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Critical Problem Of Optimal Stabilization Without Control Constraints

Volodymyr Kapustyan [1], Anna Sukretna [2,*], Zhanna Chernousova [3] and Yuriy Kharkevych [4]

[1] National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute", Kyiv, Ukraine
[2] Taras Shevchenko National University of Kyiv, Kyiv, Ukraine
[3] National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute", Kyiv, Ukraine
[4] Lesya Ukrainka Volyn National University, Lutsk, Ukraine
* Correspondence: sukretna.a.v@knu.ua

Abstract

The article analyzes the linear-quadratic optimal stabilization problem in the so-called "critical case", namely, the situation is considered when the spectrum of the system matrix contains purely imaginary eigenvalues or when the standard conditions of positive definiteness of the weight matrices of the quality functional are violated. Methods for regularizing critical problems by perturbing the system matrices and the functional are investigated, and algorithms for decomposing multidimensional problems into a set of one-dimensional canonical systems are proposed. The results are of practical importance for constructing optimal synthesis in various engineering and economic systems, in particular, the results can be used for stabilizing unmanned aerial vehicles, robotic complexes and intelligent power grids.

Keywords: optimal stabilization problem; optimal synthesis; feedback control; linear-quadratic regulator (LQR); critical case of optimal stabilization problem; regularization

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1. Introduction

The problem of finding optimal controllers and optimal stabilization in the linear-quadratic model (LQR-problem) is one of the most relevant problems in the theory of optimal control. At the theoretical level, this problem, using the Bellman dynamic programming method, is reduced to constructing a linear controller, for which it is necessary to solve the matrix Ricatti equation [1–3]. At the same time, the practical solvability of this equation strongly depends on the properties of the system matrix and their dimensions. Today, optimal stabilization problems are studied for processes of the most diverse nature, in particular, with distributed parameters [4–6], with impulse and random disturbances [7–9], etc., while linear-quadratic problems serve as the theoretical basis of these studies. In addition, LQR problems are used in mathematical modeling of complex processes in the aerospace industry, robotics, energy, are a standard method for stabilizing unmanned aerial vehicles (UAVs), in particular, quadcopters, etc. [10–14]

The main obstacle to using LQR is the need for an accurate linear model of the object. Such models are obtained either by linearizing complex nonlinear systems, or using various integrations of LQR with artificial intelligence methods and control of complex network objects. In recent years, there has been a new wave of interest in linear-quadratic problems due to their combination with artificial intelligence methods. Current research focuses on "safe reinforcement learning" (Safe RL), where LQR acts as a stabilizing framework for neural network controllers. In addition, data-based control methods (model-free LQR) are actively developing, which allow the synthesis of optimal strategies without exact knowledge of the object matrices [11,15,16].

Despite significant achievements, the analysis of critical states, when the traditional conditions of the absence of imaginary values in the spectrum of the system matrices and the positive definiteness of the matrices in the quality criterion are not met, remains a difficult mathematical problem. This work focuses on the spectral analysis of such systems and the development of methods for their regularization, which allows finding a solution in extended function spaces, even when the classical problem does not have admissible controls.

2. Problem Formulation

It is known [1–3] that the classical problem of optimal stabilization is follows: to find the optimal control by the feedback principle (optimal synthesis)

$$u = u[x(t)], \tag{1}$$

where the function $x(t) : R \to R^n$ describes the controlled process and satisfies the Cauchy problems

$$\dot{x}(t) = Ax(t) + Bu[x(t)], \quad 0 = t_0 < t \le \infty, \tag{2}$$

$$x(0) = x^0, \tag{3}$$

and the optimality criterion

$$J(u) = \int_0^\infty \Big(x^T(t)Qx(t) + u^T(t)Hu(t) \Big) dt \tag{4}$$

takes the smallest possible value. Note here that the vector $x^0 \in R^n$ is fixed, but arbitrary.

In the problem (1) – (4) $A,\ B,\ Q,\ H$ are matrices of dimensions $n \times n,\ n \times r,\ n \times n,\ r \times r$ respectively; matrices $Q,\ H$ are additionally symmetric and positive definite; the pair of matrices (A, B) is completely controllable, that is, the Kalman criterion is fulfilled

$$rang\Big(B, AB, A^2B, \dots, A^{n-1}B\Big) = n.$$

The solution of the optimal stabilization problem is searched in the following class of functions:

$$\Upsilon(x, u) \equiv \Big\{ \big(x(t), u(t)\big) :\ x(t) \in W_2^{1,n}(0, \infty),\ u(t) \in L_2^r(0, \infty) \Big\}, \tag{5}$$

where

$$L_2^r(0, \infty) \equiv \left\{ u(t) : \int_0^\infty \sum_{j=1}^r u_j^2(t)dt < \infty \right\},$$

$$W_2^{1,n}(0, \infty) \equiv \left\{ x(t) : \int_0^\infty \sum_{j=1}^n x_j^2(t)dt < \infty,\ \int_0^\infty \sum_{j=1}^n \dot{x}_j^2(t)dt < \infty. \right\}$$

Then [1–3] there is a unique solution to the problem (1) – (4) and it is determined by the control

$$u_* = u_*[x_*(t)] = -H^{-1}B^T K_* x_*(t), \tag{6}$$

where the symmetric matrix K_* is the unique positive definite solution of the algebraic Riccati equation

$$KA + A^T K - KBH^{-1}B^T K + Q \ = \ 0. \tag{7}$$

The optimal value of the quality criterion will be as follows:

$$J(u_*) = (x^0)^T K_* x^0. \tag{8}$$

From now on, we will not put the $*$ icon on optimal solutions.

In [1] it was noted that this result remains true in the case of symmetric non-negative matrix Q. However, in the general case such a statement is incorrect. In [3] the existence and sufficient conditions for the validity of such a generalization are found.

In this work we focus attention on the critical case of optimal stabilization problem when some inclusions from (5) are violated.

3. Spectral Analysis of the Optimal Stabilization Problem

Let

$$G(x(t), u(t)) = x^T(t)Qx(t) + u^T(t)Hu(t). \tag{9}$$

We extend the quadratic form $G(x, u)$ to the complex values of the vectors x, u while preserving Hermitianity

$$G(\tilde{x}, \tilde{u}) \; = \tilde{x}^\star Q\tilde{x} + \tilde{u}^\star H\tilde{u}. \tag{10}$$

Hereinafter, the asterisk $\star$ will denote Hermitian conjugation: transpose and complex conjugation.

In (10) $\tilde{x}, \tilde{u}$ are arbitrary complex vectors. Let them satisfy the equation

$$\imath\,\omega\,\tilde{x} = A\tilde{x} + B\tilde{u}, \tag{11}$$

where ω is an arbitrary real number, $\imath^2 = -1$.

For the program control of optmal control problem (2) – (4) the necessary optimality conditions in the form of the maximum principle Pontryagin hold. To formulate them, we write the Hamilton function

$$\mathcal{H}(x, u, \psi) = -G(x(t), u(t)) + \psi^T(Ax + Bu). \tag{12}$$

Then, if the pair $(x = x(t),\ u = u(t))$ is the solution of problem (2) – (4) in the class of functions $W_2^{1,n}(0,\infty) \times L_2^r(0,\infty)$, then there exists a function $\psi(t) \in W_2^{1,n}(0,\infty)$, such that pair (x, ψ) is satisfied the equations

$$\frac{dx}{dt} = \left(\frac{\partial \mathcal{H}}{\partial \psi}\right)^T, \quad \frac{d\psi}{dt} = -\left(\frac{\partial \mathcal{H}}{\partial x}\right)^T,$$

and for optimal control we have

$$\frac{\partial \mathcal{H}}{\partial u} = 0 \quad \Rightarrow \quad u = -\frac{1}{2}H^{-1}B^T\,\psi. \tag{13}$$

The system (13) can be written as a matrix equation

$$\widehat{J}\,\frac{dz}{dt} = \widehat{K}\,z, \tag{14}$$

where $z^T = (x, \psi)$,

$$\widehat{J} = \begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}, \quad \widehat{K} = \begin{pmatrix} -2Q & A^T \\ A & \frac{1}{2}BH^{-1}B^T \end{pmatrix}.$$

Let the system (14) have no purely imaginary eigenvalues, i.e.

$$\varphi(\imath\,\omega) = \det\left(\widehat{J}^{-1}\widehat{K} - \imath\,\omega\,E_{2n}\right) \neq 0 \;\text{ for all }\; \omega \in R^1. \tag{15}$$

In [3] it was established that the condition (15) is equivalent to the condition

$$\begin{gathered} G(\tilde{x}, \tilde{u}) > 0 \;\text{ for all }\; \omega \in R^1 \\ \text{and for all }\; \tilde{x},\ \tilde{u} \text{ such that } |\tilde{x}|^2 + |\tilde{u}|^2 \neq 0: \quad \imath\,\omega\,\tilde{x} = A\tilde{x} + B\tilde{u}. \end{gathered} \tag{16}$$

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In the case where the matrix A has no purely imaginary eigenvalues, condition (16) is equivalent to the condition

$$G\Big((\imath\,\omega\,E_n - A)^{-1}B\tilde{u},\,\tilde{u}\Big) > 0, \quad \forall\,\omega \in R^1, \quad \forall\,\tilde{u} \neq 0. \tag{17}$$

From (17) follows the inequality

$$G(\tilde{x},\tilde{u}) \geq \tilde{u}^\star\,H\,\tilde{u} > 0, \quad \forall \tilde{u} \neq 0. \tag{18}$$

Let the symmetric matrix $\Pi(\imath\,\omega)$ of dimension $r \times r$ be defined by the equality

$$G\Big((\imath\,\omega\,E_n - A)^{-1}B\tilde{u},\,\tilde{u}\Big) = \tilde{u}^\star\,\Pi(\imath\,\omega)\,\tilde{u}. \tag{19}$$

Then we have the representation

$$\varphi(\imath\,\omega) = 2^r(\det H)^{-1}\big|\delta(\imath\,\omega)\big|^2\,\Pi(\imath\,\omega), \tag{20}$$

where

$$\delta(\lambda) = \det(\lambda E_n - A).$$

If the matrix A has purely imaginary eigenvalues, then the problem (2) – (4) may not have solutions in the class of functions $W_2^{1,n}(0,\infty) \times L_2^r(0,\infty)$.

Let all purely imaginary eigenvalues $\imath\,\omega_j$ of the matrix A be distinct. Let p_j are corresponding eigenvectors, i.e.

$$Ap_j = \imath\,\omega_j\,p_j, \quad p_j \neq 0. \tag{21}$$

Let us show that in the case of the form (9) the necessary and sufficient condition for the existence of optimal control is the fulfillment of the inequalities

$$p_j^\star\,Q\,p_j > 0 \quad \text{for all} \quad j. \tag{22}$$

Indeed, from (16) for $\omega = \omega_j,\ \tilde{x} = p_j$ and $\tilde{u} = 0$ it follows (22).

Let us further assume that condition (22) is satisfied, but condition (16) is violated:

$$\exists\,\omega_0,\,\tilde{x}_0,\,\tilde{u}_0: \quad \imath\,\omega_0\,\tilde{x}_0 = A\tilde{x}_0 + B\tilde{u}_0, \quad |\tilde{x}_0|^2 + |\tilde{u}_0|^2 \neq 0, \quad G(\tilde{x}_0,\tilde{u}_0) = 0.$$

Then $Q\tilde{x}_0 = 0$ and $\tilde{u}_0 = 0$. Hence $\omega_0 = \omega_j$ and $\tilde{x}_0 = p_j$ for some j, so $G(\tilde{x}_0,\tilde{u}_0) = p_j^\star\,Q\,p_j = 0$, which contradicts (22).

In the general case, taking into account the representation (20), the condition (16) is equivalent to the inequalities

$$\det\Pi(\imath\,\omega) > 0, \quad \lim_{\omega\to\omega_j}\Big|\det\big(\imath\,\omega\,E_n - A\big)\Big|^2\,\det\Pi(\imath\,\omega) > 0. \tag{23}$$

Thus, the necessary and sufficient conditions for the existence of a solution to the optimal control problem (2) – (4) in the case of a matrix A with an imaginary spectrum and a non-negative symmetric matrix Q are reduced to the fulfillment of all inequalities (22), or the second inequality from (23). The optimal stabilization problem (2) – (4) in the case of a matrix A with an imaginary spectrum will not have a solution if the inequalities from (22) or (23) are violated, or

$$\varphi(\imath\,\omega_j) = 0 \tag{24}$$

at least for one imaginary eigenvalue of the matrix A.

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4. Optimal Stabilization in the Case of $n = 2,\ r = 1$

Consider the optimal stabilization problem with the following data

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix}, \quad Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{pmatrix}, \quad H = 1.$$

The matrix A has imaginary roots, i.e. the characteristic equation for the matrix A has the form

$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0,$$

and its coefficients satisfy the conditions

$$a_{11} + a_{22} = 0, \quad a_{11}^2 + a_{12}a_{21} \le 0.$$

Let's

$$\omega_{1,2} = \pm\sqrt{|a_{11}^2 + a_{12}a_{21}|}. \tag{25}$$

The pair of matrices (A, B) is completely controllable, i.e.

$$rank[B,\ AB] = 2. \tag{26}$$

The matrix Q is symmetric and non-negative. Necessary and sufficient conditions for this are

$$q_{ii} \ge 0,\ i = \overline{1,2}, \qquad q_{11} + q_{22} \ne 0, \qquad q_{11}q_{22} - q_{12}^2 \ge 0. \tag{27}$$

Then

$$\widehat{J} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \widehat{K} = \begin{pmatrix} -2q_{11} & -2q_{12} & a_{11} & a_{21} \\ -2q_{12} & -2q_{22} & a_{12} & a_{22} \\ a_{11} & a_{12} & \frac{1}{2}b_{1,1}^2 & \frac{1}{2}b_{11}b_{21} \\ a_{21} & a_{22} & \frac{1}{2}b_{11}b_{21} & \frac{1}{2}b_{21}^2 \end{pmatrix},$$

$$\varphi(\imath\omega) = \det\left(\widehat{J}^{-1}\widehat{K} - \imath\omega E_4\right).$$

To simplify the calculations and analysis, we set

$$a_{11} = a_{22} = 0,\ a_{12} = 1,\ a_{21} = -\alpha,\ \alpha \ge 0; \quad q_{11} = q_{12} = 0,\ q_{22} = \beta > 0; \quad H = 1.$$

Thus, the above conditions for the matrices A, B, Q are fulfilled, i.e. we have a critical case of the optimal stabilization problem. Then the system (2) in the field of complex numbers will correspond to the system

$$\imath\omega\,\tilde{x}_1 = \tilde{x}_2, \quad \imath\omega\,\tilde{x}_2 = -\alpha\tilde{x}_1 + \tilde{u}, \tag{28}$$

the solution of which has the form

$$\tilde{x}_1 = \left(\alpha - \omega^2\right)^{-1}\tilde{u}, \quad \tilde{x}_2 = \imath\omega\left(\alpha - \omega^2\right)^{-1}\tilde{u}.$$

Due to the fact that $G(\tilde{x}, \tilde{u}) = \beta|\tilde{x}_2|^2 + |\tilde{u}|^2 = \Pi(\imath\omega)|\tilde{u}|^2$ we have

$$\Pi(\imath\omega) = 1 + \beta\omega^2\left(\alpha - \omega^2\right)^{-2}. \tag{29}$$

Since $\delta(\lambda) = \lambda^2 + \alpha$, we get

$$\varphi(\imath\omega) = \Pi(\imath\omega)|\delta(\imath\omega)|^2 = \left(\alpha - \omega^2\right)^2 + \beta\omega^2. \tag{30}$$

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The condition $\alpha > 0$ guarantees that the condition

$$\varphi(\imath\omega) > 0 \quad \forall\omega \in R^1,$$

that is, it ensures the existence of a solution to the optimal stabilization problem.

In this case, the system (7) will have the form

$$\begin{gathered} K_{12}^2 + 2\alpha K_{12} = 0, \\ K_{11} - \alpha K_{22} - K_{12}K_{22} = 0, \\ 2K_{1,2} - K_{2,2}^2 + \beta = 0. \end{gathered} \tag{31}$$

Its only positive definite solution is given by the matrix

$$K = \begin{pmatrix} \alpha\sqrt{\beta} & 0 \\ 0 & \sqrt{\beta} \end{pmatrix}. \tag{32}$$

The optimal control will be

$$u = -H^{-1}B^T Kx = -\sqrt{\beta}x_2. \tag{33}$$

The closed system will take the form

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\alpha x_1 - \sqrt{\beta}x_2. \tag{34}$$

This system is corresponded to the characteristic equation

$$\lambda^2 + \sqrt{\beta}\lambda + \alpha = 0 \tag{35}$$

with roots

$$\lambda_{1,2} = -\frac{\sqrt{\beta}}{2} \pm \sqrt{\frac{\beta}{4} - \alpha}.$$

Depending on the correlation between the parameters α and β, three variants of the roots are possible, but all of them will have a negative real part. For certainty, let us consider the variant of real non-coincident roots, i.e., let the parameters satisfy the inequality $\frac{\beta}{4} - \alpha > 0$. Then

$$\begin{gathered} x_1(t) = \frac{x_2^0 - \lambda_2 x_1^0}{\lambda_1 - \lambda_2}\exp(\lambda_1 t) - \frac{x_2^0 - \lambda_1 x_1^0}{\lambda_1 - \lambda_2}\exp(\lambda_2 t), \\ x_2(t) = \lambda_2\frac{x_2^0 - \lambda_2 x_1^0}{\lambda_1 - \lambda_2}\exp(\lambda_1 t) - \lambda_2\frac{x_2^0 - \lambda_1 x_1^0}{\lambda_1 - \lambda_2}\exp(\lambda_2 t), \quad \lambda_2 > \lambda_1. \end{gathered} \tag{36}$$

Let $\alpha = 0$. Then for $\omega = 0$ we have

$$\varphi(\imath\omega) = 0.$$

Let us check the condition (23) of the solution existence for the optimal stabilization problem

$$\lim_{\omega\to\omega_j} \left|\det\left(\imath\omega E_n - A\right)\right|^2 \det\Pi(\imath\omega) = \lim_{\omega\to 0}\omega^4(1 + \beta\omega^{-2}) = 0.$$

Thus, the optimal stabilization problem in this case has no solution.

Having an analytical solution for the optimal stabilization problem with $\alpha > 0$, we will clarify the qualitative transition to the case $\alpha = 0$.

The matrix K will then have the form

$$K = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\beta} \end{pmatrix}. \tag{37}$$

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The control is given by the expression (33). The following solution of the system (34) corresponds to it

$$x_1(t) = x_1^0 + \frac{x_2^0}{\sqrt{\beta}}(1 - \exp(-\sqrt{\beta}t)), \qquad x_2(t) = x_2^0 \exp(-\sqrt{\beta}t). \tag{38}$$

From this example we can draw the following ***conclusions***:

1. the problem (1) – (4) with matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix}, \quad H = 1$$

has no solution in the space $Y(x,u)$ for $n = 2,\ r = 1$, but this problem has a solution constructed above, which belongs to the spaces $u(t) \in L_2(0,\infty),\ x_2(t) \in W_2^{1,1}(0,\infty),\ \dot{x}_1(t) \in L_2(0,\infty)$, $x_1(t) \in C[0,\infty)$, where $C[0,\infty)$ is space of continuous functions;

2. for arbitrary $\xi, x_2^0 \in R^1$ there exists a number $x_1^0 = \xi - \frac{x_2^0}{\sqrt{\beta}}$ such that

$$\lim_{t\to\infty} x_1(t) = \xi, \qquad \lim_{t\to\infty} x_2(t) = 0,$$

in particular, for $x_1^0 + \frac{x_2^0}{\sqrt{\beta}} = 0$ we will have $\xi = 0$.

5. Regularization of the Optimal Stabilization Problem in the Critical Case

Thus, the optimal stabilization problem (1) – (4) with a symmetric non-negative matrix Q and a matrix A whose spectrum is not purely imaginary, always has a unique solution in the form of an optimal regulator (6) and at the same time the inclusion (5) is valid. If the spectrum of the matrix A is purely imaginary, then the optimal stabilization problem also has a unique solution in the previous version, when either the conditions (22) or the boundary inequalities (23) are additionally fulfilled. In other cases, the problem (1) – (4) has no solutions on the set (5), i.e. we have a critical case.

Let us denote the space of pairs $\big(x(t),\ u(t) = u[x(t)] \in L_2(0,\infty)\big)$ on which $J(u) < \infty$, by $Y_1(x,u)$. Due to the strictly convexity of the functional, the problem will have a unique solution, but it will not belong to the space $Y(x,u)$.

There are two ways to treat the critical case:

1. consider it degenerate, when we are talking about the solution of the optimal stabilization problem in the space $Y(x,u)$ for all $x^0 \in R^n$;
2. find additional conditions on the input parameters of the problem (1) – (4), when it will still have a solution in the space $Y(x,u)$ (or in the space $Y_1(x,u)$.)

First, let's consider the first possibility. Then the problem (1) – (4) in the critical case can be regularized in two ways:

1. by perturbing the matrix Q of the functional;
2. by perturbing the matrix A in the equations of the original system.

5.1. Regularization by Perturbation of the Matrix Q

Let us replace the problem (1) – (4) with the following problem: for a sufficiently small real number $\varepsilon \in (0, \hat{\varepsilon}]$ find feedback control

$$u^\varepsilon = u^\varepsilon[x^\varepsilon(t)], \tag{39}$$

where the function $x^\varepsilon(t)$ describes the controlled process and satisfies the Cauchy problems

$$\dot{x^\varepsilon}(t) = Ax^\varepsilon(t) + Bu^\varepsilon[x^\varepsilon(t)], \quad 0 < t \leq \infty, \tag{40}$$

$$x^\varepsilon(0) = x^0, \tag{41}$$

and the optimality criterion in this case

$$J^\varepsilon(u^\varepsilon) = \int\limits_0^\infty \Big(\big(x^\varepsilon(t)\big)^T Q^\varepsilon x^\varepsilon(t) + \big(u^\varepsilon(t)\big)^T H u^\varepsilon(t) \Big)\, dt \tag{42}$$

would take the smallest value and the following inequalities were satisfied

$$\begin{aligned} |J^\varepsilon(u^\varepsilon) - J(u)| &\leq C_1\varepsilon, \\ ||u^\varepsilon(t) - u(t)||_{L_2(0,\infty)} &\leq C_2\varepsilon. \end{aligned} \tag{43}$$

In the estimates (43) C_i, $i = \overline{1,2}$, are positive real constants that do not depend on ε.

In problem (39) – (42) the matrix Q_ε is symmetric, positive defined, smooth on the parameter ε and such that $Q_0 = Q$. Then in problem (39) – (42) the optimal controller will have the form

$$u^\varepsilon = u^\varepsilon[x^\varepsilon(t)] = -H^{-1}B^T K^\varepsilon x^\varepsilon(t), \tag{44}$$

where the matrix K^ε is the unique positive defined solution of the algebraic Riccati equation

$$K^\varepsilon A + A^T K^\varepsilon - K^\varepsilon B H^{-1} B^T K^\varepsilon + Q^\varepsilon = 0. \tag{45}$$

The optimal value of the quality criterion will be

$$J^\varepsilon(u^\varepsilon) = (x^0)^T K^\varepsilon x^0. \tag{46}$$

Let us obtain the first inequality in (43).

$$|J^\varepsilon(u^\varepsilon) - J(u)| = |(x^0)^T (K^\varepsilon - K) x^0| \leq \|K^\varepsilon - K\| \|x^0\|^2,$$

where for the matrix W and the vector v we use the norms

$$\|W\| = \max_{i,j=\overline{1,n}} |w_{i,j}|, \quad \|v\| = \max_{i=\overline{1,n}} |v_i|.$$

In the case when solving the equation (45) causes difficulties, we can find its approximate solution in the form of a series

$$K^\varepsilon = \sum_{i=0}^\infty \hat{K}_i \varepsilon^i, \tag{47}$$

where the matrices $\hat{K}_i$ are solutions of linear matrix equations

$$\hat{K}_i A + A^T \hat{K}_i - \sum_{j=0}^{i} \hat{K}_j B H^{-1} B^T \hat{K}_{i-j} + \frac{1}{i!} \frac{d^i Q^\varepsilon}{d\varepsilon^i}\bigg|_{\varepsilon=0} = 0. \tag{48}$$

For example from Section 4 let us set $Q^\varepsilon = Q + \varepsilon^2 E_2$, $\varepsilon \in (0, \hat{\varepsilon}]$. Then the system (45) will take the form

$$\begin{aligned} (K^\varepsilon_{12})^2 - \varepsilon^2 &= 0, \\ K^\varepsilon_{11} - K^\varepsilon_{12} K^\varepsilon_{22} &= 0, \\ 2K^\varepsilon_{12} - (K^\varepsilon_{22})^2 + \beta + \varepsilon^2 &= 0. \end{aligned} \tag{49}$$

The system (49) has a unique positive definite solution

$$K^{\varepsilon} = \begin{pmatrix} \varepsilon\sqrt{\beta + 2\varepsilon + \varepsilon^2} & \varepsilon \\ \varepsilon & \sqrt{\beta + 2\varepsilon + \varepsilon^2} \end{pmatrix}. \tag{50}$$

Then

$$K^{\varepsilon} - K = \varepsilon \begin{pmatrix} \sqrt{\beta + 2\varepsilon + \varepsilon^2} & 1 \\ 1 & \dfrac{\varepsilon + 2}{\sqrt{\beta + 2\varepsilon + \varepsilon^2} + \sqrt{\beta}} \end{pmatrix},$$

and therefore

$$\|K^{\varepsilon} - K\| \le \varepsilon \max\left\{ \sqrt{\beta + 2\hat{\varepsilon} + \hat{\varepsilon}^2},\ 1,\ \frac{\hat{\varepsilon} + 2}{2\sqrt{\beta}} \right\}, \tag{51}$$
$$|J_{\varepsilon}(u_{\varepsilon}) - J(u)| \le C_1 \varepsilon,$$

where $C_1 = \max\left\{ \sqrt{\beta + 2\hat{\varepsilon} + \hat{\varepsilon}^2},\ 1,\ \dfrac{\hat{\varepsilon} + 2}{2\sqrt{\beta}} \right\} \|x^0\|^2$.

The regularized optimal control will be of the form

$$u^{\varepsilon} = -\varepsilon x_1^{\varepsilon}(t) - \sqrt{\beta + 2\varepsilon + \varepsilon^2}\, x_2^{\varepsilon}(t), \tag{52}$$

where $x^{\varepsilon}(t)$ is solution of the closed-loop regularized system

$$\begin{gathered} \dot{x}_1^{\varepsilon}(t) = x_2^{\varepsilon}(t), \\ \dot{x}_2^{\varepsilon}(t) = -\varepsilon x_1^{\varepsilon}(t) - \sqrt{\beta + 2\varepsilon + \varepsilon^2}\, x_2^{\varepsilon}(t), \\ x_1^{\varepsilon}(0) = x_1^0, \quad x_2^{\varepsilon}(0) = x_2^0. \end{gathered} \tag{53}$$

The characteristic equation corresponding to the system (53) is

$$(\lambda^{\varepsilon})^2 + \sqrt{\beta + 2\varepsilon + \varepsilon^2}\lambda^{\varepsilon} + \varepsilon = 0,$$

which has for sufficiently small ε two real negative roots

$$\lambda_{1,2}^{\varepsilon} = -\frac{\sqrt{\beta + 2\varepsilon + \varepsilon^2}}{2} \mp \frac{\sqrt{\beta - 2\varepsilon + \varepsilon^2}}{2}, \quad \lambda_1^{\varepsilon} < \lambda_2^{\varepsilon}.$$

The solution to the Cauchy problem (53) will be

$$\begin{aligned} x_1^{\varepsilon}(t) &= -\frac{x_2^0 - \lambda_2^{\varepsilon} x_1^0}{\lambda_2^{\varepsilon} - \lambda_1^{\varepsilon}} \exp\left(\lambda_1^{\varepsilon} t\right) + \frac{x_2^0 - \lambda_1^{\varepsilon} x_1^0}{\lambda_2^{\varepsilon} - \lambda_1^{\varepsilon}} \exp\left(\lambda_2^{\varepsilon} t\right), \\ x_2^{\varepsilon}(t) &= -\frac{\lambda_1^{\varepsilon}\left(x_2^0 - \lambda_2^{\varepsilon} x_1^0\right)}{\lambda_2^{\varepsilon} - \lambda_{\varepsilon}^1} \exp\left(\lambda_1^{\varepsilon} t\right) + \frac{\lambda_2^{\varepsilon}\left(x_2^0 - \lambda_1^{\varepsilon} x_1^0\right)}{\lambda_2^{\varepsilon} - \lambda_1^{\varepsilon}} \exp\left(\lambda_2^{\varepsilon} t\right). \end{aligned} \tag{54}$$

It is obvious that

$$\lim_{t \to \infty} x_i^{\varepsilon}(t) = 0, \quad i = \overline{1,2}, \quad \varepsilon > 0.$$

In this example, it is possible to investigate the limit transition of the solution (54) to the solution of the critical system (38) for $\varepsilon \to 0$.

First,

$$\lim_{\varepsilon \to 0} \lambda_1^{\varepsilon} = -\sqrt{\beta}, \qquad \lim_{\varepsilon \to 0} \lambda_2^{\varepsilon} = 0.$$

Secondly,

$$\lim_{\varepsilon\to 0} x_2^\varepsilon(t) = -\lim_{\varepsilon\to 0}\frac{\lambda_1^\varepsilon(x_2^0-\lambda_2^\varepsilon x_1^0)}{\lambda_2^\varepsilon-\lambda_1^\varepsilon}\exp\left(\lambda_1^\varepsilon t\right)+\lim_{\varepsilon\to 0}\frac{\lambda_2^\varepsilon(x_2^0-\lambda_1^\varepsilon x_1^0)}{\lambda_2^\varepsilon-\lambda_1^\varepsilon}\exp\left(\lambda_2^\varepsilon t\right)=$$
$$= x_2^0\exp(-\sqrt{\beta}t),$$
$$\lim_{\varepsilon\to 0} x_1^\varepsilon(t) = -\lim_{\varepsilon\to 0}\frac{x_2^0-\lambda_2^\varepsilon x_1^0}{\lambda_2^\varepsilon-\lambda_1^\varepsilon}\exp\left(\lambda_1^\varepsilon t\right)+\lim_{\varepsilon\to 0}\frac{x_2^0-\lambda_1^\varepsilon x_1^0}{\lambda_2^\varepsilon-\lambda_1^\varepsilon}\exp\left(\lambda_2^\varepsilon t\right)=$$
$$= -x_2^0\frac{\exp\left(-\sqrt{\beta}t\right)}{\sqrt{\beta}}+\frac{x_2^0+\sqrt{\beta}x_1^0}{\sqrt{\beta}}=x_1^0+\frac{x_2^0}{\sqrt{\beta}}\Big(1-\exp\left(-\sqrt{\beta}\right)t\Big).$$

From (52) and (33) we will have

$$\begin{aligned} u^\varepsilon[x^\varepsilon(t)]-u[x(t)] &= -\varepsilon x_1^\varepsilon(t)-\sqrt{\beta+2\varepsilon+\varepsilon^2}x_2^\varepsilon(t)+\sqrt{\beta}x_2(t)\\ &= -\varepsilon x_1^\varepsilon(t)-\left(\sqrt{\beta+2\varepsilon+\varepsilon^2}-\sqrt{\beta}\right)x_2^\varepsilon(t)-\sqrt{\beta}y_2^\varepsilon(t)\end{aligned} \tag{55}$$

where deviation $y_i^\varepsilon(t)=x_i^\varepsilon(t)-x_i(t),\ i=\overline{1,2}$ is a solution to the system of differential equations

$$\begin{aligned} \dot y_1^\varepsilon(t) &= y_{2,\varepsilon}(t),\\ \dot y_2^\varepsilon(t) &= -\varepsilon x_1^\varepsilon(t)-\left(\sqrt{\beta+2\varepsilon+\varepsilon^2}-\sqrt{\beta}\right)x_2^\varepsilon(t)-\sqrt{\beta}y_2^\varepsilon(t),\\ y_i^\varepsilon(0) &= 0,\quad i=\overline{1,2}.\end{aligned} \tag{56}$$

From the second equation of the system (56) we can write that

$$y_2^\varepsilon(t)=\int_0^t \exp\left(-\sqrt{\beta}(t-\tau)\right)f^\varepsilon(\tau)d\tau, \tag{57}$$

where

$$f^\varepsilon(t)=-\varepsilon x_1^\varepsilon(t)-\frac{2\varepsilon+\varepsilon^2}{\sqrt{\beta+2\varepsilon+\varepsilon^2}+\sqrt{\beta}}x_2^\varepsilon(t).$$

The inclusion $f^\varepsilon(t)\in L_2(0,\infty)$ is true and the following inequality holds

$$\|f^\varepsilon(\cdot)\|_{L_2(0,\infty)}\le C\varepsilon. \tag{58}$$

Really,

$$\|f^\varepsilon(\cdot)\|_{L_2(0,\infty)}\le\varepsilon\|x_1^\varepsilon(\cdot)\|_{L_2(0,\infty)}+\frac{2\varepsilon+\varepsilon^2}{\sqrt{\beta+2\varepsilon+\varepsilon^2}+\sqrt{\beta}}\|x_2^\varepsilon(\cdot)\|_{L_2(0,\infty)}\le C\varepsilon.$$

Further, using the representation (57) and Cauchy's inequality, we have

$$\left(y_2^\varepsilon(t)\right)^2\le\int_0^t\exp\left(-2\sqrt{\beta}(t-\tau)\right)d\tau\int_0^t\left(f^\varepsilon(\tau)\right)^2d\tau\le$$
$$\le\int_0^t\exp\left(-2\sqrt{\beta}(t-\tau)\right)d\tau\|f^\varepsilon(\cdot)\|^2_{L_2(0,\infty)}.$$

Hence, taking into account formulas (57), we obtain that

$$\|y_2^\varepsilon(\cdot)\|_{L_2(0,\infty)}\le C\varepsilon. \tag{59}$$

So from the representation (55) we finally have the estimate

$$\|u^\varepsilon[x^\varepsilon(t)]-u[x(t)]\|_{L_2(0,\infty)}\le C\varepsilon. \tag{60}$$

As for the deviation $y_1^\varepsilon(t)$, it will not belong to the space $L_2(0,\infty)$,, but the limit equality take place

$$\lim_{t\to\infty} y_1^\varepsilon(t) = x_1^0 + \frac{x_2^0}{\sqrt{\beta}}, \quad \varepsilon > 0. \tag{61}$$

Let $\mathcal{A}^\varepsilon = (J^\varepsilon,\, u^\varepsilon(t),\, x^\varepsilon(t))$ is solution of the regularized (perturbed) problem of optimal stabilization (39) – (42), and $\mathcal{A}^0 = (J,\, u(t),\, x(t))$ is solution of the problem of optimal stabilization in the critical case (degenerate). Then, according to the terminology of the work [17] and the equality (61), the regularized problem will be *singularly perturbed*. Another thing is that here we do not need to look for a uniform in ε approximation $x_1^\varepsilon(t)$ to $x_1(t)$.

5.2. Regularization by Perturbation of the Matrix A

Let us replace the problem (1) – (4) with the following problem: for a sufficiently small real number $\varepsilon \in (0,\, \hat{\varepsilon}]$ find

$$u^\varepsilon = u^\varepsilon[x^\varepsilon(t)], \tag{62}$$

where the function $x^\varepsilon(t)$ describes the controlled process and satisfies the Cauchy problems

$$\dot{x}^\varepsilon(t) = A^\varepsilon x^\varepsilon(t) + Bu^\varepsilon[x^\varepsilon(t)], \quad 0 < t < \infty, \tag{63}$$

$$x^\varepsilon(0) = x^0, \tag{64}$$

and the optimality criterion in this case

$$J^\varepsilon(u^\varepsilon) = \int\limits_0^\infty (x^\varepsilon(t))^T Q x^\varepsilon(t) + (u^\varepsilon(t))^T H u^\varepsilon(t))dt \tag{65}$$

would take the smallest value and the inequalities (43) would hold.

In problem (62) – (65) the matrix A^ε is smooth on the parameter ε, such that $A^0 = A$ and its spectrum is not imaginary. Then in problem (62) – (65) the optimal controller will have the form

$$u^\varepsilon = u^\varepsilon[x^\varepsilon(t)] = -H^{-1}B^T K^\varepsilon x^\varepsilon(t), \tag{66}$$

where the matrix K^ε is the only positive definite solution of the algebraic Riccati equation

$$K^\varepsilon A^\varepsilon + (A^\varepsilon)^T K^\varepsilon - K^\varepsilon B H^{-1} B^T K^\varepsilon + Q = 0. \tag{67}$$

The optimal value of the quality criterion will be appearance

$$J^\varepsilon(u^\varepsilon) = (x^0)^T K^\varepsilon x^0. \tag{68}$$

Let us return to the example of Section 4 and put $A^\varepsilon = A + \varepsilon E_2,\ \varepsilon \in (0, \hat{\varepsilon}]$. Then the system (67) will take the form

$$\begin{aligned} 2\varepsilon K_{11}^\varepsilon - (K_{12}^\varepsilon)^2 = 0, \\ K_{11}^\varepsilon + 2\varepsilon K_{12}^\varepsilon - K_{12}^\varepsilon K_{22}^\varepsilon = 0, \\ 2K_{12}^\varepsilon + 2\varepsilon K_{22}^\varepsilon - (K_{22}^\varepsilon)^2 + \beta = 0. \end{aligned} \tag{69}$$

The system (69) has a unique positive definite solution

$$K^\varepsilon = \begin{pmatrix} 2\varepsilon\left(\varepsilon + \sqrt{\beta+\varepsilon^2}\right)^2 & 2\varepsilon\left(\varepsilon + \sqrt{\beta+\varepsilon^2}\right) \\ 2\varepsilon\left(\varepsilon + \sqrt{\beta+\varepsilon^2}\right) & 3\varepsilon + \sqrt{\beta+\varepsilon^2} \end{pmatrix}. \tag{70}$$

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So,

$$K^{\varepsilon}-K=\varepsilon\begin{pmatrix}2\left(\varepsilon+\sqrt{\beta+\varepsilon^2}\right)^2 & 2\left(\varepsilon+\sqrt{\beta+\varepsilon^2}\right)\\ 2\left(\varepsilon+\sqrt{\beta+\varepsilon^2}\right) & 3+\frac{\varepsilon}{\sqrt{\beta+\varepsilon^2}+\sqrt{\beta}}\end{pmatrix},$$
$$\|K^{\varepsilon}-K\|\leq\varepsilon\max\left\{2\left(\hat{\varepsilon}+\sqrt{\beta+\hat{\varepsilon}^2}\right)^2,\ 2\left(\hat{\varepsilon}+\sqrt{\beta+\hat{\varepsilon}^2}\right),\ 3+\frac{\hat{\varepsilon}}{2\sqrt{\beta}}\right\}\tag{71}$$
$$\left|J^{\varepsilon}(u^{\varepsilon})-J(u)\right|\leq C_1\varepsilon,$$

where

$$C_1\max\left\{2\left(\hat{\varepsilon}+\sqrt{\beta+\hat{\varepsilon}^2}\right)^2,\ 2\left(\hat{\varepsilon}+\sqrt{\beta+\hat{\varepsilon}^2}\right),\ 3+\frac{\hat{\varepsilon}}{2\sqrt{\beta}}\right\}\|x^0\|^2.$$

The regularized optimal control will be

$$u_{\varepsilon}=-2\varepsilon\left(\varepsilon+\sqrt{\beta+\varepsilon^2}\right)x_1^{\varepsilon}(t)-\left(3\varepsilon+\sqrt{\beta+\varepsilon^2}\right)x_2^{\varepsilon}(t),\tag{72}$$

where $x_{\varepsilon}(t)$ is solution of the closed-loop regularized system

$$\begin{gathered}\dot{x}_1^{\varepsilon}(t)=\varepsilon x_1^{\varepsilon}(t)+x_2^{\varepsilon}(t),\\ \dot{x}_2^{\varepsilon}(t)=-2\varepsilon\left(\varepsilon+\sqrt{\beta+\varepsilon^2}\right)x_1^{\varepsilon}(t)-\left(2\varepsilon+\sqrt{\beta+\varepsilon^2}\right)x_2^{\varepsilon}(t),\\ x_1^{\varepsilon}(0)=x_1^0,\quad x_2^{\varepsilon}(0)=x_2^0.\end{gathered}\tag{73}$$

The characteristic equation corresponding to the system (73) is

$$(\lambda^{\varepsilon})^2+\left(\varepsilon+\sqrt{\beta+\varepsilon^2}\right)\lambda^{\varepsilon}+\varepsilon\sqrt{\beta+\varepsilon^2}=0,$$

which has two real negative roots

$$\lambda_1^{\varepsilon}=-\sqrt{\beta+\varepsilon^2},\qquad\lambda_2^{\varepsilon}=-\varepsilon,\qquad\lambda_1^{\varepsilon}<\lambda_2^{\varepsilon}.$$

From (72) and (33) we will have

$$\begin{gathered}u^{\varepsilon}[x^{\varepsilon}(t)]-u[x(t)]=-2\varepsilon\left(\varepsilon+\sqrt{\beta+\varepsilon^2}\right)x_1^{\varepsilon}(t)-\\ -\left(3\varepsilon+\sqrt{\beta+\varepsilon^2}-\sqrt{\beta}\right)x_2^{\varepsilon}(t)-\sqrt{\beta}y_2^{\varepsilon}(t),\end{gathered}\tag{74}$$

where deviations $y_i^{\varepsilon}(t)=x_i^{\varepsilon}(t)-x_i(t),\quad i=\overline{1,2}$ are solutions of the system of differential equations

$$\begin{gathered}\dot{y}_1^{\varepsilon}(t)=\varepsilon x_{1,\varepsilon}(t)+y_2^{\varepsilon}(t),\\ \dot{y}_2^{\varepsilon}(t)=-2\varepsilon\left(\varepsilon+\sqrt{\beta+\varepsilon^2}\right)x_1^{\varepsilon}(t)-\left(2\varepsilon+\sqrt{\beta+\varepsilon^2}-\sqrt{\beta}\right)x_2^{\varepsilon}(t)-\sqrt{\beta}y_2^{\varepsilon}(t),\\ y_i^{\varepsilon}(0)=0,\quad i=\overline{1,2}.\end{gathered}\tag{75}$$

The estimate $\|u^{\varepsilon}[x^{\varepsilon}(\cdot)]-u[x(\cdot)]\|_{L_2(0,\infty)}$ is established here similarly to the corresponding estimate for the case of regularization by perturbation of the matrix Q.

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However, in the case of regularization by perturbation of the matrix A, we are more interested in the estimate $\|u^\varepsilon[x^\varepsilon(\cdot)] - u[x(\cdot)]\|_{L_2(0,\infty)}$, when the vector $x^\varepsilon(t)$ is the solution of the closed-loop system generated by the critical system and the regularized control (72), i.e.

$$\begin{gathered} \dot{x}_1^\varepsilon(t) = x_{2,\varepsilon}(t), \\ \dot{x}_2^\varepsilon(t) = -2\varepsilon\left(\varepsilon + \sqrt{\beta+\varepsilon^2}\right)x_1^\varepsilon(t) - \left(3\varepsilon + \sqrt{\beta+\varepsilon^2}\right)x_2^\varepsilon(t), \\ x_1^\varepsilon(0) = x_1^0, \quad x_2^\varepsilon(0) = x_2^0. \end{gathered} \tag{76}$$

The characteristic equation corresponding to the system (76) is

$$(\lambda^\varepsilon)^2 + \left(3\varepsilon + \sqrt{\beta+\varepsilon^2}\right)\lambda^\varepsilon + 2\varepsilon\left(\varepsilon + \sqrt{\beta+\varepsilon^2}\right) = 0,$$

which has two real negative roots

$$\lambda_1^\varepsilon = -\left(\varepsilon + \sqrt{\beta+\varepsilon^2}\right), \qquad \lambda_2^\varepsilon = -2\varepsilon, \qquad \lambda_1^\varepsilon < \lambda_2^\varepsilon.$$

Further, all calculations that were characteristic of the regularization of the functional are repeated. As a result, we obtain the estimate (60), and from it the estimate

$$|J(u^\varepsilon) - J(u)| \le C\varepsilon.$$

6. Special System of Optimal Stabilization in the Critical Case

Let us consider in the general case the possibilities of solving the optimal stabilization problem in the critical case (see example Section 4).

6.1. Solving the Optimal Stabilization Problem for a Canonical System

Let us first consider the optimal stabilization problem for the canonical system

$$\begin{gathered} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = x_3(t), \\ \dots\dots\dots\dots\dots \\ \dot{x}_{n-1}(t) = x_n(t), \\ \dot{x}_n(t) = u(t), \\ x_i(0) = x_i^0, \quad i = \overline{1,n}, \end{gathered} \tag{77}$$

with the optimality criterion

$$J(u) = \int_0^\infty \left(x^T(t)Qx(t) + u^2(t)\right)dt, \tag{78}$$

where the matrix Q is symmetric, non-negative and has dimension $n \times n$.

In the work [18] the system (77) was the basis for constructing algorithms of constrained nonlinear synthesis using the controllability function. There the complete controllability of the canonical system was also proved. We will assume that the optimal stabilization problem (77) – (78) is critical.

6.1.1. The "Weakest" Optimal Stabilization is One-Coordinate

Let the matrix Q have the following elements:

$$q_{ij} = 0,\ i = \overline{1,n-1},\ j = \overline{1,n}; \qquad q_{nj} = 0,\ j = \overline{1,n-1}; \qquad q_{nn} > 0.$$

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Then for the coordinate $x_n(t)$ we will have a one-dimensional optimal stabilization problem: find the control $u = u[x_n(t)]$, which minimizes the functional

$$J(u) = \int_0^\infty \left(q_{nn} x_n^2(t) + u^2(t) \right) dt \tag{79}$$

on the solutions of the Cauchy problem

$$\dot{x}_n(t) = u(t), \quad x_n(0) = x_n^0. \tag{80}$$

The desired control has the form

$$u = -\sqrt{q_{nn}}\, x_n(t). \tag{81}$$

It corresponds to the solution of the Cauchy problem (80)

$$x_n(t) = x_n^0 \exp\left(-\sqrt{q_{nn}}\, t \right) \tag{82}$$

and the optimal value of the functional

$$J(u) = \sqrt{q_{nn}} \left(x_n^0 \right)^2. \tag{83}$$

In this case, the following inclusions will take place: $x_n(t) \in L_2(0, \infty),\ u(t) \in L_2(0, \infty)$.

According to the canonical system (77), the two previous coordinates will take the form

$$\begin{aligned} x_{n-1}(t) &= x_{n-1}^0 + \frac{x_n^0}{\sqrt{q_{nn}}} \left(1 - \exp\left(-\sqrt{q_{nn}}\, t \right) \right), \\ x_{n-2}(t) &= x_{n-2}^0 + \left(x_{n-1}^0 + \frac{x_n^0}{\sqrt{q_{nn}}} \right) t - \frac{x_n^0}{q_{nn}} \left(1 - \exp\left(-\sqrt{q_{nn}}\, t \right) \right). \end{aligned} \tag{84}$$

Here we have the inclusions $x_{n-1}(t) \in C[0, \infty),\ x_{n-2}(t) \notin C[0, \infty)$. In order for $x_{n-2}(t) \in C[0, \infty)$ we need to put

$$x_{n-1}^0 + \frac{x_n^0}{\sqrt{q_{nn}}} = 0. \tag{85}$$

Then the coordinates (84) will have the form

$$\begin{aligned} x_{n-1}(t) &= -\frac{x_n^0}{\sqrt{q_{nn}}} \exp\left(-\sqrt{q_{nn}}\, t \right), \\ x_{n-2}(t) &= x_{n-2}^0 - \frac{x_n^0}{q_{nn}} \left(1 - \exp\left(-\sqrt{q_{nn}}\, t \right) \right). \end{aligned} \tag{86}$$

Continuing this process, at the k-th step we will have

$$\begin{aligned} x_{n-k}(t) &= (-1)^k \frac{x_n^0}{\left(\sqrt{q_{nn}} \right)^k} \exp\left(-\sqrt{q_{nn}}\, t \right), \\ x_{n-k-1}(t) &= x_{n-k-1}^0 - (-1)^{k+1} \frac{x_n^0}{\left(\sqrt{q_{nn}} \right)^{k+1}} \left(1 - \exp\left(-\sqrt{q_{nn}}\, t \right) \right), \quad k = \overline{2, n-2}, \end{aligned} \tag{87}$$

where

$$x_{n-k}^0 + (-1)^{k+1} \frac{x_n^0}{\left(\sqrt{q_{nn}} \right)^k} = 0. \tag{88}$$

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Thus, the optimal solution of the canonical system has the form

$$x_{n-k}(t) = (-1)^k \frac{x_n^0}{\left(\sqrt{q_{nn}}\right)^k} \exp\left(-\sqrt{q_{nn}}\, t\right), \quad k = \overline{0, n-2}, \tag{89}$$

$$x_{n-j}^0 + (-1)^{j+1} \frac{x_n^0}{\left(\sqrt{q_{nn}}\right)^j} = 0, \quad j = \overline{1, n-2}, \tag{90}$$

$$x_1(t) = x_1^0 - (-1)^{n-1} \frac{x_n^0}{\left(\sqrt{q_{nn}}\right)^{n-1}} \left(1 - \exp\left(-\sqrt{q_{nn}}\, t\right)\right). \tag{91}$$

From (89) follows the inclusions $x_j(t) \in L_2(0,\infty)$, $j = \overline{2,n}$, but not for any x_j^0, $j = \overline{2, n-1}$, but only for those that satisfy the condition (90). The coordinate $x_1(t)$ for $t \to \infty$ can take any values $c_1 \in R^1$ by virtue of the formula

$$x_1^0 - (-1)^{n-1} \frac{x_n^0}{\left(\sqrt{q_{nn}}\right)^{n-1}} = c_1. \tag{92}$$

If $c_1 \neq 0$, then $x_1(t) \in C[0,\infty)$. In the case $c_1 = 0$ it will have the inclusion $x_1(t) \in L_2(0,\infty)$. Thus, in the case $c_1 = 0$ the above solution of the optimal stabilization problem will be called *one-coordinate partial stabilization*. If conditions (90) are violated, the optimal stabilization problem does not even have a one-coordinate partial solution.

6.1.2. Multi-Coordinate Optimal Stabilization

Let the matrix Q have the following elements: $q_{ij} = 0$, $i, j = \overline{1, n-2}$, and the numbers $q_{n-1\,n-1}$, q_{nn}, $q_{n-1\,n} = q_{n\,n-1}$ form a positive definite matrix. Further, to reduce the calculations, we will assume that $q_{n-1\,n} = q_{n\,n-1} = 0$. Then for the coordinates $x_{n-1}(t)$, $x_n(t)$ we will have a two-dimensional optimal stabilization problem: find the control $u = u[x_{n-1}(t), x_n(t)]$, which minimizes the functional

$$J(u) = \int_0^\infty \left(q_{n-1\,n-1} x_{n-1}^2(t) + q_{nn} x_n^2(t) + u^2(t)\right) dt \tag{93}$$

on the solutions of the Cauchy problem

$$\begin{gathered} \dot{x}_{n-1}(t) = x_n(t), \\ \dot{x}_n(t) = u(t), \\ x_{n-1}(0) = x_{n-1}^0, \quad x_n(0) = x_n^0, \end{gathered} \tag{94}$$

where the real numbers x_{n-1}^0, x_n^0 are fixed, but arbitrary.

The system of algebraic Riccati equations will have the form

$$\begin{gathered} \left(K_{n-1\,n}\right)^2 - q_{n-1\,n-1} = 0, \\ K_{n-1\,n-1} - K_{n-1\,n} K_{nn} = 0, \\ 2K_{n-1\,n} - (K_{nn})^2 + q_{nn} = 0. \end{gathered} \tag{95}$$

The system (95) has a unique positive definite solution. It should be noted here that the optimal stabilization problem (92) - (94) has a solution also in the case $q_{nn} = 0$, i.e. when the matrix for the phase coordinates in the criterion (92) is only non-negative. The difference between the case of a positive definite matrix and a non-negative one is only in the magnitude of the stability threshold: in the first case it is higher.

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Let us return to the case when $q_{n-1\,n-1} > 0,\ q_{n,n} > 0$. Then the desired control will have the form

$$u = -K_{n\,n-1}x_{n-1}(t) - K_{nn}x_n(t). \tag{96}$$

It corresponds to the solution of the Cauchy problem (94)

$$\begin{aligned} x_{n-1}(t) = \frac{x_n^0 - \lambda_{n-1\,n}^+ x_{n-1}^0}{\lambda_{n-1\,n}^- - \lambda_{n-1\,n}^+}\exp\left(\lambda_{n-1\,n}^- t\right) - \frac{x_n^0 - \lambda_{n-1\,n}^- x_{n-1}^0}{\lambda_{n-1\,n}^- - \lambda_{n-1\,n}^+}\exp\left(\lambda_{n-1\,n}^+ t\right), \\ x_n(t) = \lambda_{n-1\,n}^- \frac{x_n^0 - \lambda_{n-1\,n}^+ x_{n-1}^0}{\lambda_{n-1\,n}^- - \lambda_{n-1\,n}^+}\exp\left(\lambda_{n-1\,n}^- t\right) - \\ -\lambda_{n-1\,n}^+ \frac{x_n^0 - \lambda_{n-1\,n}^- x_{n-1}^0}{\lambda_{n-1\,n}^- - \lambda_{n-1\,n}^+}\exp\left(\lambda_{n-1\,n}^+ t\right), \end{aligned} \tag{97}$$

where

$$\lambda_{n-1\,n}^{\pm} = -\frac{\sqrt{q_{nn} + 2\sqrt{q_{n-1\,n-1}}}}{2} \pm \frac{\sqrt{q_{nn} - 2\sqrt{q_{n-1\,n-1}}}}{2}.$$

In the formulas (97) a condition is adopted for simplification

$$q_{nn} - 2\sqrt{q_{n-1\,n-1}} > 0. \tag{98}$$

The optimal value of the functional (92) will be

$$\begin{aligned} J(u) = \sqrt{q_{n-1\,n-1}}\sqrt{q_{nn} + 2\sqrt{q_{n-1\,n-1}}}\left(x_{n-1}^0\right)^2 + 2\sqrt{q_{n-1\,n-1}}x_{n-1}^0 x_n^0 + \\ + \sqrt{q_{nn} + 2\sqrt{q_{n-1\,n-1}}}\left(x_n^0\right)^2. \end{aligned} \tag{99}$$

In this case, the following inclusions will take place: $x_{n-1}(t),\ x_n(t) \in L_2(0,\infty),\ u(t) \in L_2(0,\infty)$. According to the canonical system (77), the two previous coordinates will take the form

$$\begin{aligned} x_{n-2}(t) = x_{n-2}^0 + \frac{x_n^0 - \lambda_{n-1\,n}^+ x_{n-1}^0}{\lambda_{n-1\,n}^-\left(\lambda_{n-1\,n}^- - \lambda_{n-1\,n}^+\right)}\Big(\exp\left(\lambda_{n-1\,n}^- t\right) - 1\Big) - \\ - \frac{x_n^0 - \lambda_{n-1\,n}^- x_{n-1}^0}{\lambda_{n-1\,n}^+\left(\lambda_{n-1\,n}^- - \lambda_{n-1\,n}^+\right)}\Big(\exp\left(\lambda_{n-1\,n}^+ t\right) - 1\Big), \\ x_{n-3}(t) = x_{n-3}^0 + \left(x_{n-2}^0 - \frac{x_n^0 - \lambda_{n-1\,n}^+ x_{n-1}^0}{\lambda_{n-1\,n}^-\left(\lambda_{n-1\,n}^- - \lambda_{n-1\,n}^+\right)} + \frac{x_n^0 - \lambda_{n-1\,n}^- x_{n-1}^0}{\lambda_{n-1\,n}^+\left(\lambda_{n-1\,n}^- - \lambda_{n-1\,n}^+\right)}\right)t + \\ + \frac{x_n^0 - \lambda_{n-1\,n}^+ x_{n-1}^0}{\left(\lambda_{n-1\,n}^-\right)^2\left(\lambda_{n-1\,n}^- - \lambda_{n-1\,n}^+\right)}\Big(\exp\left(\lambda_{n-1\,n}^-\right) - 1\Big) - \\ - \frac{x_n^0 - \lambda_{n-1\,n}^- x_{n-1}^0}{\left(\lambda_{n-1\,n}^+\right)^2\left(\lambda_{n-1\,n}^- - \lambda_{n-1\,n}^+\right)}\Big(\exp\left(\lambda_{n-1\,n}^+ t\right) - 1\Big). \end{aligned}$$

In order to $x_{n-2}(t) \in L_2(0,\infty),\ x_{n-3}(t) \in C[0,\infty)$ the initial condition for the coordinate $x_{n-2}(t)$ must be determined from the equation

$$x_{n-2}^0 - \frac{x_n^0 - \lambda_{n-1\,n}^+ x_{n-1}^0}{\lambda_{n-1\,n}^-\left(\lambda_{n-1\,n}^- - \lambda_{n-1\,n}^+\right)} + \frac{x_n^0 - \lambda_{n-1\,n}^- x_{n-1}^0}{\lambda_{n-1\,n}^+\left(\lambda_{n-1\,n}^- - \lambda_{n-1\,n}^+\right)} = 0. \tag{100}$$

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Thus, the optimal solution of the canonical system (in our case *two-coordinate partial stabilization*) has the form

$$x_{n-k}(t) = \frac{x_n^0 - \lambda_{n-1\,n}^+ x_{n-1}^0}{\left(\lambda_{n-1\,n}^-\right)^{k-1}\left(\lambda_{n-1\,n}^- - \lambda_{n-1\,n}^+\right)} \exp\left(\lambda_{n-1\,n}^- t\right) - \\ - \frac{x_n^0 - \lambda_{n-1\,n}^- x_{n-1}^0}{\left(\lambda_{n-1\,n}^+\right)^{k-1}\left(\lambda_{n-1\,n}^- - \lambda_{n-1\,n}^+\right)} \exp\left(\lambda_{n-1\,n}^+ t\right), \qquad k = \overline{0, n-1}. \tag{101}$$

In this case, for numbers $2 \le j \le n-1$ the following equalities must hold

$$x_{n-j}^0 - \frac{x_n^0 - \lambda_{n-1\,n}^+ x_{n-1}^0}{\left(\lambda_{n-1\,n}^-\right)^{j-1}\left(\lambda_{n-1\,n}^- - \lambda_{n-1\,n}^+\right)} + \frac{x_n^0 - \lambda_{n-1\,n}^- x_{n-1}^0}{\left(\lambda_{n-1\,n}^+\right)^{j-1}\left(\lambda_{n-1\,n}^- - \lambda_{n-1\,n}^+\right)} = 0. \tag{102}$$

From (101) follows the inclusions $x_j(t) \in L_2(0, \infty),\ j = \overline{2, n}$, but not for any $x_j^0,\ j = \overline{2, n-1}$, only for those that satisfy condition (102). If conditions (102) are violated, the optimal stabilization problem does not even have a two-coordinate partial solution.

From the above considerations it follows that to formulate the problem $(i+1)$-coordinate partial optimal stabilization $(0 \le i \le n-2)$ it is necessary:

1. to formulate the optimal stabilization problem with the quality criterion

$$J_{n-i\,n}(u) = \int_0^\infty \left(\sum_{l=n-i}^{n} \sum_{k=0}^{i} q_{n-k\,l} x_{n-k} x_l + u^2(t) \right) dt, \tag{103}$$

on the solutions of the Cauchy problem

$$\begin{aligned} \dot{x}_{n-i}(t) &= x_{n-i+1}(t), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \dot{x}_{n-1}(t) &= x_n(t), \\ \dot{x}_n(t) &= u(t), \\ x_{n-i}(0) &= x_{n-i}^0. \end{aligned} \tag{104}$$

The numbers $q_{kj},\ k, j = \overline{n-i\,n}$ in the criterion (103) form a positive definite symmetric matrix – this is the "strongest" $(i+1)$-coordinate partial stabilization. If in the criterion (103) we limit ourselves only to the positive number $q_{n-i\,n-i},\ i \ge 2$, considering all other elements as zero, then we will obtain the variant of the "weakest" $(i+1)$-coordinate partial stabilization; we will denote the corresponding matrix by $Q^{n-i\,n}$;

2. find the unique positive definite solution of the matrix algebraic Riccati equation

$$K^{n-i\,n} A^{n-i\,n} + \left(A^{n-i\,n}\right)^T K^{n-i\,n} - K^{n-i\,n} b^{n-i\,n} \left(b^{n-i\,n}\right)^T K^{n-i\,n} + Q^{n-i\,n} = 0, \tag{105}$$

where $A^{n-i\,n}$ is numerical square matrix of dimension $(i+1) \times (i+1)$, in which all elements are 0, and there are 1 above the main diagonal; $b^{n-i\,n} \in R^{i+1}$ and $\left(b^{n-i\,n}\right)^T = (0, \ldots, 0, 1)$, i.e. from number $n-i$ to number $n-1$ there are 0, and in the last position there is 1;

3. the optimal stabilizing control has the form

$$u = -\sum_{j=n-i}^{n} K_{nj}^{n-i\,n} x_j(t), \tag{106}$$

where $x_j(t),\ j = \overline{n-i, n},$ are solutions of the closed-loop system

$$\begin{gathered} \dot{x}_{n-i}(t) = x_{n-i+1}(t), \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \dot{x}_{n-1}(t) = x_n(t), \\ \dot{x}_n(t) = -\sum_{j=n-i}^{n} K_{nj}^{n-i\,n} x_j(t), \\ x_j(0) = x_j^0, \quad j = \overline{n-i, n}; \end{gathered} \tag{107}$$

4. starting from the number $j = n - i + 1$ we integrate sequentially the system of differential equations

$$\begin{gathered} \dot{x}_{n-i+1}(t) = x_{n-i}(t), \\ x_{n-i+1}(0) = x_{n-i+1}^0, \qquad i \le n-2; \end{gathered} \tag{108}$$

5. the numbers x_{n-i+1}^0 from the previous point are chosen from the condition

$$x_{n-i+1}(t) \in L_2(0, \infty), \quad i \le n-2. \tag{109}$$

6.2. Solving the Critical Optimal Stabilization Problem for a Linear System with One-Dimensional Control

Let us consider the optimal stabilization problem in the critical case, i.e. let the phase vector of the controlled system $x(t) \in R^n,\ t > 0$, satisfy the Cauchy problems

$$\begin{gathered} \dot{x}(t) = Ax(t) + Bu(t), \\ x(0) = x^0, \end{gathered} \tag{110}$$

where $B^T = (b_1, \ldots, b_n)$.

The quality criterion has the form (78). We assume that the system (110) is completely controllable, i.e. the columns of the matrix

$$\mathcal{K} = [B, AB, \ldots, A^{n-1}B] \tag{111}$$

are linearly independent. Next, we use the algorithm from [18] to reduce the Cauchy problem (110) to the canonical form in new variables. In our case, the optimality criterion (78) is also subject to transformation. For this purpose, we choose a vector c, which is orthogonal to the vectors $B, AB, \ldots, A^{n-2}B$ and satisfies the condition $(c, A^{n-1}B) = 1$, i.e. we will find the vector c as a solution to the system of equations

$$\mathcal{K}^T c = e_n, \tag{112}$$

where $e_n^T = (0, \ldots, 0, 1)$.

The system (112) has a unique solution, since its determinant is not zero. Let us introduce a new variable

$$z = Lx, \tag{113}$$

where the matrix L has the form

$$L = \begin{pmatrix} c^T \\ c^T A \\ \vdots \\ c^T A^{n-1} \end{pmatrix}. \tag{114}$$

Let us show that the matrix L is nondegenerate. Indeed, let

$$\sum_{i=0}^{n-1} \alpha_i c^T A^i = 0.$$

Multiplying this equality successively by the vectors $B, AB, \ldots, A^{n-1}B$ and taking into account the system (112), we will have
$$\alpha_{n-1} = \alpha_{n-2} = \ldots = \alpha_0 = 0,$$
i.e. the matrix L is nondegenerate.

Let's take the k-th component of the vector z and find its derivative by the system (110)

$$\dot{z}_k = c^T A^k x + c^T A^{k-1} B u = c^T A^k x = z_{k+1}, \quad k = \overline{1, n-1},$$

$$\dot{z}_n = c^T A^n x + c^T A^{n-1} B u = \sum_{i=1}^{n} p_i c' A^{i-1} x + u.$$

Here we use the equality $A^n = \sum\limits_{i=1}^{n} p_i A^{i-1}$, where p_i are coefficients of the matrix A characteristic polynomial $\lambda^n - p_n \lambda^{n-1} - \ldots - p_1 = 0$.

Thus, the system (110) by virtue of the substitution (**??**) will take the form

$$\begin{gathered} \dot{z}_1 = z_2, \\ \ldots\ldots\ldots\ldots\ldots \\ \dot{z}_n = p_1 z_1 + \ldots + p_n z_n + u = z^T \mathcal{P} + u, \end{gathered} \tag{115}$$

where $\mathcal{P} = (p_1, \ldots, p_n)^T$.

Let us introduce a new control according to the formula

$$v = z^T \mathcal{P} + u. \tag{116}$$

With the new variables (z, v) the optimality criterion (78) will take the form

$$\mathcal{J}(v) = \int_0^\infty \Big(z^T(t) \mathcal{Q} z(t) - 2 z^T(t) \mathcal{P} v(t) + v^2(t) \Big) dt, \tag{117}$$

where
$$\mathcal{Q} = (L^{-1})^T Q L^{-1} + \mathcal{P}(\mathcal{P})^T,$$
and the vector $z(t)$ satisfies the canonical system

$$\begin{gathered} \dot{z}_1 = z_2, \\ \ldots\ldots\ldots\ldots\ldots \\ \dot{z}_{n-1} = z_n, \\ \dot{z}_n = v \end{gathered} \tag{118}$$

with initial conditions
$$z(0) = z^0 = L x^0. \tag{119}$$

Due to the non-degeneracy of the matrices $\mathcal{K}$ and L, the criticality property of the original optimal stabilization problem is invariant with respect to the change of coordinates, i.e. we will fall into one of the situations of the previous subsection, but now the quality criterion will contain a quadratic form with respect to the phase vector and control. Then, in the new variables (z, v) in the corresponding constructions, it is necessary to take into account that the solution of the problem of optimal stabilization (118), (117) will be given by the linear form

$$v = h^T z(t), \quad h = -(KB + \mathcal{P}), \tag{120}$$

where the positive definite matrix K is determined from the Riccati equation

$$KA + A^T K + \mathcal{Q} = KBB^T K + KB\mathcal{P}^T + \mathcal{P}B^T K + \mathcal{P}\mathcal{P}^T, \tag{121}$$

$$B^T = (0, \dots, 0, 1), \quad A \text{ is the matrix of the canonical system.}$$

The optimal value of the quality criterion is given by the quadratic form

$$J(u) = (z^0)^T K z^0. \tag{122}$$

6.3. Solving the Critical Optimal Stabilization Problem for a Linear System with Multidimensional Control

Let us consider the problem of optimal stabilization in the critical case, i.e. let the phase vector of the controlled system $x(t) \in R^n$, $t > 0$, satisfy the Cauchy problems

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ x(0) &= x^0, \end{aligned} \tag{123}$$

where $B = (b_{i,j})_{i,j=1}^{n,r}$. The quality criterion has the form

$$J(u) = \int_0^\infty \Big(x^T(t) Q x(t) + u^T(t) H u(t) \Big) dt, \tag{124}$$

where the matrix Q is symmetric, non-negative and has dimension $n \times n$, the matrix H is symmetric, positive definite and has dimension $r \times r$.

We assume that the system (123) is completely controllable, i.e.

$$rang\, \mathcal{K} = rang\, [B, AB, \dots, A^{n-1}B] = n. \tag{125}$$

Next, we use the algorithm from [18] and reduce of the Cauchy problem by means of a variables change and the introduction of a new control to r systems of canonical form with one-dimensional controls. In our case, the optimality criterion (124) is also subject to transformation.

Next, we assume, without loss of generality, that the rank of the matrix B is r. Consider the vectors

$$b_1,\ b_2,\ \dots,\ b_r,\ Ab_1,\ \dots,\ Ab_r,\ \dots,\ A^{n-1}b_1,\ \dots,\ A^{n-1}b_r, \tag{126}$$

where b_i is i-th column of the matrix B.

Let us form a new sequence of vectors, in which the first r vectors will be the vectors $b_1, b_2, \dots, b_r$. Next, taking vectors from the set (126) sequentially, we check, starting from the vector Ab_1, whether it will be linearly independent on the vectors $b_1, b_2, \dots, b_r$. If the vector Ab_1 is linearly independent of the vectors $b_1, b_2, \dots, b_r$, then we add it to them and obtain a new sequence of linearly independent vectors $b_1, b_2, \dots, b_r, Ab_1$. If the vector Ab_1 turns out to be linearly dependent on the vectors $b_1, b_2, \dots, b_r$, then we remove all vectors of the form $A^j b_1$, $j \geq 1$, from the sequence (126) and do not consider them further. In the general case, if we have considered the first k vectors from the set (126) and constructed a new sequence of s vectors from them, then in the next step we consider the vector following k-th vector from the set (126), which was not excluded from consideration in the previous step, and check whether it was linearly independent of s vectors of the constructed sequence. If it is linearly independent, then we add it as $(s+1)$-th vector to the new sequence. If it is linearly dependent, then this vector ω and all vectors of the form $A^m \omega$, $m \geq 1$, are removed from the set (126) during further consideration. As a result, by permuting, and possibly renumbering, the vectors (126), we obtain the sequence of vectors

$$b_1, \ldots, A^{n_1-1}b_1, \ldots, b_r, \ldots, A^{n_r-1}b_r, \qquad \sum_{i=1}^{r} n_i = n, \tag{127}$$

where $n_1 \geq n_2 \geq \ldots \geq n_r$.

Let us choose the vector c_k, $k = \overline{1,r}$, so that it satisfies the equality

$$\left(c_k, A^{n_k-1}b_k\right) = 1 \tag{128}$$

and is orthogonal to all vectors of the sequence (127), i.e.

$$\left(c_k, A^i b_j\right) = \begin{cases} 1, & i = n_k - 1, \quad j = k, \\ 0, & i \neq n_k - 1, \quad j \neq k. \end{cases} \tag{129}$$

Let

$$L = \begin{pmatrix} L_1 \\ \vdots \\ L_r \end{pmatrix}, \tag{130}$$

where

$$L_k = \begin{pmatrix} c_k^T \\ \vdots \\ c_k^T A^{n_k-1} \end{pmatrix}, \quad k = \overline{1,r}.$$

Let us perform the coordinate transformation

$$z = Lx. \tag{131}$$

Using the same considerations as in the previous subsection, we verify that the matrix L is not degenerate. As a result of the replacement (131), we obtain the canonical form of the system (123)

$$\begin{aligned} \dot{z}_{s_{i-1}+j} &= z_{s_{i-1}+j+1}, \quad j = \overline{1, n_i - 1}, \\ \dot{z}_{s_i} &= \sum_{j=1}^{n} \tilde{a}_{s_i j} z_j + u_i + \sum_{j=i+1}^{r} m_{ij} u_j, \end{aligned} \tag{132}$$

where $s_0 = 0$, $s_i = \sum_{k=1}^{i} n_k$, $m_{ij} = c_i^T A^{n_i-1} b_j$, $j = \overline{i+1, r}$, $i = \overline{1,r}$.

In vector form, the system (132) will take the form

$$\dot{z} = \tilde{A}z + B_0 M u, \tag{133}$$

where $\tilde{A} = LAL^{-1}$ is matrix of dimension $n \times n$, in which the row with the number s_i has the form $\left(\tilde{a}_{s_i 1}, \ldots, \tilde{a}_{s_i n}\right)$, $i = \overline{1,r}$, the elements of the matrix

$$\tilde{a}_{(s_{i-1}+j)\,(s_{i-1}+j+1)} = 1, \quad j = \overline{1, n_i - 1}, \quad i = \overline{1,r},$$

and all other elements matrices are zero; B_0 is matrix of dimension $n \times r$, in which i-th element s_i-th row is equal to one, $i = \overline{1,r}$, and all other elements of the matrix are zero; M is upper triangular matrix of dimension $r \times r$ of the form

$$M = \begin{pmatrix} 1 & c_1^T A^{n_1-1} b_2 & c_1^T A^{n_1-1} b_3 & \dots & c_1^T A^{n_1-1} b_r \\ 0 & 1 & c_2^T A^{n_2-1} b_3 & \dots & c_2^T A^{n_2-1} b_r \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Let us introduce a new control

$$v = B_0^T \tilde{A} z + M u. \tag{134}$$

Then the system (132) will take the form

$$\begin{aligned} \dot{z}_{s_{i-1}+j} &= z_{s_{i-1}+j+1}, \quad j = \overline{1, n_i - 1}, \\ \dot{z}_{s_i} &= v_i, \quad i = \overline{1,r}, \end{aligned} \tag{135}$$

or in vector form

$$\dot{z} = A_0 z + B_0 v, \tag{136}$$

where the matrix A_0 of dimension $n \times n$ has the form $A_0 = diag\Big(A_{01}, \dots, A_{0r}\Big)$, A_{0i} is matrix of dimension $n_i \times n_i$, the elements of the first atop-diagonal of which are equal to one and all other elements are equal to zero.

From (131) and (134) we obtain

$$x = L^{-1} z, \quad u = M^{-1}\Big(v - B_0^T \tilde{A} z\Big). \tag{137}$$

Then the optimality criterion (124) will take the form

$$J(u) = \int_0^\infty \Big(z^T(t)\mathcal{Q}z(t) - 2z^T(t)\mathcal{G}v(t) + v^T(t)\mathcal{H}v(t)\Big)dt, \tag{138}$$

where

$$\mathcal{Q} = (L^{-1})^T Q L^{-1} + \tilde{A}^T B_0 (M^{-1})^T H M^{-1} B_0^T \tilde{A},$$

$$\mathcal{G} = \tilde{A}^T B_0 (M^{-1})^T H M^{-1}, \qquad \mathcal{H} = (M^{-1})^T H\, M^{-1}.$$

Due to the non-degeneracy of the above transformations, the original optimal stabilization problem in the critical case decomposes into r one-dimensional optimal stabilization problems for the systems

$$\begin{aligned} \dot{z}_{s_{i-1}+j} &= z_{s_{i-1}+j+1}, \quad j = \overline{1, n_i - 1}, \\ \dot{z}_{s_i} &= v_i, \end{aligned} \tag{139}$$

with quality criteria

$$\begin{aligned} J_{s_i}(u_i) &= \int_0^\infty \Big(z_{s_i}^T(t)\mathcal{Q}_{s_i} z_{s_i}(t) - 2z_{s_i}^T(t)\mathcal{G}_{s_i} v_{s_i}(t) + v_{s_i}^T(t)\mathcal{H}_{s_i} v_{s_i}(t)\Big)dt, \quad i = \overline{1,r}, \\ J(u) &= \sum_{i=1}^r J_{s_i}(u_i), \end{aligned} \tag{140}$$

where the content of all vectors and matrices in the representation (140) is determined after performing the above transformations.

To clarify the subtleties of the "work" of the given algorithm, let us consider an example.

Let the dynamics of the controlled system satisfy the system [18]

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2 + x_3 + u_1 + u_2, \\ \dot{x}_2 &= x_1 - x_2 + x_3, \\ \dot{x}_3 &= -x_1 + x_2 - x_3 - u_2, \end{aligned} \tag{141}$$

with initial conditions

$$x_i(0) = x_i^0, \quad i = \overline{1,3}. \tag{142}$$

Here the matrices A and B have the form

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & -1 \end{pmatrix}.$$

In this case, the set of vectors (127) has the form

$$b_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad Ab_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad n_1 = 2,\ n_2 = 1,\ rang\,\mathcal{K} = 3.$$

On the solutions of the system (141) we consider the quality criterion

$$J(u) = \int_0^\infty \left(x_2^2(t) + x_3^2(t) + u_1^2(t) + u_2^2(t) \right) dt. \tag{143}$$

Let us choose the vectors $c_1^T = (c_1^1, c_2^1, c_3^1)$ and $c_2^T = (c_1^2, c_2^2, c_3^2)$ from the conditions

$$\begin{cases} (c_1, b_1) = 0, \\ (c_1, Ab_1) = 1, \\ (c_1, b_2) = 0, \end{cases} \qquad \begin{cases} (c_2, b_1) = 0, \\ (c_2, Ab_1) = 0, \\ (c_2, b_2) = 1. \end{cases}$$

The written systems have solutions $c_1^T = (0, 1, 0)$, $c_2^T = (0, -1, -1)$. Then the change of variables has the form

$$z = Lx = \begin{pmatrix} c_1^T \\ c_1^T A \\ c_2^T \end{pmatrix} x = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{144}$$

Furthermore,

$$L^{-1} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix}.$$

Then the system (141) in the new variables takes the form (133), i.e.

$$\dot{z} = \tilde{A}z + B_0 M u,$$

where

$$\tilde{A} = LAL^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$B_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & c_1^T A b_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

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Thus, in the new variables we obtain the system

$$\begin{cases} \dot{z}_1 = z_2, \\ \dot{z}_2 = 2z_1 - z_2 + u_1, \\ \dot{z}_3 = u_2 \end{cases} \tag{145}$$

By introducing the substitution

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2z_1 - z_2 + u_1, \\ u_2 \end{pmatrix}, \tag{146}$$

we obtain that the system (141) together with the initial conditions decomposes into two controlled systems

$$\begin{cases} \dot{z}_1 = z_2, \\ \dot{z}_2 = v_1, \end{cases} \\ z_1(0) = z_1^0, \quad z_2(0) = z_2^0, \tag{147}$$

$$\dot{z}_3 = v_2, \quad z_3(0) = z_3^0. \tag{148}$$

Criterion (143) will appear

$$J(u) = \int_0^\infty \Big(x_2^2(t) + x_3^2(t) + u_1^2(t) + u_2^2(t) \Big) dt =$$

$$= \int_0^\infty \Big(z_2^2(t) + (z_1(t) + z_3(t))^2 + (v_1(t) - 2z_1(t) + z_2(t))^2 + v_2^2(t) \Big) dt = J_{s_1}(v_1) + J_{s_2}(v_2).$$

As follows the initial non-critical optimal stabilization problem is decomposed into two:
1) minimize the criterion

$$J_{s_2}(v_2) = \int_0^\infty \Big(z_3^2(t) + v_2^2(t) \Big) dt,$$

at the solutions of the system

$$\dot{z}_3 = v_2, \quad z_3(0) = z_3^0;$$

2) minimize the criterion

$$J_{s_1}(v_1) = \int_0^\infty \Big(z_2^2(t) + z_1^2(t) + 2z_1(t)z_3(t) + (v_1(t) - 2z_1(t) + z_2(t))^2 \Big) dt$$

at the solutions of the system

$$\begin{cases} \dot{z}_1 = z_2, \\ \dot{z}_2 = v_1, \end{cases} \\ z_1(0) = z_1^0, \quad z_2(0) = z_2^0.$$

7. Discussion and Conclusions

The paper considers the critical case of optimal stabilization for a linear-quadratic problem, when the system matrix has a number of imaginary eigenvalues and the matrices in the quality criterion are not necessarily strictly positive definite. For this case, necessary and sufficient conditions for the existence of a solution are given, which consist in the fact that the weight matrix of the functional must "penalize" precisely those states that correspond to the imaginary eigenvalues of the matrix. The paper also demonstrates two regularization methods: through perturbation of the functional matrix $Q_\varepsilon = Q + \varepsilon^2 E_n$ and through perturbation of the system matrix $A_\varepsilon = A + \varepsilon E_n$. Both approaches allow obtaining a stabilizing control that, as $\epsilon \to 0$, asymptotically approaches the solution of the critical problem with an accuracy of the order of $O(\epsilon)$. The obtained estimates provide a clear tool for estimating the error when tuning controllers in practice. A constructive algorithm for reducing multidimensional critical problems to a set of one-dimensional canonical systems is also proposed.

These materials are the theoretical foundation for the development of reliable control systems in robotics, aviation and other industries, where the physical parameters of the object can change under the influence of external factors, introducing the system into critical operating modes.

Further research can be aimed at extending the obtained results to stochastic systems and systems with impulse effects. And the developed algorithms for reduction to the canonical form can become a tool for the design of complex controlled systems in specialized application software packages.

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