-1}}{\Gamma(p-q)}\Phi(\ell)d\ell + b_3\sum_{i=1}^{m-2}\rho_i\int_0^{\sigma_i}\frac{(\sigma_i-\ell)^{p-q-1}}{\Gamma(p-q)}\Phi(\ell)d\ell \\ +b_4\sum_{j=1}^{p-2}\gamma_j\int_\alpha^\beta\int_0^\tau \frac{(\tau-\ell)^{p-q-1}}{\Gamma(p-q)}\Phi(\ell)d\ell d\tau \end{array}\right] \\
&\quad +\frac{(1+\jmath)}{\hbar_1}\left[\begin{array}{c} -a_2\int_0^1 \frac{(1-\ell)^{q-1}}{\Gamma(q)}\Phi(\ell)d\ell + a_3\sum_{i=1}^{m-2}\psi_i\int_0^{\sigma_i}\frac{(\sigma_i-\ell)^{q-1}}{\Gamma(q)}\Phi(\ell)d\ell \\ +a_4\sum_{j=1}^{p-2}\lambda_j\int_\alpha^\beta\int_0^\tau \frac{(\tau-\ell)^{q-1}}{\Gamma(q)}\Phi(\ell)d\ell d\tau \end{array}\right]
\end{aligned}
$$

$$ u(\jmath) = \int_0^1 G(\jmath,\ell)\Phi(\ell)d\ell + \delta(\jmath) $$

where

$$ G(\jmath,\ell) = \begin{cases} \frac{\hbar_1(\jmath-\ell)^{q-1} - a_2(1+\jmath)(1-\ell)^{q-1}}{\hbar_1\Gamma(q)} + \frac{b_2\hbar_2(1-\ell)^{p-q-1}}{\hbar_3\hbar_1\Gamma(p-q)}, & \ell \le \jmath \\ \frac{b_2\hbar_2(1-\ell)^{p-q-1}}{\hbar_3\hbar_1\Gamma(p-q)} - \frac{a_2(1+\jmath)(1-\ell)^{q-1}}{\hbar_1\Gamma(q)} & \jmath \le 1 \end{cases} $$

$$
\begin{aligned}
\delta(\jmath) &= -\frac{1}{\hbar_3}\left[\frac{\hbar_2}{\hbar_1}\right] b_3\sum_{i=1}^{m-2}\rho_i\int_0^{\sigma_i}\frac{(\sigma_i-\ell)^{p-q-1}}{\Gamma(p-q)}\Phi(\ell)d\ell \\
&\quad -\frac{1}{\hbar_3}\left[\frac{\hbar_2}{\hbar_1}\right] b_4\sum_{j=1}^{p-2}\gamma_j\int_\alpha^\beta\int_0^\tau\frac{(\tau-\ell)^{p-q-1}}{\Gamma(p-q)}\Phi(\ell)d\ell d\tau \\
&\quad +\frac{(1+\jmath)}{\hbar_1}a_3\sum_{i=1}^{m-2}\psi_i\int_0^{\sigma_i}\frac{(\sigma_i-\ell)^{q-1}}{\Gamma(q)}\Phi(\ell)d\ell \\
&\quad +\frac{(1+\jmath)}{\hbar_1}a_4\sum_{j=1}^{p-2}\lambda_j\int_\alpha^\beta\int_0^\tau\frac{(\tau-\ell)^{q-1}}{\Gamma(q)}\Phi(\ell)d\ell d\tau
\end{aligned}
$$

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and

$$K_1 = a_3 \sum_{i=1}^{m-2} \psi_i + a_4 \sum_{j=1}^{p-2} \lambda_j (\beta - \alpha)$$

$$K_2 = a_3 \sum_{i=1}^{m-2} \psi_i d_2(\sigma_i) + a_4 \sum_{j=1}^{p-2} \lambda_j d_2 \frac{(\beta-\alpha)^2}{2}$$

$$K_3 = b_3 \sum_{i=1}^{m-2} \rho_i \frac{(\sigma_i)^{1-p}}{\Gamma(2-p)} + b_4 \sum_{j=1}^{p-2} \gamma_j \frac{(\beta-\alpha)^{2-p}}{\Gamma(3-p)}$$

$$\hbar_1 = a_1 + a_2 - K_1$$

$$\hbar_2 = a_2 - K_2$$

$$\hbar_3 = b_2 \frac{1}{\Gamma(2-p)} - K_3$$

$$\zeta = b_2 - b_3 \sum_{i=1}^{m-2} \rho_i - b_4 \sum_{j=1}^{p-2} \gamma_j.$$

$$\eta = \frac{\hbar_1}{2} - a_2 + a_3 \sum_{i=1}^{m-2} \psi_i + a_4 \sum_{j=1}^{p-2} \lambda_j (\beta - \alpha).$$

Proof. The general solution for linear problem

$$^cD^q u(\jmath) = \Phi(\jmath),\ 0 < \jmath < 1 \text{ and } 1 < q \le 2,$$

is written as

$$u(\jmath) = \int_0^\jmath \frac{(\jmath - \ell)^{q-1}}{\Gamma(q)} \Phi(\ell) d\ell + d_1 + d_2 \jmath$$

for some real constants d_1 and d_2. Now using first boundary condition from (2.2), i.e.,

$$a_1 u(0) + a_2 u(1) = a_3 \sum_{i=1}^{m-2} \psi_i u(\sigma_i) + a_4 \sum_{j=1}^{p-2} \lambda_j \int_\alpha^\beta u(\tau) d\tau$$

and obtain,

$$\begin{aligned} & d_1 a_1 + a_2 \int_0^1 \frac{(1-\ell)^{q-1}}{\Gamma(q)} \Phi(\ell) d\ell + d_1 a_2 + d_2 a_2 \\ = \quad & a_3 \sum_{i=1}^{m-2} \psi_i \left[\int_0^{\sigma_i} \frac{(\sigma_i - \ell)^{q-1}}{\Gamma(q)} \Phi(\ell) d\ell + d_1 + d_2(\sigma_i) \right] \\ & a_4 \sum_{j=1}^{p-2} \lambda_j \int_\alpha^\beta \left[\int_0^\tau \frac{(\tau-\ell)^{q-1}}{\Gamma(q)} \Phi(\ell) d\ell + d_1 + d_2 \tau \right] d\tau \end{aligned}$$

$$\begin{aligned} & d_1 a_1 + a_2 \int_0^1 \frac{(1-\ell)^{q-1}}{\Gamma(q)} \Phi(\ell) d\ell + d_1 a_2 + d_2 a_2 \\ = \quad & +a_3 \sum_{i=1}^{m-2} \psi_i \int_0^{\sigma_i} \frac{(\sigma_i - \ell)^{q-1}}{\Gamma(q)} \Phi(\ell) d\ell + a_3 \sum_{i=1}^{m-2} \psi_i d_1 + a_3 \sum_{i=1}^{m-2} \psi_i d_2(\sigma_i) \\ & +a_4 \sum_{j=1}^{p-2} \lambda_j \int_\alpha^\beta \int_0^\tau \frac{(\tau-\ell)^{q-1}}{\Gamma(q)} \Phi(\ell) d\ell d\tau + a_4 \sum_{j=1}^{p-2} \lambda_j d_1 \int_\alpha^\beta d\tau + a_4 \sum_{j=1}^{p-2} \lambda_j d_2 \int_\alpha^\beta \tau d\tau \end{aligned}$$

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simplifying above equation, we have Further simplification gives

$$\begin{aligned} & d_1 a_1 + a_2 \int_0^1 \frac{(1-\ell)^{q-1}}{\Gamma(q)} \Phi(\ell) d\ell + d_1 a_2 + d_2 a_2 \\ = \; & a_3 \sum_{i=1}^{m-2} \psi_i \int_0^{\sigma_i} \frac{(\sigma_i-\ell)^{q-1}}{\Gamma(q)} \Phi(\ell) d\ell + a_3 \sum_{i=1}^{m-2} \psi_i d_1 + + a_3 \sum_{i=1}^{m-2} \psi_i d_2(\sigma_i) \\ & a_4 \sum_{j=1}^{p-2} \lambda_j \int_\alpha^\beta \int_0^\tau \frac{(\tau-\ell)^{q-1}}{\Gamma(q)} \Phi(\ell) d\ell d\tau + a_4 \sum_{j=1}^{p-2} \lambda_j d_1 (\beta-\alpha) + a_4 \sum_{j=1}^{p-2} \lambda_j d_2 \frac{(\beta-\alpha)^2}{2} \end{aligned}$$

$$\begin{aligned} & d_1 a_1 + d_1 a_2 - a_3 \sum_{i=1}^{m-2} \psi_i d_1 - a_4 \sum_{j=1}^{p-2} \lambda_j d_1 (\beta-\alpha) \\ = \; & -a_2 \int_0^1 \frac{(1-\ell)^{q-1}}{\Gamma(q)} \Phi(\ell) d\ell + a_3 \sum_{i=1}^{m-2} \psi_i \int_0^{\sigma_i} \frac{(\sigma_i-\ell)^{q-1}}{\Gamma(q)} \Phi(\ell) d\ell + a_3 \sum_{i=1}^{m-2} \psi_i d_2(\sigma_i) \\ & -d_2 a_2 + a_4 \sum_{j=1}^{p-2} \lambda_j \int_\alpha^\beta \int_0^\tau \frac{(\tau-\ell)^{q-1}}{\Gamma(q)} \Phi(\ell) d\ell d\tau + a_4 \sum_{j=1}^{p-2} \lambda_j d_2 \frac{(\beta-\alpha)^2}{2} \end{aligned}$$

$$\begin{aligned} & d_1 \left[a_1 + a_2 - a_3 \sum_{i=1}^{m-2} \psi_i - a_4 \sum_{j=1}^{p-2} \lambda_j (\beta-\alpha) \right] \\ = \; & -a_2 \int_0^1 \frac{(1-\ell)^{q-1}}{\Gamma(q)} \Phi(\ell) d\ell + a_3 \sum_{i=1}^{m-2} \psi_i \int_0^{\sigma_i} \frac{(\sigma_i-\ell)^{q-1}}{\Gamma(q)} \Phi(\ell) d\ell + a_3 \sum_{i=1}^{m-2} \psi_i d_2(\sigma_i) \\ & -d_2 a_2 + a_4 \sum_{j=1}^{p-2} \lambda_j \int_\alpha^\beta \int_0^\tau \frac{(\tau-\ell)^{q-1}}{\Gamma(q)} \Phi(\ell) d\ell d\tau + a_4 \sum_{j=1}^{p-2} \lambda_j d_2 \frac{(\beta-\alpha)^2}{2} \end{aligned}$$

$$\begin{aligned} & d_1 [a_1 + a_2 - K_1] \\ = \; & -a_2 \int_0^1 \frac{(1-\ell)^{q-1}}{\Gamma(q)} \Phi(\ell) d\ell + a_3 \sum_{i=1}^{m-2} \psi_i \int_0^{\sigma_i} \frac{(\sigma_i-\ell)^{q-1}}{\Gamma(q)} \Phi(\ell) d\ell + a_3 \sum_{i=1}^{m-2} \psi_i d_2(\sigma_i) \\ & -d_2 a_2 + a_4 \sum_{j=1}^{p-2} \lambda_j \int_\alpha^\beta \int_0^\tau \frac{(\tau-\ell)^{q-1}}{\Gamma(q)} \Phi(\ell) d\ell d\tau + + a_4 \sum_{j=1}^{p-2} \lambda_j d_2 \frac{(\beta-\alpha)^2}{2} \end{aligned}$$

$$\begin{aligned} d_1 &= \frac{1}{\hbar_1} \left[-d_2 a_2 + a_3 \sum_{i=1}^{m-2} \psi_i d_2(\sigma_i) + a_4 \sum_{j=1}^{p-2} \lambda_j d_2 \frac{(\beta-\alpha)^2}{2} \right] \\ & + \frac{1}{\hbar_1} \left[\begin{array}{c} -a_2 \int_0^1 \frac{(1-\ell)^{q-1}}{\Gamma(q)} \Phi(\ell) d\ell + a_3 \sum_{i=1}^{m-2} \psi_i \int_0^{\sigma_i} \frac{(\sigma_i-\ell)^{q-1}}{\Gamma(q)} \Phi(\ell) d\ell \\ + a_4 \sum_{j=1}^{p-2} \lambda_j \int_\alpha^\beta \int_0^\tau \frac{(\tau-\ell)^{q-1}}{\Gamma(q)} \Phi(\ell) d\ell d\tau \end{array} \right] \end{aligned}$$

$$\begin{aligned} d_1 &= \frac{-d_2}{\hbar_1} \left[a_2 - a_3 \sum_{i=1}^{m-2} \psi_i d_2(\sigma_i) - a_4 \sum_{j=1}^{p-2} \lambda_j d_2 \frac{(\beta-\alpha)^2}{2} \right] \\ & + \frac{1}{\hbar_1} \left[\begin{array}{c} -a_2 \int_0^1 \frac{(1-\ell)^{q-1}}{\Gamma(q)} \Phi(\ell) d\ell + a_3 \sum_{i=1}^{m-2} \psi_i \int_0^{\sigma_i} \frac{(\sigma_i-\ell)^{q-1}}{\Gamma(q)} \Phi(\ell) d\ell \\ + a_4 \sum_{j=1}^{p-2} \lambda_j \int_\alpha^\beta \int_0^\tau \frac{(\tau-\ell)^{q-1}}{\Gamma(q)} \Phi(\ell) d\ell d\tau \end{array} \right] \end{aligned}$$

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$$d_1 = \frac{-d_2}{\hbar_1}\left[a_2 - a_3\sum_{i=1}^{m-2}\psi_i d_2(\sigma_i) - a_4\sum_{j=1}^{p-2}\lambda_j d_2\frac{(\beta-\alpha)^2}{2}\right]$$
$$+\frac{1}{\hbar_1}\left[\begin{array}{c} -a_2\int_0^1 \frac{(1-\ell)^{q-1}}{\Gamma(q)}\Phi(\ell)d\ell + a_3\sum_{i=1}^{m-2}\psi_i\int_0^{\sigma_i}\frac{(\sigma_i-\ell)^{q-1}}{\Gamma(q)}\Phi(\ell)d\ell \\ +a_4\sum_{j=1}^{p-2}\lambda_j\int_\alpha^\beta\int_0^\tau\frac{(\tau-\ell)^{q-1}}{\Gamma(q)}\Phi(\ell)d\ell d\tau \end{array}\right]$$

$$d_1 = -d_2\left[\frac{a_2-K_2}{\hbar_1}\right]$$
$$+\frac{1}{\hbar_1}\left[\begin{array}{c} -a_2\int_0^1 \frac{(1-\ell)^{q-1}}{\Gamma(q)}\Phi(\ell)d\ell + a_3\sum_{i=1}^{m-2}\psi_i\int_0^{\sigma_i}\frac{(\sigma_i-\ell)^{q-1}}{\Gamma(q)}\Phi(\ell)d\ell \\ +a_4\sum_{j=1}^{p-2}\lambda_j\int_\alpha^\beta\int_0^\tau\frac{(\tau-\ell)^{q-1}}{\Gamma(q)}\Phi(\ell)d\ell d\tau \end{array}\right]$$

$$d_1 = -d_2\left[\frac{\hbar_2}{\hbar_1}\right]$$
$$+\frac{1}{\hbar_1}\left[\begin{array}{c} -a_2\int_0^1 \frac{(1-\ell)^{q-1}}{\Gamma(q)}\Phi(\ell)d\ell + a_3\sum_{i=1}^{m-2}\psi_i\int_0^{\sigma_i}\frac{(\sigma_i-\ell)^{q-1}}{\Gamma(q)}\Phi(\ell)d\ell \\ +a_4\sum_{j=1}^{p-2}\lambda_j\int_\alpha^\beta\int_0^\tau\frac{(\tau-\ell)^{q-1}}{\Gamma(q)}\Phi(\ell)d\ell d\tau \end{array}\right]$$

$$u(\jmath) = \int_0^\jmath \frac{(\jmath-\ell)^{q-1}}{\Gamma(q)}\Phi(\ell)d\ell + d_1 + d_2\jmath$$

$$^cD^p u(\jmath) = \int_0^\jmath \frac{(\jmath-\ell)^{p-q-1}}{\Gamma(p-q)}\Phi(\ell)d\ell + d_2\frac{\jmath^{(1-p)}}{\Gamma(2-p)}$$

with second boundary condition

$$b_1\,{}^cD^p u(0) + b_2\,{}^cD^p u(1) = b_3\sum_{\imath=1}^{m-2}\rho_i\,{}^cD^p u(\sigma_i) + b_4\sum_{j=1}^{p-2}\gamma_j\int_\alpha^\beta {}^cD^p u(\tau)d\tau,\ 0<p\le 1.$$

$$b_2\left[\int_0^1\frac{(1-\ell)^{p-q-1}}{\Gamma(p-q)}\Phi(\ell)d\ell + d_2\frac{1}{\Gamma(2-p)}\right]$$
$$= b_3\sum_{i=1}^{m-2}\rho_i\left[\int_0^{\sigma_i}\frac{(\sigma_i-\ell)^{p-q-1}}{\Gamma(p-q)}\Phi(\ell)d\ell + d_2\frac{(\sigma_i)^{1-p}}{\Gamma(2-p)}\right]$$
$$+b_4\sum_{j=1}^{p-2}\gamma_j\int_\alpha^\beta\left[\int_0^\tau\frac{(\tau-\ell)^{p-q-1}}{\Gamma(p-q)}\Phi(\ell)d\ell + d_2\frac{(\tau)^{1-p}}{\Gamma(2-p)}\right]d\tau$$

$$b_2\int_0^1\frac{(1-\ell)^{p-q-1}}{\Gamma(p-q)}\Phi(\ell)d\ell + b_2d_2\frac{1}{\Gamma(2-p)}$$
$$= b_3\sum_{i=1}^{m-2}\rho_i\int_0^{\sigma_i}\frac{(\sigma_i-\ell)^{p-q-1}}{\Gamma(p-q)}\Phi(\ell)d\ell + d_2b_3\sum_{i=1}^{m-2}\rho_i\frac{(\sigma_i)^{1-p}}{\Gamma(2-p)}$$
$$+b_4\sum_{j=1}^{p-2}\gamma_j\int_\alpha^\beta\int_0^\tau\frac{(\tau-\ell)^{p-q-1}}{\Gamma(p-q)}\Phi(\ell)d\ell d\tau + d_2b_4\sum_{j=1}^{p-2}\gamma_j\int_\alpha^\beta\frac{(\tau)^{1-p}}{\Gamma(2-p)}d\tau$$

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$$\begin{aligned}
& b_2 \int_0^1 \frac{(1-\ell)^{p-q-1}}{\Gamma(p-q)} \Phi(\ell) d\ell + b_2 d_2 \frac{1}{\Gamma(2-p)} \\
= \; & b_3 \sum_{i=1}^{m-2} \rho_i \int_0^{\sigma_i} \frac{(\sigma_i-\ell)^{p-q-1}}{\Gamma(p-q)} \Phi(\ell) d\ell + d_2 b_3 \sum_{i=1}^{m-2} \rho_i \frac{(\sigma_i)^{1-p}}{\Gamma(2-p)} \\
& + b_4 \sum_{j=1}^{p-2} \gamma_j \int_\alpha^\beta \int_0^\tau \frac{(\tau-\ell)^{p-q-1}}{\Gamma(p-q)} \Phi(\ell) d\ell d\tau + d_2 b_4 \sum_{j=1}^{p-2} \gamma_j \frac{(\beta-\alpha)^{2-p}}{\Gamma(3-p)}
\end{aligned}$$

$$\begin{aligned}
& b_2 d_2 \frac{1}{\Gamma(2-p)} - d_2 b_3 \sum_{i=1}^{m-2} \rho_i \frac{(\sigma_i)^{1-p}}{\Gamma(2-p)} - d_2 b_4 \sum_{j=1}^{p-2} \gamma_j \frac{(\beta-\alpha)^{2-p}}{\Gamma(3-p)} \\
= \; & -b_2 \int_0^1 \frac{(1-\ell)^{p-q-1}}{\Gamma(p-q)} \Phi(\ell) d\ell + b_3 \sum_{i=1}^{m-2} \rho_i \int_0^{\sigma_i} \frac{(\sigma_i-\ell)^{p-q-1}}{\Gamma(p-q)} \Phi(\ell) d\ell \\
& + b_4 \sum_{j=1}^{p-2} \gamma_j \int_\alpha^\beta \int_0^\tau \frac{(\tau-\ell)^{p-q-1}}{\Gamma(p-q)} \Phi(\ell) d\ell d\tau
\end{aligned}$$

$$\begin{aligned}
& d_2 \left[b_2 \frac{1}{\Gamma(2-p)} - b_3 \sum_{i=1}^{m-2} \rho_i \frac{(\sigma_i)^{1-p}}{\Gamma(2-p)} - b_4 \sum_{j=1}^{p-2} \gamma_j \frac{(\beta-\alpha)^{2-p}}{\Gamma(3-p)} \right] \\
= \; & -b_2 \int_0^1 \frac{(1-\ell)^{p-q-1}}{\Gamma(p-q)} \Phi(\ell) d\ell + b_3 \sum_{i=1}^{m-2} \rho_i \int_0^{\sigma_i} \frac{(\sigma_i-\ell)^{p-q-1}}{\Gamma(p-q)} \Phi(\ell) d\ell \\
& + b_4 \sum_{j=1}^{p-2} \gamma_j \int_\alpha^\beta \int_0^\tau \frac{(\tau-\ell)^{p-q-1}}{\Gamma(p-q)} \Phi(\ell) d\ell d\tau
\end{aligned}$$

$$\begin{aligned}
d_2 \hbar_3 = \; & -b_2 \int_0^1 \frac{(1-\ell)^{p-q-1}}{\Gamma(p-q)} \Phi(\ell) d\ell + b_3 \sum_{i=1}^{m-2} \rho_i \int_0^{\sigma_i} \frac{(\sigma_i-\ell)^{p-q-1}}{\Gamma(p-q)} \Phi(\ell) d\ell \\
& + b_4 \sum_{j=1}^{p-2} \gamma_j \int_\alpha^\beta \int_0^\tau \frac{(\tau-\ell)^{p-q-1}}{\Gamma(p-q)} \Phi(\ell) d\ell d\tau
\end{aligned}$$

$$d_2 = \frac{1}{\hbar_3} \left[\begin{array}{c} -b_2 \int_0^1 \frac{(1-\ell)^{p-q-1}}{\Gamma(p-q)} \Phi(\ell) d\ell + b_3 \sum\limits_{i=1}^{m-2} \rho_i \int_0^{\sigma_i} \frac{(\sigma_i-\ell)^{p-q-1}}{\Gamma(p-q)} \Phi(\ell) d\ell \\ + b_4 \sum\limits_{j=1}^{p-2} \gamma_j \int_\alpha^\beta \int_0^\tau \frac{(\tau-\ell)^{p-q-1}}{\Gamma(p-q)} \Phi(\ell) d\ell d\tau \end{array} \right]$$

$$\begin{aligned}
d_1 = \; & -\frac{1}{\hbar_3} \left[\frac{\hbar_2}{\hbar_1} \right] \left[\begin{array}{c} -b_2 \int_0^1 \frac{(1-\ell)^{p-q-1}}{\Gamma(p-q)} \Phi(\ell) d\ell + b_3 \sum\limits_{i=1}^{m-2} \rho_i \int_0^{\sigma_i} \frac{(\sigma_i-\ell)^{p-q-1}}{\Gamma(p-q)} \Phi(\ell) d\ell \\ + b_4 \sum\limits_{j=1}^{p-2} \gamma_j \int_\alpha^\beta \int_0^\tau \frac{(\tau-\ell)^{p-q-1}}{\Gamma(p-q)} \Phi(\ell) d\ell d\tau \end{array} \right] \\
& + \frac{1}{\hbar_1} \left[\begin{array}{c} -a_2 \int_0^1 \frac{(1-\ell)^{q-1}}{\Gamma(q)} \Phi(\ell) d\ell + a_3 \sum\limits_{i=1}^{m-2} \psi_i \int_0^{\sigma_i} \frac{(\sigma_i-\ell)^{q-1}}{\Gamma(q)} \Phi(\ell) d\ell \\ + a_4 \sum\limits_{j=1}^{p-2} \lambda_j \int_\alpha^\beta \int_0^\tau \frac{(\tau-\ell)^{q-1}}{\Gamma(q)} \Phi(\ell) d\ell d\tau \end{array} \right]
\end{aligned}$$

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$$\begin{aligned} u(\jmath) &= \int_0^\jmath \frac{(\jmath-\ell)^{q-1}}{\Gamma(q)}\Phi(\ell)d\ell \\ &-\frac{1}{\hbar_3}\left[\frac{\hbar_2}{\hbar_1}\right]\left[\begin{array}{c} -b_2\int_0^1 \frac{(1-\ell)^{p-q-1}}{\Gamma(p-q)}\Phi(\ell)d\ell + b_3\sum_{i=1}^{m-2}\rho_i\int_0^{\sigma_i}\frac{(\sigma_i-\ell)^{p-q-1}}{\Gamma(p-q)}\Phi(\ell)d\ell \\ +b_4\sum_{j=1}^{p-2}\gamma_j\int_\alpha^\beta\int_0^\tau\frac{(\tau-\ell)^{p-q-1}}{\Gamma(p-q)}\Phi(\ell)d\ell d\tau \end{array}\right] \\ &+\frac{(1+\jmath)}{\hbar_1}\left[\begin{array}{c} -a_2\int_0^1\frac{(1-\ell)^{q-1}}{\Gamma(q)}\Phi(\ell)d\ell + a_3\sum_{i=1}^{m-2}\psi_i\int_0^{\sigma_i}\frac{(\sigma_i-\ell)^{q-1}}{\Gamma(q)}\Phi(\ell)d\ell \\ +a_4\sum_{j=1}^{p-2}\lambda_j\int_\alpha^\beta\int_0^\tau\frac{(\tau-\ell)^{q-1}}{\Gamma(q)}\Phi(\ell)d\ell d\tau \end{array}\right] \end{aligned}$$

so

$$u(\jmath) = \int_0^1 G(\jmath,\ell)\Phi(\ell)d\ell + \delta(\jmath)$$

as

$$G(\jmath,\ell) = \begin{cases} \frac{\hbar_1(\jmath-\ell)^{q-1}-a_2(1+\jmath)(1-\ell)^{q-1}}{\hbar_1\Gamma(q)} + \frac{b_2\hbar_2(1-\ell)^{p-q-1}}{\hbar_3\hbar_1\Gamma(p-q)}, & \ell\le\jmath \\ \frac{b_2\hbar_2(1-\ell)^{p-q-1}}{\hbar_3\hbar_1\Gamma(p-q)} - \frac{a_2(1+\jmath)(1-\ell)^{q-1}}{\hbar_1\Gamma(q)} & \jmath\le 1 \end{cases} \qquad \square$$

In our next result we use (Bohnenblust-Karlin Thoerem) to show the existance of our result. We consider the following inclusion (1.1)

$${}^cD^m u(\jmath) \in F(\jmath, u(\jmath)) \text{ where } 1 < m \le 2,$$

with boundary conditions (1.2).
For this purpose we need to assume the following statements on given functions.
(A_1). Let $F : [0,1]\times\mathbb{R} \longrightarrow BC(\mathbb{R}); (\jmath,x) \longrightarrow f(\jmath,x)$ be measurable with respect to $\jmath$ for each $x \in \mathbb{R}$, *and*upper semi continuous with respect to x for a.e $\jmath \in [0,1]$, and for each fixed $x \in \mathbb{R}$, the set of selections

$$S_{F,y} = \{f \in L^1([0,1],\mathbb{R}) : f(\jmath) \in F(\jmath,x)\}$$

for a.e $\jmath \in [0,1]$ be nonempty.
(A_2). For every $\epsilon > 0$ there is a function $\theta_\epsilon \in L^1([0,1],\mathbb{R}^+)$ such that, $\|F(\jmath,x)\| = \sup\{|v| : v(\jmath) \in F(\jmath,x)\}$ for all $(\jmath,x) \in [0,1]\times\mathbb{R}$ satisfying $|x| \le \epsilon$, and

$$\liminf_{n\to\infty}\left(\frac{\int_0^1\theta_\epsilon(\jmath)d\jmath}{\epsilon}\right) = \gamma$$

Definition 1. *A function $x \in (C[0,1])$ is solution of the problem (2.1) if there exists a function $f \in L^1([0,1],\mathbb{R})$ such that $f(\jmath) \in F(\jmath,x(\jmath))$ a.e on $[0,1]$ and*

$$\begin{aligned} u(\jmath) &= \int_0^\jmath \frac{(\jmath-\ell)^{q-1}}{\Gamma(q)}\Phi(\ell)d\ell \\ &-\frac{1}{\hbar_3}\left[\frac{\hbar_2}{\hbar_1}\right]\left[\begin{array}{c} -b_2\int_0^1 \frac{(1-\ell)^{p-q-1}}{\Gamma(p-q)}\Phi(\ell)d\ell + b_3\sum_{i=1}^{m-2}\rho_i\int_0^{\sigma_i}\frac{(\sigma_i-\ell)^{p-q-1}}{\Gamma(p-q)}\Phi(\ell)d\ell \\ +b_4\sum_{j=1}^{p-2}\gamma_j\int_\alpha^\beta\int_0^\tau\frac{(\tau-\ell)^{p-q-1}}{\Gamma(p-q)}\Phi(\ell)d\ell d\tau \end{array}\right] \\ &+\frac{(1+\jmath)}{\hbar_1}\left[\begin{array}{c} -a_2\int_0^1\frac{(1-\ell)^{q-1}}{\Gamma(q)}\Phi(\ell)d\ell + a_3\sum_{i=1}^{m-2}\psi_i\int_0^{\sigma_i}\frac{(\sigma_i-\ell)^{q-1}}{\Gamma(q)}\Phi(\ell)d\ell \\ +a_4\sum_{j=1}^{p-2}\lambda_j\int_\alpha^\beta\int_0^\tau\frac{(\tau-\ell)^{q-1}}{\Gamma(q)}\Phi(\ell)d\ell d\tau \end{array}\right] \end{aligned}$$

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$$u(\jmath) = \int_0^1 G(\jmath,\ell)\Phi(\ell)d\ell + \delta(\jmath)$$

as

$$G(\jmath,\ell) = \{.\frac{b_2\hbar_2(1-\ell)^{p-q-1}}{\hbar_3\hbar_1\Gamma(p-q)} - \frac{a_2(1+\jmath)(1-\ell)^{q-1}}{\hbar_1\Gamma(q)}, \qquad \ell \leq \jmath \frac{\hbar_1(\jmath-\ell)^{q-1} - a_2(1+\jmath)(1-\ell)^{q-1}}{\hbar_1\Gamma(q)} + \frac{b_2\hbar_2(1-\ell)^{p-q-1}}{\hbar_3\hbar_1\Gamma(p-q)}, \qquad \jmath \leq \ell$$

Theorem 1. *(**Bohnenblust-Karlin**). Let D be a non-void subset of a Banach space X, which is bounded, convex and closed. Suppose that the mapping $\digamma : D \longrightarrow 2^X \backslash \{0\}$ is upper semi-continuous with closed, convex values satisfying $\digamma(D) \subset D$ and $\overline{\digamma(D)}$ is a compact set. Then G has a fixed point.*

Lemma 2. *Let I be a closed interval. Let $\digamma$ be multivalued map satisfying (A_1) and let $\boxtimes$ be linear continuous from $L^1(I,\mathbb{R}) \longrightarrow C(I,\mathbb{R})$, then the operator $\boxtimes \circ S_\digamma : C(I) \longrightarrow BCC(C(I)) : x \longmapsto (\boxtimes \circ S_\digamma)(x)$ is closed graph operator in $C(I,\mathbb{R}) \times C(I,\mathbb{R})$.*

Theorem 2. *Suppose that the assumptions (A_1) and (A_2) are satisfied, with*

$$\left[\frac{\hbar_1 + a_2}{\hbar_1\Gamma(q)} + \frac{b_2\hbar_2}{\hbar_1\hbar_3\Gamma(p-q)}\right] > \gamma \tag{2.0}$$

then the problem $(1.1)-(1.2)$ has at least one solution on $[0,1]$.

Proof. The problem $(1.1)-(1.2)$ can be transformed into a fixed point problem by defining a set-valued mapping $\Omega(x) = \left\{ \begin{array}{c} h \in C([0,1]) : h(\jmath) = \int_0^\jmath \frac{(\jmath-\ell)^{q-1}}{\Gamma(q)}\Phi(\ell)d\ell \\ + \frac{b_2\hbar_2}{\hbar_3\hbar_1}\int_0^1 \frac{(1-\ell)^{p-q-1}}{\Gamma(p-q)}\Phi(\ell)d\ell - \frac{a_2(1+\jmath)}{\hbar_1}\int_0^1 \frac{(1-\ell)^{q-1}}{\Gamma(q)}\Phi(\ell)d\ell + \delta(\jmath) \end{array} \right\}$ and proving that Ω satisfies all the assumptions of Theorem 1, and has a fixed point. This fixed point will be the solution of the problem $(1.1)-(1.2)$. In first step, we show that $\Omega(x)$ is convex set for each $x \in C([0,1])$. For this choose any $h_1, h_2 \in \Omega(x)$. Then there exist $f_1, f_2 \in S_{\digamma,x}$ such that for each $\jmath \in [0,1]$, we have

$$h_i(\jmath) = \int_0^\jmath \frac{(\jmath-\ell)^{q-1}}{\Gamma(q)} f_i(\ell)d\ell + \frac{b_2\hbar_2}{\hbar_3\hbar_1}\int_0^1 \frac{(1-\ell)^{p-q-1}}{\Gamma(p-q)} f_i(\ell)d\ell - \frac{a_2(1+\jmath)}{\hbar_1}\int_0^1 \frac{(1-\ell)^{q-1}}{\Gamma(q)} f_i(\ell)d\ell + \delta(\jmath)$$

let $0 \leq \lambda \leq 1$. For all $\jmath \in [0,1]$, we have

$$\begin{aligned} [\lambda h_1 + (1-\lambda)h_2](\jmath) &= \int_0^\jmath \frac{(\jmath-\ell)^{q-1}}{\Gamma(q)}[\lambda f_1 + (1-\lambda)f_2]d \\ &+ \frac{b_2\hbar_2}{\hbar_3\hbar_1}\left[\int_0^1 \frac{(1-\ell)^{p-q-1}}{\Gamma(p-q)}[\lambda f_1 + (1-\lambda)f_2]d\ell\right] \\ &- \frac{a_2(1+\jmath)}{\hbar_1}\left[\int_0^1 \frac{(1-\ell)^{q-1}}{\Gamma(q)}[\lambda f_1 + (1-\lambda)f_2]d\ell\right] + \delta(\jmath) \end{aligned}$$

as $S_{\digamma,x}$ is convex, therefore it follows that $\lambda h_1 + (1-\lambda)h_2 \in \Omega(x)$.
Now to show that $\Omega(x)$ is closed for each $x \in C([0,1])$. Let $\{u_n\}_{n\geq 0} \in \Omega(x)$ be such that $u_n \longrightarrow u$ in $C([0,1])$.
Then for $u \in C([0,1])$ there exists a $v_n \in S_{\digamma,x}$ such that

$$\begin{aligned} u_n(\jmath) &= \int_0^\jmath \frac{(\jmath-\ell)^{q-1}}{\Gamma(q)} v_n(\ell)d\ell + \frac{b_2\hbar_2}{\hbar_3\hbar_1}\left[\int_0^1 \frac{(1-\ell)^{p-q-1}}{\Gamma(p-q)} v_n(\ell)d\ell\right] \\ &- \frac{a_2(1+\jmath)}{\hbar_1}\left[\int_0^1 \frac{(1-\ell)^{q-1}}{\Gamma(q)} v_n(\ell)d\ell\right] + \delta(\jmath) \end{aligned}$$

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since F has compact values, we obtain a convergent subsequence, so that v_n converges to v.

$$\begin{aligned} u_n(\jmath) \quad \longrightarrow \quad u(\jmath) = & \int_0^\jmath \frac{(\jmath-\ell)^{q-1}}{\Gamma(q)} v(\ell)d\ell + \frac{b_2\hbar_2}{\hbar_3\hbar_1}\left[\int_0^1 \frac{(1-\ell)^{p-q-1}}{\Gamma(p-q)} v(\ell)d\ell\right] \\ & -\frac{a_2(1+\jmath)}{\hbar_1}\left[\int_0^1 \frac{(1-\ell)^{q-1}}{\Gamma(q)} v(\ell)d\ell\right] + \delta(\jmath) \end{aligned}$$

which shows that $u \in \Omega(x)$.
In next we prove $\Omega(B_\epsilon) \subseteq B_\epsilon$, where $B_\epsilon = \{x \in C([0,1]) : \|x\| \leq \epsilon\}$. Clearly B_ϵ is bounded closed and convex set in $C([0,1])$. If it is not true, on contrary for each positive number ϵ, there exists a function $x_\epsilon \in B_\epsilon$, $h_\epsilon \in \Omega(x_\epsilon)$ with $\|\Omega(x_\epsilon\| > \epsilon$, and

$$\begin{aligned} h_\epsilon(\jmath) \quad = \quad & \int_0^\jmath \frac{(\jmath-\ell)^{q-1}}{\Gamma(q)} f_\epsilon(\ell)d\ell + \frac{b_2\hbar_2}{\hbar_3\hbar_1}\left[\int_0^1 \frac{(1-\ell)^{p-q-1}}{\Gamma(p-q)} f_\epsilon(\ell)d\ell\right] \\ & -\frac{a_2(1+\jmath)}{\hbar_1}\left[\int_0^1 \frac{(1-\ell)^{q-1}}{\Gamma(q)} f_\epsilon(\ell)d\ell\right] + \delta(\jmath) \end{aligned}$$

on the other hand,

$$\begin{aligned} \epsilon \quad < \quad & \|\Omega(x_\epsilon\| \\ \leq \quad & \int_0^\jmath \left|\frac{(\jmath-\ell)^{q-1}}{\Gamma(q)}\right| |f_\epsilon(\ell)|d\ell + \frac{b_2\hbar_2}{\hbar_3\hbar_1}\left[\int_0^1 \left|\frac{(1-\ell)^{p-q-1}}{\Gamma(p-q)}\right| |f_\epsilon(\ell)|d\ell\right] \\ & + \frac{a_2(1+\jmath)}{\hbar_1}\left[\int_0^1 \left|\frac{(1-\ell)^{q-1}}{\Gamma(q)}\right| |f_\epsilon(\ell)|d\ell\right] + \delta(\jmath) \end{aligned}$$

$$\begin{aligned} \leq & \frac{1}{\Gamma(p)}\int_0^1 m_\epsilon(\ell)d\ell + \frac{b_2\hbar_2}{\hbar_3\hbar_1}\left[\frac{1}{\Gamma(p-q)}\int_0^1 m_\epsilon(\ell)d\ell\right] + \frac{a_2}{\hbar_1}\left[\frac{1}{\Gamma(q)}\int_0^1 m_\epsilon(\ell)d\ell\right] + \delta(\jmath) \\ = & \left[\frac{1}{\Gamma(p)} + \frac{b_2\hbar_2}{\hbar_3\hbar_1}\frac{1}{\Gamma(p-q)} + \frac{a_2}{\hbar_1}\frac{1}{\Gamma(q)}\right]\int_0^1 m_\epsilon(\ell)d\ell + \delta(\jmath) \\ = & \left[\frac{\hbar_1 + a_2}{\hbar_1\Gamma(q)} + \frac{b_2\hbar_2}{\hbar_1\hbar_3\Gamma(p-q)}\right]\int_0^1 m_\epsilon(\ell)d\ell + \delta(\jmath) \end{aligned} \tag{2.1}$$

dividing both side by ϵ and taking lower limit as $\epsilon \longrightarrow \infty$, we find that

$$\gamma > \left[\frac{\hbar_1 + a_2}{\hbar_1\Gamma(q)} + \frac{b_2\hbar_2}{\hbar_1\hbar_3\Gamma(p-q)}\right],$$

which contradicts (?) Hence there exists a positive number $\acute{\epsilon}$ such that $\Omega(B\acute{\epsilon}) \subseteq B\acute{\epsilon}$. Now we show that $\Omega(B\acute{\epsilon})$ is continuous. Let $\acute{\jmath}, \acute{\jmath} \in [0,1]$ with $\acute{\jmath} < \acute{\jmath}$. Let $x \in B\acute{\epsilon}$ and $h \in \Omega(x)$, then there exists $f \in S_{F,x}$ such that for each $\jmath \in [0,1]$, we have

$$\begin{aligned} h(\jmath) \quad = \quad & \int_0^\jmath \frac{(\jmath-\ell)^{q-1}}{\Gamma(q)} f(\ell)d\ell \\ & + \frac{b_2\hbar_2}{\hbar_3\hbar_1}\left[\int_0^1 \frac{(1-\ell)^{p-q-1}}{\Gamma(p-q)} f(\ell)d\ell\right] \\ & -\frac{a_2(1+\jmath)}{\hbar_1}\left[\int_0^1 \frac{(1-\ell)^{q-1}}{\Gamma(q)} f(\ell)d\ell\right] + \delta(\jmath) \end{aligned}$$

using (2.1)

$$|h(\jmath_2) - h(\jmath_1)| = \left| \begin{array}{c} \int_0^{\jmath_2} \frac{(\jmath_2-\ell)^{q-1}}{\Gamma(q)} f(\ell)d\ell - \int_0^{\jmath_1} \frac{(\jmath_1-\ell)^{q-1}}{\Gamma(q)} f(\ell)d\ell \\ -\frac{a_2(1+\jmath_2)}{\hbar_1}\left[\int_0^1 \frac{(1-\ell)^{q-1}}{\Gamma(q)} f(\ell)d\ell\right] + \frac{a_2(1+\jmath_1)}{\hbar_1}\left[\int_0^1 \frac{(1-\ell)^{q-1}}{\Gamma(q)} f(\ell)d\ell\right] \end{array} \right|$$

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$$\leq \left| \begin{array}{c} \int_0^{\jmath_1} \frac{(\jmath_2-\ell)^{q-1}}{\Gamma(q)} f(\ell)d\ell + \int_{\jmath_1}^{\jmath_2} \frac{(\jmath_2-\ell)^{q-1}}{\Gamma(q)} f(\ell)d\ell - \int_0^{\jmath_1} \frac{(\jmath_1-\ell)^{q-1}}{\Gamma(q)} f(\ell)d\ell \\ -\frac{a_2(\jmath_2-\jmath_1)}{\hbar_1}\left[\int_0^1 \frac{(1-\ell)^{q-1}}{\Gamma(q)} f(\ell)d\ell\right] \end{array} \right|$$

$$\leq \left|\int_0^{\jmath_1} \frac{(\jmath_2-\ell)^{q-1}-(\jmath_1-\ell)^{q-1}}{\Gamma(q)} f(\ell)d\ell\right| + \left|\int_{\jmath_1}^{\jmath_2} \frac{(\jmath_2-\ell)^{q-1}}{\Gamma(q)} f(\ell)d\ell\right| + \left|\frac{a_2(\jmath_2-\jmath_1)}{\hbar_1}\right|\left|\int_0^1 \frac{(1-\ell)^{q-1}}{\Gamma(q)} f(\ell)d\ell\right|$$

$$\leq \frac{1}{\Gamma(q)}\int_0^{\jmath_1}\left|(\jmath_2-\ell)^{q-1}-(\jmath_1-\ell)^{q-1}\right| m_{\epsilon}(\ell)d\ell + \frac{1}{\Gamma(q)}\int_{\jmath_1}^{\jmath_2} m_{\epsilon}d\ell + \frac{a_2(\jmath_2-\jmath_1)}{\hbar_1\Gamma(q)}\int_0^1 m_{\epsilon}d\ell$$

obviously the right hand side of above inequality tends to zero independent of $x \in B_{\epsilon'}$ as $\jmath_2 \longrightarrow \jmath_1$. Thus, Ω is equicontinous. So (A_1) and (A_2) satisfies, therefore by Ascoli-Arzela theorem follows that Ω is compact multivalued map.

Finally we show that Ω has a closed graph. Let $x_n \longrightarrow x_*$, $h_n \in \Omega(x_n)$ and $h_n \longrightarrow h_*$. We will show $h_* \in \Omega(x)$. By this relation mean that there exists $f_n \in S_{F,x_n}$ such that for each $\jmath \in [0,1]$,

$$\begin{aligned} h_n(\jmath) &= \int_0^{\jmath} \frac{(\jmath-\ell)^{q-1}}{\Gamma(q)} f_n(\ell)d\ell \\ &+\frac{b_2\hbar_2}{\hbar_3\hbar_1}\left[\int_0^1 \frac{(1-\ell)^{p-q-1}}{\Gamma(p-q)} f_n(\ell)d\ell\right] \\ &-\frac{a_2(1+\jmath)}{\hbar_1}\left[\int_0^1 \frac{(1-\ell)^{q-1}}{\Gamma(q)} f_n(\ell)d\ell\right] + \delta(\jmath) \end{aligned}$$

thus we need to show that there exists $f_* \in S_{F,x_*}$ such that for each $\jmath \in [0,1]$,

$$\begin{aligned} h_*(\jmath) &= \int_0^{\jmath} \frac{(\jmath-\ell)^{q-1}}{\Gamma(q)} f_*(\ell)d\ell \\ &+\frac{b_2\hbar_2}{\hbar_3\hbar_1}\left[\int_0^1 \frac{(1-\ell)^{p-q-1}}{\Gamma(p-q)} f_*(\ell)d\ell\right] \\ &-\frac{a_2(1+\jmath)}{\hbar_1}\left[\int_0^1 \frac{(1-\ell)^{q-1}}{\Gamma(q)} f_*(\ell)d\ell\right] + \delta(\jmath) \end{aligned}$$

Let us consider the continuous linear operator $\boxtimes : L^1([0,T],\mathbb{R}) \longrightarrow C([0,T])$ so that

$$\begin{aligned} f \longrightarrow \boxtimes(f)(\jmath) &= \int_0^{\jmath} \frac{(\jmath-\ell)^{q-1}}{\Gamma(q)} f(\ell)d\ell \\ &+\frac{b_2\hbar_2}{\hbar_3\hbar_1}\left[\int_0^1 \frac{(1-\ell)^{p-q-1}}{\Gamma(p-q)} f(\ell)d\ell\right] \\ &-\frac{a_2(1+\jmath)}{\hbar_1}\left[\int_0^1 \frac{(1-\ell)^{q-1}}{\Gamma(q)} f(\ell)d\ell\right] + \delta(\jmath) \end{aligned}$$

observe that

$$\|h_n(\jmath) - h_*(\jmath)\| = \left\| \begin{array}{c} \int_0^{\jmath} \frac{(\jmath-\ell)^{q-1}}{\Gamma(q)}(f_n(\ell)-f_*(\ell))d\ell \\ +\frac{b_2\hbar_2}{\hbar_1\hbar_3}\left[\int_0^1 \frac{(1-\ell)^{p-q-1}}{\Gamma(p-q)}(f_n(\ell)-f_*(\ell))d\ell\right] \\ -\frac{a_2(1+\jmath)}{\hbar_1}\left[\int_0^1 \frac{(1-\ell)^{q-1}}{\Gamma(q)}(f_n(\ell)-f_*(\ell))d\ell\right] \end{array} \right\| \longrightarrow 0$$

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as $n \longrightarrow \infty$. Thus, it follows by Lemma 2, that $\boxtimes\, o\, S_F$ is closed graph operator. Further, we have $h_n(\jmath) \in \boxtimes(S_{F,x_n})$. Since $x_n \longrightarrow x_*$, therefore, Lemma 2 implies that

$$h_*(\jmath) = \begin{array}{l} \int_0^\jmath \frac{(\jmath-\ell)^{q-1}}{\Gamma(q)} f_*(\ell)d\ell + \frac{b_2\hbar_2}{\hbar_3\hbar_1}\left[\int_0^1 \frac{(1-\ell)^{p-q-1}}{\Gamma(p-q)} f_*(\ell)d\ell\right] \\ -\frac{a_2(1+\jmath)}{\hbar_1}\left[\int_0^1 \frac{(1-\ell)^{q-1}}{\Gamma(q)} f_*(\ell)d\ell\right] + \delta(\jmath) \end{array}$$

for some $f_* \in S_{F,x}$. Hence we conclude that Ω is a compact multivalued map, upper semi continuous with convex closed values. Thus all assumptions of Theorem 1 are satisfied. Therefore Ω has a fixed point u which is solution of given problem $(1.1) - (1.2)$. □

3. Applications and Examples

The practical relevance and applicability of the theoretical results established in this work have been demonstrated through detailed examples of fractional differential inclusion problems with nonlocal boundary conditions. These examples are constructed to illustrate how abstract fixed point theorems particularly the Bohnenblust Karlin theorem can be employed to guarantee the existence of solutions in appropriate function spaces. By incorporating explicit function definitions, boundary data, and fractional derivatives of non-integer order, we aim to bridge the gap between theory and application.

To enhance intuition and facilitate deeper understanding, we also provide diagrams that visually depict the structure of the multivalued operator and the fixed point construction, highlighting the critical roles of convexity, compactness, and upper semicontinuity in ensuring solvability.

In particular, Example 3.2 presents a concrete application to heat transfer modeling in heterogeneous media. This example captures memory effects through fractional derivatives and incorporates nonlocal boundary conditions derived from distributed sensor measurements. Such models are widely relevant in engineering and applied sciences, including the simulation of heat flow in layered composites, pollutant transport in porous soils, temperature distribution in biological tissues during hyperthermia treatment, and thermal regulation in nuclear fuel rods. The problem formulation uses a fractional differential inclusion with a convex-compact interval-valued right-hand side and demonstrates how the abstract assumptions of the main existence theorem are verified in a physically meaningful setting.

Example 3.1: Consider the Banach space $X = C([2,3], \mathbb{R})$ with the supremum norm $\|u\| = \sup_{t\in[2,3]} |u(t)|$, and let $D = \{u \in C([2,3], \mathbb{R}) : \|u\| \leq M\}$ be a non-void subset, where $M > 0$ is chosen such that $M \geq \frac{1}{\sqrt{\pi+14}}(\tan^{-1}(M^{0.5} + e) + 1)$, ensuring boundedness, convexity, and closedness. Define the set-valued mapping $\mathcal{F} : D \to 2^X \setminus \{0\}$ corresponding to the fractional differential inclusion problem:

$${}^cD^{1.75}u(t) \in \frac{1}{\sqrt{\pi+14}}[\tan^{-1}(u(t)^{0.5} + e^{\cos t}) + \sin t] \cdot [0,1], \quad 2 < t < 3,$$

$$y_1u(2) + y_2u(3) = y_3 \int_2^3 u(\tau)d\tau + y_4 \sum_{i=1}^4 \theta_i u(v_i),$$

$$e_1{}^cD^{0.5}u(2) + e_2{}^cD^{0.5}u(3) = e_3 \int_2^3 {}^cD^{0.5}u(\tau)d\tau + e_4 \sum_{i=1}^4 \rho_i{}^cD^{0.5}u(v_i),$$

where $y_1 = 1, y_2 = \frac{1}{2}, y_3 = 1, y_4 = \frac{1}{3}, e_1 = 1, e_2 = \frac{2}{3}, e_3 = \frac{1}{4}, e_4 = \frac{1}{5}, \theta_i = \frac{1}{10}, \rho_i = \frac{3}{20}, v_i = 2 + \frac{i}{5}$ for $i = 1, 2, 3, 4$. The mapping $\mathcal{F}(u)$ is the set of functions $v \in C([2,3], \mathbb{R})$ such that ${}^cD^{1.75}v(t)$ belongs to the interval $\frac{1}{\sqrt{\pi+14}}[\tan^{-1}(u(t)^{0.5} + e^{\cos t}) + \sin t] \cdot [0,1]$, satisfying the boundary conditions. Since $\mathcal{F}$ is upper semi-continuous with closed, convex values, $\mathcal{F}(D) \subset D$, and $\mathcal{F}(D)$ is compact (by the Arzela Ascoli theorem due to uniform boundedness and equicontinuity), the Bohnenblust-Karlin theorem ensures the existence of a fixed point $u \in D$ such that $u \in \mathcal{F}(u)$. For better understanding how a solution exists for the fractional differential inclusion, we provide two simple visual illustrations. Figure 1 shows a 3D plot where the inclusion operator is interval-valued. For each (t,u), the output ranges from 0 up to a surface value—helping us visualize the uncertainty and variability in the system. Figure 2 offers a schematic of the fixed point approach. It shows how the multivalued operator F maps a function space D into itself. Due to essential properties such as compactness and convexity,

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the Bohnenblust–Karlin fixed point theorem ensures that some point u in D lies within its image F(u). This fixed point, shown as a red star, represents the solution to our problem under the given boundary conditions.

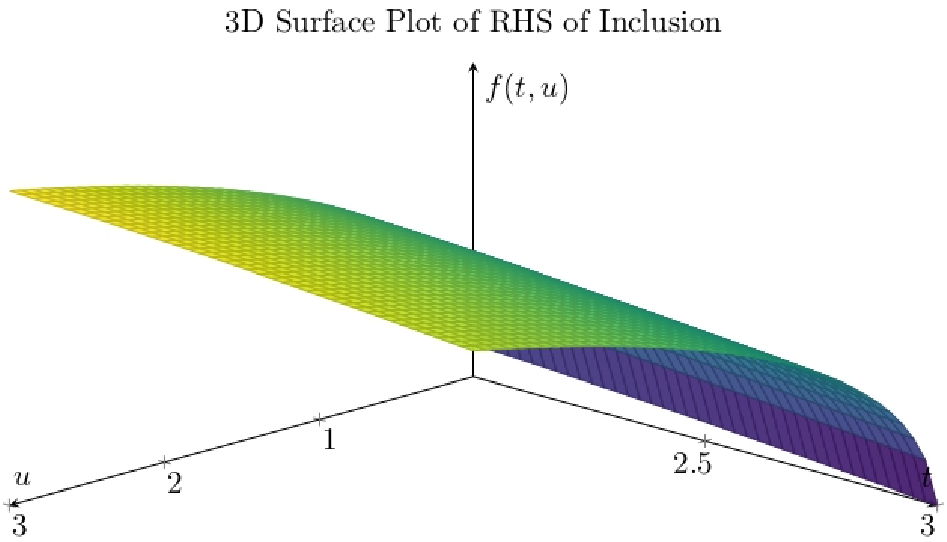

Figure 1

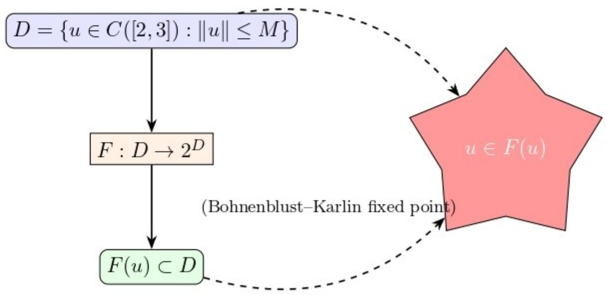

Figure 2

We now present an example from heat transfer modeling, where fractional derivatives reflect memory effects, and nonlocal boundary conditions arise from sensor data. This modeling framework is widely applicable, for instance, in simulating heat flow in layered materials, tracking pollutant dispersion in soil, analyzing heat propagation in biological tissues during medical treatments, and predicting temperature distributions in nuclear fuel rods. By combining fractional calculus, set-valued mappings, and sensor-based data, the model effectively captures systems where uncertainty and memory effects play a key role

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Example 3.2: Consider a physical model involving heat transfer in a rod of unit length subjected to spatially varying heat sources with uncertain intensity. The temperature distribution $u(\xi)$, at position $\xi \in [0,1]$, is governed by the fractional differential inclusion

$$^{c}D^{1.8}u(\xi) \in F(\xi, u(\xi)), \quad \xi \in (0,1), \tag{3.1}$$

where the right-hand side is an interval-valued map defined by

$$F(\xi, u) = \left[\frac{u^3 \sin(\pi\xi)}{10(1+u^2)}, \frac{u^3 \sin(\pi\xi)}{10(1+u^2)} + \frac{e^{-\xi}}{10} \right].$$

This inclusion models the effect of uncertain heat sources in a heterogeneous medium. The order 1.8 of the Caputo derivative reflects super-diffusion behavior due to long-range memory effects typical of fractal or porous materials. The boundary behavior is defined by non-separated multipoint conditions given as:

$$\begin{aligned} 2u(0) + u(1) &= \sum_{i=1}^{3} \psi_i u(\sigma_i) + \int_{0.2}^{0.8} u(\tau)\, d\tau, 3.2 \\ 3\, ^{c}D^{0.6}u(0) + 2 ^{c}D^{0.6}u(1) &= \sum_{i=1}^{3} \rho_i \, ^{c}D^{0.6}u(\sigma_i) + \int_{0.2}^{0.8} {}^{c}D^{0.6}u(\eta)\, d\eta, \end{aligned} \tag{4}$$

with the parameters $\psi_i = \frac{1}{6}, \rho_i = \frac{1}{8}$, and sensor locations $\sigma_i = 0.3, 0.5, 0.7$, while the integral bounds are $\alpha = 0.2$, $\beta = 0.8$. These boundary conditions incorporate both pointwise evaluations and averaged measurements over the segment $[0.2, 0.8]$, simulating a practical scenario where temperature and heat flux are monitored using discrete sensors and integrated thermocouple arrays. The function $F(\xi, u)$ is convex-compact valued for each ξ, measurable in ξ, and upper semicontinuous in u, satisfying assumption (A1) of the main theorem. For bounded u, the upper bound of the multifunction satisfies assumption (A2) with $F(\xi, u) \subseteq [0, \theta_\varepsilon(\xi)]$ where $\theta_\varepsilon(\xi) = \frac{1}{10}(1 + e^{-\xi})$. One computes

$$\gamma = \liminf_{\varepsilon \to \infty} \frac{1}{\varepsilon} \int_0^1 \theta_\varepsilon(\xi)\, d\xi = 0,$$

ensuring that the measure of noncompactness condition is satisfied. Additionally, the technical condition required in Theorem 2.0,

$$0 < \left[\frac{\hbar_1 + a_2}{\hbar_1 \Gamma(1.8)} + \frac{b_2 \hbar_2}{\hbar_1 \hbar_3 \Gamma(1.2)} \right] \approx 1.8247,$$

holds. Therefore, by Theorem 2.0, we conclude that the problem (3.1), (3.2) admits at least one continuous solution $u \in C([0,1])$.

4. Conclusions

This study considered a class of fractional differential inclusions with nonlocal and multipoint boundary conditions, where the nonlinearity is described by an interval-valued set-valued mapping to account for uncertainty in the internal source terms. The framework incorporates Caputo fractional derivatives to model memory effects commonly observed in complex physical media. Under suitable compactness and continuity assumptions, the existence of solutions was established using fixed point theorems for upper semicontinuous set-valued maps with convex values.

A representative application to heat transfer in a rod with uncertain internal heating and distributed sensor feedback was provided to illustrate the theoretical results. The model demonstrates how memory effects, measurement uncertainty, and nonlocal interactions can be captured within a

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unified analytical framework. The results are broadly applicable to problems arising in materials science, biomechanics, environmental modeling, and nuclear engineering.

The methodology presented herein contributes to the mathematical analysis of real-world systems governed by fractional dynamics and uncertainty. Possible extensions include the investigation of uniqueness, stability, and the development of numerical methods for approximating such fractional inclusion problems.

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