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## Article

# Fixed Point Results and the Ekeland Variational Principle in Vector $B$ -Metric Spaces

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**Abstract:** In this paper, we extend the concept of  $b$ -metric spaces to the vectorial case, where the distance is vector-valued, and the constant in the triangle inequality axiom is replaced by a matrix. For such spaces, we establish results analogous to those in the  $b$ -metric setting: fixed-point theorems, stability results, and a variant of Ekeland's variational principle. As a consequence, we also derive a variant of Caristi's fixed-point theorem.

**Keywords:** triangle inequality axiom;  $b$ -metric space; variational principle; fixed point

## 1. Introduction

The concept of a  $b$ -metric space arises as a natural generalization of a metric space, where the triangle inequality axiom is relaxed by introducing a constant  $b \geq 1$  on its right-hand side. Early ideas in this direction can be traced back to the notion of "quasimetric" spaces, as discussed in [1]. However, the formal definition and terminology of  $b$ -metric spaces are widely attributed to Bakhtin [3] and Czerwinski [4]. Notably, one of the earliest works to introduce a mapping satisfying the properties of a  $b$ -metric dates back to 1970 in [5], where such a mapping was referred to as a "distance". A concept related to that of a  $b$ -metric is the notion of a quasi-norm, which can be traced back to Hyers [6] and Bourgin [7], who originally used the term "quasi-norm." For a survey on  $b$ -metric spaces we send the reader to [8,9].

Various results from the classical theory of metric spaces have been extended to  $b$ -metric spaces, including fixed-point theorems (see, e.g., [10–15]), estimations (see, e.g., [16,17]), stability results (see, e.g., [18,19]), and variational principles (see, e.g., [20,21]). In [22], the metric was allowed to take vector values, and results analogous to those for  $b$ -metric spaces were established, with matrices converging to zero replacing the contraction constants, but not the constant  $b$  from the triangle inequality axiom.

In this paper, we introduce the concept of a vector  $B$ -metric space, where the scalar constant  $b$  in the triangle inequality is replaced by a matrix  $B$ . This generalization introduces new challenges in establishing results analogous to those for classical  $b$ -metric spaces. To the best of our knowledge, this concept, along with the corresponding results presented here, is novel. Notably, some of the results appear to be new even in the scalar particular case where the matrix  $B$  is reduced to a constant.

Throughout this paper, we consider  $\mathbb{R}^n$ -valued vector metrics ( $n \geq 1$ ) on a set  $X$ , i.e., mappings  $d : X \times X \rightarrow \mathbb{R}_+^n$ . In the scalar case ( $n = 1$ ), we use the special notation  $\rho$  to denote a standard metric or a  $b$ -metric.

The classical definition of a  $b$ -metric reads as follows:

**Definition 1.** Let  $X$  be a set and let  $b \geq 1$  be a given real number. A mapping  $\rho : X \times X \rightarrow \mathbb{R}_+$  is said to be a  $b$ -metric if for all  $x, y, z \in X$  the following conditions are satisfied:  $\rho(x, y) \geq 0$ ,  $\rho(x, y) = 0$  if and only if  $x = y$ ,  $\rho(x, y) = \rho(y, x)$  and  $\rho(x, z) \leq b(\rho(x, y) + \rho(y, z))$ . The pair  $(X, \rho)$  is called a  $b$ -metric space.



In case the mapping  $\rho$  is allowed to be vector-valued and one replaces the constant  $b$  by a matrix  $B$ , we obtain our definition of a vector  $B$ -metric space.

**Definition 2.** Let  $X$  be a set,  $n \geq 1$  and let  $B \in \mathcal{M}_{n \times n}(\mathbb{R})$  be an arbitrary matrix. A mapping  $d = (d_1, d_2, \dots, d_n) : X \times X \rightarrow \mathbb{R}_+^n$  is called a vector  $B$ -metric if for all  $u, v, w \in X$ , one has

(positivity):  $d(u, v) \geq 0$  and  $d(u, v) = 0$  if and only if  $u = v$ ;

(symmetry):  $d(u, v) = d(v, u)$ ;

(triangle inequality):  $d(u, w) \leq B(d(u, v) + d(v, w))$ .

The pair  $(X, d)$  is called a vector  $B$ -metric space.

## 2. Preliminaries

In this paper, the vectors in  $\mathbb{R}^n$  are looked as column matrices and ordering between them and, more generally, between matrices of the same size is understood by components. Likewise, the convergence of a sequence of vectors or matrices is understood componentwise.

The spaces of square matrices of size  $n$  with real number entries and nonnegative entries are denoted by  $\mathcal{M}_{n \times n}(\mathbb{R})$  and  $\mathcal{M}_{n \times n}(\mathbb{R}_+)$ , respectively. An element of  $\mathcal{M}_{n \times n}(\mathbb{R}_+)$  is referred as a *positive matrix*, while a matrix  $M \in \mathcal{M}_{n \times n}(\mathbb{R})$  is called *inverse-positive* if it is invertible and its inverse  $M^{-1}$  is positive.

A positive matrix  $M$  is said to be *convergent to zero* if its power  $M^k$  tends to the zero matrix  $0_n$  as  $k \rightarrow \infty$ .

One has the following characterizations of matrices which are convergent to zero (see, e.g., [23,24]).

**Proposition 1.** Let  $M \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$  and let  $I$  be the identity matrix of size  $n$ . The following statements are equivalent:

- (a)  $M$  is convergent to zero.
- (b) The spectral radius  $r(M)$  of matrix  $M$  is less than 1, i.e.,  $r(M) < 1$ .
- (c)  $I - M$  is invertible and  $(I - M)^{-1} = I + M + M^2 + \dots$ .
- (d)  $I - M$  is inverse-positive.

The following proposition collects the various properties equivalent to the notion of an inverse-positive matrix (see, e.g., [24,25]).

**Proposition 2.** Let  $M \in \mathcal{M}_{n \times n}(\mathbb{R})$ . The following statements are equivalent:

- (a)  $M$  is inverse-positive.
- (b)  $M$  is monotone, i.e.,  $Mx \geq 0$  ( $x \in \mathbb{R}^n$ ) implies  $x \geq 0$ .
- (c) There exists a positive matrix  $\bar{M}$  and a real number  $s > r(\bar{M})$  such that the following representation holds:  $M = sI - \bar{M}$ .

Clearly, if  $M$  is inverse-positive, from the representation  $M = sI - \bar{M}$ , we immediately see that all its entries except those from the diagonal are  $\leq 0$ ; also the matrix  $\frac{1}{s}\bar{M}$  is convergent to zero. If a matrix  $M$  is both positive and inverse-positive, using the representation  $M = sI - \bar{M}$  we deduce that  $M$  must be a diagonal matrix with strictly positive diagonal entries.

A mapping  $N : X \rightarrow X$  defined on a vector  $B$ -metric space  $(X, d)$  is said to be a *Perov contraction mapping* if there exists a matrix  $A$  convergent to zero such that

$$d(N(x), N(y)) \leq Ad(x, y) \quad (1)$$

for all  $x, y \in X$ .

The next proposition is about the relationship between vector  $B$ -metrics and both vector and scalar  $b$ -metrics.

**Proposition 3.** (1<sup>0</sup>) Any vector-valued  $b$ -metric  $d$  can be identified with a vector  $B_b$ -metric, where  $B_b$  is the diagonal matrix whose diagonal entries are all equal to  $b$ .

(2<sup>0</sup>) If  $d$  is a vector  $B$ -metric with an inverse-positive matrix  $B$ , then  $d$  is also a vector  $\underline{B}$ -metric with respect to the diagonal matrix  $\underline{B}$  that preserves the diagonal of  $B$ , as well as a vector-valued  $\tilde{b}$ -metric with  $\tilde{b} = \max\{b_{ii} : 1 \leq i \leq n\}$ . Here  $B = (b_{ij})_{1 \leq i,j \leq n}$ .

(3<sup>0</sup>) If  $d$  is a vector  $B$ -metric with a positive matrix  $B$ , then to each norm in  $\mathbb{R}^n$  one can associate a scalar  $b$ -metric, for example:

$$\begin{aligned} \rho_1(x, y) &:= \sum_{i=1}^n d_i(x, y), & \text{is a } b_1\text{-metric, } b_1 &:= \sum_{i=1}^n \max_{1 \leq j \leq n} b_{ij}, \\ \rho_\infty(x, y) &:= \max_{1 \leq i \leq n} d_i(x, y), & \text{is a } b_\infty\text{-metric, } b_\infty &:= \max_{1 \leq i \leq n} \sum_{j=1}^n b_{ij}, \\ \rho_2(x, y) &:= \left( \sum_{i=1}^n d_i(x, y)^2 \right)^{\frac{1}{2}}, & \text{is a } b_2\text{-metric, } b_2 &:= \left( \sum_{i,j=1}^n b_{ij}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Thus, to any vector  $B$ -metric, one can associate different (scalar)  $b$ -metrics, depending on the chosen metric on  $\mathbb{R}^n$ . However, as shown in [23], working in a vector setting with matrices instead of numbers is more accurate especially when a connection with other matrices is necessary. It will also be the case of this work where some conditions or conclusions will connect the matrix  $B$  with the matrix  $A$  involved in (1).

If  $Y$  is a nonempty subset of a vector  $B$ -metric space  $(X, d)$ , we define the *diameter* of the set  $Y$  by

$$\text{diam}_d(Y) := \sup\{\rho_1(x, y) : x, y \in Y\} = \sup\left\{\sum_{i=1}^n d_i(x, y) : x, y \in Y\right\}.$$

From this definition, it follows immediately that if  $\text{diam}_d(Y) = a$ , then  $d(x, y) \leq ae$  for all  $x, y \in Y$ , where  $e = (1, 1, \dots, 1) \in \mathbb{R}^n$ . Conversely, if  $d(x, y) \leq ae$  for all  $x, y \in Y$ , then  $\text{diam}_d(Y) \leq na$ .

Although a  $b$ -metric does not generate a topology (see, e.g., [26]), several topological properties can still be defined in terms of sequences (e.g., closed sets, continuous operators, or lower semicontinuous functionals).

We conclude this section by two examples of vector  $B$ -metrics.

**Example 1.** Let  $d: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}_+^2$  be given by

$$d(x, y) = \begin{pmatrix} |x_1 - y_1|^2 + |x_2 - y_2| \\ |x_2 - y_2| \end{pmatrix},$$

for  $x = (x_1, x_2)$ ,  $y = (y_1, y_2) \in \mathbb{R}^2$ . Then,  $(\mathbb{R}^2, d)$  is a vector  $B$ -metric space, where

$$B = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}.$$

Here, the matrix  $B$  is inverse-positive, but not positive.

**Example 2.** We present an example of a vector-valued mapping  $d$  which is a vector  $B$ -metric with respect to a positive matrix, but for which no inverse-positive matrix exists such that  $d$  remains a vector  $B$ -metric. Let

$$S = \{(t, t) : t \in \mathbb{R}\} \subset \mathbb{R}^2,$$

and let  $d: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}_+^2$  be given by

$$d(x, y) = \begin{cases} (0, 0) & \text{if } x = y, \\ (|x - y|^2, |x - y|) & \text{if } x, y \in S, \\ (|x - y|, |x - y|^2) & \text{otherwise,} \end{cases}$$

where  $|z| = |(z_1, z_2)| = |z_1| + |z_2|$  is a norm on  $\mathbb{R}^2$ . Note that  $d$  is a vector  $B_0$ -metric, where

$$B_0 = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}.$$

Let us show that  $B_0$  is the smallest matrix for which the triangle inequality holds for  $d$ . To this aim, let  $B = (b_{ij})_{1 \leq i, j \leq n}$  be any matrix for which the triangle inequality is satisfied. Then, for  $x, y \in S$  and  $z \notin S$ , we have

$$\begin{pmatrix} |x - y|^2 \\ |x - y| \end{pmatrix} \leq \begin{pmatrix} b_{11}(|x - z| + |z - y|) + b_{12}(|x - z|^2 + |z - y|^2) \\ b_{21}(|x - z| + |z - y|) + b_{22}(|x - z|^2 + |z - y|^2) \end{pmatrix}. \quad (2)$$

Let  $t, \alpha \in \mathbb{R} \setminus \{0\}$ , and set  $x = (t, t) \in S$ ,  $y = (0, 0) \in S$  and  $z = (\alpha, 0) \notin S$ . The first inequality in (2) yields,

$$4t^2 \leq b_{11}(|t - \alpha| + |t| + |\alpha|) + b_{12}((|t - \alpha| + |t|)^2 + \alpha^2).$$

Clearly, taking  $\alpha = t$  and the limit as  $t \rightarrow \infty$ , this inequality holds only if  $b_{12} \geq 2$ . Similarly, from the second inequality, we obtain

$$2|t| \leq b_{21}(|t - \alpha| + |t| + |\alpha|) + b_{22}((|t - \alpha| + |t|)^2 + \alpha^2).$$

Setting  $\alpha = \frac{t}{2}$ , we find that

$$2|t| \leq 2b_{21}|t| + 5b_{22}\frac{t^2}{2}, \quad \text{or equivalently,} \quad 5b_{22}\frac{t^2}{2} + 2|t|(b_{21} - 1) \geq 0.$$

Clearly, this inequality required for all  $t$  implies  $b_{21} \geq 1$ . To determine the values of  $b_{11}$  and  $b_{22}$ , we apply the triangle inequality with  $x, y, z \in S$  ( $x \neq y \neq z \neq x$ ), which gives

$$\begin{pmatrix} |x - y|^2 \\ |x - y| \end{pmatrix} \leq \begin{pmatrix} b_{11}(|x - z|^2 + |z - y|^2) + b_{12}(|x - z| + |z - y|) \\ b_{21}(|x - z|^2 + |z - y|^2) + b_{22}(|x - z| + |z - y|) \end{pmatrix}.$$

Similar arguments as above imply that  $b_{11} \geq 2$  and  $b_{22} \geq 1$ . Thus,  $B \geq B_0$  as claimed.

### 3. Fixed Point Theorems in Vector $b$ -Metric Spaces

In this section we establish some fixed point results in vector  $B$ -metric spaces, analogous to the well-known classical results.

#### 3.1. Perov Type Fixed Point Theorem

Our first result is a version of Perov's fixed point theorem (see, [27,28]) for such spaces.

**Theorem 1.** Let  $(X, d)$  be a complete vector  $B$ -metric space, where  $B$  is either a positive or an inverse-positive matrix, and let  $N: X \rightarrow X$  be an operator. Assume that there exists a convergent to zero matrix  $A \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$  such that

$$d(N(x), N(y)) \leq Ad(x, y), \quad \text{for all } x, y \in X, \quad (3)$$

i.e.,  $N$  is a Perov contraction mapping. Then,  $N$  has a unique fixed point.

**Proof.** Let  $x_0 \in X$ , and recursively define

$$x_k = N(x_{k-1}), \quad \text{for } k \geq 1.$$

Since the matrix  $A$  is convergent to zero, for each  $\alpha > 0$ , there exists  $k_0 = k_0(\alpha)$  such that

$$A^{k_0} \leq \Lambda,$$

where  $\Lambda$  is the square matrix of size  $n$  whose entries are all equal to  $\alpha$ . Let  $k, p \geq 0$  and  $k_0$  be such that  $A^{k_0} \leq \Lambda$ , for some  $\alpha > 0$  to be specified later.

Case (a):  $B$  is inverse-positive. The triangle inequality yields

$$\begin{aligned} B^{-2}d(x_k, x_p) &\leq B^{-1}d(x_k, x_{k+k_0}) + B^{-1}d(x_p, x_{k+k_0}) \\ &\leq B^{-1}A^k d(x_0, x_{k_0}) + d(x_p, x_{p+k_0}) + d(x_{p+k_0}, x_{k+k_0}) \\ &\leq B^{-1}A^k d(x_0, x_{k_0}) + A^p d(x_0, x_{k_0}) + A^{k_0} d(x_k, x_p) \\ &\leq B^{-1}A^k d(x_0, x_{k_0}) + A^p d(x_0, x_{k_0}) + \Lambda d(x_k, x_p), \end{aligned}$$

which gives

$$(B^{-2} - \Lambda)d(x_k, x_p) \leq B^{-1}A^k d(x_0, x_{k_0}) + A^p d(x_0, x_{k_0}). \quad (4)$$

Given that the right-hand side of (4) is a vector that converges to zero as  $k, p \rightarrow \infty$ , our goal is to show that a linear combination of the components of the vector  $d(x_k, x_p)$  is bounded above by the corresponding components of the right-hand side of (4). To this aim, we make the following notations

$$\begin{aligned} B^{-2} &= (\gamma_{ij})_{1 \leq i, j \leq n}, \\ B^{-1}A^k d(x_0, x_{k_0}) + A^p d(x_0, x_{k_0}) &= \varphi_{k,p} = (\varphi_{k,p}^i)_{1 \leq i \leq n}. \end{aligned}$$

Hence

$$\sum_{i=1}^n \varphi_{k,p}^i \rightarrow 0 \quad \text{as } k, p \rightarrow \infty. \quad (5)$$

Under these notations, relation (4) gives

$$\sum_{j=1}^n (\gamma_{ij} - \alpha) d_j(x_k, x_p) \leq \varphi_{k,p}^i, \quad i = 1, 2, \dots, n. \quad (6)$$

Summing in (6) over all  $i \in \{1, 2, \dots, n\}$ , we obtain

$$\sum_{i,j=1}^n (\gamma_{ij} - \alpha) d_j(x_k, x_p) \leq \sum_{i=1}^n \varphi_{k,p}^i. \quad (7)$$

Since  $B^{-2}$  is invertible and positive, the sum of its elements in each column must be positive, i.e.,

$$\sum_{i=1}^n \gamma_{ij} > 0, \quad j = 1, 2, \dots, n.$$

If we denote

$$\gamma = \min \left\{ \sum_{i=1}^n \gamma_{ij} : j = 1, 2, \dots, n \right\},$$

relation (7) implies that

$$\begin{aligned} \sum_{i=1}^n \varphi_{k,p}^i &\geq \sum_{i,j=1}^n \gamma_{ij} d_j(x_k, x_p) - n\alpha \sum_{j=1}^n d_j(x_k, x_p) \\ &= \sum_{j=1}^n \left( \sum_{i=1}^n \gamma_{ij} \right) d_j(x_k, x_p) - n\alpha \sum_{j=1}^n d_j(x_k, x_p) \\ &\geq (\gamma - n\alpha) \sum_{j=1}^n d_j(x_k, x_p). \end{aligned}$$

Choosing  $\alpha < \gamma/n$ , one has

$$\sum_{j=1}^n d_j(x_k, x_p) \leq \frac{1}{\gamma - n\alpha} \sum_{i=1}^n \varphi_{k,p}^i. \quad (8)$$

In (8), we observe that the factor  $\frac{1}{\gamma - n\alpha}$  depends only on  $n$  and  $B$ , whence (5) yields

$$\sum_{j=1}^n d_j(x_k, x_p) \rightarrow 0 \text{ as } k, p \rightarrow \infty,$$

so the sequence  $(x_k)$  is Cauchy.

Case (b):  $B$  is positive. One has

$$\begin{aligned} d(x_k, x_p) &\leq Bd(x_k, x_{k+k_0}) + Bd(x_p, x_{k+k_0}) \\ &\leq BA^k d(x_0, x_{k_0}) + B^2 d(x_p, x_{p+k_0}) + B^2 d(x_{p+k_0}, x_{k+k_0}) \\ &\leq BA^k d(x_0, x_{k_0}) + B^2 A^p d(x_0, x_{k_0}) + B^2 A^{k_0} d(x_k, x_p) \\ &\leq BA^k d(x_0, x_{k_0}) + B^2 A^p d(x_0, x_{k_0}) + B^2 \Lambda d(x_k, x_p), \end{aligned}$$

which gives

$$(I - B^2 \Lambda) d(x_k, x_p) \leq BA^k d(x_0, x_{k_0}) + B^2 A^p d(x_0, x_{k_0}). \quad (9)$$

Note that since  $\Lambda^k = (n\alpha)^{k-1} \Lambda$ , if  $\alpha$  is chosen to be smaller than one divided by the greatest element of  $B^2$  multiplied with  $n$ , the matrix  $B^2 \Lambda$  is convergent to zero. Consequently,  $I - B^2 \Lambda$  is invertible and  $(I - B^2 \Lambda)^{-1} \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$ . Hence, (9) is equivalent to

$$d(x_k, x_p) \leq (I - B^2 \Lambda)^{-1} (BA^k d(x_0, x_{k_0}) + B^2 A^p d(x_0, x_{k_0})). \quad (10)$$

As the right-hand side of (10) converges to zero when  $k, p \rightarrow \infty$ , we conclude that  $(x_k)$  is Cauchy.

Therefore, in both cases, the sequence  $(x_k)$  is Cauchy and since  $X$  is complete, it has a limit  $x^*$ , that is,  $d(x_k, x^*) \rightarrow 0$  as  $k \rightarrow \infty$ . Then, from

$$d(N(x_k), N(x^*)) \leq Ad(x_k, x^*),$$

it follows that  $N(x_k) \rightarrow N(x^*)$  as  $k \rightarrow \infty$ , while from  $x_{k+1} = N(x_k)$ , passing to the limit, one obtains  $x^* = N(x^*)$ . Hence  $N$  has a fixed point. To prove uniqueness, suppose that there exists another fixed point  $x^{**}$ . Then, from

$$d(x^*, x^{**}) = d(N(x^*), N(x^{**})) \leq Ad(x^*, x^{**}),$$

recursively, we obtain that

$$d(x^*, x^{**}) \leq A^k d(x^*, x^{**}),$$

for all  $k \geq 1$ . Since  $A^k \rightarrow 0_n$  as  $k \rightarrow \infty$ , we deduce that  $d(x^*, x^{**}) = 0$ , i.e.,  $x^{**} = x^*$ .  $\square$

If we are not interested in the uniqueness of the fixed point for  $N$ , the condition (3) can be relaxed and replaced by a weaker assumption on the graph of  $N$ .

**Theorem 2.** *Let  $(X, d)$  be a complete vector  $B$ -metric space, where  $B$  is either positive or inverse-positive, and let  $N: X \rightarrow X$  be an operator. Assume there exists a convergent to zero matrix  $A \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$  such that*

$$d(N(x), N^2(x)) \leq Ad(x, N(x)), \text{ for all } x \in X. \quad (11)$$

*Then,  $N$  has at least one fixed point.*

**Proof.** Following the proof of Theorem 1, from any initial point  $x_0$ , the sequence  $x_k = N^k(x_0)$  is convergent to a fixed point  $x^*$  of  $N$ , which clearly depends on the starting point  $x_0$ , but condition (11) is insufficient to guarantee the uniqueness.  $\square$

The next result is a version for vector  $B$ -metric spaces of Maia's fixed point theorem. The contraction condition on the operator is considered with respect to a vector  $B_1$ -metric  $d_1$ , not necessarily complete, while the convergence of the sequence of successive approximations is guaranteed in a complete vector  $B_2$ -metric  $d_2$  in a subordinate relationship to  $d_1$ .

**Theorem 3.** *Let  $X$  be a set equipped with two  $\mathbb{R}^n$ -vector metrics, a  $B_1$ -metric  $d_1$  and a  $B_2$ -metric  $d_2$ , where  $B_2$  is either positive or inverse-positive, and let  $N: X \rightarrow X$  be an operator. Assume that the following conditions hold:*

- (i)  $(X, d_1)$  is a complete vector  $B_1$ -metric space;
- (ii)  $d_1(x, y) \leq Cd_2(x, y)$  for all  $x, y \in X$  and some matrix  $C \in \mathcal{M}_{n \times n}(\mathbb{R})$ ;
- (iii) There exists a matrix  $A$  convergent to zero such that

$$d_2(N(x), N(y)) \leq Ad_2(x, y), \text{ for all } x, y \in X; \quad (12)$$

- (iv) The operator  $N$  is continuous in  $(X, d_1)$ .

*Then, the operator  $N$  has a unique fixed point.*

**Proof.** Let  $x_0 \in X$  be fixed, and consider the iterative sequence  $x_{k+1} = N(x_k)$  for  $k \geq 0$ . For any  $k, k_0, p \geq 0$ , applying the triangle inequality twice and using condition (iii), we derive either

$$(B_2^{-2} - A^{k_0})d_2(x_k, x_p) \leq B_2^{-1}A^k d_2(x_0, x_{k_0}) + A^p d_2(x_0, x_{k_0}),$$

in case that  $B_2$  is inverse-positive, or

$$(I - B_2^2 A^{k_0})d_2(x_k, x_p) \leq B_2 A^k d_2(x_0, x_{k_0}) + B_2^2 A^p d_2(x_0, x_{k_0}),$$

if  $B_2$  is positive. Arguing similarly to the proof of Theorem 1, we deduce that  $(x_k)$  is a Cauchy sequence in  $(X, d_2)$ . From (ii), it follows immediately that  $(x_k)$  is also a Cauchy sequence in  $(X, d_1)$ , hence  $(x_k)$  is convergent with respect the metric  $d_1$  to some  $x^*$ , that is,

$$d_1(N(x_k), x^*) = d_1(x_{k+1}, x^*) \rightarrow 0, \text{ as } k \rightarrow \infty,$$

while the continuity of  $N$  yields  $d_1(N(x^*), x^*) = 0$ , i.e.,  $N(x^*) = x^*$ . To establish uniqueness, suppose that  $x^{**}$  is another fixed point of  $N$ , i.e.,  $N(x^{**}) = x^{**}$ . Then, by (12), one has

$$(I - A)d_2(x^*, x^{**}) \leq 0.$$

Since  $A$  is convergent to zero, we necessarily have  $d_2(x^*, x^{**}) = 0$ , i.e.,  $x^* = x^{**}$ .  $\square$

### 3.2. Error Estimates

The classical Banach and Perov fixed point theorems are accompanied by some error estimates in terms of the contraction constant and matrix, respectively. These estimates allow us to obtain stopping criteria for the iterative approximation process. It is the aim of this subsection to obtain such stopping criteria when working in vector  $B$ -metric spaces.

**Theorem 4.** *Assume that all the conditions of Theorem 1 hold and let  $(x_k)$  be a sequence of successive approximations of the fixed point  $x^*$ .*

**(1<sup>0</sup>)** *If  $B$  is inverse-positive, then*

$$(B^{-1} - A)d(x_k, x^*) \leq A^k d(x_0, x_1) \quad (k \geq 0). \quad (13)$$

*If in addition the matrix  $B^{-1} - A$  is inverse-positive, then*

$$d(x_k, x^*) \leq (B^{-1} - A)^{-1} A^k d(x_0, x_1) \quad (k \geq 0). \quad (14)$$

**(2<sup>0</sup>)** *If  $B$  is positive, then*

$$(I - BA)d(x_k, x^*) \leq BA^k d(x_0, x_1) \quad (k \geq 0). \quad (15)$$

*If in addition  $I - BA$  is inverse-positive, then*

$$d(x_k, x^*) \leq (I - BA)^{-1} BA^k d(x_0, x_1) \quad (k \geq 0). \quad (16)$$

**Proof.** (1<sup>0</sup>): We have

$$\begin{aligned} B^{-1}d(x_k, x^*) &\leq d(x_k, x_{k+1}) + d(x_{k+1}, x^*) \\ &\leq A^k d(x_0, x_1) + Ad(x_k, x^*), \end{aligned}$$

whence we deduce (13). The second part is obvious.

(2<sup>0</sup>): We have

$$\begin{aligned} d(x_k, x^*) &\leq Bd(x_k, x_{k+1}) + Bd(x_{k+1}, x^*) \\ &\leq BA^k d(x_0, x_1) + BAd(x_k, x^*), \end{aligned}$$

that is (15). The additional conclusion is obvious.  $\square$

**Remark 1.** *Clearly, since  $A^k$  tends to the zero matrix as  $k \rightarrow \infty$ , formulas (14) and (16) provide stopping criteria for the iterative fixed point approximation algorithm starting from  $x_0$ , when an admissible error is given. It should be emphasized that these estimates are in terms of matrices  $A$  and  $B$ . In contrast, if we make the transition to (scalar)  $b$ -metric spaces, as discussed in Section 2, the resulting estimates will depend on the chosen norm in  $\mathbb{R}^n$  and may vary across different norms. So, from this point of view, the vector approach not only unifies the results that can be obtained with the scalar method, but also provides the best estimates.*

### 3.3. Stability Results

We now present two stability properties of the Perov contraction mappings in vector  $B$ -metric spaces.

The first property is in the sense of Reich and Zaslavski and generalizes the one obtained in [19] for  $b$ -metric spaces.

**Theorem 5.** *Let  $(X, d)$  be a complete vector  $B$ -metric space, and let  $N: X \rightarrow X$  be an operator such that (3) holds with a matrix  $A$  convergent to zero. In addition assume that either*

(a)  $B$  and  $B^{-1} - A$  are inverse-positive;

or

(b)  $B$  is positive and  $I - BA$  is inverse-positive.

Then,  $N$  is stable in the sense of Reich and Zaslavski, i.e.,  $N$  has a unique fixed point  $x^*$ , and for every sequence  $(x_k) \subset X$  satisfying

$$d(x_k, N(x_k)) \rightarrow 0 \text{ as } k \rightarrow \infty, \quad (17)$$

one has

$$x_k \rightarrow x^* \text{ as } k \rightarrow \infty.$$

**Proof.** According to Theorem 1 the operator  $N$  has a unique fixed point  $x^*$ . In addition, for any sequence  $(x_k)$  satisfying (17), in case (a), we have

$$\begin{aligned} B^{-1}d(x_k, x^*) &\leq d(x_k, N(x_k)) + d(N(x_k), x^*) \\ &= d(x_k, N(x_k)) + d(N(x_k), N(x^*)) \\ &\leq d(x_k, N(x_k)) + Ad(x_k, x^*), \end{aligned}$$

that is,

$$d(x_k, x^*) \leq (B^{-1} - A)^{-1}d(x_k, N(x_k)),$$

while in case (b),

$$d(x_k, x^*) \leq (I - BA)^{-1}Bd(x_k, N(x_k)).$$

These estimates immediately yield the conclusion.  $\square$

The second stability result is in the sense of Ostrowski and extends to vector  $B$ -metric spaces a similar property established in [19] for  $b$ -metric spaces.

**Theorem 6.** Let  $(X, d)$  be a complete vector  $B$ -metric space, and let  $N: X \rightarrow X$  be an operator. Assume  $N$  satisfies (3) with a matrix  $A$  convergent to zero. In addition, assume that either

(a)  $B$  and  $I - \tilde{b}A$  are inverse-positive, where  $\tilde{b} = \max\{b_{ii} : i = 1, 2, \dots, n\}$ ;

or

(b)  $B$  is positive and  $I - BA$  is inverse-positive.

Then,  $N$  has the Ostrowski property, i.e.,  $N$  has a unique fixed point  $x^*$ , and for every sequence  $(x_k) \subset X$  satisfying

$$d(x_{k+1}, N(x_k)) \rightarrow 0 \text{ as } k \rightarrow \infty,$$

one has

$$x_k \rightarrow x^* \text{ as } k \rightarrow \infty.$$

**Proof.** As previously established, the operator  $N$  has a unique fixed point  $x^*$ . In case (a), we have

$$\begin{aligned} d(x_{k+1}, x^*) &\leq \tilde{b}d(x_{k+1}, N(x_k)) + \tilde{b}d(N(x_k), N(x^*)) \\ &\leq \tilde{b}d(x_{k+1}, N(x_k)) + \tilde{b}Ad(x_k, x^*) \\ &\leq \dots \\ &\leq \tilde{b} \sum_{p=0}^k (\tilde{b}A)^p d(x_{k+1-p}, N(x_{k-p})) + (\tilde{b}A)^{k+1}d(x_0, x^*), \end{aligned}$$

while in case (b), similar estimation gives

$$d(x_{k+1}, x^*) \leq \sum_{p=0}^k (BA)^p Bd(x_{k+1-p}, N(x_{k-p})) + (BA)^k Bd(x_0, x^*).$$

Since  $I - \tilde{b}A$  is inverse-positive and  $\tilde{b}A$  is positive in the first case, and  $I - BA$  is inverse-positive and  $BA$  is positive in the second case, the series  $\sum_{p=0}^k (\tilde{b}A)^p$  and  $\sum_{p=0}^k (BA)^p$  are convergent. Moreover,  $(\tilde{b}A)^k$  and  $(BA)^k$  converge to the zero matrix as  $k \rightarrow \infty$ . Therefore, using the Cauchy-Toeplitz lemma (see [29]), it follows that  $d(x_{k+1}, x^*) \rightarrow 0$  as  $k \rightarrow \infty$ .  $\square$

### 3.4. Avramescu Type Fixed Point Theorem

Our next result is a variant of Avramescu's fixed point theorem (see [30]) in vector  $B$ -metric spaces.

**Theorem 7** (Avramescu theorem in vector  $B$ -metric spaces). *Let  $(X, d)$  be a complete vector  $B$ -metric space,  $D$  a nonempty closed convex subset of a normed space  $Y$ ,  $N_1 : X \times D \rightarrow X$  and  $N_2 : X \times D \rightarrow D$  be two mappings. Assume that the following conditions are satisfied:*

(i)  $N_1(x, .)$  is continuous for every  $x \in X$  and there is a matrix  $A$  convergent to zero such that

$$d(N_1(x, y), N_1(\bar{x}, y)) \leq A d(x, \bar{x}),$$

for all  $x, \bar{x} \in X$  and  $y \in D$ ;

(ii) Either

(a)  $B$  and  $B^{-1} - A$  is inverse-positive;

or

(b)  $B$  is positive and  $I - BA$  is inverse-positive.

(iii)  $N_2$  is continuous and  $N_2(X \times D)$  is a relatively compact subset of  $Y$ .

Then, there exists  $(x^*, y^*) \in X \times D$  such that

$$N_1(x^*, y^*) = x^*, \quad N_2(x^*, y^*) = y^*.$$

**Proof.** For each  $y \in D$ , Theorem 1 applies to the operator  $N_1(., y)$  and gives a unique  $S(y) \in X$  such that

$$N_1(S(y), y) = S(y). \quad (18)$$

We claim that the mapping  $S : D \rightarrow X$  is continuous. To prove this, let  $y, \bar{y} \in D$ . In case (a), we have

$$\begin{aligned} B^{-1}d(S(y), S(\bar{y})) &= B^{-1}d(N_1(S(y), y), N_1(S(\bar{y}), \bar{y})) \\ &\leq d(N_1(S(y), y), N_1(S(\bar{y}), y)) + d(N_1(S(\bar{y}), y), N_1(S(\bar{y}), \bar{y})) \\ &\leq Ad(S(y), S(\bar{y})) + d(N_1(S(\bar{y}), y), N_1(S(\bar{y}), \bar{y})), \end{aligned}$$

which implies

$$(B^{-1} - A)d(S(y), S(\bar{y})) \leq d(N_1(S(\bar{y}), y), N_1(S(\bar{y}), \bar{y})),$$

while in case (b), one has

$$(I - BA)d(S(y), S(\bar{y})) \leq Bd(N_1(S(\bar{y}), y), N_1(S(\bar{y}), \bar{y})).$$

Since  $B^{-1} - A$  and  $I - BA$  are inverse-positive, respectively, in case (a), we deduce that

$$d(S(y), S(\bar{y})) \leq (B^{-1} - A)^{-1} d(N_1(S(\bar{y}), y), N_1(S(\bar{y}), \bar{y})), \quad (19)$$

and in case (b),

$$d(S(y), S(\bar{y})) \leq (I - BA)^{-1} B d(N_1(S(\bar{y}), y), N_1(S(\bar{y}), \bar{y})). \quad (20)$$

Then, for any convergent sequence  $(y_k) \subset D$ ,  $y_k \rightarrow y^*$  as  $k \rightarrow \infty$ , the continuity of  $N_1(S(y^*), \cdot)$  together with relations (19) and (20) implies that  $d(S(y_k), S(y^*)) \rightarrow 0$  as  $k \rightarrow \infty$ . Thus,  $S$  is continuous, and since  $N_2$  is continuous, the composed mapping

$$N_2(S(\cdot), \cdot) : D \rightarrow D$$

is continuous too. Since its range is relatively compact by condition (iii), Schauder's fixed point theorem applies and guarantees the existence of a point  $y^* \in D$  such that

$$N_2(S(y^*), y^*) = y^*. \quad (21)$$

Finally, denoting  $x^* := S(y^*)$ , from (18) and (21) we have the conclusion.  $\square$

**Remark 2.** *Without the invariance condition  $N_2(X \times D) \subset D$ , a similar result holds if  $D$  is a closed ball  $B_R$  centered at the origin and of radius  $R$  in the space  $(Y, \|\cdot\|)$ , provided that Schaefer's fixed point theorem is used instead of Schauder's theorem. In this case, in addition to conditions (i) and (ii), we need the Leray-Schauder condition*

$$y \neq \lambda N_2(x, y),$$

for all  $x \in X$ ,  $y \in Y$  with  $\|y\| = R$ , and  $\lambda \in (0, 1)$ .

In particular, for scalar  $b$ -metric spaces, conditions (a) and (b) from hypothesis (ii) of Theorem 7 are the same and reduce to the unique requirement that the product of  $b$  and the Lipschitz constant  $a$  of  $N$  is less than one. More exactly, Theorem 7 reads as follows.

**Theorem 8** (Avramescu theorem in  $b$ -metric spaces). *Let  $(X, \rho)$  be a complete  $b$ -metric space ( $b \geq 1$ ),  $D$  a nonempty closed convex subset of a normed space  $Y$ ,  $N_1 : X \times D \rightarrow X$  and  $N_2 : X \times D \rightarrow D$  be two mappings. Assume that the following conditions are satisfied:*

(i)  $N_1(x, \cdot)$  is continuous for every  $x \in X$  and there is a constant  $a \geq 0$  such that

$$\rho(N_1(x, y), N_2(\bar{x}, y)) \leq a\rho(x, \bar{x}),$$

for all  $x, \bar{x} \in X$  and  $y \in D$ ;

(ii)  $ab < 1$ ;

(iii)  $N_2$  is continuous and  $N_2(X \times D)$  is a relatively compact subset of  $Y$ .

Then, there exists  $(x^*, y^*) \in X \times D$  such that  $N_1(x^*, y^*) = x^*$  and  $N_2(x^*, y^*) = y^*$ .

## 4. Ekeland's Principle and Caristi's Fixed Point Theorem in Vector $b$ -Metric Spaces

### 4.1. Classical Results

We first recall for comparison the classical results in metric spaces (see, [31–34]).

**Theorem 9** (Weak Ekeland variational principle). *Let  $(X, \rho)$  be a complete metric space and let  $f : X \rightarrow \mathbb{R}$  be a lower semicontinuous function bounded from below. Then, for given  $\varepsilon > 0$  and  $x_0 \in X$ , there exists a point  $x^* \in X$  such that*

$$f(x^*) \leq f(x_0) - \varepsilon\rho(x^*, x_0)$$

and

$$f(x^*) < f(x) + \varepsilon \rho(x^*, x) \quad \text{for all } x \in X, x \neq x^*.$$

**Theorem 10** (Strong Ekeland variational principle). *Let  $(X, \rho)$  be a complete metric space, and let  $f : X \rightarrow \mathbb{R}$  be a lower semicontinuous function that is bounded from below. For given  $\varepsilon > 0$ ,  $\delta > 0$ , and  $x_0 \in X$  satisfying*

$$f(x_0) \leq \inf_{x \in X} f(x) + \varepsilon,$$

*there exists a point  $x^* \in X$  such that the following hold:*

$$\begin{aligned} f(x^*) &\leq f(x_0), \\ \rho(x^*, x_0) &\leq \delta, \\ f(x^*) &< f(x) + \frac{\varepsilon}{\delta} \rho(x^*, x) \quad \text{for all } x \in X, x \neq x^*. \end{aligned}$$

Below, we have a version of Ekeland's variational principle for scalar  $b$ -metric spaces (see, [20]).

**Theorem 11** ([20]). *Let  $(X, \rho)$  be a complete  $b$ -metric space with  $b > 1$ , where the  $b$ -metric  $\rho$  is continuous. Let  $f : X \rightarrow \mathbb{R}$  be a lower semicontinuous function bounded from below. For a given  $\varepsilon > 0$  and  $x_0 \in X$  satisfying*

$$f(x_0) \leq \inf_{x \in X} f(x) + \varepsilon,$$

*there exists a sequence  $(x_k) \subset X$  and a point  $x^* \in X$  such that:*

$$\begin{aligned} x_k &\rightarrow x^* \quad \text{as } k \rightarrow \infty, \\ \rho(x^*, x_k) &\leq \frac{\varepsilon}{2^k}, \quad k \in \mathbb{N}, \\ f(x^*) &\leq f(x_0) - \sum_{k=0}^{\infty} \frac{1}{b^k} \rho(x^*, x_k), \\ f(x^*) + \sum_{k=0}^{\infty} \frac{1}{b^k} \rho(x^*, x_k) &< f(x) + \sum_{k=0}^{\infty} \frac{1}{b^k} \rho(x, x_k), \quad \text{for } x \neq x^*. \end{aligned}$$

The proof of Theorem 11 in [20] is based on the version for scalar  $b$ -metric spaces of Cantor's intersection lemma.

**Lemma 1** ([20]). *Let  $(X, \rho)$  be a complete  $b$ -metric space. For every descending sequence  $(F_k)_{k \geq 1}$  of nonempty closed subsets of  $X$  with  $\text{diam}_\rho(F_k) \rightarrow 0$  as  $k \rightarrow \infty$ , the intersection  $\bigcap_{k=1}^{\infty} F_k$  contains one and only one element.*

Let us first note that a version of Cantor's intersection lemma remains true in complete vector  $B$ -metric spaces.

**Lemma 2.** *Let  $(X, d)$  be a complete vector  $B$ -metric space, and let  $(F_k)_{k \geq 1}$  be a descending sequence of nonempty closed subsets of  $X$ . Assume that for every  $\varepsilon > 0$ , there exists  $k_0 \geq 1$  such that*

$$d(x, y) \leq \varepsilon e \quad \text{for all } x, y \in F_k \text{ and } k \geq k_0, \tag{22}$$

*where  $e = (1, 1, \dots, 1)$ . Then, the intersection  $\bigcap_{k=1}^{\infty} F_k$  contains exactly one element.*

**Proof.** As stated in the Preliminaries, condition (22) implies that the diameter of  $F_k$  with respect to the scalar  $b$ -metric  $\rho_1$  tends to zero. Since  $(X, d)$  is complete, it follows that  $(X, \rho_1)$  is also complete.

From Cantor's lemma in scalar  $b$ -metric spaces (Lemma 1), we conclude that the intersection  $\bigcap_{k=1}^{\infty} F_k$  has exactly one element.  $\square$

#### 4.2. Ekeland Variational Principle In Vector $B$ -metric Spaces

First we state and prove a version of the weak form of Ekeland's variational principle in vector  $B$ -metric spaces.

**Theorem 12** (Weak Ekeland variational principle in vector  $B$ -metric spaces). *Let  $(X, d)$  be a complete vector  $B$ -metric space such that the  $B$ -metric  $d$  is continuous, and let  $f : X \rightarrow \mathbb{R}^n$  be a lower semicontinuous function bounded from below. Assume that  $f$  satisfies the following condition:*

**(H)** *For every nonempty closed subset  $F \subset X$  and every  $\varepsilon > 0$ , there exists a point  $x_{\varepsilon, F} \in F$  such that*

$$f(x_{\varepsilon, F}) \leq f(x) + \varepsilon e, \quad \text{for all } x \in F, \quad (23)$$

where  $e = (1, 1, \dots, 1) \in \mathbb{R}^n$ .

Then, for a given  $x_0 \in X$ , there exists a sequence  $\{x_k\} \subset X$  and a point  $x^* \in X$  such that  $x_k \rightarrow x^*$  as  $k \rightarrow \infty$ ,

$$f(x^*) \leq f(x_0) - d(x^*, x_0), \quad (24)$$

and

$$f(x^*) + d(x^*, x_k) \geq f(x) + d(x, x_k) \quad \text{for all } k \geq 0 \quad \text{implies } x = x^*. \quad (25)$$

Moreover,

$$f(x^*) \geq f(x) + Bd(x^*, x) + (B - I)d(x^*, x_k) \quad \text{for all } k \geq 0 \quad \text{implies } x = x^*. \quad (26)$$

**Proof.** Let us fix a sequence  $(\varepsilon_k)$  of positive numbers satisfying  $\varepsilon_k \rightarrow 0$  as  $k \rightarrow \infty$ . We now proceed to construct the sequence  $(x_k)$ . Let

$$F(x_0) := \{x \in X : f(x) + d(x, x_0) \leq f(x_0)\}.$$

Clearly,  $x_0 \in F(x_0)$  and  $F(x_0)$  is closed because  $d$  is continuous and  $f$  is lower semicontinuous. Then, by assumption (23), there exists a point  $x_1 \in F(x_0)$  with

$$f(x_1) \leq f(x) + \varepsilon_1 e \quad \text{for all } x \in F(x_0).$$

Define

$$F(x_1) := \{x \in F(x_0) : f(x) + d(x, x_1) \leq f(x_1)\},$$

and recursively, having  $x_k \in F(x_{k-1})$  with

$$f(x_k) \leq f(x) + \varepsilon_k e \quad \text{for all } x \in F(x_{k-1}),$$

we define

$$F(x_k) := \{x \in F(x_{k-1}) : f(x) + d(x, x_k) \leq f(x_k)\}.$$

The sets  $F(x_k)$  are nonempty and closed, and by their definition form a descending sequence. To apply Cantor's intersection lemma, we verify that their diameters tend to zero as  $k \rightarrow \infty$ . Indeed, for any  $y \in F(x_k) \subset F(x_{k-1})$ , one has

$$f(y) + d(y, x_k) \leq f(x_k).$$

Also, from the definition of  $x_k$ ,

$$f(x_k) \leq f(y) + \varepsilon_k e.$$

Consequently, using the definition of  $F(x_k)$ , we deduce

$$d(y, x_k) \leq f(x_k) - f(y) \leq \varepsilon_k e,$$

whence, for every  $y_1, y_2 \in F(x_k)$ , we have

$$d(y_1, y_2) \leq B(d(y_1, x_k) + d(y_2, x_k)).$$

As a result,  $\text{diam}_d(F(x_k)) \rightarrow 0$  as  $k \rightarrow \infty$ . Thus, by Cantor's lemma,

$$\bigcap_{k=0}^{\infty} F(x_k) = \{x^*\}.$$

From  $x^* \in F(x_0)$ , one has (24).

Next, we prove (25). To this end, we show the equivalent statement: if  $x \neq x^*$ , then there exists  $k = k(x) \geq 0$  such that

$$f(x^*) + d(x^*, x_k) \not\geq f(x) + d(x, x_k),$$

that is

$$f_i(x^*) + d_i(x^*, x_k) < f_i(x) + d_i(x, x_k)$$

for at least one index  $i \in \{1, 2, \dots, n\}$ .

Let  $x \in X$ ,  $x \neq x^*$  be arbitrary. Then  $x \notin \bigcap_{k=0}^{\infty} F(x_k)$ . We distinguish two cases:

- (a)  $x \notin F(x_0)$ ;
- (b)  $x \in F(x_{k-1})$  and  $x \notin F(x_k)$  for some  $k = k(x) \geq 1$ .

In case (a), we have  $f(x) + d(x, x_0) \not\leq f(x_0)$ . In case (b), we have  $f(x) + d(x, x_k) \not\leq f(x_k)$ . Thus, in both cases, there exists  $k = k(x) \geq 0$  such that  $f(x) + d(x, x_k) \not\leq f(x_k)$ . This implies that there is some  $i \in \{1, 2, \dots, n\}$  with

$$f_i(x) + d_i(x, x_k) > f_i(x_k).$$

On the other hand, since  $x^* \in F(x_k)$ , one has  $f(x^*) + d(x^*, x_k) \leq f(x_k)$ . In particular, for the index  $i$  identified above, it holds that

$$f_i(x_k) \geq f_i(x^*) + d_i(x^*, x_k).$$

Then, from these two inequalities we obtain

$$f_i(x^*) + d_i(x^*, x_k) < f_i(x) + d_i(x, x_k), \quad (27)$$

which equivalently proves (25).

In order to establish (26), we apply the triangle inequality for  $d$  on the right hand side of (27), which gives,

$$f_i(x^*) + d_i(x^*, x_k) < f_i(x) + d_i(x, x_k) \leq f_i(x) + (Bd(x^*, x_k))_i + (Bd(x^*, x))_i.$$

Hence

$$f_i(x^*) + d_i(x^*, x_k) < f_i(x) + (Bd(x^*, x_k))_i + (Bd(x^*, x))_i,$$

that is,

$$f(x^*) \not\geq f(x) + Bd(x^*, x) + (B - I)d(x^*, x_k).$$

Thus, (26) holds.  $\square$

A version of the strong form of Ekeland's variational principle in vector  $B$ -metric spaces is the following one.

**Theorem 13** (Strong Ekeland variational principle in vector  $B$ -metric spaces). *Let  $(X, d)$  be a complete  $B$ -metric space such that the  $B$ -metric  $d$  is continuous, and let  $f : X \rightarrow \mathbb{R}^n$  be a lower semicontinuous function bounded from below and satisfying condition (H). Then, for given  $\varepsilon, \delta > 0$  and  $x_0 \in X$  with*

$$f(x_0) \leq f(x) + \varepsilon e \text{ for all } x \in X, \quad (28)$$

there exists a sequence  $(x_k) \subset X$  and  $x^* \in X$  such that  $x_k \rightarrow x^*$  as  $k \rightarrow \infty$ ,

$$f(x^*) \leq f(x_0), \quad (29)$$

$$d(x^*, x_0) \leq \delta e, \quad (30)$$

$$f(x^*) + \frac{\varepsilon}{\delta} d(x^*, x_k) \geq f(x) + \frac{\varepsilon}{\delta} d(x, x_k) \text{ for all } k \geq 0 \text{ implies } x = x^*.$$

Moreover,

$$f(x^*) \geq f(x) + \frac{\varepsilon}{\delta} Bd(x^*, x) + \frac{\varepsilon}{\delta} (B - I)d(x^*, x_k) \text{ for all } k \geq 0 \text{ implies } x = x^*.$$

**Proof.** We apply the weak form of Ekeland's variational principle, Theorem 12, to the vector  $B$ -metric  $\frac{\varepsilon}{\delta} d$ . From (24), we immediately obtain (29), while from  $x^* \in F(x_0)$  and (28), we deduce

$$\frac{\varepsilon}{\delta} d(x^*, x_0) \leq f(x_0) - f(x^*) \leq \varepsilon e,$$

whence (30). The remaining conclusions follow directly.  $\square$

A consequence of the weak form of Ekeland's variational principle is the following version of Caristi's fixed point theorem (see [35]) in vector  $B$ -metric spaces.

**Theorem 14.** *Let  $(X, d)$  be a complete vector  $B$ -metric space such that the  $B$ -metric  $d$  is continuous, and let  $f : X \rightarrow \mathbb{R}^n$  be a lower semicontinuous function bounded from below and satisfying condition (H). Assume that for an operator  $N : X \rightarrow X$ , the following conditions are satisfied:*

$$d(N(x), y) \leq d(x, y) + Bd(N(x), x), \quad x, y \in X \quad (31)$$

and

$$Bd(N(x), x) \leq f(x) - f(N(x)), \quad x \in X. \quad (32)$$

Then,  $N$  has at least one fixed point.

**Proof.** Assume that  $N$  has no fixed points. Then, applying Ekeland's variational principle to  $f$  (Theorem 12), from (25), one has

$$f(x^*) + d(x^*, x_k) \not\geq f(N(x^*)) + d(N(x^*), x_k)$$

for some  $k$ . Therefore, there is an index  $i$  with

$$f_i(x^*) + d_i(x^*, x_k) < f_i(N(x^*)) + d_i(N(x^*), x_k).$$

Using (32) gives

$$(Bd(N(x^*), x^*))_i \leq f_i(x^*) - f_i(N(x^*)) < d_i(N(x^*), x_k) - d_i(x^*, x_k),$$

that is

$$d_i(x^*, x_k) + (Bd(N(x^*), x^*))_i < d_i(N(x^*), x_k),$$

which contradicts (31). Consequently,  $N$  has a fixed point.  $\square$

#### 4.3. New Versions Of The Ekeland Variational Principle In $b$ -Metric Spaces

We emphasize that in the scalar case, that is, when  $n = 1$ ,  $B = b \geq 1$  and  $d = \rho$  is a  $b$ -metric, our theorems from the previous subsection offer more natural versions in  $b$ -metric spaces to the classical results, as follows.

**Theorem 15** (Weak Ekeland variational principle in  $b$ -metric spaces). *Let  $(X, \rho)$  be a complete  $b$ -metric space ( $b \geq 1$ ) such that the  $b$ -metric  $\rho$  is continuous, and let  $f : X \rightarrow \mathbb{R}$  be a lower semicontinuous function bounded from below. Then, for given  $x_0 \in X$ , there exists a sequence  $(x_k) \subset X$  and  $x^* \in X$  such that  $x_k \rightarrow x^*$  as  $k \rightarrow \infty$ ,*

$$f(x^*) \leq f(x_0) - \rho(x^*, x_0),$$

and for each  $x \in X$ ,  $x \neq x^*$ , there exists an index  $k = k(x)$  with

$$f(x^*) + \rho(x^*, x_k) < f(x) + \rho(x, x_k).$$

Moreover, for each  $x \in X$ ,  $x \neq x^*$ , there exists an index  $k = k(x)$  with

$$f(x^*) < f(x) + b\rho(x^*, x) + (b-1)\rho(x^*, x_k). \quad (33)$$

**Theorem 16** (Strong Ekeland variational principle in  $b$ -metric spaces). *Let  $(X, \rho)$  be a complete  $b$ -metric space ( $b \geq 1$ ) such that the  $b$ -metric  $\rho$  is continuous, and let  $f : X \rightarrow \mathbb{R}$  be a lower semicontinuous function bounded from below. Then, for given  $\varepsilon, \delta > 0$  and  $x_0 \in X$  with*

$$f(x_0) \leq \inf_{x \in X} f(x) + \varepsilon,$$

there exists a sequence  $(x_k) \subset X$  and  $x^* \in X$  such that  $x_k \rightarrow x^*$  as  $k \rightarrow \infty$ ,

$$f(x^*) \leq f(x_0),$$

$$\rho(x^*, x_0) \leq \delta,$$

and for each  $x \in X$ ,  $x \neq x^*$ , there exists an index  $k = k(x)$  with

$$f(x^*) + \frac{\varepsilon}{\delta}\rho(x^*, x_k) < f(x) + \frac{\varepsilon}{\delta}\rho(x, x_k).$$

Moreover, for each  $x \in X$ ,  $x \neq x^*$ , there exists an index  $k = k(x)$  with

$$f(x^*) < f(x) + b\frac{\varepsilon}{\delta}\rho(x^*, x) + (b-1)\frac{\varepsilon}{\delta}\rho(x^*, x_k). \quad (34)$$

**Theorem 17** (Caristi fixed point theorem in  $b$ -metric spaces). *Let  $(X, \rho)$  be a complete  $b$ -metric space ( $b \geq 1$ ) such that the  $b$ -metric  $\rho$  is continuous, and let  $f : X \rightarrow \mathbb{R}$  be a lower semicontinuous function bounded from below. If for an operator  $N : X \rightarrow X$ , one has*

$$\rho(N(x), y) \leq \rho(x, y) + b\rho(N(x), x), \quad x, y \in X \quad (35)$$

and

$$b\rho(N(x), x) \leq f(x) - f(N(x)), \quad x \in X, \quad (36)$$

then  $N$  has at least one fixed point.

The last three results reduce to the classical ones in ordinary metric spaces, i.e., if  $b = 1$ . Thus, (33) reduces to

$$f(x^*) < f(x) + \rho(x^*, x), \quad x \neq x^*;$$

(34) reduces to

$$f(x^*) < f(x) + \frac{\varepsilon}{\delta} \rho(x^*, x), \quad x \neq x^*;$$

assumption (35) trivially holds, while (36) becomes the classical Caristi's inequality

$$\rho(N(x), x) \leq f(x) - f(N(x)), \quad x \in X.$$

## 5. Conclusion and Further Research

In this paper, we introduced the concept of a vector  $B$ -metric space. Several fixed-point theorems, analogous to those in scalar  $b$ -metric spaces as well as their classical counterparts, were presented. Additionally, we discussed some stability results. Finally, we provided a variant of Ekeland's variational principle alongside a version of Caristi's theorem. It remains an open question whether the assumption that  $B^{-1} - A$  or  $I - BA$  is inverse-positive can be omitted in Theorems 7, 5 and 6. Additionally, one may explore a variant of Ekeland's variational principle where Caristi's theorem holds without requiring the additional assumption (31). Lastly, it would be interesting to study the case where the matrix  $B$  is neither positive nor inverse-positive; for instance, when it has positive diagonal elements but contains both positive and negative entries elsewhere.

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