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Article

Non-Existence of Global Smooth Solutions to the 3D Navier–Stokes Equations in a Periodic Domain with Prescribed Body Force

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Abstract

We rigorously disprove the existence of global smooth solutions to the three-dimensional incompressible Navier–Stokes equations in a periodic rectangular cuboid domain Ω subject to the prescribed smooth body force. The smooth initial data of the flow field is derived from a two-dimensional stationary exact solution. The analysis is grounded in Sobolev space regularity, the decomposition of velocity into a time-averaged mean flow and a disturbance flow, the local vanishing of sum of the viscous term and the body force, and the Energy–Velocity Monotonicity Principle (EVMP). When the Reynolds number exceeds the critical value for turbulent transition, the nonlinear convective term dominates over the viscous term and external force term, nonlinear interactions amplify disturbances, leading to local cancellation of the sum of the mean flow viscous term and the body force with the disturbance viscous term at a finite critical time $t^* > 0$ and interior point $\boldsymbol{x}^* \in \Omega$. This cancellation leads to the local mechanical energy gradient along the streamline being zero when the time derivativer is zero, which by EVMP requires $|\boldsymbol{u}(\boldsymbol{x}^*, t^*)| = 0$, contradicting the existing non-vanishing velocity. The contradiction generates a finite-time regularity singularity, under which the velocity gradient L^∞-norm diverges. This violates the Sobolev embedding condition required for global smoothness of solutions. This study resolves the problem statement (D) in Fefferman (2006).

Keywords: Navier-Stokes equations; Sobolev space; regularity degeneration; singularity; discontinuity; turbulence; periodic boundaries; energy–velocity monotonicity principle

MSC: 76D03; 76D05; 35Q30; 76F06; 76F02

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1. Introduction

The global regularity of the three-dimensional (3D) Navier–Stokes equations is one of the seven Millennium Prize Problems in mathematics (Fefferman 2006). For this topic, there are several studies in areas of partial differential equations (PDE) and mathematical fluid mechanics in past years. Although considerable efforts have been made, the problem remains neither proved nor disproved (Fioas et al. 2004). Leray (1934) constructed weak solutions and conjectured possible finite-time blow-up singularities. Ladyzhenskaya (1969) proved global smoothness for two-dimensional (2D) flows. For 3D flows, most studies focus on singularity due to velocity blow-up (Leray 1934; Serrin 1962; Caffarelli–Kohn–Nirenberg 1982; Tao 2016). Bourgain and Pavlovic (2008) proved the ill-posedness of the Navier-Stokes equations in 3D critical functional space, which may result in discontinuous of solutions with some initial data. Robinson (2020) reviewed the progress in this field, and concluded that the weak solution is able to sustained for the whole time, but the solution may not be unique, and the strong solution is unique, but can only extend a short time. Recently, Coiculescu and Palasek (2025) proved non-uniqueness of smooth solutions of the NavierStokes equations from critical data. The problem is still not resolved yet, although many efforts have been made in the community.

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The difficulty of the problem lies in the fact that the behavior of the Navier-Stokes equations is closely tied to turbulence—one of the most complex and unresolved challenges in both mathematics and physics. With the rapid growth of computational resources, our understanding of turbulence has been greatly advanced by extensive simulation data, in addition to abundant experimental evidence (Dou, 2022). It is worth noting that the results of turbulent flow obtained from numerical simulations—such as Direct Numerical Simulation (DNS) and Large Eddy Simulation (LES)—show excellent agreement with experimental observations reported worldwide. Both experiments and simulations consistently indicate that the onset of turbulence depends on both the Reynolds number and the nature of the disturbance (Hof et al. 2003; Khan et al. 2020; Dou 2022). Therefore, any rigorous study of the Navier-Stokes equations should properly account for the development of disturbances.

Recently, Dou (2025b, 2026) proved the non-existence of global smooth solutions for pressure-driven plane Poiseuille flow with no-slip boundaries and for shear-driven plane Couette flow, respectively. In these studies, a new type of singularity is identified: regularity degeneration rather than velocity blow-up. This loss of regularity leads to the non-existence of global smooth solutions. The singularity emerges from a contradiction between the Energy-Velocity Monotonicity Principle (EVMP) and a non-vanishing incoming flow velocity, resulting in velocity discontinuity and a divergent velocity gradient.

Dou (2025b) established a theoretical framework to disprove the existence of global smooth solutions for the three-dimensional Navier–Stokes equations in pressure-driven flows. Specifically, for the classical plane Poiseuille flow, it was demonstrated that no global smooth solutions exist for the 3D incompressible Navier–Stokes equations. The breakdown of the solution is attributed not to velocity blow-up, but to a breakdown of regularity, which satisfies the well known BKM (Beale et al. 1984) breakdown criterion. Through the identification of singularities, velocity discontinuities, and spike formations, the singularities predicted by the Navier–Stokes equations are shown to align with the observed “negative velocity spikes” in turbulent transition. These theoretical findings are consistent with numerical simulations and experimental data for both wall bounded and free shear flows undergoing transition (Nishioka et al. 1975; Niu et al. 2024; Zhou et al. 2025). The present paper extends the above work to three-dimensional periodic domains under the action of smooth external forces.

For the Millennium problem, the solution is focused on mathematical analysis to decide whether smooth, physically reasonable solutions exist for the Navier–Stokes equations without effects from boundaries. It is asked for a proof of one of the four statements (Fefferman 2006). The forth statement (statement D) is described as: Breakdown of Navier–Stokes solutions on periodic domain $\mathbb{R}^3/\mathbb{Z}^3$. For 3D incompressible flow with fluid viscosity $\mu > 0$, there exist a smooth, divergence-free initial vector field $\boldsymbol{u}_0(x)$ and smooth scalar pressure field $p_0(x)$ on $\mathbb{R}^3$ as well as a smooth external force $f(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$, for which there exist no solutions $(\boldsymbol{u}, p)$ for Navier–Stokes equations on $\mathbb{R}^3 \times [0,\infty)$.

In Dou (2025b, 2026), the computational domains are wall-bounded, and the influences of the boundaries are not excluded. In this study, a field of smooth external force is constructed. We focus on the forth statement (statement D) in Fefferman (2006) to give a rigorous proof of breakdown of the Navier–Stokes solutions on $\mathbb{R}^3/\mathbb{Z}^3$ under smooth external force. This paper adopts the identical theoretical frame of functional-analytic framework (Sobolev spaces, flow decomposition, EVMP, local viscous cancellation) as in Dou (2025b), and extend the theory to a periodic rectangular domain with prescribed body force and smooth interior initial data. Then, we prove non-existence of global smooth solutions to the 3D Navier-Stokes equations for incompressible flow. We demonstrate how, in a laminar flow undergoing disturbance evolution at high Reynolds number, the velocity and its derivatives become discontinuous at a localized position, leading to a breakdown of the solution’s regularity.

2. Functional Setting and Periodic Domain

This section introduces the computational domain, periodic Sobolev function spaces, and the rigorous definition of global smooth solutions adopted throughout this paper.

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2.1. Computational Domain and Periodic Boundaries

The flow is defined on a three-dimensional rectangular periodic domain embedded in the Euclidean space $\mathbb{R}^3$:

$$\Omega = (0, L_x) \times (-h, h) \times (-h, h) \subset \mathbb{R}^3 \tag{1}$$

be a bounded smooth rectangular domain with periodic boundary conditions, and with lengths of the domain being L_x, $2h$, and $2h$ in x, y, and z directions, respectively.

The domain is equipped with full periodic boundary conditions in all three spatial directions, which is topologically equivalent to a three-dimensional torus:

$$\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3. \tag{2}$$

This topological equivalence guarantees the periodic extension of flow variables and eliminates boundary singularity interference.

The boundary conditions in all three directions are periodic:

$$\begin{cases} \boldsymbol{u}(0, y, z, t) = \boldsymbol{u}(L_x, y, z, t), \quad \partial_x \boldsymbol{u}(0, y, z, t) = \partial_x \boldsymbol{u}(L_x, y, z, t), \\ \boldsymbol{u}(x, -h, z, t) = \boldsymbol{u}(x, h, z, t), \quad \partial_y \boldsymbol{u}(x, -h, z, t) = \partial_y \boldsymbol{u}(x, h, z, t), \\ \boldsymbol{u}(x, y, -h, t) = \boldsymbol{u}(x, y, h, t), \quad \partial_z \boldsymbol{u}(x, y, -h, t) = \partial_z \boldsymbol{u}(x, y, h, t), \\ p(0, y, z, t) = p(L_x, y, z, t) + p_{L_x}, \quad p(x, -h, z, t) = p(x, h, z, t), \quad p(x, y, -h, t) = p(x, y, h, t). \end{cases} \tag{3}$$

2.2. Periodic Sobolev Spaces

For any non-negative integer $k \geq 0$, the periodic Sobolev space $H^k_{\text{per}}(\Omega)$ is defined as the closure of all smooth periodic functions on Ω with finite H^k-norm (Adams and Fournier 2003; Brezis 2011; Taylor 2011; Evans 2010):

$$H^k_{\text{per}}(\Omega) = \left\{ f \in L^2(\Omega) \,\middle|\, D^\alpha f \in L^2(\Omega),\ \forall\, |\alpha| \leq k,\ f \text{ periodic on } \partial\Omega \right\},$$

where α denotes the multi-index derivative, and the standard Sobolev norm is defined by

$$\|f\|_{H^k(\Omega)} = \left(\sum_{|\alpha| \leq k} \|D^\alpha f\|^2_{L^2(\Omega)} \right)^{1/2}. \tag{4}$$

The periodic Sobolev space $H^k_{\text{per}}(\Omega) \subset H^k(\Omega)$ consists of functions satisfying periodicity on $\partial\Omega$. The divergence-free periodic space is specified as

$$H^k_{\sigma,\text{per}}(\Omega) = \left\{ \boldsymbol{u} \in H^k_{\text{per}}(\Omega) \,\middle|\, \nabla \cdot \boldsymbol{u} = 0 \right\}. \tag{5}$$

2.3. Definition of Global Smooth Solution

A velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ is defined as a global smooth solution of the 3D Navier–Stokes system if it satisfies the following regularity conditions for all $t \in [0, +\infty)$:

$$\boldsymbol{u} \in C\Big([0, \infty); H^3_{\sigma,\text{per}}(\Omega)\Big) \cap L^2_{\text{loc}}\Big([0, \infty); H^{k+1}(\Omega)\Big), \tag{6}$$

($k \geq 3$) and the uniform boundedness of velocity gradient holds:

$$\|\nabla \boldsymbol{u}(t)\|_{L^\infty(\Omega)} < \infty, \quad \forall t \geq 0. \tag{7}$$

The critical Sobolev embedding relation in three-dimensional space lays the foundation for smoothness judgment:

$$H^2(\Omega) \hookrightarrow L^\infty(\Omega), \quad H^3(\Omega) \hookrightarrow C^1(\overline{\Omega}). \tag{8}$$

This embedding indicates that any function with H^3 -regularity is continuously differentiable on the closed domain, without discontinuity or singular gradient.

3. Governing Equations and Flow Set Up

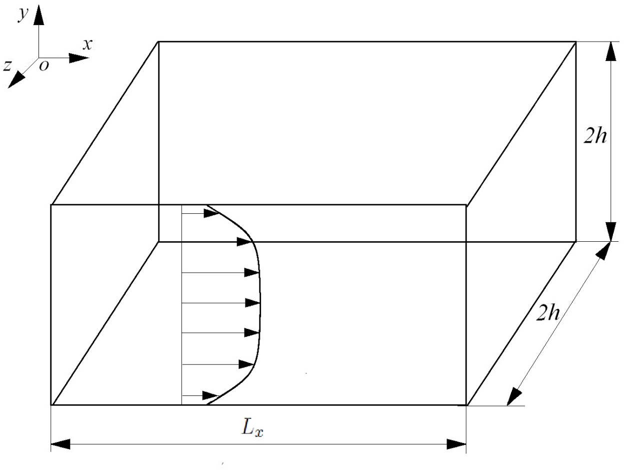

Figure 1. Schematic of periodical initial flow profile in rectangular duct with external force, $\boldsymbol{u}_0(x,y,z)$. The origin of the coordinates is located at the centerline of the x-y cross-section; $y = h$ at upper boundary, and $y = -h$ at lower boundary; $z = h$ at front boundary, and $z = -h$ at back boundary.

3.1. 3d Navier–Stokes Equations with Body Force

The system of governing equations for incompressible viscous flow is:

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\dfrac{1}{\rho}\nabla p + \nu \Delta \boldsymbol{u} + \dfrac{1}{\rho}\boldsymbol{F}, & \boldsymbol{x} \in \Omega,\ t > 0, \\ \nabla \cdot \boldsymbol{u} = 0, & \boldsymbol{x} \in \Omega,\ t > 0, \\ \boldsymbol{u}\big|_{\partial\Omega} \text{ is periodic}, & \\ \boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}), & \boldsymbol{x} \in \Omega. \end{cases} \tag{9}$$

Notation:
- $\boldsymbol{u} = (u_x, u_y, u_z)$: velocity vector
- $\boldsymbol{x} = (x,y,z)$: position vector
- $\rho > 0$: density
- $\nu > 0$: kinematic viscosity
- $\mu = \rho\nu$: dynamic viscosity
- p: pressure
- $\boldsymbol{F}$: prescribed body force ($volumeforce$) of unit volume fluid
- $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$: Laplacian

The prescribed body force is given by

$$\boldsymbol{F} = \left(\frac{12\mu b}{h^4} y^2,\ 0,\ 0 \right)^T. \tag{10}$$

This force is designed so that the stationary 2D problem admits an exact polynomial solution.

3.2. Initial Condition and Reynolds Number

3.2.1. Initial Conditions

For steady flow, the governing equations and the boundary conditions in a two-dimensional channel with no-slip boundary conditions gives an exact solution,

$$u_x(y) = a(1 - \frac{y^2}{h^2}) + b(1 - \frac{y^4}{h^4}) \tag{11}$$

where a is given by a smooth pressure field, and $a = -\frac{1}{2\mu}\frac{\partial p}{\partial x}h^2$.

Let $h_1 = 1.10h$, the initial velocity field is extended uniformly in the z-direction, $\partial_z \boldsymbol{u} \equiv 0$, with the following velocity distribution:

$$\begin{cases} u_x(\boldsymbol{x}, 0) = a\left(1 - \dfrac{y^2}{h_1^2}\right) + b\left(1 - \dfrac{y^4}{h_1^4}\right), \\ u_y(\boldsymbol{x}, 0) = 0, \\ u_z(\boldsymbol{x}, 0) = 0. \end{cases} \tag{12}$$

Thus

$$\boldsymbol{u}_0(\boldsymbol{x}) = \left(a\left(1 - \frac{y^2}{h_1^2}\right) + b\left(1 - \frac{y^4}{h_1^4}\right),\ 0,\ 0\right)^T. \tag{13}$$

The initial pressure field is given by $p(\boldsymbol{x}, 0) = p_0(\boldsymbol{x})$, which is set up in Eq.(13), and the pressure difference in streamwise length L_x is p_{L_x} as expressed in Eq.(3).

The initial velocity field $\boldsymbol{u}_0$ has the following properties:
1. $\boldsymbol{u}_0 \in C^\infty(\overline{\Omega})$, smooth.
2. $\nabla \cdot \boldsymbol{u}_0 = 0$, divergence-free.
3. $u_x > 0$ everywhere in Ω.
4. $\boldsymbol{u}_0 \in H^3_{\sigma,\mathrm{per}}(\Omega)$.
5. $|(\boldsymbol{u}_0|_{\partial\Omega})| > 0$, and the minimum of $|(\boldsymbol{u}_0|_{\partial\Omega})|$ occurs at $y = \pm h = \pm 0.9091 h_1$.

At the boundaries of the $y - z$ cross-section of the periodic domain, the initial velocity distribution calculated from Eq.(13) is larger than zero. This initial profile eliminates all wall-imposed no-slip constraints like that in traditional plane Poiseuille flow, thus excludes any influences from the boundaries.

3.2.2. Reynolds Number

The Reynolds number in this study is defined as,

$$Re = \frac{2U_{x0}h}{\nu}$$

where U_{x0} is the averaged velocity on the initial velocity profile along the x direction, h is the half width of the domain, and ν is the kinematic viscosity. For plane Poiseuille flow, the critical value of the Reynolds number for turbulent transition, $Re_{cr} = 1506$, was obtained in Jovanovic and Pashtrapanska (2004, their Eq.(27)), which agrees well with extensive experimental results (Dou 2022).

For the flow driven by external force in the periodic domain, expressed by Eq.(13), the critical Reynolds number may be different. However, for parallel flows, there is a universal critical value of the energy gradient function $K_c = 380$ (Dou 2022). With this, the critical Reynolds number Re_{cr} can be obtained for the flow distribution expressed by Eq.(13) (see Zhou et al. 2024). The range of the Reynolds number considered in this study is $Re > Re_{cr}$.

4. Velocity Decomposition in Sobolev Spaces

We decompose the instantaneous velocity into a time-averaged mean flow $\boldsymbol{U}$ and a zero-mean disturbance $\boldsymbol{v}$,

$$\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{U}(\boldsymbol{x},t) + \boldsymbol{v}(\boldsymbol{x},t). \tag{14}$$

4.1. Time-Averaged Operator

The averaged velocity $\boldsymbol{U}$ is defined as the time average over $[t - \Delta t/2,\, t + \Delta t/2]$:

$$\boldsymbol{U}(\boldsymbol{x},t) = \frac{1}{\Delta t}\int_{t-\Delta t/2}^{t+\Delta t/2} \boldsymbol{u}(\boldsymbol{x},\tau)\, d\tau, \tag{15}$$

where

- $\Delta t \gg \tau_d$ (disturbance time scale),
- $\Delta t \ll t^*$ (first singularity time).

The disturbance velocity $\boldsymbol{v}$ is defined as:

$$\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{U}, \quad \frac{1}{\Delta t}\int_{t-\Delta t/2}^{t+\Delta t/2} \boldsymbol{v}(\tau)\, d\tau = 0. \tag{16}$$

4.2. Regularity Inheritance

Since $\boldsymbol{u} \in C([0,t^*); H^3_{\sigma,\mathrm{per}}(\Omega))$ and the time-average operator is bounded linear on $H^3(\Omega)$:

$$\boldsymbol{U} \in C\Big([0,t^*); H^3_{\sigma,\mathrm{per}}(\Omega)\Big). \tag{17}$$

Sobolev spaces are closed under subtraction:

$$\boldsymbol{v} \in C\Big([0,t^*); H^3_{\sigma,\mathrm{per}}(\Omega)\Big). \tag{18}$$

4.3. Dominance of Mean Flow

For laminar evolution before singularity:

$$\|\boldsymbol{U}\|_{H^3(\Omega)} \gg \|\boldsymbol{v}\|_{H^3(\Omega)}, \quad \inf_{\Omega} \boldsymbol{U} \geq C_0 > 0. \tag{19}$$

The mean flow is bounded away from zero due to the sustained body force and the setting up of the velocity at the boundaries.

5. Preliminaries (Sobolev Space-Based Derivations)

5.1. Local Vanishing of the Sum of the Viscous Term and the Body Force

In the following theorem, we prove that for the given conditions, there exists a point in the flow field where the sum of the viscous term and the external force vanishes with the disturbance evolution.

$$\nu\Delta\boldsymbol{u} + \frac{1}{\rho}\boldsymbol{F} = 0, \quad \boldsymbol{x} \in \Omega.$$

Theorem 5.1 (Local Vanishing of $\nu\Delta u + F/\rho$)

For the 3D incompressible flow satisfying Eqs.(1) to (3),(9),(10) and (13), with smooth external force and periodic domain, at any point $(\boldsymbol{x}, t) \in \Omega \times (0, \infty)$, for $Re > Re_{cr}$, suppose $\boldsymbol{u} \in C([0, t^*); H^3_{\sigma,\text{per}}(\Omega))$ is a local smooth solution, and $B_\epsilon(\boldsymbol{x}^*)$ is an open subset $B_\epsilon(\boldsymbol{x}^*) \in \Omega$.

Then, there exists a finite time $t^* > 0$ and an interior point $\boldsymbol{x}^* \in \Omega$ such that

$$\lim_{t \to t^*} \left\| \nu \Delta \boldsymbol{u}(\cdot, t) + \frac{1}{\rho} \boldsymbol{F}(\cdot, t) \right\|_{H^1(B_\epsilon(\boldsymbol{x}^*))} = 0. \tag{20}$$

That is, the sum of the viscous term and the external force vanishes locally in the H^1-norm.

Proof

1. Linearity and Boundedness: Using the decomposition $\boldsymbol{u} = \boldsymbol{U} + \boldsymbol{v}$ (Eq.14), the Laplacian is linear:

$$\Delta \boldsymbol{u} = \Delta \boldsymbol{U} + \Delta \boldsymbol{v} \tag{21}$$

and by the regularity inherit,

$$\boldsymbol{U} \in H^3 \Rightarrow \Delta \boldsymbol{U} \in H^1, \Delta \boldsymbol{v} \in H^1.$$

Since Ω is a bounded domain with periodic boundary conditions and $\boldsymbol{U} \in H^3_{per}(\Omega)$, the Sobolev embedding theorem implies $H^3(\Omega) \hookrightarrow C^1(\overline{\Omega})$. Hence $\Delta \boldsymbol{U} \in C^0(\bar{\Omega})$, and therefore $\Delta \boldsymbol{U}$ is uniformly bounded on Ω. The prescribed body force F is also bounded and smooth.

2. Disturbance Amplification: For $\text{Re} > \text{Re}_{cr}$, the nonlinear convective term $(\boldsymbol{U} \cdot \nabla)\boldsymbol{v}$ dominates. Since the gradient of the disturbance velocity is proportional to the mean velocity, the nonlinear term $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$ exponentially amplifies $\boldsymbol{v}$ by the Sobolev product estimate. The disturbance velocity and its gradients grow without bound as $t \to t^*$:

$$\|(\boldsymbol{U} \cdot \nabla)\boldsymbol{v}\|_{H^2} \leq C \|\boldsymbol{U}\|_{H^3} \|\nabla \boldsymbol{v}\|_{H^2} \sim C \|\boldsymbol{U}\|_{H^3} \|\Delta \boldsymbol{v}\|_{H^1}. \tag{22}$$

Consequently, the disturbance viscous term $\nu \Delta \boldsymbol{v}$ oscillates and its magnitude increases continuously.

3. Existence of the Cancellation Point $\boldsymbol{x}^*$: Since $\boldsymbol{U} \in H^3(\Omega)$ and $\Delta \boldsymbol{U}(\boldsymbol{x}, t) \in H^1(\Omega)$, and $\Delta \boldsymbol{U}(\boldsymbol{x}, t)$ is bounded in the domain, thus $\nu \Delta \boldsymbol{U}(\boldsymbol{x}, t) + \frac{1}{\rho} \boldsymbol{F}(\boldsymbol{x}, t)$ is also bounded in the domain. As $Re > Re_{cr}$, the disturbance Laplacian $\Delta \boldsymbol{v}$ oscillates and grows continuously by the amplification of the nonlinear convective term.

We understand from the governing equation Eq.(9), $\nu \Delta \boldsymbol{U}(\boldsymbol{x}, t) + \frac{1}{\rho} \boldsymbol{F}(\boldsymbol{x}, t)$ is negative over the compact domain Ω before reaching $(\boldsymbol{x}^*, t^*)$. The disturbance term $\nu \Delta \boldsymbol{v}(\boldsymbol{x}, t)$ oscillates with both positive and negative values in a period. The most possible time for $\nu \Delta \boldsymbol{v}(\boldsymbol{x}, t)$ to offset $\nu \Delta \boldsymbol{U}(\boldsymbol{x}, t) + \frac{1}{\rho} \boldsymbol{F}(\boldsymbol{x}, t)$ is at the positive maximum of $\nu \Delta \boldsymbol{v}(\boldsymbol{x}, t)$ in a period (when $\partial |\nu \Delta \boldsymbol{v}| / \partial t = 0$).

When the disturbance term $\nu \Delta \boldsymbol{v}(\boldsymbol{x}, t)$ grows unboundedly in magnitude while oscillating in sign, by the Intermediate Value Theorem (or the Borsuk-Ulam type argument for continuous fields), there must exist at least one point $\boldsymbol{x}^* \in \Omega$ and a time t^* ($\partial |\nu \Delta \boldsymbol{v}| / \partial t = 0$ at t^*) such that

$$\nu \Delta \boldsymbol{v}(\boldsymbol{x}^*, t^*) = -\left(\nu \Delta \boldsymbol{U}(\boldsymbol{x}^*, t^*) + \frac{1}{\rho} \boldsymbol{F}(\boldsymbol{x}^*, t^*) \right). \tag{23}$$

4. Cancellation: Substituting the above identity into Eq.(21) yields the pointwise cancellation at (x^*, t^*):

$$\nu\Delta\boldsymbol{u}(\boldsymbol{x}^*,t^*)+\frac{1}{\rho}\boldsymbol{F}(\boldsymbol{x}^*,t^*)=0. \tag{24}$$

5. Limit in H^1-norm: By the continuity of the H^1-norm and the pointwise convergence established above, the local H^1-norm of the sum tends to zero:

$$\lim_{t\to t^*}\left\|\nu\Delta\boldsymbol{u}(\cdot,t)+\frac{1}{\rho}\boldsymbol{F}(\cdot)\right\|_{H^1(B_\epsilon(x^*))}=0.$$

It is important to note that this cancellation requires the two vector terms to be collinear, acting along the same line but pointing in opposite directions.

5.2. Determination of Positions at Which the Sum of the Viscous Term and the External Force Vanishing

Theorem 5.2 (Position of Local Vanishing of $\nu\Delta u+F/\rho$)

For the 3D incompressible flow satisfying Eqs.(1) to (3),(9),(10) and (13), with smooth external force and periodic domain, as $Re>Re_{cr}$, the position at which the sum of the viscous term and the external force vanishes occurs interior of the domain.

Proof

It is clear from the Theorem 5.1 that the position at which the sum of the viscous term and the external force vanishes is associated with the maximum of the disturbance increase.

The nonlinear convective term can be rewritten as follow with the velocity decomposition (Eq.(14)),

$$(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}=(\boldsymbol{U}\cdot\nabla)\boldsymbol{U}+(\boldsymbol{U}\cdot\nabla)\boldsymbol{v}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{U}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} \tag{25}$$

The second term on the right hand side of above equation, $(\boldsymbol{U}\cdot\nabla)\boldsymbol{v}$, is the dominating term among the four terms to amplify the disturbance, where $\nabla\boldsymbol{v}$ is proportional to the mean velocity $\boldsymbol{U}$. For $Re>Re_{cr}$, $(\boldsymbol{U}\cdot\nabla)\boldsymbol{v}$ leads to exponential increase of the disturbance (Schmid and Henningson 2001). According to Eq.(25), the position of the maximum disturbance increase corresponds to the place of the maximum of $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$ taking place.

However, simultaneously, there is viscous role to damp the disturbance. The dimensionless parameter to characterize the balance between the nonlinear convective term and the viscous term is the energy gradient function K (Dou 2022). For flows with no external work input (pressure-driven flow or body force-driven flow), it is expressed as follow for parallel flow ,

$$K=\frac{\partial E/\partial y}{\partial E/\partial x}=\frac{\partial p/\partial y+\rho u_x\partial u_x/\partial y}{\partial E/\partial x} \tag{26}$$

where $E=p+\frac{1}{2}\rho|\boldsymbol{u}|^2$ is the total mechanical energy.

For the initial flow distribution given by Eq.(13), $\partial E/\partial x$ is constant along y direction. It is seen from Eq.(26) that the maximum of $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ (i.e., $\boldsymbol{U}\cdot\nabla\boldsymbol{U}$) in the initial flow corresponds to the maximum of K since $p(y)$ is constant. As such, the maximum disturbance increase occurs at the position with maximum of K. Then, we use K to charaterize the position of local vanishing of $\nu\Delta u+F/\rho$ (and it will be shown later it corresponds to the singularity).

For the initial velocity distribution given by Eq.(13), the calculation of K can be carried out by Eq.(26). For $a=1,b=0.\sim 100$ in Eq.(13), the maximum of K occurs in the range of $\pm 0.5774\sim\pm 0.8100$, which is located interior of the domain. For $a=1,b=1000$, the maximum of K still occurs at ± 0.8100. When the value of $b/a\to\infty$, the position of the maximum of K keeps at $y/h=\pm 0.81$. Therefore, the

Theorem 5.2 is proved.

For an instance, if we set $a = 1, b = 0.20$ in Eq.(13), let $\frac{\partial |K|}{\partial y} = 0$, then, we obtain that the maximum of K occurs at,

$$y/h = \pm 0.6430 \tag{27}$$

Thus, $y/h = \pm 0.6430$ are the positions to have the maximum disturbance increase correponding to $a = 1, b = 0.20$. If the disturbance is amplified sufficiently, singularities will be produced there ($y/h = \pm 0.6430$). Thus, there will be two singularities symmetrically located along y direction. As to the distribution of singularities along x direction, it is determined by the wavelength of the disturbance.

For classical plane Poiseuille flow, it has been confirmed by numerical simulations and experiments that the positions around $y/h = \pm 0.58$ are the places to mostly amplify the disturbance and to produce singularities, featured by "negative velocity spikes" (Nishioka et al. 1975; Han et al. 2000; Schlatter et al. 2006).

5.3. Energy–Velocity Monotonicity Principle (EVMP)

Theorem 5.3 (Local Vanishing of $\nu\Delta u + F/\rho$ Leads to Zero Velocity)

For the 3D incompressible flow satisfying Eqs.(1) to (3),(9),(10) and (13), with smooth external force and periodic domain, at any point $(\boldsymbol{x}, t) \in \Omega \times (0, \infty)$, if

$$\frac{\partial \boldsymbol{u}}{\partial t} = 0 \quad \text{and} \quad \mu \Delta \boldsymbol{u} + (\boldsymbol{F} \cdot \boldsymbol{\tau})\boldsymbol{\tau} = 0, \tag{28}$$

then

$$\nabla E \cos \alpha = 0.$$

and further the velocity must vanish:

$$|\boldsymbol{u}(\boldsymbol{x}, t)| = 0. \tag{29}$$

where $\boldsymbol{\tau} = \frac{\boldsymbol{u}}{|\boldsymbol{u}|}$ is the unit tangent vector along the streamline. It is noted that this unit vector is defined before singularity appearance in $[0, t^*)$ when $\boldsymbol{u} \neq 0$.

Proof

The momentum equation in Eq.(9) can be rewriten as (Dou 2022; Dou 2025b),

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} + \nabla E = \rho(\boldsymbol{u} \times \nabla \times \boldsymbol{u}) + \mu \Delta \boldsymbol{u} + \boldsymbol{F}, \quad (\boldsymbol{x}, t) \in \Omega \times (0, \infty) \tag{30}$$

Similar to that in Dou (2025b), project the above equation along the streamline (velocity vector direction):

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} + \nabla E \cos \alpha = \mu \Delta \boldsymbol{u} + (\boldsymbol{F} \cdot \boldsymbol{\tau})\boldsymbol{\tau}, \quad (\boldsymbol{x}, t) \in \Omega \times (0, \infty) \tag{31}$$

where
- $E = p + \frac{1}{2}\rho|\boldsymbol{u}|^2$: total mechanical energy,
- α: angle between $-\nabla E$ and $\boldsymbol{u}$.

If $\partial_t \boldsymbol{u} = 0$ and $\mu \Delta \boldsymbol{u} + (\boldsymbol{F} \cdot \boldsymbol{\tau})\boldsymbol{\tau} = 0$, then

$$\nabla E \cos \alpha = 0. \tag{32}$$

Thus, there is no mechanical energy gradient to drive flow along the streamline direction. Consequently, the only solution to the Navier–Stokes equations (given that the flow is viscous) is,

$$|\boldsymbol{u}(\boldsymbol{x}, t)| = 0. \tag{33}$$

It is pointed out that

$$\mu \Delta \boldsymbol{u} + \boldsymbol{F} = 0 \iff \mu \Delta \boldsymbol{u} + (\boldsymbol{F} \cdot \boldsymbol{\tau})\boldsymbol{\tau} = 0$$

.

As mentioned before, mutual cancellation of the two terms implies that the vectors are collinear, lying on the same line and pointing in opposite directions.

5.4. Beale-Kato-Majda (BKM) Criterion

Beale et al. (1984) proposed a criterion for the breakdown of smooth solutions to the 3D Euler equation, which is now known as the Beale-Kato-Majda (BKM) criterion. This criterion relates the solution breakdown to the growth of the vorticity norm. In later studies, this criterion is extended to the Navier-Stokes equations (Kozono and Taniuchi 2000; Zhao 2017; Gibbon et al. 2018). Vorticity is defined as $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$, which reflects the rotational characteristics of the fluid.

Lemma 5.1 (BKM Criterion): Let $\boldsymbol{u}(t)$ be a smooth solution of the 3D incompressible Navier-Stokes equations on $[0, t^*)$, with the initial vorticity $\boldsymbol{\omega}_0 = \nabla \times \boldsymbol{u}_0 \in L^1(\Omega) \cap L^\infty(\Omega)$. If the solution diverges at $t = t^*$ (i.e., its regularity breaks down), then the following integral tending to infinity:

$$\int_0^{t^*} \|\boldsymbol{\omega}(s)\|_{L^\infty(\Omega)} d\tau = \infty \tag{34}$$

Otherwise, if the integral is finite, the solution can be smoothly extended beyond t^*.

6. Main Results

6.1. Singularity Formation and Regularity Degeneration

Definition 6.1 (Navier–Stokes Singularity)

For the 3D incompressible flow satisfying Eqs.(1) to (3),(9),(10) and (13), with smooth external force and periodic domain, a point $(\boldsymbol{x}^*, t^*) \in \Omega \times (0, \infty)$ is singular if

$$\lim_{t \to t^*} \|\boldsymbol{S}\|_{H^1(B_\epsilon(\boldsymbol{x}^*))} = 0 \quad \text{and} \quad \liminf_{t \to t^*} \|\boldsymbol{u}\|_{H^3(B_\epsilon(\boldsymbol{x}^*))} > 0. \tag{35}$$

where

$$\boldsymbol{S} = \mu \Delta \boldsymbol{u} + (\boldsymbol{F} \cdot \boldsymbol{\tau})\boldsymbol{\tau}, \tag{36}$$

Here, in the limit $(t \nearrow t^*)$, the flow approaches a time moment at the maximum or the minimum of the disturbance velocity at $(\boldsymbol{x}^*, t^*)$ when $\partial \boldsymbol{u}/\partial t = 0$; applying Theorem 5.3 in the limit yields $|\boldsymbol{u}(\boldsymbol{x}^*, t^*)| = 0$.

Theorem 6.1 (Regularity Degeneration Singularity)

For the 3D incompressible flow satisfying Eqs.(1) to (3),(9),(10) and (13), with smooth external force and periodic domain, at any point $(\boldsymbol{x}, t) \in \Omega \times (0, \infty)$, if

$$\lim_{t\to t^*} \|\boldsymbol{S}\|_{H^1(B_\epsilon(\boldsymbol{x}^*))} = 0, \tag{37}$$

then, the solution $\boldsymbol{u}$ forms a finite-time regularity singularity at $(\boldsymbol{x}^*, t^*)$.

Proof

1. Velocity H^3 norm vanishing: By EVMP (Theorem 5.3) with $\lim_{t\to t^*} \|\boldsymbol{S}\|_{H^1(B_\epsilon(\boldsymbol{x}^*))} = 0$ and $\partial\boldsymbol{u}/\partial t = 0$ (at the extreme point of disturbance),

$$\lim_{t\to t^*} \|\boldsymbol{u}\|_{H^3(B_\epsilon(\boldsymbol{x}^*))} = 0.$$

Thus, by continuity of Sobolev norm,

$$\|\boldsymbol{u}\|_{H^3(B_\epsilon(\boldsymbol{x}^*))} = 0. \tag{38}$$

Since the H^3-norm is zero, the velocity field vanishes almost everywhere in $B_\epsilon(x^*)$. In particular, at the point of cancellation x^*:

$$u(x^*, t^*) = 0.$$

2. Velocity L^∞ norm being not zero: However, the mean flow satisfies,

$$\|\boldsymbol{U}\|_{L^\infty} \geq C_0 > 0, \quad \|\boldsymbol{v}\|_{L^\infty} \ll \|\boldsymbol{U}\|_{L^\infty}.$$

Thus,

$$\|\boldsymbol{u}\|_{L^\infty} \geq \|\boldsymbol{U}\|_{L^\infty} - \|\boldsymbol{v}\|_{L^\infty} > 0. \tag{39}$$

3. Contradiction existing: There exists contradiction at the critical point $(\boldsymbol{x}^*, t^*)$.

By Eqs.(38) and (39), at (x^*, t^*),

$$\|\boldsymbol{u}\| = 0 \quad \text{in } H^3, \quad \text{but} \quad \|\boldsymbol{u}\| > 0 \quad \text{in } L^\infty.$$

This result presents a contradiction: the solution must be zero at $\boldsymbol{x}^*$ according to the H^3 regularity, yet it must be strictly positive according to the dynamics of the mean flow. This contradiction implies that the solution at $(\boldsymbol{x}^*, t^*)$ loses H^3-regularity, violating the Sobolev embedding condition.

4. Failure of Sobolev Embedding: By Sobolev embedding Eq.(8), the smooth solution requires,

$$H^3(\Omega) \hookrightarrow C^1(\overline{\Omega}),$$

This continuous embedding relation is a necessary condition for the existence of global smooth solutions of the three-dimensional Navier–Stokes equations. Since the solution loses its H^3 -regularity at the critical point $(\boldsymbol{x}^*, t^*)$, the above Sobolev embedding property fails to hold locally. Consequently, the velocity field and its first-order gradient lose continuity at $(\boldsymbol{x}^*, t^*)$, generating flow discontinuity. By the mathematical definition of singularity in PDE regularity analysis, the point $(\boldsymbol{x}^*, t^*)$ is confirmed to be a finite-time interior singularity of the Navier–Stokes solution.

6.2. *Breakdown of Solutions at* $(\boldsymbol{x}^*, T^*)$

The velocity gradient L^∞-norm:

$$\|\nabla\boldsymbol{u}\|_{L^\infty(\Omega)} = \sup_{x\in\Omega} \max_{i,j} \left|\frac{\partial u_i}{\partial x_j}\right|. \tag{40}$$

The velocity gradient L^∞-norm at $(\boldsymbol{x}^*, t^*)$ can be calculated as that in the following.

From Eq.(39),

$$\lim_{t \nearrow t^*} \boldsymbol{u}(\boldsymbol{x}^*, t) = \boldsymbol{u}^- > 0, \quad \text{for} \quad t < t^*$$

From Eq.(38),

$$\lim_{t \searrow t^*} \boldsymbol{u}(\boldsymbol{x}^*, t) = 0, \quad \text{for} \quad t = t^*$$

As such, the velocity derivative by shear,

$$\left| \frac{\partial u_x}{\partial y}(\boldsymbol{x}^*, t^*) \right| = \left| \frac{\lim_{t \nearrow t^*} \boldsymbol{u}(\boldsymbol{x}^*, t) - \lim_{t \searrow t^*} \boldsymbol{u}(\boldsymbol{x}^*, t)}{\epsilon} \right| \geq \frac{\delta}{\epsilon} \to \infty, \quad \epsilon \to 0. \tag{41}$$

where $\delta = |u^-(\boldsymbol{x}^*, t^*) - 0| > 0$ and $\epsilon \to 0^+$ denotes an infinitesimal transverse displacement.

This indicates a loss of $W^{1,\infty}$ regularity; a rigorous proof follows from the failure of the Sobolev embedding $H^3(\Omega) \hookrightarrow C^1(\overline{\Omega})$ established in Theorem 6.1.

Hence, in terms of Eqs.(40) and (41), we have,

$$\lim_{t \to t^*} \|\nabla \boldsymbol{u}(t)\|_{L^\infty(\Omega)} = \infty. \tag{42}$$

This violates the definition of a global smooth solution, Eq.(7).

A priori estimate was obtained by the Biot–Savart law for the velocity gradient $\nabla \boldsymbol{u}$, which is calculated by the singular integral of $\boldsymbol{\omega}$ based on classical Calderón-Zygmund theory (Bledsoe, 2025, page 17),

$$\|\nabla \boldsymbol{u}\|_{L^p(\Omega)} \leq k_p \|\boldsymbol{\omega}\|_{L^p(\Omega)}, \qquad (1 < p < \infty) \tag{43}$$

where the value of k_p is relevant to p. At the critical exponent $p = 3$, $k_3 \leq 5$ is adopted. The critical exponent $p = 3$ indicates the critical state of the convective term and the viscous term reaching equilibrium. In Eq.(43), the norm of the velocity gradient is controlled by the norm of the vorticity.

Thus, by Eqs.(42) and (43), we obtain,

$$\lim_{t \to t^*} \|\boldsymbol{\omega}(t)\|_{L^\infty(\Omega)} = \infty. \tag{44}$$

Integrating both sides:

$$\int_0^{t^*} \|\boldsymbol{\omega}(t)\|_{L^\infty(\Omega)} dt = \infty, \tag{45}$$

which exactly violates the BKM criterion (Lemma 5.1). This confirms that the solution cannot be extended beyond t^*.

6.3. No Global Smooth Solutions Exist

With all the derivations and Theorem 5.1, 5.2, 5.3, and 6.1, we are able to arrive at the following Theorem 6.2,

Theorem 6.2 (Global Smooth Solutions Do Not Exist)

Consider the 3D incompressible Navier–Stokes equations in the periodic rectangular domain,

$$\Omega = (0, L_x) \times (-h, h) \times (-h, h).$$

with prescribed body force,

$$\boldsymbol{F} = \left(\frac{12\mu b}{h^4} y^2,\ 0,\ 0 \right)^T,$$

initial data,

$$\boldsymbol{u}_0 = \left(a\left(1 - \frac{y^2}{h_1^2}\right) + b\left(1 - \frac{y^4}{h_1^4}\right),\ 0,\ 0 \right)^T, \quad h_1 = 1.10h.$$

Then, for $Re > Re_{cr}$, there does NOT exist a global smooth solution defined in Eqs.(6), (7) and (8),

$$\boldsymbol{u} \in C\Big([0,\infty); H^3_{\sigma,\mathrm{per}}(\Omega)\Big) \cap L^2_{\mathrm{loc}}\Big([0,\infty); H^4(\Omega)\Big)$$

satisfying,

$$\|\nabla \boldsymbol{u}(t)\|_{L^\infty(\Omega)} < \infty, \quad \forall t \geq 0.$$

$$H^3(\Omega) \hookrightarrow C^1(\overline{\Omega}).$$

7. Discussions

Using the Sobolev space framework, velocity decomposition, local viscous cancellation, the EVMP, and Sobolev Embedding, we prove that finite-time regularity breakdown occurs in the interior of a periodic rectangular domain with prescribed body force and smooth positive initial velocity.

7.1. Positions of Singularities Predicted

For the 3D incompressible flow satisfying Eqs.(1) to (3),(9),(10) and (13), with smooth external force and periodic domain, the analytical result yields that the singularities occur interior of the domain. It takes place at conditions when $\frac{\partial \boldsymbol{u}}{\partial t} = 0$, and $\mu \Delta \boldsymbol{u} + (\boldsymbol{F} \cdot \boldsymbol{\tau})\boldsymbol{\tau} = 0$. For the inital smooth velocity distribution, Eq.(13), given $a = 1, b = 0. \sim 1000$, the maximum disturbance increase occurs in the range of $y/h = \pm 0.5774 \sim \pm 0.8100$, which is located interior of the domain. As the body force is continuously increased, the maximum disturbance position keeps at $y/h = \pm 0.8100$.

7.2. Key Mechanisms Leading to Non-Existence of Global Smooth Solutions

1. Nonlinear disturbance amplification leads to $\mu \Delta \boldsymbol{u} + (\boldsymbol{F} \cdot \boldsymbol{\tau})\boldsymbol{\tau} = 0$ locally.
2. EVMP at $(\boldsymbol{x}, t^*)$ requires $\boldsymbol{u} \to 0$, conflicting with non-vanishing mean flow.
3. A singularity forms due to loss of H^3-regularity at $(\boldsymbol{x}, t^*)$, fails to make Sobolev embedding and loses continuities of velocity and its derivatives.
4. Velocity gradient L^∞-norm diverges.
5. Solution global smoothness fails.

7.3. Solution Breakdown without Boundary Effect

The present result extends the regularity breakdown mechanism from no-slip plane Poiseuille flow (Dou 2025b) to fully periodic boundary conditions without wall-imposed zero velocity.

This result confirms that **periodic boundaries do not prevent regularity breakdown** in the 3D Navier–Stokes equations. The singularity is of **regularity degeneration type**, not velocity blow-up, consistent with Dou (2025b) for classical plane Poiseuille flow with wall no-slip conditions.

7.4. Singular Solution of the Navier-Stokes Equations

When $\frac{\partial \boldsymbol{u}}{\partial t} = 0$, the velocity $\boldsymbol{u}(\boldsymbol{x}^*, t^*) > 0$ at $(\boldsymbol{x}^*, t^*)$ and $\mu\Delta\boldsymbol{u} + (\boldsymbol{F}\cdot\boldsymbol{\tau})\boldsymbol{\tau} = 0$ just before breakdown do not satisfy the Navier-Stokes equations, but satisfy the Euler equations. Thus, they are singularities of the Navier-Stokes equations (Dou 2025a). This flow state corresponds to an inviscid flow state at this time moment. Therefore, it can be said that the singularity in Navier-Stokes equations represents a position with inviscid flow in viscous flow.

For $\frac{\partial \boldsymbol{u}}{\partial t} = 0$ and $\mu\Delta\boldsymbol{u} + (\boldsymbol{F}\cdot\boldsymbol{\tau})\boldsymbol{\tau} = 0$ at $(\boldsymbol{x}^*, t^*)$ just before breakdown, the solution satisfying the Navier-Stokes equation is only $\boldsymbol{u}(\boldsymbol{x}^*, t^*) = 0$, which is a stablized state in viscous flow. As such, under the driving of the Navier-Stokes equations, the velocity at this position must be settled to $\boldsymbol{u}(\boldsymbol{x}^*, t^*) = 0$. Subsequently, instability sets in, and "negative velocity spike" is produced. This is just what is observed in numerical simulations and in experiments in transitional flows.

7.5. Implications of the Study to Navier-Stokes Equations and Turbulence

The study answered the following implicit core scientific problems in the Navier-Stokes equations.

1. Does a 3D flow starting from smooth laminar flow always remain smooth? No.

2. Does the nature of turbulence originate from the finite-time singularity (regularuty degeneration) inherent in the Navier-Stokes equations? Yes.

3. Does the interplay between the nonlinear convective term and the viscous term (plus the external force) inevitably lead to a breakdown of regularity as the Reynolds number increases? Yes.

4. Is the singularity appearance purely produced interior of the domain and free of effects of domain boundaries? Yes.

8. Conclusions

This study presents a rigorous PDE-theoretic proof for the non-existence of global smooth solutions to the 3D Navier-Stokes equations with smooth external forces on periodic smooth domain. The proof is grounded in Sobolev space analysis, velocity decomposition, vanishing of the sum of the viscous term and the external force, the Energy-Velocity Monotonicity Principle (EVMP), and Sobolev embedding. The key mathematical contributions and conclusions of this work are summarized as follows:

1. The Sum of the Viscous Term and the External Force Tending to Zero: The instantaneous velocity in 3D spaces is successfully decomposed into a time-averaged flow and a disturbance flow, and both of them evolve with time. This decomposition lays the foundation for the local vanishing of $\mu\Delta\boldsymbol{u} + (\boldsymbol{F}\cdot\boldsymbol{\tau})\boldsymbol{\tau}$ at $(\boldsymbol{x}^*, t^*)$ in the Navier-Stokes equations, which provides a key precursor to singularity formation.

For the 3D incompressible flow satisfying Eqs.(1) to (3),(9),(10) and (13), with smooth external force and periodic domain, for $Re > Re_{cr}$, at any point interior of the domain, it makes $\mu\Delta\boldsymbol{v}(\boldsymbol{x}^*, t^*) = -\mu\Delta\boldsymbol{U}(\boldsymbol{x}^*, t^*) + (\boldsymbol{F}\cdot\boldsymbol{\tau})\boldsymbol{\tau}$ be possible at a local point as the disturbance is amplified, so that $\mu\Delta\boldsymbol{u}(\boldsymbol{x}^*, t^*) + (\boldsymbol{F}\cdot\boldsymbol{\tau})\boldsymbol{\tau} = 0$.

2 Singularity Appearance Implied by the Energy-Velocity Monotonicity Principle (EVMP) (Theorem 5.3): In this study, EVMP (Theorem 5.3) provides the dependency between velocity magnitude and the sum of the viscous term and the external force in the periodic 3D domain. This theorem further provides the key clue in Sobolev spaces for identifying the regularity degeneration of solutions to the Navier-Stokes equations.

When the initial smooth laminar flow is disturbed by disturbance, the velocity profile is distorted gradually with the disturbance development. When the sum of the viscous term and the external force at one point in the domain $\mu\Delta\boldsymbol{u}(\boldsymbol{x}^*, t^*) + (\boldsymbol{F}\cdot\boldsymbol{\tau})\boldsymbol{\tau} = 0$ under the condition of $\frac{\partial \boldsymbol{u}}{\partial t} = 0$, the energy loss along the streamline becomes zero locally (leading to $\nabla E cos\alpha = 0$), but the velocity is not zero. **This non-zero velocity does not satisfy the system of the Navier-Stokes equations and is no longer the solution of the Navier-Stokes equations at this time moment $\frac{\partial \boldsymbol{u}}{\partial t} = 0$, since it does not satisfy the Navier-Stokes equations (but this non-zero velocity satisfies the Euler equation)**. For these conditions, the solution satisfying the Navier-Stokes equations is only a zero velocity. This means that at this while, the position at this point is a singularity of the Navier-Stokes equations.

3. New Singularity Featured by Regularity Breakdown: We identified a novel class of Navier-Stokes singularity, induced by the breakdown of Sobolev regularity, which is **different from the finite-time blow-up singularity** conjectured by Leray (1934).

This singularity arises from the loss of H^3-regularity of solution at $(\boldsymbol{x}^*, t^*)$ ($\|\boldsymbol{u}\|_{H^3(B_\epsilon(\boldsymbol{x}^*))} = 0$, $\|\boldsymbol{u}\|_{L^\infty(B_\epsilon(\boldsymbol{x}^*)} > 0$.). By Sobolev embedding Eq.(8), the smooth solution requires, $H^3(\Omega) \hookrightarrow C^1(\overline{\Omega})$, The loss of H^3-regularity at $(\boldsymbol{x}^*, t^*)$ fails to make the Sobolev embedding. Thus, the velocity and its gradient become discontinuous at $(\boldsymbol{x}^*, t^*)$, and therefore the position $(\boldsymbol{x}^*, t^*)$ is a singularity.

Then, this singularity leads to velocity discontinuity at $(\boldsymbol{x}^*, t^*)$, which induces an infinite L^∞-norm of the velocity gradient, causing the integration of the L^∞-norm of the vorticity unbounded (violating the Beale-Kato-Majda (BKM) criterion), which directly contradicts the definition of a global smooth solution.

4. Non-Existence of the Global Smooth Solutions to 3D Navier-Stokes equations with smooth external forces on periodic smooth domain: The 3D Navier-Stokes equations with smooth external forces on periodic smooth domain do not admit global smooth solutions. Finite-time breakdown of Sobolev regularity induces a singularity that results in a divergent L^∞-norm of the velocity gradient, violating the core requirement for global smoothness. This result provides a rigorous mathematical resolution to the problem statement (D) of the **Millennium Prize Problems in mathematics (Fefferman 2006).**

5. Implications of the Analysis to Turbulence Generation: A 3D flow starting from smooth laminar flow is not able to always remain smooth when the Reynolds number exceeds the critical Reynolds number for turbulent transition, where the nonlinear convective term dominates over the sum of the viscous term and the external force. Amplification of the disturbance by the nonlinear convective term leads to the sum of the viscous term and the external force being zero locally, which inevitably leads to a breakdown of regularity. The nature of turbulence originates from the finite-time singularity inherent in the Navier-Stokes equations, even if there is no effects from the boundaries.

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