SOME IDENTITIES FOR A SEQUENCE OF UNNAMED POLYNOMIALS CONNECTED WITH THE BELL POLYNOMIALS

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Abstract. In the paper, using two inversion theorems for the Stirling numbers and binomial coefficients, employing properties of the Bell polynomials of the second kind, and utilizing a higher order derivative formula for the ratio of two differentiable functions, the authors present two explicit formulas, a determinantal expression, and a recursive relation for a sequence of unnamed polynomials, derive two identities connecting the sequence of unnamed polynomials with the Bell polynomials, and recover a known identity connecting the sequence of unnamed polynomials with the Bell polynomials.

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1. Motivations

In [4, pp. 257–258] and [5], the functions in

$$F(t,x)=\frac{1}{\sqrt{1-t}}\exp\biggl[x\biggl(\frac{1}{\sqrt{1-t}}-1\biggr)\biggr]=\sum_{n=0}^{\infty}h_n(x)\frac{t^n}{n!} \tag{1.1}$$

were considered. In [4, pp. 257–258], it was pointed out that the unnamed polynomials $h_n(x)$ satisfy

$$h_n(x)=\frac{1}{n!}\frac{x}{e^x}\biggl[\frac{\mathrm{d}}{\mathrm{d}(x^2)}\biggr]^n\bigl(x^{2n-1}e^x\bigr). \tag{1.2}$$

In [5], it was pointed out that the unnamed polynomials $h_n(x)$ and the Bell polynomials $B_n(x)$ are connected by the identity

$$\sum_{k=0}^{n}\binom{n}{k}B_k(x)=(-2)^n\sum_{k=0}^{n}(-1)^k h_k(x)S(n,k), \quad n\ge0, \tag{1.3}$$

where the Bell polynomials $B_k(x)$ can be generated by

$$e^{x(e^t-1)}=\sum_{k=0}^{\infty}B_k(x)\frac{t^k}{k!}$$

and the Stirling numbers of the second kind $S(n,k)$ can be generated by

$$\frac{(e^x-1)^k}{k!}=\sum_{n=k}^{\infty}S(n,k)\frac{x^n}{n!}, \quad k\in\{0\}\cup\mathbb{N}.$$

It was pointed out in [4, pp. 257–258] that the expression (1.2) had been applied in 1937 to the theory of hyperbolic differential equations. It was also pointed out in [10] that there have been some studies on interesting applications of Bell polynomials $B_k(x)$ in soliton theory, including links with bilinear and trilinear forms of nonlinear differential equations which possess soliton solutions. See, for example, [6, 7, 8]. Therefore, applications of the Bell polynomials $B_k(x)$ to integrable nonlinear equations are greatly expected and any amendment on multi-linear forms of soliton equations, even on exact solutions, would be beneficial to interested audiences in the research community.

To simplify main results in [5], among other things, the following conclusions were established in the newly published paper [10].

Theorem 1.1 ([10, Theorem 1]). *For $n\ge0$, the nth derivative of the generating function $F(t,x)$ of the polynomials $h_n(x)$ in (1.1) can be computed by*

$$\frac{\mathrm{d}^n F(t,x)}{\mathrm{d}\,t^n}=\frac{F(t,x)}{2^n}\sum_{k=0}^{n}\Biggl[\sum_{\ell=k}^{n}\ell!(2n-2\ell-1)!!\binom{2n-\ell-1}{2(n-\ell)}\binom{\ell}{k}\Biggr]\frac{x^k}{k!(1-t)^{n+k/2}},$$

where the double factorial of negative odd integers $-(2n+1)$ is defined by

$$(-2n-1)!!=\frac{(-1)^n}{(2n-1)!!}=(-1)^n\frac{2^n n!}{(2n)!}, \quad n\ge0$$

and the conventions $\binom{0}{0}=1$ and $\binom{p}{q}=0$ for $q>p\ge0$ are adopted. Consequently, the polynomials $h_n(x)$ for $n\ge0$ can be expressed as

$$h_n(x)=\frac{1}{2^n}\sum_{k=0}^{n}\Biggl[\sum_{\ell=k}^{n}\ell!(2n-2\ell-1)!!\binom{2n-\ell-1}{2(n-\ell)}\binom{\ell}{k}\Biggr]\frac{x^k}{k!}.$$

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Theorem 1.2 ([10, Theorem 3])**.** *For $n \ge 0$, the Bell polynomials $B_n(x)$ and the polynomials $h_n(x)$ can be expressed each other by*

$$B_n(x) = (-1)^n \sum_{\ell=0}^{n} (-1)^\ell \left[\sum_{k=\ell}^{n} 2^k \binom{n}{k} S(k,\ell)\right] h_\ell(x) \tag{1.4}$$

and

$$h_n(x) = \frac{1}{2^n} \sum_{\ell=0}^{n} \left[\sum_{k=\ell}^{n} \binom{n}{k} (2n-2k-1)!!2^{k-\ell} s(k,\ell)\right] B_\ell(x), \tag{1.5}$$

where $s(n,k)$ for $n \ge k \ge 0$, which can be generated by

$$\frac{[\ln(1+x)]^k}{k!} = \sum_{n=k}^{\infty} s(n,k) \frac{x^n}{n!}, \quad |x| < 1,$$

stand for the Stirling numbers of the first kind.

Theorem 1.3 ([10, Theorem 4])**.** *For $n \ge 0$, the Bell polynomials $B_n(x)$ and the polynomials $h_n(x)$ satisfy the identities*

$$\sum_{k=0}^{n} \binom{n}{k} 2^k (2n-2k-3)!! h_k(x) = -2^n \sum_{k=0}^{n} \frac{s(n,k)}{2^k} B_k(x) \tag{1.6}$$

and

$$\sum_{k=0}^{n} \binom{n}{k} \left[\sum_{\ell=0}^{n-k} \frac{s(n-k,\ell)}{2^\ell} B_\ell(-x)\right] h_k(x) = \frac{(2n-1)!!}{2^n}. \tag{1.7}$$

In this paper, using two inversion theorems for the Stirling numbers $s(n,k)$ and $S(n,k)$ and binomial coefficients $\binom{n}{k}$, employing properties of the Bell polynomials of the second kind $\mathrm{B}(n,k)$, and utilizing a higher order derivative formula for the ratio of two differentiable functions, the authors present two explicit formulas, a determinantal expression, and a recursive relation for $h_n(x)$, derive two identities connecting $h_n(x)$ with $B_n(x)$, and recover the identity (1.3).

Our main results can be stated as the following four theorems.

Theorem 1.4. *For $n \ge 0$, the polynomials $h_n(x)$ satisfy*

$$h_n(x) = \frac{1}{n!} \sum_{k=0}^{n} \frac{(-1)^k}{2^k} \binom{n}{k} \left\langle n - \frac{1}{2} \right\rangle_{n-k} \sum_{\ell=0}^{k} (-1)^\ell [2(k-\ell)-1]!! \binom{2k-\ell-1}{2(k-\ell)} x^\ell,$$

where

$$\langle x \rangle_n = \prod_{k=0}^{n-1} (x-k) = \begin{cases} x(x-1)\cdots(x-n+1), & n \ge 1 \\ 1, & n = 0 \end{cases}$$

is the falling factorial.

Theorem 1.5. *For $n \ge 0$, the Bell polynomials $B_n(x)$ and $h_n(x)$ satisfy*

$$B_n(x) = -2^n \sum_{k=0}^{n} \frac{S(n,k)}{2^k} \sum_{\ell=0}^{k} \binom{k}{\ell} 2^\ell (2k-2\ell-3)!! h_\ell(x). \tag{1.8}$$

Theorem 1.6. *For $n \ge 0$, the identity (1.3) is valid and*

$$h_n(x) = \sum_{k=0}^{n} (-1)^{n-k} \frac{s(n,k)}{2^k} \sum_{\ell=0}^{k} \binom{k}{\ell} B_\ell(x). \tag{1.9}$$

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Theorem 1.7. *For $n \ge 0$, the polynomials $h_n(x)$ can be computed by the determinantal expression*

$$h_n(x) = (-1)^n \begin{vmatrix} 1 & \lambda_0(x) & 0 & \cdots & 0 & 0 \\ \frac{1!!}{2} & \lambda_1(x) & \lambda_0(x) & \cdots & 0 & 0 \\ \frac{3!!}{2^2} & \lambda_2(x) & \binom{2}{1}\lambda_1(x) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{(2n-5)!!}{2^{n-2}} & \lambda_{n-2}(x) & \binom{n-2}{1}\lambda_{n-3}(x) & \cdots & \lambda_0(x) & 0 \\ \frac{(2n-3)!!}{2^{n-1}} & \lambda_{n-1}(x) & \binom{n-1}{1}\lambda_{n-2}(x) & \cdots & \binom{n-1}{n-2}\lambda_1(x) & \lambda_0(x) \\ \frac{(2n-1)!!}{2^n} & \lambda_n(x) & \binom{n}{1}\lambda_{n-1}(x) & \cdots & \binom{n}{n-2}\lambda_2(x) & \binom{n}{n-1}\lambda_1(x) \end{vmatrix}, \tag{1.10}$$

by the recursive relation

$$\sum_{r=0}^{n} \binom{n}{r} \lambda_{n-r}(x) h_r(x) = -\frac{(2n-1)!!}{2^n}, \tag{1.11}$$

and by the explicit formula

$$h_n(x) = n! \sum_{k=0}^{n} \frac{[2(n-k)-1]!!}{[2(n-k)]!!} \frac{\lambda_k(-x)}{k!} \tag{1.12}$$

for $n \ge 0$, where

$$\lambda_k(x) = \frac{1}{2^k} \sum_{\ell=0}^{k} \left[\sum_{p=0}^{\ell} (-1)^p \binom{\ell}{p} \prod_{q=0}^{k-1} (p+2q)\right] \frac{x^\ell}{\ell!}, \quad k \ge 0.$$

2. Lemmas

In order to prove our main results, we recall several lemmas below.

Lemma 2.1 ([11, Theorem 1.4])**.** *For $n \ge 0$, we have*

$$\frac{d^n\left(e^{\sqrt{u}}\right)}{d\,u^n} = e^{\sqrt{u}} \frac{(-1)^n}{(2u)^n} \sum_{k=0}^{n} (-1)^k (2n-2k-1)!! \binom{2n-k-1}{2(n-k)} u^{k/2}.$$

Lemma 2.2 ([14, p. 171, Theorem 12.1])**.** *If b_α and a_k are a collection of constants independent of n, then*

$$a_n = \sum_{\alpha=0}^{n} S(n,\alpha) b_\alpha \quad \text{if and only if} \quad b_n = \sum_{k=0}^{n} s(n,k) a_k.$$

Lemma 2.3 ([14, p. 83, Eq. (7.12)])**.** *If a_k and b_k for $k \ge 0$ are a collection of constants independent of n such that $n \ge k \ge 0$, then*

$$a(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} b(k) \quad \text{if and only if} \quad b(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a(k).$$

Lemma 2.4 ([3, pp. 134 and 139])**.** *For $n \ge k \ge 0$, the Bell polynomials of the second kind, or say, partial Bell polynomials, denoted by $\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1})$, are defined by*

$$\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{1 \le i \le n, \ell_i \in \{0\} \cup \mathbb{N} \\ \sum_{i=1}^{n} i\ell_i = n \\ \sum_{i=1}^{n} \ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left(\frac{x_i}{i!}\right)^{\ell_i}.$$

The Faà di Bruno formula can be described in terms of the Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$ *by*

$$\frac{\mathrm{d}^n}{\mathrm{d}\,t^n}f\circ h(t)=\sum_{k=0}^{n}f^{(k)}(h(t))\mathrm{B}_{n,k}\bigl(h'(t),h''(t),\dots,h^{(n-k+1)}(t)\bigr). \tag{2.1}$$

Lemma 2.5 ([3, p. 135])**.** *For* $n\ge k\ge0$*, we have*

$$\mathrm{B}_{n,k}\bigl(abx_1,ab^2x_2,\dots,ab^{n-k+1}x_{n-k+1}\bigr)=a^kb^n\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1}), \tag{2.2}$$

where a *and* b *are any complex numbers.*

Lemma 2.6 ([9, Remark 1])**.** *For* $n\ge k\ge0$*, we have*

$$\mathrm{B}_{n,k}\left(1,1-\lambda,(1-\lambda)(1-2\lambda),\dots,\prod_{\ell=0}^{n-k}(1-\ell\lambda)\right)\\ =\frac{(-1)^k}{k!}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\prod_{q=0}^{n-1}(\ell-q\lambda). \tag{2.3}$$

Lemma 2.7 ([1, p. 40, Entry 5])**.** *Let* $p=p(x)$ *and* $q=q(x)\ne0$ *be two differentiable functions. Then*

$$\left[\frac{p(x)}{q(x)}\right]^{(k)}=\frac{(-1)^k}{q^{k+1}}\begin{vmatrix}
p & q & 0 & \dots & 0 & 0\\
p' & q' & q & \dots & 0 & 0\\
p'' & q'' & \binom{2}{1}q' & \dots & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
p^{(k-2)} & q^{(k-2)} & \binom{k-2}{1}q^{(k-3)} & \dots & q & 0\\
p^{(k-1)} & q^{(k-1)} & \binom{k-1}{1}q^{(k-2)} & \dots & \binom{k-1}{k-2}q' & q\\
p^{(k)} & q^{(k)} & \binom{k}{1}q^{(k-1)} & \dots & \binom{k}{k-2}q'' & \binom{k}{k-1}q'
\end{vmatrix} \tag{2.4}$$

for $k\ge0$*. In other words, the formula* (2.4) *can be rewritten as*

$$\frac{\mathrm{d}^k}{\mathrm{d}\,x^k}\left[\frac{p(x)}{q(x)}\right]=\frac{(-1)^k}{q^{k+1}(x)}\bigl|W_{(k+1)\times(k+1)}(x)\bigr|, \tag{2.5}$$

where $|W_{(k+1)\times(k+1)}(x)|$ *denotes the determinant of the* $(k+1)\times(k+1)$ *matrix*

$$W_{(k+1)\times(k+1)}(x)=\begin{pmatrix}U_{(k+1)\times1}(x) & V_{(k+1)\times k}(x)\end{pmatrix},$$

the quantity $U_{(k+1)\times1}(x)$ *is a* $(k+1)\times1$ *matrix whose elements* $u_{\ell,1}(x)=p^{(\ell-1)}(x)$ *for* $1\le\ell\le k+1$*, and* $V_{(k+1)\times k}(x)$ *is a* $(k+1)\times k$ *matrix whose elements*

$$v_{i,j}(x)=\begin{cases}\dbinom{i-1}{j-1}q^{(i-j)}(x), & i-j\ge0\\ 0, & i-j<0\end{cases}$$

for $1\le i\le k+1$ *and* $1\le j\le k$*.*

Lemma 2.8 ([2, p. 222, Theorem] and [12, Remark 3]). *Let* $M_0 = 1$ *and*

$$M_n = \begin{vmatrix} m_{1,1} & m_{1,2} & 0 & \cdots & 0 & 0 \\ m_{2,1} & m_{2,2} & m_{2,3} & \cdots & 0 & 0 \\ m_{3,1} & m_{3,2} & m_{3,3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ m_{n-2,1} & m_{n-2,2} & m_{n-2,3} & \cdots & m_{n-2,n-1} & 0 \\ m_{n-1,1} & m_{n-1,2} & m_{n-1,3} & \cdots & m_{n-1,n-1} & m_{n-1,n} \\ m_{n,1} & m_{n,2} & m_{n,3} & \cdots & m_{n,n-1} & m_{n,n} \end{vmatrix}$$

for $n \in \mathbb{N}$. *Then the sequence* M_n *for* $n \ge 0$ *satisfies* $M_1 = m_{1,1}$ *and*

$$M_n = m_{n,n} M_{n-1} + \sum_{r=1}^{n-1} (-1)^{n-r} m_{n,r} \left(\prod_{j=r}^{n-1} m_{j,j+1} \right) M_{r-1}, \quad n \ge 2. \tag{2.6}$$

3. Proofs of main results

Now we are in a position to prove our main results.

Proof of Theorem 1.4. By virtue of Lemma 2.1, the formula (1.2) implies that

$$\begin{aligned} h_n(x) &= \frac{1}{n!} \frac{x}{e^x} \left(\frac{\mathrm{d}}{\mathrm{d}\, u} \right)^n \left(u^{n-1/2} e^{\sqrt{u}} \right), \quad u = x^2 \\ &= \frac{1}{n!} \frac{x}{e^x} \sum_{k=0}^{n} \binom{n}{k} \left(u^{n-1/2} \right)^{(n-k)} \left(e^{\sqrt{u}} \right)^{(k)} \\ &= \frac{1}{n!} \frac{x}{e^x} \sum_{k=0}^{n} \binom{n}{k} \left\langle n - \frac{1}{2} \right\rangle_{n-k} u^{k-1/2} \frac{(-1)^k}{(2u)^k} \\ &\quad \times e^{\sqrt{u}} \sum_{\ell=0}^{k} (-1)^\ell [2(k-\ell)-1]!! \binom{2k-\ell-1}{2(k-\ell)} u^{\ell/2} \\ &= \frac{1}{n!} \sum_{k=0}^{n} \frac{(-1)^k}{2^k} \binom{n}{k} \left\langle n - \frac{1}{2} \right\rangle_{n-k} \sum_{\ell=0}^{k} (-1)^\ell [2(k-\ell)-1]!! \binom{2k-\ell-1}{2(k-\ell)} x^\ell. \end{aligned}$$

The proof of Theorem 1.4 is complete. □

Proof of Theorem 1.5. The identity (1.6) in Theorem 1.3 can be rearranged as

$$-\frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} 2^k (2n-2k-3)!! h_k(x) = \sum_{k=0}^{n} s(n,k) \frac{B_k(x)}{2^k}.$$

Further utilizing Lemma 2.2 yields

$$\frac{B_n(x)}{2^n} = -\sum_{k=0}^{n} S(n,k) \frac{1}{2^k} \sum_{\ell=0}^{k} \binom{k}{\ell} 2^\ell (2k-2\ell-3)!! h_\ell(x).$$

The proof of Theorem 1.5 is complete. □

Proof of Theorem 1.6. Interchanging the order of sums in the right hand side of the identity (1.4) in Theorem 1.2 arrives at

$$(-1)^n B_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (-2)^k \sum_{\ell=0}^{k} S(k,\ell)(-1)^\ell h_\ell(x).$$

Further applying Lemma 2.3 leads to

$$(-2)^n\sum_{\ell=0}^{n}S(n,\ell)(-1)^\ell h_\ell(x)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}(-1)^kB_k(x)=\sum_{k=0}^{n}\binom{n}{k}B_k(x)$$

which is equivalent to (1.3).

Employing Lemma 2.2 in (1.3) produces

$$(-1)^nh_n(x)=\sum_{k=0}^{n}s(n,k)\frac{(-1)^k}{2^k}\sum_{\ell=0}^{k}\binom{k}{\ell}B_\ell(x).$$

The proof of Theorem 1.6 is complete. □

Proof of Theorem 1.7. By straightforward computation, we obtain

$$\begin{aligned}\frac{\mathrm{d}^k}{\mathrm{d}\,t^k}\bigl[(1-t)^{-1/2}\bigr]&=\left\langle-\frac12\right\rangle_k(-1)^k(1-t)^{-1/2-k}\\ &\to\left\langle-\frac12\right\rangle_k(-1)^k=\left(\frac12\right)_k=\frac{(2k-1)!!}{2^k},\quad t\to0\end{aligned}$$

and, when denoting $u=u(t)=(1-t)^{-1/2}$ and making use of the formulas (2.1) and (2.2),

$$\begin{aligned}\frac{\mathrm{d}^k\exp\bigl[-x(1-t)^{-1/2}\bigr]}{\mathrm{d}\,t^k}&=\sum_{\ell=0}^{k}\bigl(e^{-xu}\bigr)^{(\ell)}\mathrm{B}_{k,\ell}\bigl(u'(t),u''(t),\dots,u^{(k-\ell+1)}(t)\bigr)\\ &=\sum_{\ell=0}^{k}(-x)^\ell e^{-xu}\mathrm{B}_{k,\ell}\Biggl(\left\langle-\frac12\right\rangle_1\frac{-1}{(1-t)^{3/2}},\left\langle-\frac12\right\rangle_2\frac{1}{(1-t)^{5/2}},\dots,\\ &\qquad\left\langle-\frac12\right\rangle_{k-\ell+1}\frac{(-1)^{k-\ell+1}}{(1-t)^{k-\ell+3/2}}\Biggr)\\ &\to e^{-x}\sum_{\ell=0}^{k}(-x)^\ell\mathrm{B}_{k,\ell}\biggl(\frac{1!!}{2},\frac{3!!}{2^2},\dots,\frac{(2k-2\ell+1)!!}{2^{k-\ell+1}}\biggr),\quad t\to0\\ &=\frac{e^{-x}}{2^k}\sum_{\ell=0}^{k}(-x)^\ell\mathrm{B}_{k,\ell}(1!!,3!!,\dots,(2k-2\ell+1)!!).\end{aligned}$$

Taking $\lambda=-2$ in (2.3) gives

$$\mathrm{B}_{n,k}(1!!,3!!,\dots,(2n-2k+1)!!)=\frac{(-1)^k}{k!}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\prod_{q=0}^{n-1}(\ell+2q)$$

for $n\ge k\ge0$. Therefore, we obtain

$$\lim_{t\to0}\frac{\mathrm{d}^k\exp\bigl[-x(1-t)^{-1/2}\bigr]}{\mathrm{d}\,t^k}=\frac{e^{-x}}{2^k}\sum_{\ell=0}^{k}\frac{x^\ell}{\ell!}\sum_{p=0}^{\ell}(-1)^p\binom{\ell}{p}\prod_{q=0}^{k-1}(p+2q)=e^{-x}\lambda_k(x).$$

The equation (1.1) means that

$$h_n(x)=\lim_{t\to0}\frac{\mathrm{d}^n}{\mathrm{d}\,t^n}\frac{(1-t)^{-1/2}}{\exp\bigl[x\bigl(1-(1-t)^{-1/2}\bigr)\bigr]}=e^{-x}\lim_{t\to0}\frac{\mathrm{d}^n}{\mathrm{d}\,t^n}\frac{(1-t)^{-1/2}}{\exp\bigl[-x(1-t)^{-1/2}\bigr]}.$$

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Further applying the formula (2.5) to

$$p(t)=(1-t)^{-1/2}\quad\text{and}\quad q(t)=\exp\bigl[-x(1-t)^{-1/2}\bigr]$$

results in

$$\begin{aligned}
h_n(x)&=(-1)^n e^{-x}\lim_{t\to0}\exp\bigl[(n+1)x(1-t)^{-1/2}\bigr]\\
&\quad\times\lim_{t\to0}\begin{vmatrix}
p & q & 0 & \cdots & 0 & 0\\
p' & q' & q & \cdots & 0 & 0\\
p'' & q'' & \binom{2}{1}q' & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
p^{(n-2)} & q^{(n-2)} & \binom{n-2}{1}q^{(n-3)} & \cdots & q & 0\\
p^{(n-1)} & q^{(n-1)} & \binom{n-1}{1}q^{(n-2)} & \cdots & \binom{n-1}{n-2}q' & q\\
p^{(n)} & q^{(n)} & \binom{n}{1}q^{(n-1)} & \cdots & \binom{n}{n-2}q'' & \binom{n}{n-1}q'
\end{vmatrix}\\
&=(-1)^n e^{nx}\begin{vmatrix}
1 & q_0(x) & 0 & \cdots & 0 & 0\\
\frac{1!!}{2} & q_1(x) & q_0(x) & \cdots & 0 & 0\\
\frac{3!!}{2^2} & q_2(x) & \binom{2}{1}q_1(x) & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
\frac{(2n-5)!!}{2^{n-2}} & q_{n-2}(x) & \binom{n-2}{1}q_{n-3}(x) & \cdots & q_0(x) & 0\\
\frac{(2n-3)!!}{2^{n-1}} & q_{n-1}(x) & \binom{n-1}{1}q_{n-2}(x) & \cdots & \binom{n-1}{n-2}q_1(x) & q_0(x)\\
\frac{(2n-1)!!}{2^{n}} & q_{n}(x) & \binom{n}{1}q_{n-1}(x) & \cdots & \binom{n}{n-2}q_2(x) & \binom{n}{n-1}q_1(x)
\end{vmatrix}\\
&=(-1)^n\begin{vmatrix}
1 & \lambda_0(x) & 0 & \cdots & 0 & 0\\
\frac{1!!}{2} & \lambda_1(x) & \lambda_0(x) & \cdots & 0 & 0\\
\frac{3!!}{2^2} & \lambda_2(x) & \binom{2}{1}\lambda_1(x) & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
\frac{(2n-5)!!}{2^{n-2}} & \lambda_{n-2}(x) & \binom{n-2}{1}\lambda_{n-3}(x) & \cdots & \lambda_0(x) & 0\\
\frac{(2n-3)!!}{2^{n-1}} & \lambda_{n-1}(x) & \binom{n-1}{1}\lambda_{n-2}(x) & \cdots & \binom{n-1}{n-2}\lambda_1(x) & \lambda_0(x)\\
\frac{(2n-1)!!}{2^{n}} & \lambda_{n}(x) & \binom{n}{1}\lambda_{n-1}(x) & \cdots & \binom{n}{n-2}\lambda_2(x) & \binom{n}{n-1}\lambda_1(x)
\end{vmatrix},
\end{aligned}$$

where $q_k(x)=e^{-x}\lambda_k(x)$ for $k\ge0$. The determinantal expression (1.10) is proved.

Applying (2.6) to the determinantal expression (1.10) and considering $\lambda_0(x)=1$ derive

$$\begin{aligned}
(-1)^n h_n(x)&=n\lambda_1(x)(-1)^{n-1}h_{n-1}(x)+(-1)^{n-1}\frac{(2n-1)!!}{2^n}\\
&\quad+\sum_{r=2}^{n}(-1)^{n-r+1}\binom{n}{r-2}\lambda_{n-r+2}(x)(-1)^{r-2}h_{r-2}(x),\quad n\ge2.
\end{aligned}$$

This can be reformulated as (1.11).

From the equation (1.1) and the above arguments, it follows that

$$\begin{aligned}
h_n(x)&=e^{-x}\lim_{t\to0}\frac{\mathrm{d}^n}{\mathrm{d}\,t^n}\biggl[\frac{1}{\sqrt{1-t}}\exp\biggl(\frac{x}{\sqrt{1-t}}\biggr)\biggr]\\
&=e^{-x}\lim_{t\to0}\sum_{k=0}^{n}\binom{n}{k}\frac{\mathrm{d}^{n-k}}{\mathrm{d}\,t^{n-k}}\frac{1}{\sqrt{1-t}}\frac{\mathrm{d}^k}{\mathrm{d}\,t^k}\exp\biggl(\frac{x}{\sqrt{1-t}}\biggr)
\end{aligned}$$

$$= e^{-x}\sum_{k=0}^{n}\binom{n}{k}\lim_{t\to0}\frac{\mathrm{d}^{n-k}}{\mathrm{d}\,t^{n-k}}\frac{1}{\sqrt{1-t}}\lim_{t\to0}\frac{\mathrm{d}^{k}}{\mathrm{d}\,t^{k}}\exp\biggl(\frac{x}{\sqrt{1-t}}\biggr)$$

$$= e^{-x}\sum_{k=0}^{n}\binom{n}{k}\frac{(2n-2k-1)!!}{2^{n-k}}\frac{e^{x}}{2^{k}}\sum_{\ell=0}^{k}\frac{(-x)^{\ell}}{\ell!}\sum_{p=0}^{\ell}(-1)^{p}\binom{\ell}{p}\prod_{q=0}^{k-1}(p+2q)$$

$$= \frac{1}{2^{n}}\sum_{k=0}^{n}\binom{n}{k}(2n-2k-1)!!\sum_{\ell=0}^{k}\frac{(-1)^{\ell}}{\ell!}x^{\ell}\sum_{p=0}^{\ell}(-1)^{p}\binom{\ell}{p}\prod_{q=0}^{k-1}(p+2q)$$

which can be rewritten as (1.12). The proof of Theorem 1.7 is complete. □

4. Remarks

Finally we list several remarks about our main results and other things.

Remark 4.1. Recently a new inversion theorem was discovered in [13, Theorem 4.3] which can be reformulated as that

$$s_n=\sum_{k=1}^{n}\binom{k}{n-k}S_k \quad\text{if and only if}\quad (-1)^{n}nS_n=\sum_{k=1}^{n}(-1)^{k}k\binom{2n-k-1}{n-1}s_k,$$

where s_k and S_k are two sequences independent of n such that $n\ge k\ge 1$.

Remark 4.2. The identity (1.4) is slightly different from (1.8).

Remark 4.3. The identity (1.9) is obviously simpler than (1.5) in their forms.

Remark 4.4. Comparing (1.7) with (1.11) reveals that

$$\sum_{\ell=0}^{n-k}\frac{s(n-k,\ell)}{2^{\ell}}B_{\ell}(-x)=-\lambda_{n-k}(x),$$

that is,

$$\sum_{\ell=0}^{n}s(n,\ell)\frac{B_{\ell}(x)}{2^{\ell}}=-\lambda_{n}(-x).$$

Further considering Lemma 2.2 deduces

$$B_n(x)=-2^{n}\sum_{\ell=0}^{n}S(n,\ell)\lambda_{\ell}(-x).$$

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