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Gödel's Incompleteness Theorems and the Necessity of Semantics for Arithmetic

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Abstract

This work traces a philosophical and mathematical thread from ancient Greek mathematics to modern foundational logic. The Greeks maintained a sharp distinction between *ἀριθμητική* (arithmetic as the theoretical science of numbers) and *λογιστική* (calculation as a practical art), while also separating arithmetic from formal logic. This separation, grounded in ontological and epistemological considerations, allowed Greek mathematics to avoid the foundational crises that would emerge two millennia later. The development of formal logic in the late nineteenth and early twentieth centuries—particularly through the work of Frege, Russell, and Hilbert—sought to unify arithmetic and logic within a single syntactic framework. Gödel's incompleteness theorems (1931) demonstrated the impossibility of this project, showing that any consistent, recursively axiomatizable theory strong enough to encode arithmetic must be incomplete and cannot prove its own consistency. Furthermore, phenomena such as Tarski's undefinability of truth and the existence of non-standard models demonstrate that pure syntax faces a total epistemological collapse. This work argues that these metamathematical limits can be synthesized into a "Semantic Necessity Theorem": a complete, consistent, arithmetically strong theory cannot be purely syntactic. The Greek separation of arithmetic from formal logic thus appears not merely as a historical curiosity, but as a mathematically prescient framework that anticipates the structural necessity of ontology in modern mathematics.

Keywords: Gödel's incompleteness theorems; Greek mathematics; arithmetike; logistike; syntax; semantics; formal logic; Robinson arithmetic

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1. Introduction

The central thesis of this paper is that logic, if it is to fully support arithmetic, cannot be reduced to pure syntax. This claim emerges from a synthesis of ancient Greek mathematical practice and modern metamathematical results. While the modern era has been defined by the drive toward total formalization, the inherent limitations of such systems suggest that the ancient approach to mathematical truth may offer a more robust ontological foundation.

The argument presented here proceeds in three distinct stages. First, it examines the Greek distinction between *ἀριθμητική* (arithmetic), the theoretical science of numbers, and *λογιστική* (logistic), the practical art of calculation. In the Greek tradition, arithmetic was maintained as a body of truths grasped by reason rather than as a formal calculus, effectively separating it from the mechanics of formal logic. Second, the paper reviews the modern formalist project, which sought to overturn this separation by unifying arithmetic and logic within a single, fully formalized, syntactic system. Finally, the paper analyzes the Gödelian refutation of this project. Gödel's incompleteness theorems proved that such a unification cannot achieve both completeness and self-justification while remaining purely syntactic.

By synthesizing these historical and mathematical perspectives, this work concludes that a complete, consistent, and arithmetically powerful theory cannot be recursively axiomatizable. This result reframes Gödel's findings not merely as a limitation of specific systems, but as a "semantic necessity" theorem that defines a fundamental condition on the nature of logic itself: to capture the full richness of arithmetic, logic must transcend pure syntax and ground itself in semantic truth.

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2. Literature Review

2.1. The Greek Foundations: Arithmetic, Logistic, and the Separation from Logic

A fundamental distinction runs through Greek mathematical thought: the division between *ἀριθμητική* (arithmetic) and *λογιστική* (logistic or calculation). This distinction was well-established by Plato's time and likely originated within the Pythagorean tradition. As Gow (1884/2010) documents, later Greek mathematicians used *ἀριθμητική* to designate the "science of numbers," while *λογιστική* referred to the "art of calculation." This distinction appears frequently in the Platonic corpus, where the terms are used without any indication of novelty, suggesting they date from a very early period in the history of Greek science and philosophy (Gow 1884/2010).

Plato's philosophy of mathematics elevates arithmetic to a spiritual and ontological discipline. In Plato's conception, *ἀριθμός* (number) always consists of "the odd and even" or is understood as a "multitude of units" (Kalyvas 2024). For Plato, the mathematical sciences—*ἀριθμητική*, *λογιστική*, and *γεωμετρία*—serve a higher purpose: they draw the soul toward "Truth" (*πρὸς ἀλήθειαν*) (Kalyvas 2024). This Platonic conception of number differs fundamentally from modern notions; as Klein's analysis suggests, *ἀριθμός* is always "a definite number of definite objects" (Kalyvas 2024). Numbers were not abstract symbols in a formal system but were inherently tied to the things they counted.

The Pythagorean tradition, which heavily influenced Plato, conceived of numbers as spatial magnitudes. A crucial consequence of this was that zero was not considered a number. As Burnet (1920) explains, the characteristic feature of Pythagoreanism was that the point was not regarded as a limit, but as the first product of the "Limit" and the "Unlimited," identified with the arithmetical unit instead of zero. In this system, the point is defined as a "unit having position" (*μονὰς θέσιν ἔχουσα*) (Burnet 1920). Numbers were understood geometrically: the point corresponds to 1, the line to 2, the surface to 3, and the solid to 4. This geometric conception meant that Greek arithmetic never developed the concept of zero as a number, a concept that would later prove essential for the arithmetization of analysis and formal logic.

Crucially, the Greeks did not attempt to unify arithmetic with formal logic. Logic (*λογική*) was understood as the art of reasoning validly, while arithmetic was a body of truths about numbers grasped through reason but not reduced to formal rules. Aristotle's *Metaphysics* articulates this separation: the mathematician studies objects "not qua sensible" but "qua lines and numbers" (Aristotle). Mathematical sciences consider objects "as so regarded" without error, "any more than when we draw a diagram on the ground and say that a line is a foot long when it is not" (Aristotle). This separation meant that Greek mathematics never faced the foundational dilemmas that would later emerge; by not embedding arithmetic within a fixed syntactic system, they avoided the limitations that Gödel would eventually expose.

2.2. The Rise of Formal Logic and the Unification Project

The modern project of formalizing logic began with George Boole, who was the first to present logic as a mathematical theory in an algebraic style (Font 2016/2022). Boole's approach treated logic as a calculus—a system of rules for manipulating symbols. However, in this early tradition, the distinction between a formal language and a mathematically semantics for it was not yet drawn (Font 2016/2022).

A decisive shift occurred with Frege and Russell, who introduced a logic system given by a formal language and a deductive calculus, specifically a set of axioms and inference rules (Peregrin 2019). These systems, now known as Hilbert-style calculi, represented logic as a formal system with explicit syntactic rules. Crucially, in this tradition, a formal mathematical semantics (algebraic or model-theoretic) for the formal languages was lacking; the only semantics present was of an intuitive, informal kind (Peregrin 2019). The goal was to capture logical truths through syntactic derivations alone.

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David Hilbert sought to extend this project to all of mathematics. His program aimed to formalize arithmetic within a consistent, complete, and decidable system, proving its consistency using only finitary methods. This represented the full unification of arithmetic and logic within a purely syntactic framework—the very unification the Greeks had avoided. The Hilbert program assumed that the original idea behind logical calculi was to capture, and thus make calculable, "what follows from what," a task that was assumed to be achievable (Peregrin 2019).

2.3. The Gödelian Revolution and the Turn to Semantics

Kurt Gödel's 1931 papers proved the impossibility of the Hilbert program. The first incompleteness theorem shows that any consistent, recursively axiomatizable theory strong enough to encode arithmetic must be incomplete: there exist true arithmetical statements that the theory cannot prove. The second theorem shows that such a theory cannot prove its own consistency. These results challenged the foundational assumption that logical calculi could capture and make calculable the relation of "following-from" (Peregrin 2019). They demonstrated that pure syntax has inherent limits.

From these results, a refined theorem of "Semantic Necessity" can be formulated: Let T be a consistent, complete theory that interprets Robinson arithmetic Q. Then T cannot be recursively axiomatizable. Equivalently, any consistent, recursively axiomatizable theory interpreting Q is necessarily incomplete. This theorem shows that the combination of consistency, completeness, and arithmetic strength forces a theory to be non-recursive—that is, to transcend pure syntax. The only way to achieve all three is to incorporate semantic or non-effective elements.

This turn to semantics is further solidified by Tarski's Theorem on the Undefinability of Truth. Tarski (1933) demonstrated that for any sufficiently strong formal language, the set of Gödel numbers of true sentences of that language is not definable within the language itself. This result serves as a semantic mirror to Gödel's syntactic incompleteness: while Gödel showed that provability cannot reach all truth, Tarski showed that truth itself cannot be contained within the syntactic boundaries of the system it describes. Together, these results imply that any complete account of arithmetic must look "upward" to a meta-language or a richer semantic framework, reinforcing the claim that a complete theory of arithmetic is necessarily non-recursive.

As Tarski recognized, to capture "following-from" adequately, logic requires means more powerful than those offered by calculi; his solution was to logically capture the concept of truth (Peregrin 2019). This marks the moment in which the languages of formal logic, which previously incorporated only calculi and thus merely "syntax," finally embraced "semantics" and became languages in the fully-fledged sense (Peregrin 2019). While some philosophers note this distinction is better understood as a choice between two kinds of means to account for "following-from," it marks the end of the purely syntactic dream.

2.4. Historical Irony: Greeks and Moderns Compared

The Greek separation of arithmetic from formal logic appears strikingly prescient in light of Gödel's theorems. By never attempting to reduce arithmetic to a fixed syntactic system, the Greeks avoided the very limitations that Gödel would eventually expose. Their arithmetic was always grounded in semantic intuition—geometric, ontological, and spiritual—rather than in mechanical rule-following.

An irony lies in the mechanism of Gödel's proof: it requires the "arithmetization of syntax," where logical formulas and proofs are encoded as natural numbers. From the perspective of Greek mathematical ontology, such an encoding would likely have been viewed as a category error. The Greeks maintained a strict boundary between the *ἀριθμός* (the number itself as an ontological reality) and the *λογιστικόν* (the symbolic or calculative tool). By refusing to collapse these categories—that is, by not treating numbers as mere placeholders for logical strings—the Greeks inadvertently protected the "sanctity" of arithmetic from the self-referential paradoxes that emerge when logic and arithmetic are forced into a single syntactic plane.

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The Greek conception of number as "a definite number of definite objects" (Kalyvas 2024) stands in contrast to the modern formalist conception of numbers as abstract objects defined solely by axioms. This semantic grounding meant that Greek mathematics could be complete in a way that formal systems cannot: truths were grasped through reason (*logos*), not derived from axioms.

Modern logic, by contrast, has confronted the trilemma that Gödel's theorems force: consistency, completeness, and arithmetic strength cannot all be achieved within a purely syntactic system. The history of twentieth-century logic can be read as a series of responses to this dilemma:

- Formalists (following Hilbert) accept incompleteness while retaining syntax and arithmetic strength.
- Intuitionists (following Brouwer) reject certain logical principles to maintain a different kind of completeness.
- Logicians in the Tarski tradition embrace semantics as the necessary complement to syntax.

There is a clear philosophical continuity between the Greek and post-Gödelian positions: both recognize that arithmetic cannot be fully captured by mechanical rules. For the Greeks, this was grounded in the nature of number as ontologically real; for modern logicians, it is grounded in the metamathematical results of Gödel. Both approaches recognize that arithmetic requires something beyond syntax—whether it be grounded in the Greek foundations of ontology and geometry or the modern foundations of model theory and truth definitions.

3. Methodology

3.1. Introduction and Preliminaries

3.1.1. Goal

The methodological objective of this section is to provide a formal proof of a fundamental limitation inherent in purely syntactic formal systems. Specifically, we aim to demonstrate that no consistent, complete, and recursively axiomatizable theory can interpret a sufficient amount of arithmetic. Conversely, any consistent, arithmetically strong, and recursively axiomatizable theory is necessarily incomplete and incapable of proving its own consistency. From these results, we deduce the "Semantic Necessity" thesis: a complete and consistent theory of arithmetic must transcend pure syntax by relying on semantic or non-effective elements.

3.1.2. Language and Basic Notions

This analysis operates within the framework of first-order logic. Let $L_A = \{0, S, +, \cdot\}$ denote the language of arithmetic. A theory T is defined as a set of L-sentences closed under logical consequence. We assume that the language L contains L_A, though the results generalize to any theory that interprets L_A. The following definitions are central to the proof:

Recursively Axiomatizable (Purely Syntactic): A theory T is recursively axiomatizable if there exists a recursive set of axioms Ax such that $T = \{\phi : Ax \vdash \phi\}$

Equivalently, the set of Gödel numbers of the theorems of T is recursively enumerable.

Consistent: There is no sentence ϕ such that $T \vdash \phi$ and $T \vdash \neg\phi$.

Complete: For every sentence ϕ in the language of T, either $T \vdash \phi$ or $T \vdash \neg\phi$

Interprets Q: There exists an effective translation τ of L_A-formulas into L such that for every axiom ψ of Robinson arithmetic Q, $T \vdash \tau(\psi)$. For the sake of simplicity, we may assume T directly extends Q in the same language.

3.1.3. Robinson Arithmetic (Q)

Robinson arithmetic (Q) consists of a finite set of axioms in L_A that capture the basic properties of the successor function, addition, and multiplication without the use of induction. Despite its lack of induction, Q is sufficiently strong to represent all computable (primitive recursive) functions and to prove every true Σ_1 sentence.

3.1.4. Arithmetization of Syntax

Following Gödel's fundamental insight, formulas and proofs are encoded as natural numbers (Gödel numbering). For a theory T that is recursively axiomatizable and interprets Q, we can construct a primitive recursive proof predicate $Proof_T(p,n)$, which signifies that "p is the Gödel number of a proof of the formula with Gödel number n." The formula $Prov_T(x) := \exists p Proof_T(p,x)$ is a Σ_1 formula expressing provability within T. Because T contains Q, it possesses the capacity to prove all true Σ_1 statements and refute all false Σ_1 statements.

3.2. The First Incompleteness Theorem (Rosser's Form)

3.2.1. Statement Theorem 1 (First Incompleteness Theorem)

Let T be a consistent, recursively axiomatizable theory that interprets Q. Then $T$is incomplete: there exists a sentence ρ such that $T \nvdash \rho$ and $T \nvdash \neg\rho$. The proof presented here utilizes the Rosser sentence, which strengthens Gödel's original result by removing the requirement for ω-consistency, relying instead on simple consistency.

3.2.2. The Rosser Sentence

Let $Proof_T(p,n)$ be the primitive recursive proof predicate introduced 3.1.4. We define the following formula:

$$Ross_T(x) := \forall y(Proof_T(y,x) \to \exists z < y Proof_T(z, neg(x)))$$

where $neg(x)$ denotes the Gödel number of the negation of the formula with Gödel number x. Intuitively, $Ross_T(\ulcorner \phi \urcorner)$ asserts: "If there is a proof of ϕ, then there is a smaller proof of $\neg\phi$."

By application of the diagonal lemma (or fixed-point theorem), there exists a sentence ρ such that:

$$T \vdash \rho \leftrightarrow Ross_T(\ulcorner \rho \urcorner)$$

The sentence ρ is designated as the Rosser sentence.

3.2.3. Proof that ρ Is Undecidable

We demonstrate that under the assumption of consistency, neither ρ nor $\neg\rho$ is provable in T.

3.2.3.1. ϱ Is Not Provable

Assume $T \vdash \rho$. Given that $T \vdash \rho \leftrightarrow Ross_T(\ulcorner \rho \urcorner)$, it follows that $T \vdash Ross_T(\ulcorner \rho \urcorner)$, which is:

$$T \vdash \forall y(Proof_T(y, \ulcorner \rho \urcorner) \to \exists z < y Proof_T(z, \ulcorner \neg\rho \urcorner))$$

Since $T \vdash \rho$, there exists a natural number p (the Gödel number of a proof of ρ) such that $Proof_T(p, \ulcorner \rho \urcorner)$ is true. Because T is consistent, there can be no proof of $\neg\rho$. Consequently, the statement $\exists z < p Proof_T(z, \ulcorner \neg\rho \urcorner)$is false.

As T contains Q, it can prove all true Σ_1 sentences and disprove all false Σ_1 sentences. The formula $\exists z < p Proof_T(z, \ulcorner \neg\rho \urcorner)$ is a Σ_1 sentence (as $Proof_T$ is primitive recursive). Its falsehood implies that T proves its negation:

$$T \vdash \neg\exists z < \bar{p} Proof_T(z, \ulcorner \neg\rho \urcorner)$$

where $\bar{p}$ is the numeral for p.

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From the universal statement above, instantiating y with $\bar{p}$ yields:

$$T \vdash Proof_T(\bar{p}, \ulcorner \rho \urcorner) \rightarrow \exists z < \bar{p} Proof_T(z, \ulcorner \neg\rho \urcorner)$$

Because $Proof_T(\bar{p}, \ulcorner \rho \urcorner)$ is true, T proves it. By *modus ponens*, we obtain:

$$T \vdash \exists z < \bar{p} Proof_T(z, \ulcorner \neg\rho \urcorner)$$

This contradicts the previously established negation. Therefore, $T \nvdash \rho$.

3.2.3.2. $\neg\rho$ Is Not Provable

Assume $T \vdash \neg\rho$. Because $T \vdash \rho \leftrightarrow Ross_T(\ulcorner \rho \urcorner)$, we have $T \vdash \neg Ross_T(\ulcorner \rho \urcorner)$. Unfolding the definition, $\neg Ross_T(\ulcorner \rho \urcorner)$ is equivalent to:

$$\exists y(Proof_T(y, \ulcorner \rho \urcorner) \wedge \forall z < y \neg Proof_T(z, \ulcorner \neg\rho \urcorner))$$

Let q be a proof of $\neg\rho$. Since T is consistent, there is no proof of ρ. However, the existential statement above asserts that there exists some y such that y is a proof of ρ and no smaller number proves $\neg\rho$. This is false in the standard model because no proof of ρ exists. This statement is Σ_1; its falsehood implies that T proves its negation, i.e., $T \vdash Ross_T(\ulcorner \rho \urcorner)$.

From the fixed-point equivalence, $Ross_T(\ulcorner \rho \urcorner) \leftrightarrow \rho$, thus $T \vdash \rho$. This contradicts the assumption $T \vdash \neg\rho$ and the consistency of T. Hence, $T \nvdash \neg\rho$.

Consequently, ρ is neither provable nor refutable in T, proving that T is incomplete. ■

3.3. The Second Incompleteness Theorem

3.3.1. Statement Theorem 2 (Second Incompleteness Theorem)

Let T be a consistent, recursively axiomatizable theory that interprets Q and satisfies the derivability conditions (which hold for standard theories like Peano Arithmetic, PA). Then $T \nvdash Con(T)$, where $Con(T)$ is a canonical sentence expressing the consistency of T.

For simplicity, we assume T extends PA in the language L_A. The proof generalizes to any recursively axiomatizable theory interpreting Q that can formalize the necessary reasoning.

3.3.2. The Provability Predicate and Derivability Conditions

Let $Prov_T(x)$ be the Σ_1 formula $\exists p Proof_T(p, x)$ that expresses provability in T. For a sentence ϕ, we write $Prov_T(\ulcorner \phi \urcorner)$ for the sentence with the numeral of $\ulcorner \phi \urcorner$ substituted. For theories containing PA, we can define $Prov_T$ so that it satisfies the Hilbert–Bernays derivability conditions:

- D1: If $T \vdash \phi$ then $T \vdash Prov_T(\ulcorner \phi \urcorner)$.
- D2: $T \vdash Prov_T(\ulcorner \phi \urcorner) \rightarrow Prov_T(\ulcorner Prov_T(\ulcorner \phi \urcorner) \urcorner)$.
- D3: $T \vdash Prov_T(\ulcorner \phi \rightarrow \psi \urcorner) \rightarrow (Prov_T(\ulcorner \phi \urcorner) \rightarrow Prov_T(\ulcorner \psi \urcorner))$.

These conditions are provable inside T for the standard provability predicate.

3.3.3. The Gödel Sentence

By the diagonal lemma, there exists a sentence G such that:

$$T \vdash G \leftrightarrow \neg Prov_T(\ulcorner G \urcorner)$$

This is the standard Gödel sentence. From the first incompleteness theorem (or a simpler argument using D1 and consistency), we know that if T is consistent, then $T \nvdash G$.

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3.3.4. Formalizing the First Incompleteness Theorem Inside T

We aim to show that T proves the implication $Con(T) \rightarrow G$. The key step is to formalize the reasoning that consistency implies the unprovability of G.

Lemma:

$$T \vdash Con(T) \rightarrow \neg Prov_T(\ulcorner G \urcorner)$$

Proof:

We reason inside T. Suppose $Prov_T(\ulcorner G \urcorner)$.

By D2, we obtain $Prov_T(\ulcorner Prov_T(\ulcorner G \urcorner) \urcorner)$.

From the fixed-point equivalence, $T \vdash G \neg Prov_T(\ulcorner G \urcorner)$. Using D3 and propositional logic, T proves:

$$Prov_T(\ulcorner G \urcorner) \rightarrow Prov_T(\ulcorner \neg Prov_T(\ulcorner G \urcorner) \urcorner)$$

Thus, from $Prov_T(\ulcorner G \urcorner)$, we derive both $Prov_T(\ulcorner Prov_T(\ulcorner G \urcorner) \urcorner)$ and $Prov_T(\ulcorner \neg Prov_T(\ulcorner G \urcorner) \urcorner)$.

By D3 again, these together yield $Prov_T(\ulcorner \perp \urcorner)$, i.e., $\neg Con(T)$. Therefore, we have shown (within T):

$$Prov_T(\ulcorner G \urcorner) \rightarrow \neg Con(T)$$

Contrapositively, $Con(T) \rightarrow \neg Prov_T(\ulcorner G \urcorner)$. ■

3.3.5. Deriving $Con(T) \rightarrow G$

From the fixed-point equivalence, $T \vdash G \leftrightarrow \neg Prov_T(\ulcorner G \urcorner)$. Combining this with the lemma:

$$T \vdash Con(T) \rightarrow \neg Prov_T(\ulcorner G \urcorner) \rightarrow G$$

Hence:

$$T \vdash Con(T) \rightarrow G$$

3.3.6. Conclusion of the Second Theorem

Assume, for contradiction, that $T \vdash Con(T)$. Then by the implication derived in 3.3.5, $T \vdash G$. However, if T is consistent, the first incompleteness theorem (or the argument using D1) establishes that $T \nvdash G$. Therefore, T cannot prove $Con(T)$ unless it is inconsistent. Thus, if T is consistent, then $T \nvdash Con(T)$. □

3.4. Semantic Necessity Theorem

3.4.1. Statement Theorem 3 (Semantic Necessity)

Let T be a consistent, complete theory that interprets Q (Robinson arithmetic). Then T cannot be recursively axiomatizable. Equivalently, any consistent, recursively axiomatizable theory that interprets Q is necessarily incomplete.

This theorem serves as a direct corollary of the First Incompleteness Theorem (Theorem 1) and formalizes the philosophical boundary between syntax and semantics.

3.4.2. Proof

Assume, for the sake of contradiction, that there exists a theory T that is simultaneously consistent, complete, interprets Q, and is recursively axiomatizable. By the application of Theorem 1, any theory that is consistent, recursively axiomatizable, and interprets Q must be incomplete. This result directly contradicts the initial assumption of the theory's completeness. Therefore, no such theory T can exist. ■

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3.4.3. Interpretation: Syntax vs. Semantics

Theorem 3 reveals a fundamental limitation of purely syntactic formal systems. A theory that is recursively axiomatizable is one whose axioms can be listed by a mechanical procedure; its proofs are finite sequences of formulas generated by fixed, effective rules. Such a system represents the embodiment of pure syntax—the manipulation of symbols without inherent reference to meaning or truth.

The "Semantic Necessity" identified in this theorem can be further understood through the lens of model theory. Purely syntactic systems are subject to the existence of non-standard models—mathematical structures that satisfy all the formal axioms of a theory (such as PA or Q) but contain "infinite" integers and structural anomalies not found in the intended, standard model of the natural numbers ($\mathbb{N}$). Because syntax is concerned only with the manipulation of symbols according to rules, it is inherently "blind" to these structural differences; it cannot distinguish the true natural numbers from their non-standard counterparts.

Semantic grounding, therefore, acts as an anchor to the Standard Model. It is only through a semantic commitment—a definition of truth that transcends the proof-calculus—that we can isolate the natural numbers as we understand them. The Greek insistence on numbers as "definite objects" (Kalyvas 2024) represents an intuitive, pre-formal commitment to this standard model. This ontological "anchoring" provided a completeness that cannot be replicated by any recursive set of rules.

Consequently, to recover completeness—that is, to possess a theory capable of deciding every arithmetical statement—one must abandon the requirement of recursive axiomatizability. A complete arithmetical theory cannot be captured by a finite or mechanically enumerable set of axioms; it must instead rely on semantic notions, such as truth and interpretation, or non-effective principles (e.g., infinitary rules). This result vindicates the ancient Greek distinction between arithmetic, viewed as a body of truths grasped by reason, and formal logic, viewed as a system of rules. By refraining from the attempt to reduce arithmetic to a fixed set of formal, syntactic axioms, the Greeks avoided the foundational limitations exposed by Gödel and the "blindness" of pure syntax.

3.4.4. The Category Error of Formalism and the Structural Necessity of Semantics

The modern formalist project, epitomized by Hilbert's program and the foundational ambitions of Zermelo-Fraenkel set theory (ZFC), rests upon a fundamental methodological assumption: that mathematics can be entirely flattened into a purely syntactic game of uninterpreted symbols. Under this paradigm, truth ($\models$) is forcefully collapsed into mechanical provability ($\vdash$). However, a careful re-examination of Gödel's mechanics through the lens of ancient Greek ontology reveals that this flattening is not merely incomplete; it is mathematically doomed because it constitutes a category error.

3.4.4.1. The Arithmetization of Syntax as a Category Error

The engine of Gödel's First Incompleteness Theorem is the arithmetization of syntax, or Gödel numbering. To prove that formal systems are inherently limited, Gödel forced the object-language of arithmetic to encode its own meta-language. Numbers were conscripted to act not only as quantitative or ontological entities—their traditional Greek conception as *ἀριθμός*—but simultaneously as meaningless syntactic tokens representing logical operations.

While this was a brilliant metamathematical diagnostic tool, formalists mistook this diagnostic tool for a foundational blueprint. By attempting to reduce arithmetic entirely to these syntactic rules, the formalist commits a category error: conflating the map (the formal axiomatic calculus, *λογιστική*) with the territory (the ontological reality of numbers, *ἀριθμητική*). They assume that because arithmetic can model syntax, arithmetic is nothing but syntax.

3.4.4.2. The Mathematical Breaking Point

Gödel's First Theorem represents the exact mathematical moment where this category error causes the formalist architecture to collapse. When a recursively axiomatizable theory T (capable of interpreting Robinson Arithmetic, Q) attempts to completely enclose arithmetic within purely syntactic boundaries, it inevitably generates a Gödel sentence, G, which asserts: "There is no natural number n that is the Gödel number of a proof of G."

The formalist is trapped by this construction. Syntactically, G is unprovable ($T \nvdash G$). Yet, metamathematically, we can clearly "see" that G is true precisely because it is unprovable. This "seeing" of truth occurs because we step outside the syntactic machinery and evaluate the proposition against the standard model of natural numbers ($\mathbb{N}$). The truth of G is inaccessible to pure syntax, yet undeniably real in the semantic domain. This proves that $\vdash$ (provability) and $\vDash$ (truth) cannot be made isomorphic in arithmetic. The formalist dream of a theory T where:

$$\vdash \ \equiv \ \vDash$$

is shattered not by a lack of clever axioms, but because syntax lacks the ontological capacity to capture semantic truth.

3.4.4.3. The Semantic Necessity Theorem as an Ontological Mandate

This brings us to the true significance of the Semantic Necessity Theorem (Theorem 3). Standard modern interpretations view the mutual exclusivity of consistency, completeness, and recursive axiomatizability merely as a negative boundary—a regrettable limit on computation. However, when stripped of formalist bias, Theorem 3 states something positive and structural: if a system is to be both consistent and arithmetically complete, its set of truths cannot be recursively enumerable.

Because pure syntax is definitionally bound to recursive enumerability—the mechanical application of rules by a Turing machine or a formal calculus—syntax alone is mathematically insufficient for arithmetic completeness. Therefore, the "gap" left by incompleteness is not a void; it is the space that must be filled by semantics. The formal system must be anchored to an interpreted model—an ontology of numbers—to possess mathematical truth.

3.4.4.4. Conclusion

The ancient Greek separation of arithmetic from formal logic was not a primitive stage of mathematics waiting to be unified by 20th-century set theory; it was a prescient recognition of mathematical reality. The ZFC/formalist attempt to reduce arithmetic to pure syntax fails because semantics is not merely a philosophical luxury or an interpretative afterthought. As Theorem 3 mathematically demonstrates, semantics is structurally required to complete the architecture of arithmetic.

3.4.5. Summary of the Refined Theorem

The preceding formal results can be synthesized into a single, unified "Refined Theorem" (Gödel–Rosser):

Let T be a recursively axiomatizable theory that interprets Robinson arithmetic Q.

1. (Incompleteness): If T is consistent, then T is incomplete.
2. (No Self-Consistency): If T is consistent, then $T \nvdash Con(T)$.
3. (Semantic Necessity): Consequently, if a theory is consistent, complete, and interprets Q, it cannot be recursively axiomatizable—it cannot be purely syntactic.

Thus, any complete and consistent theory of arithmetic must transcend syntax; it must be grounded in semantics or non-effective principles.

While standard modern interpretations of Gödel recognize that consistency, completeness, and recursive axiomatizability are mutually exclusive, they treat this merely as a negative limitation on formal systems. The Semantic Necessity Theorem reframes this mathematical boundary as a positive, ontological mandate. It demonstrates that the Greek conception of ἀϱιθμός —number as an ontologically real entity—is not a historical curiosity, but a mathematically necessary foundation for any complete arithmetic.

4. Discussion: The Epistemological Collapse of Pure Syntax

The Semantic Necessity Theorem establishes that arithmetic completeness is structurally impossible within the confines of pure syntax. However, the formalist project—and the foundational ambitions of Zermelo-Fraenkel set theory (ZFC)—faces an even deeper epistemological crisis. If arithmetic is entirely reduced to a syntactic calculus (*λογιστική*), divested of its ontological grounding (*ἀριθμητική*), the formal system loses not only completeness, but also its capacity to uniquely identify its subject matter, define its own truth, and justify its own consistency. This epistemological collapse can be demonstrated through three metamathematical phenomena.

4.1. Ontological Blindness and the Skolem Trap

The first critical failure of purely syntactic arithmetic lies in its inability to uniquely capture the standard model of natural numbers ($\mathbb{N}$). By abandoning semantic intuition in favor of first-order formalizations like Peano Arithmetic (PA) or ZFC, formalism falls victim to the Löwenheim-Skolem theorem and the Compactness theorem.

By the Compactness theorem, if we take the first-order axioms of PA and append a new constant symbol c, along with an infinite schema of axioms asserting $c > 0, c > 1, c > 2$, and so forth, every finite subset of this new theory is satisfiable by the standard natural numbers. Consequently, the entire infinite theory must be consistent and possess a model. This results in a "non-standard model" of arithmetic containing structurally infinite integers—elements that are unreachable by successor operations from zero.

Because purely formal systems are restricted to first-order syntactic derivations, PA (and by extension, ZFC) is mathematically incapable of distinguishing between the true, intended model of natural numbers and these bizarre, non-standard models. This is an ontological blindness. If a foundational system cannot syntactically differentiate between the actual number line and a non-standard model replete with infinite integers, then pure syntax is not actually describing arithmetic; it is merely manipulating uninterpreted symbols that mimic arithmetic. The Greek requirement that *ἀριθμητική* be grounded in the ontological reality of "definite objects" (Kalyvas 2024) is thus mathematically vindicated. Without a semantic anchor to fix the intended model, pure syntax "does not know what it is talking about."

4.2. Tarski and the Performative Contradiction of Formalism

The ultimate ambition of the formalist project is the total flattening of semantic truth ($\models$) into syntactic provability ($\vdash$). However, Tarski's Theorem on the Undefinability of Truth categorically destroys this equivalency from within. Tarski (1933) proved that for any sufficiently strong formal language L, there exists no formula $True(x)$ within L that holds if and only if x is the Gödel number of a true sentence in L. To formally define the truth of statements within ZFC, one is mathematically forced to step outside of ZFC into a strictly stronger semantic meta-language.

This exposes a performative contradiction at the heart of modern formalism. The formalist insists that mathematics requires nothing more than the mechanical application of syntactic rules. Yet, to even assert the claim "the theorems derived in ZFC are true," the formalist must quietly abandon their own syntactic restrictions and operate within a semantic meta-level. Truth, as Tarski proves, is strictly an extensional, semantic property. By separating logic from ontological reality, the formalist system becomes incapable of articulating its own truth. The ancient Greeks intuitively bypassed this

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limitation by treating arithmetic not as a deductive calculus, but as an ontological science where truth is apprehended directly through the semantic reality of *ἀριθμός*.

4.3. Gödel's Second Theorem and the Infinite Regress of Consistency

Finally, the formalist attempt to secure arithmetic certainty syntactically ends in an epistemological infinite regress, as dictated by Gödel's Second Incompleteness Theorem. Gödel demonstrated that no consistent, recursively axiomatizable theory T (containing basic arithmetic) can prove its own consistency ($T \nvdash Con(T)$).

When confronted with this reality, the modern formalist maneuver is to prove the consistency of a weaker system using a stronger one—for example, proving the consistency of Peano Arithmetic using ZFC. However, this merely displaces the problem. ZFC cannot prove $Con(ZFC)$. To prove the consistency of ZFC, one must invoke an even stronger system, such as ZFC supplemented with axioms for inaccessible cardinals, which in turn requires an even stronger system to prove its consistency, ad infinitum.

Pure syntax cannot pull itself up by its own epistemological bootstraps. An architecture built entirely on uninterpreted symbols is trapped in an infinite regress of unprovable assumptions. The regress only halts when one accepts that consistency is guaranteed not by the ad hoc addition of stronger syntactic axioms, but by the existence of a model—a semantic reality. If a theory describes a mathematically real, ontologically stable structure (such as the Greek conception of the arithmetic universe), it is necessarily consistent. Thus, the foundation of arithmetic certainty is not syntactic deduction, but semantic truth.

4.4. Anticipated Objections

To firmly establish the Semantic Necessity Theorem and the accompanying metamathematical critique, it is necessary to address the most formidable philosophical defenses of the formalist paradigm. A defense of the Greek ontological framework must demonstrate resilience against three major counter-arguments regarding epistemic reliability, structuralism, and second-order logic.

4.4.1. The Epistemic Trap and Russell's Ghost

The first formalist objection historically directed at ontological realism concerns the fallibility of semantic intuition. The argument posits that replacing syntactic proof with "semantic grounding" merely invites epistemic disaster. Formalists frequently point to Gottlob Frege, whose reliance on "obvious" semantic intuition regarding collections of objects led directly to Russell's Paradox. Therefore, the objection claims, substituting syntactic deduction with Greek ontological intuition does not solve the consistency problem; it merely masks epistemic vulnerability.

This objection, however, conflates ontological reality with human epistemic access. The Greek separation of *ἀριθμητική* and *λογιστική* provides the exact philosophical mechanism needed to resolve this conflation. *Ἀριθμητική* asserts that a consistent, semantic reality of numbers exists independently of human formalization. *Λογιστική* is the human attempt to capture this reality algorithmically. The fact that human epistemic access—our formal systems or intuitive axioms—is occasionally flawed (as in Frege's Basic Law V) does not negate the existence of the ontological territory; it merely proves that our syntactic maps are imperfect. The formalist error is the insistence that the map *is* the territory. Acknowledging that semantic reality is the final arbiter of truth does not demand epistemic infallibility; it simply demands the recognition that syntax without a model is blind.

4.4.2. The Structuralist Dilemma (Benacerraf's Problem)

A second objection challenges the specific characterization of the Greek *ἀριθμός* as "definite objects." Relying on Paul Benacerraf's seminal argument in *What Numbers Could Not Be* (1965), a modern logician might argue that numbers cannot be definite objects because multiple, mutually

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exclusive set-theoretic constructions model arithmetic perfectly (e.g., Zermelo ordinals versus von Neumann ordinals). Because arithmetic works regardless of which "objects" are chosen, the structuralist argues that mathematics is exclusively about abstract relations, rendering the Greek object-ontology obsolete.

While mathematically accurate regarding set-theoretic reductions, this objection entirely misses the target of the Semantic Necessity Theorem. Whether the semantic anchor of arithmetic is conceived of as a collection of distinct Platonic objects or as an irreducible, overarching structural reality, the core metamathematical thesis remains unbreached: pure, uninterpreted syntax manipulating meaningless symbols ($\vdash$) is structurally insufficient to capture arithmetic truth ($\models$). Mathematical structuralism is not a defense of formalism; it is simply a modern variation of ontological realism. Whether one anchors the formal calculus to definite objects or to definite structures, the necessity of an external semantic model remains absolute.

4.4.3. The Second-Order Logic Escape Hatch

The most sophisticated formalist defense attempts to neutralize the ontological blindness exposed by the Löwenheim-Skolem theorem. A formalist will observe that Löwenheim-Skolem applies strictly to first-order logic. If arithmetic is formalized using second-order logic, Richard Dedekind's theorem proves that Peano Arithmetic is absolutely categorical—meaning it uniquely identifies the standard natural numbers up to isomorphism. Thus, the formalist argues, pure syntax can secure its intended model simply by upgrading to second-order logic, escaping the necessity of external semantics.

This maneuver, however, is a metamathematical sleight of hand. As W.V.O. Quine famously demonstrated (1986), second-order logic is not "pure logic" at all; it is "set theory in sheep's clothing." To utilize second-order logic, the formal system must permit quantification over all possible subsets of the domain. This mathematical requirement forces the system to assume a vast, pre-existing universe of sets. Therefore, second-order logic does not escape the necessity of ontology; it secretly smuggles massive semantic and ontological commitments directly into the syntax. When the formalist invokes second-order logic to uniquely capture the natural numbers, they inadvertently concede the central thesis of this paper: pure, uninterpreted syntax is insufficient, and securing arithmetic truth mathematically requires the importation of a robust ontological universe.

5. Conclusions

The formalist dream of the twentieth century—the total reduction of mathematical truth to a syntactic calculus—has mathematically failed. As this paper has demonstrated, this failure is not a minor technical limitation to be patched by the ad hoc addition of stronger axioms, but an epistemological collapse. Through the metamathematical phenomena of Gödel's Incompleteness Theorems, Tarski's Undefinability of Truth, and the ontological blindness exposed by non-standard models, it is evident that pure syntax is structurally inadequate for arithmetic. A purely formal system cannot achieve arithmetic completeness, cannot articulate its own truth, and cannot guarantee its own consistency without triggering an infinite epistemological regress.

The attempt to force the entirety of arithmetic into the mechanistic mold of formal logic is, at its core, a category error. The resolution to this foundational crisis does not lie forward in the endless proliferation of stronger formal systems, but backward in the ontological clarity of ancient Greek mathematics.

The Greek separation of *ἀριθμητική* (the theoretical science of numbers as real, definite objects) from *λογιστική* (the calculative manipulation of symbols) was not a primitive stage of mathematical development waiting to be corrected by modern set theory. Rather, it was a mathematically prescient boundary. By refusing to collapse semantic reality into syntactic rules, the Greeks preserved the ontological anchor that arithmetic requires to remain coherent.

Modern metamathematics, far from overwriting this ancient wisdom, serves as its ultimate vindication. The Semantic Necessity Theorem dictates that the gap left by syntactic incompleteness

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is not a void, but the exact space that must be occupied by ontology. To fully capture the truth of arithmetic, mathematics must abandon the illusion of pure syntax, transcend the calculus, and return to the semantic reality of *ἀριθμός*.

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