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Scattering in the Energy Space for Solutions of the Damped Nonlinear Schrödinger Equation on $\mathbb{R}^d \times \mathbb{T}$

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Abstract: We will show, in any space dimension $d \geq 1$, the decay and scattering in the energy space for the solution to the damped nonlinear Schrödinger equation posed on $\mathbb{R}^d \times \mathbb{T}$ and initial data in $H^1(\mathbb{R}^d \times \mathbb{T})$. We will derive also new bilinear Morawetz identities and corresponding localized Morawetz estimates.

Keywords: nonlinear Schrödinger equations; Schrödinger operators; scattering theory; local nonlinearity; damping

MSC: 35J10; 35Q55; 35B40; 35P25

1. Introduction

We will consider the following Cauchy problem for the nonlinear defocusing damped Schrödinger equation posed on $\mathbb{R}^d \times \mathbb{T}$, for any space dimensions $d \geq 1$:

$$\begin{cases} i\partial_t u + \Delta_{x,y} u + ib(t)u - \lambda u|u|^\beta = 0, & (t,x,y) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{T}, \\ u(0,x,y) = f(x,y) \in H^1(\mathbb{R}^d \times \mathbb{T}), \end{cases} \tag{1}$$

where

$$\Delta_{x,y} = \sum_{i=1}^d \Delta_x + \partial_y^2,$$

with $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ endowed with the flat metric and $\lambda > 0$. Here $u = u(t,x,y) : [0,\infty) \times \mathbb{R}^d \times \mathbb{T} \to \mathbb{C}$, $b : [0,\infty) \to \mathbb{C}$ is a measurable function that contains dissipative and oscillatory terms and $0 < \beta < 4/(d-1)$. We shall assume also $\Re b(t), \Im b(t) \in \mathcal{C}([0,\infty))$ with $\Re b(t) \geq 0$,

$$\mathbf{B}(t) = \int_0^t b(s)ds, \quad \inf_{t>0}\left(\frac{\Re\mathbf{B}(t)}{t}\right) = \mathbf{b} \geq 0, \tag{2}$$

$$\|e^{-\Re\mathbf{B}(t)}\|_{L^\infty([0,\infty))} < \infty. \tag{3}$$

The first two conditions above ensure, in the strict inequality regime, that every global solution of (1) behaves like the solution to the associated free equation, that is $b(t) = \lambda = 0$ if $t \to +\infty$. The main target of this paper is show the decay and scattering of the solutions to (1) in the energy space. More explicitly we will prove the following result:

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Theorem 1.1. *Let $d \geq 1$, $\lambda > 0$ and let $u \in \mathcal{C}([0,\infty); H^1(\mathbb{R}^d))$ be a global solution to (1) with initial data $f \in H^1(\mathbb{R}^d \times \mathbb{T})$ such that (2) and (3) are satisfied. Then, for $2 < r < \frac{2d+2}{d-1}$ ($2 < r < \infty$, if $d = 1$), one achieves*

$$\lim_{t\to\infty} \|e^{\mathbf{B}(t)} u(t,x,y)\|_{L^r(\mathbb{R}^d \times \mathbb{T})} = 0. \tag{4}$$

In addition, the solution to (1) scatters, that is, there exist $f_\pm \in H^1(\mathbb{R}^d \times \mathbb{T})$ such that

$$\lim_{t\to\pm\infty} \|e^{\mathbf{B}(t)} u(t,x,y) - e^{it\Delta_{x,y}} f_\pm\|_{H^1(\mathbb{R}^d \times \mathbb{T})} = 0, \tag{5}$$

if

(a) $\frac{4}{d} < \beta < \frac{4}{d-1}$;

(b) $0 < \beta \leq 4/d$ *under the condition that (2) is fulfilled with strict inequality.*

The nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u - |u|^{\tilde{\beta}} u = 0 \\ u(0) = f(x) \in H^1(\mathbb{R}^d), \end{cases} \tag{6}$$

represents a cornerstone in the mathematical physics, with deep implications across quantum mechanics, nonlinear optics, plasma physics, and fluid mechanics. In the quantum mechanics, for example, this equation provides crucial insights into Bose-Einstein condensates by modeling the self-interactions of charged particles. The scattering properties of solutions to (6) represent a classical problem in mathematical physics, thoroughly documented in [1] and references therein. Understanding the asymptotic behavior of these solutions fundamentally depends on analytical tools such as the Morawetz multiplier technique and the resulting estimates. Morawetz estimates were first established in [2] for the Klein-Gordon equation with general nonlinearity. Their significance in scattering theory became evident when they were subsequently employed to prove asymptotic completeness in noteworthy studies: first by [3] for the cubic NLS in $\mathbb{R}^3$ (that is, (6) with $\tilde{\beta} = 2$), and later by [4] for the Schrödinger equation in $\mathbb{R}^d$ with pure power nonlinearity as in (6), when $4/d < \tilde{\beta} < 4/(d-2)$. A significant methodological advancement has recently simplified scattering proofs through the development of bilinear Morawetz inequalities, also termed interaction or quadratic Morawetz inequalities. Key contributions to this approach include the studies [5] and [6] examining cubic and quintic defocusing NLS in $\mathbb{R}^3$; the analysis in [7] demonstrating interaction Morawetz properties and asymptotic completeness for cubic defocusing nonlinear Schrödinger equation in $\mathbb{R}^2$; the work [8] presenting interaction Morawetz estimates without bilaplacian involvement for L^2-supercritical H^1-subcritical nonlinear Schrödinger equation in $\mathbb{R}^d$ with $d \geq 1$, including applications to the nonlinear problem in $3d$ exterior domains; the comprehensive survey [9] establishing quadratic Morawetz estimates and scattering for the nonlinear Schrödinger equation in $L^2 - H^1$-intercritical case. In our previous work [10] (see also [11–13]), we developed a method combining Morawetz inequalities with localization steps and interpolation within a contradiction framework to demonstrate solution decay in energy space, applicable to equation (6). Motivated by this we present a generalization of this technique to the damped nonlinear Schrödinger equation defined on the wave-guide spaces $\mathbb{R}^d \times \mathbb{T}$, for any space dimensions $d \geq 1$. The nonlinear Schrödinger equation with linear damping is essential across various scientific fields, such as nonlinear optics, plasma physics, and fluid dynamics. It provides key insights into complex phenomena including optical pulse propagation in nonlinear materials, wave dynamics in plasmas subject to magnetic fields, and specific fluid flow behaviors. For further details and examples, we refer to the studies presented in [14] and [15]. Our novel contribution simplifies and extends the approaches used in [16–19]. We also point out that the methods utilized in the aforementioned papers, are circumvented here, due to the necessity of dealing directly with the complex-valued function $b(t)$ appearing in equation (1). To be more precise, here we establish new Morawetz-type identities and their interaction variants, along with corresponding inequalities, applicable to equation (1). We focus on localizing the

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nonlinear terms appearing in these Morawetz inequalities onto suitable space-time regions, where the spatial sets are specifically selected as cubes in $\mathbb{R}^d$. Finally, by employing an argument by contradiction, we deduce the decay behavior of the $L^p(\mathbb{R}^d \times \mathbb{T})$-norms of solutions to (1) when $t \to \infty$, provided that the exponent p satisfies $2 < p < 2(d+1)(d-1)$ and, as straightforward effect, the scattering in $H^1(\mathbb{R}^d \times \mathbb{T})$.

2. Preliminaries

Before outlining our main achievements, we will unveil some necessary notations and several useful results. For any two positive real numbers a, b, we write $a \lesssim b$ (resp. $a \gtrsim b$) to denote $a \leq Cb$ (resp. $Ca \geq b$), with $C > 0$, we unravel the constant only when it is necessary. We introduce the Banach spaces $L^r(\mathbb{R}^d) = L^r_{x,y}$ and $L^q_t L^r_{x,y}$, for $1 \leq r < \infty$, endowed with the norms

$$\|f\|^r_{L^r_{x,y}} = \int_{\mathbb{R}^d \times \mathbb{T}} |f(x,y)|^r \, dxdy < \infty,$$

and

$$\|f\|_{L^q_t L^r_{x,y}} = \left(\int_{\mathbb{R}} \|f(x)\|^q_{L^r_{x,y}} dt \right)^{1/q},$$

respectively, with obvious modification for $r = \infty$. We define also

$$H^{1,r}(\mathbb{R}^d \times \mathbb{T}) = H^{1,r}_{x,y} = (1 - \Delta_x - \partial_y)^{-\frac{1}{2}} L^r(\mathbb{R}^d \times \mathbb{T})$$

and

$$H^s_x H^\sigma_y = (1 - \Delta_x)^{-\frac{s}{2}} (1 - \partial^2_y)^{-\frac{\sigma}{2}} L^2_{x,y}.$$

We adopt the notation $L^\infty_{(t_1,t_2)} X$ when one restricts t in some interval having endpoints $0 \leq t_1, t_2 \leq \infty$. We itemize, at this point, a series of achievements available in [13]. We recall the following existence and uniqueness result

Theorem 2.1. *Let $d \geq 1$ and $0 < \beta < \frac{4}{d-1}$ be given. Then, the following holds:*

1. *For any initial condition $f \in H^1(\mathbb{R}^d \times \mathbb{T})$, the equation described in (1) admits a uniquely determined local-in-time solution*
$$u(t,x,y) \in \mathcal{C}([0,T); H^1(\mathbb{R}^d \times \mathbb{T})),$$
where T depends on $\|f\|_{H^1_{x,y}}$, i.e., $T = T(\|f\|_{H^1_{x,y}}) > 0$.
2. *The solution $u(t,x,y)$ admits a global extension with respect to the time variable.*

We notice that the local existence and uniqueness for (1) follows directly from the theorem above since $e^{-A(t)}$ is a bounded function on the set $[0, \infty)$. The global well-posedness instead from the inequality (25) below. We have likewise the Strichartz estimate

Proposition 2.1. *Let be $d \geq 1$, and $\sigma \in \mathbb{R}$. Then the following homogeneous estimates hold*

$$\|e^{it\Delta_{x,y}} f\|_{L^q_t L^r_x H^\sigma_y} \leq C \|f\|_{H^{s,\sigma}_{x,y}}, \tag{7}$$

when (q,r) verify the condition

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2} - s, \tag{8}$$

with $s < \frac{d}{2}$ and satisfy the following

- *$4 \leq q \leq \infty$ and $2 \leq r \leq \infty$, for $d = 1$;*
- *$2 < q \leq \infty$ and $2 \leq r \leq \infty$, for $d = 2$;*

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- $2 \le q \le \infty$ *and* $2 \le r \le \infty$, *for* $d \ge 3$.

Furthermore the following inhomogeneous inequalities are fulfilled

$$\left\| \int_0^t e^{-i(t-\tau)\Delta_{x,y}} F(\tau) d\tau \right\|_{L_t^\ell L_x^p H^\sigma} \le C \|F\|_{L_t^{\tilde{\ell}'} L_x^{\tilde{p}'} H^\sigma}, \tag{9}$$

provided that

$$\frac{2}{\ell} + \frac{d}{p} = \frac{2}{\tilde{\ell}} + \frac{d}{\tilde{p}} = \frac{d}{2}, \qquad \ell \ge 2, \quad (\ell, 2) \ne (2, 2).$$

Moreover the following extended inhomogeneous estimates are satisfied

$$\left\| \int_0^t e^{i(t-\tau)\Delta_{x,y}} F(\tau, x, y) d\tau \right\|_{L_t^q L_x^r H_y^\sigma} \le C \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'} H_y^\sigma}, \tag{10}$$

when the Schrödinger-acceptable pairs (q, r) *and* $(\tilde{q}, \tilde{r})$ *verify the condition*

$$\frac{1}{q} + \frac{1}{\tilde{q}} = \frac{d}{2}\left(\frac{1}{\tilde{r}'} - \frac{1}{r}\right), \tag{11}$$

with $2 \le q, r \le \infty$ *and* $2 \le \tilde{q}, \tilde{r} \le \infty$ *and satisfy the following*

- *for* $d = 1$, *no additional conditions are needed;*
- *for* $d = 2$, *conditions* $r < \infty$ *and* $\tilde{r} < \infty$ *are required;*
- *for* $d \ge 3$, *the further conditions*

$$\frac{1}{q} < \frac{1}{\tilde{q}'}, \qquad \frac{d-2}{d} \le \frac{r}{\tilde{r}} \le \frac{d}{d-2}, \tag{12}$$

are needed.

We have also the following (see again [13])

Lemma 2.1. *Assume that* $d \ge 1$, $\frac{4}{d} < \beta < \frac{4}{d-1}$ *and* $\sigma = \frac{\beta d - 4}{2\beta}$. *Then one can find* $\varrho \in (0,1)$ *and* $(q_\varrho, r_\varrho, \tilde{q}_\varrho, \tilde{r}_\varrho)$, *such that*

$$0 < \frac{1}{q_\varrho}, \frac{1}{r_\varrho}, \frac{1}{\tilde{q}_\varrho}, \frac{1}{\tilde{r}_\varrho} < \frac{1}{2}$$

$$\frac{1}{q_\varrho} + \frac{1}{\tilde{q}_\varrho} < 1, \qquad \frac{d-2}{d} < \frac{r_\varrho}{\tilde{r}_\varrho} < \frac{d}{d-2} \tag{13}$$

$$\frac{1}{q_\varrho} + \frac{d}{r_\varrho} < \frac{d}{2}, \qquad \frac{1}{\tilde{q}_\varrho} + \frac{d}{\tilde{r}_\varrho} < \frac{d}{2}$$

$$\frac{2}{q_\varrho} + \frac{d}{r_\varrho} = \frac{d}{2} - s, \qquad \frac{2}{q_\varrho} + \frac{d}{r_\varrho} + \frac{2}{\tilde{q}_\varrho} + \frac{d}{\tilde{r}_\varrho} = d,$$

$$\frac{1}{(\beta+1)\tilde{q}_\varrho'} = \frac{\varrho}{q_\varrho}, \qquad \frac{1}{(\beta+1)\tilde{r}_\varrho'} = \frac{\varrho}{r_\varrho} + \frac{2(1-\varrho)}{\beta d}. \tag{14}$$

For $d = 1, 2$ *we get the same conclusion provided that we drop conditions* (13).

Moreover we can also assume that

$$\frac{\beta}{q_\varrho} + \frac{\beta d}{2 r_\varrho} = 1, \qquad \frac{\beta}{r_\varrho} < 1. \tag{15}$$

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Lemma 2.2. *Assume $d \geq 1$ and $\frac{4}{d} < \beta < \frac{4}{d-1}$. Then one can find $2 < \ell \leq \infty, 2 \leq p \leq \infty$ such that:*

$$\frac{2}{\ell} + \frac{1}{p} = \frac{1}{2}, \tag{16}$$

$$\frac{1}{p'} = \frac{1}{p} + \frac{\beta}{r_\varrho}, \tag{17}$$

$$\frac{1}{\ell'} = \frac{1}{\ell} + \frac{\beta}{q_\varrho}, \tag{18}$$

where (q_ϱ, r_ϱ) is a couple given as in Proposition 2.1.

Lemma 2.3. *For every $0 < s < 1, \beta > 0$ there exist two positive constants $C_1 = C_1(\beta, s)$ and $C_2 = C_2(\beta, s)$, such that:*

$$\|u|u|^\beta\|_{\dot{H}^s_y} \leq C_1 \|u\|_{\dot{H}^s_y} \|u\|^\beta_{L^\infty} \leq C_2 \|u\|^{\beta+1}_{\dot{H}^s_y}.$$

We observe that the solutions to (1) enjoy the following conservation laws

$$\|u(t)\|_{L^2_{x,y}} = e^{-\Re \mathbf{B}(t)} \|f\|_{L^2_{x,y}}, \quad \mathcal{H}(u(t)) = \mathcal{H}(f), \tag{19}$$

where

$$H(u(t)) = e^{2\Re \mathbf{B}(t)} \int_{\mathbb{R}\times\mathbb{T}} |\nabla_{x,y} u(t,x,y)|^2 \, dxdy + \frac{2\lambda e^{2\Re \mathbf{B}(t)}}{\beta+2} \int_{\mathbb{R}\times\mathbb{T}} |u(t,x,y)|^{\beta+2} \, dxdy$$
$$+ \frac{2\beta\lambda}{\beta+2} \int_0^t \int_{\mathbb{R}\times\mathbb{T}} \Re b(s) e^{2\Re \mathbf{B}(t)} |u(t,x,y)|^{\beta+2} \, dxdy. \tag{20}$$

We utilize the change of variable

$$v(t,x,y) := e^{\mathbf{B}(t)} u(t,x,y) \tag{21}$$

and see that u satisfies (1) if v solves

$$\begin{cases} i\partial_t v + \Delta_{x,y} v = \lambda e^{-\beta \Re \mathbf{B}(t)} |v|^\beta v, & (t,x,y) \in [0,\infty) \times \mathbb{R}^d \times \mathbb{T}, \\ v(0,x,y) = f(x,y). \end{cases} \tag{22}$$

We multiply the above equation by $\bar{v}(t,x,y)$, integrate by parts w.r.t. the x-variable, achieving,

$$i\frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}\times\mathbb{T}^d} |v(t,x,y)|^2 dxdy - \int_{\mathbb{R}^d} \nabla_{x,y} \bar{v}(t,x) \nabla_{x,y} v(t,x,y) dxdy$$
$$- \int_{\mathbb{R}^d\times\mathbb{T}} \lambda e^{-\beta \Re \mathbf{B}(t)} |v(t,x,y)|^{\beta+2} dxdy.$$

Then, by taking the imaginary part we arrive at

$$\frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}^d\times\mathbb{T}} |v(t,x)|^2 dxdy = 0.$$

Thus, solutions that are local in time satisfy the conservation of mass

$$\|v(t)\|^2_{L^2_{x,y}} = \|f\|^2_{L^2_{x,y}}.$$

That is, the first identity in (19). We multiply the equation in (22) by $\partial_t \bar{v}(t,x,y)$, integrate by parts w.r.t. the x, y-variable and take the imaginary part, we have

$$\Re \int_{\mathbb{R}^d\times\mathbb{T}} \Big(\nabla_{x,y} v(t,x,y) \nabla_{x,y} \partial_t \bar{v}(t,x) + \lambda e^{-\beta \Re \mathbf{B}(t)} |v(t,x,y)|^\beta v(t,x) \partial_t \bar{v}(t,x) \Big) dxdy = 0.$$

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The previous identity enhances to the following

$$\int_{\mathbb{R}^d\times\mathbb{T}}\left(\frac{1}{2}\partial_t|\nabla_{x,y}v(t,x,y)|^2+\frac{\lambda}{\beta+2}e^{-\beta\Re\mathbf{B}(t)}\partial_t|v(t,x,y)|^{\beta+2}\right)dxdy=0$$

and then to

$$\begin{aligned}\partial_t\int_{\mathbb{R}^d\times\mathbb{T}}\left(\frac{1}{2}|\nabla_{x,y}v(t,x,y)|^2+\frac{\lambda}{\beta+2}e^{-\beta\Re\mathbf{B}(t)}|v(t,x,y)|^{\beta+2}\right)dxdy \\ =-\frac{\beta\lambda}{\beta+2}\int_{\mathbb{R}^d\times\mathbb{T}}\Re b(t)e^{-\beta\Re\mathbf{B}(t)}|v(t,x,y)|^{\beta+2}dxdy.\end{aligned}\tag{23}$$

Integrating w.r.t. the t-variable the above identity (23) we get

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathbb{T}}\left(\frac{1}{2}|\nabla_{x,y}v(t,x,y)|^2+\frac{\lambda}{\beta+2}e^{-\beta\Re\mathbf{B}(t)}|v(t,x,y)|^{\beta+2}\right)dxdy \\ +\frac{\beta\lambda}{\beta+2}\int_0^t\int_{\mathbb{R}^d\times\mathbb{T}}\Re b(t)e^{-\beta\Re\mathbf{B}(t)}|v(t,x,y)|^{\beta+2}dxdyds \\ =\int_{\mathbb{R}^d\times\mathbb{T}}\left(\frac{1}{2}|\nabla_{x,y}v(0,x,y)|^2+\frac{\lambda}{\beta+2}|v(0,x,y)|^{\beta+2}\right)dxdy.\end{aligned}\tag{24}$$

The above identity (24) indicates that the quantity

$$\begin{aligned}\widetilde{H}(v(t))=\int_{\mathbb{R}^d\times\mathbb{T}}\left(\frac{1}{2}|\nabla_{x,y}v(t,x,y)|^2+\frac{\lambda}{\beta+2}e^{-\beta\Re\mathbf{B}(t)}|v(t,x,y)|^{\beta+2}\right)dxdy \\ +\frac{\beta\lambda}{\beta+2}\int_0^t\int_{\mathbb{R}^d\times\mathbb{T}}\Re b(t)e^{-\beta\Re\mathbf{B}(t)}|v(t,x,y)|^{\beta+2}dxdyds\end{aligned}$$

is conserved. Hence, we get the local conservation of the Hamiltonian in (19) with $H(u(t))$ as in (20). The above conservation laws (19) infer also the bound

$$\left\|e^{\mathbf{B}(t)}u\right\|_{H^1_{x,y}}\lesssim\left\|e^{\mathbf{B}(t)}u\right\|_{L^2_{x,y}}+\left\|\nabla_{x,y}e^{\mathbf{B}(t)}u\right\|_{L^2_x}\lesssim H(u(0))+\|f\|_{L^2_{x,y}}.\tag{25}$$

3. Morawetz Identities and Inequalities

Our first contribution here is the Morawetz equalities associated to (1). We start with the following

Lemma 3.1 (Morawetz identities)**.** *Let $d\geq 1$ and $u\in\mathcal{C}([0,\infty);H^1_{x,y})$ be a global solution to* (1) *with initial data $f\in H^1_{x,y}$ such that* (2) *and* (3) *are satisfied. Moreover, let $\psi=\psi(x):\mathbb{R}^d\to\mathbb{R}$ be a sufficiently regular and decaying function, and denote the virial by*

$$\mathcal{V}(t):=\int_{\mathbb{R}^d\times\mathbb{T}}\psi(x)|e^{\mathbf{B}(t)}u(t,x,y)|^2\,dxdy.$$

Then the following identities hold:

$$\dot{\mathcal{V}}(t)=2\Im\int_{\mathbb{R}^d\times\mathbb{T}}e^{2\Re\mathbf{B}(t)}\bar{u}(t,x,y)\nabla_x\psi(x)\cdot\nabla_x u(t,x,y)\,dxdy\tag{26}$$

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and

$$\begin{aligned}\ddot{\mathcal{V}}(t) = -\int_{\mathbb{R}^d\times\mathbb{T}} \Delta_x^2\psi(x)|e^{\mathbf{B}(t)}u(t,x,y)|^2\,dxdy \\ +4\int_{\mathbb{R}^d\times\mathbb{T}} e^{2\Re\mathbf{B}(t)}\nabla_x u(t,x,y)D_x^2\psi(x)\cdot\overline{\nabla_x u(t,x,y)}\,dxdy \\ -4\Im\int_{\mathbb{R}^d\times\mathbb{T}} e^{2\Re\mathbf{B}(t)}u(t,x,y)\nabla_x\psi(x)\cdot B(x)\overline{\nabla_x u(t,x,y)}\,dxdy \\ +\frac{2\lambda\beta}{\beta+2}\int_{\mathbb{R}^d\times\mathbb{T}} \Delta_x e^{2\Re\mathbf{B}(t)}\psi(x)|u(t,x,y)|^{\beta+2}dxdy,\end{aligned} \tag{27}$$

where $D_x^2\psi\in\mathcal{M}_{d\times d}(\mathbb{R}^d)$ is the Hessian matrix of ψ and $\Delta_x^2\psi=\Delta_x(\Delta_x\psi)$ the Bi-Laplacian operator.

Proof. We choose a smooth rapidly decreasing solution $u=u(t,x)$. The general case $e^{\mathbf{B}(t)}u\in\mathcal{C}([0,\infty);H^1_{x,y})$ can be recovered via a classical density argument. The proof of (26) and (27) is similar to the one given in [10], for instance, since one can use the transformation (21) and then the equation (22). Thus we skip it. ☐

We continue with the following

Lemma 3.2 (Tensor Morawetz)**.** *Assume $d\geq 1$ and let $u(t,x,y)\in C([0,\infty),H^1(\mathbb{R}^d))$ be a global solution to (1) such that (2) and (3) are satisfied. Furthermore, let us denote by $z(t,x_1,x_2,y)=e^{2\mathbf{B}(t)}u_1(t,x_1,y_1)u_2(t,x_2,y_2)$ and $a(x_1,x_2)=\psi(x_1-x_2)$ and set the tensor action*

$$\begin{aligned}\mathcal{M}(t) \\ =\Im\int_{(\mathbb{R}^d\times\mathbb{T})^2}\bar{z}(t,x_1,x_2,y_1,y_2)\nabla_{x_1,x_2}z(t,x_1,x_2,y_1,y_2)\cdot\nabla_{x_1,x_2}a(x_1,x_2)\,dx_1dx_2dy_1dy_2,\end{aligned} \tag{28}$$

where $\nabla_{x_1,x_2}=(\nabla_{x_1},\nabla_{x_2})$ and $(\mathbb{R}^d\times\mathbb{T})^2=(\mathbb{R}^d\times\mathbb{T})\times(\mathbb{R}^d\times\mathbb{T})$. Then the following identity holds:

$$\begin{aligned}\dot{\mathcal{M}}(t) = 2\int_{(\mathbb{R}^d\times\mathbb{T})^2} D^2_{x_1x_2}a(x_1,x_2)\nabla_{x_1}|e^{\mathbf{B}(t)}u_1(t,x_1)|^2\nabla_{x_2}|e^{\mathbf{B}(t)}u_2(t,x_2)|^2\,dx_1dx_2dy_1dy_2 \\ +\frac{4\beta}{\beta+1}\int_{(\mathbb{R}^d\times\mathbb{T})^2} e^{4\Re\mathbf{B}(t)}|u_1(t,x_1)|^{\beta+2}|u_2(t,x_2)|^2\Delta_x a(x_1,x_2)\,dx_1dx_2dy_1dy_2 \\ +4\Re\int_{(\mathbb{R}^d\times\mathbb{T})^2} e^{4\Re\mathbf{B}(t)}\nabla_{x_1}u(x_1)D^2_{x_1x_1}a(x_1,x_2)\nabla_{x_1}\bar{u}(x_1)|u(x_2)|^2\,dx_1dx_2dy_1dy_2 \\ +4\Re\int_{(\mathbb{R}^d\times\mathbb{T})^2} e^{4\Re\mathbf{B}(t)}\nabla_{x_2}u(x_2)D^2_{x_2x_2}a(x_1,x_2)\nabla_{x_2}\bar{u}(x_2)|u(x_1)|^2\,dx_1dx_2dy_1dy_2.\end{aligned} \tag{29}$$

Proof. As above, we choose a smooth, decaying solution to (1). From now on we hide the variables t, y_1 and y_2 to simplify the calculations. Note that,

$$i\partial_t z(x,y)+\Delta_{x_1,x_2}z(x_1,x_2)=e^{2\mathbf{B}(t)}u_1(x_1)|u_1(x_1)|^\beta u_2(x_2)+e^{2\mathbf{B}(t)}u_1(x_1)|u_2(x_2)|^\beta u_2(x_2), \tag{30}$$

with $\Delta_{x_1,x_2}=\Delta_{x_1,y_1}+\Delta_{x_2,y_2}$. Then, we differentiate the action (28) w.r.t. time variable, obtaining

$$\begin{aligned}\dot{\mathcal{M}}(t)=2\Re\int_{(\mathbb{R}^d\times\mathbb{T})^2}\Delta_{x_1,x_2}z(x_1,x_2)][(\Delta_{x_1,x_2}a(x_1,x_2)\bar{z}(x_1,x_2)]\,dx_1dx_2dy_1dy_2 \\ +4\Re\int_{(\mathbb{R}^d\times\mathbb{T})^2}[\Delta_{x_1,x_2}z(x_1,x_2)][(\nabla_{x_1},\nabla_{x_2})a(x_1,x_2)\cdot(\nabla_{x_1},\nabla_{x_2})\bar{z}(x_1,x_2)]\,dx_1dx_2dy_1dy_2 \\ +\mathcal{N}^\beta(t),\end{aligned} \tag{31}$$

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where, by the identity (30) and after exploiting the symmetry of $a(x_1,x_2)$ in combination with Fubini's Theorem, we have

$$\begin{aligned} &\mathcal{N}^{\beta}(t) \\ &= \frac{2\beta}{\beta+1}\int_{(\mathbb{R}^d\times\mathbb{T})^2} e^{4\Re\mathbf{B}(t)}|u_1(x_1)|^{\beta+2}|u_2(x_2)|^2\Delta_{x_1}a(x_1,x_2)\,dx_1dx_2dy_1dy_2 \\ &+\frac{2\beta}{\beta+1}\int_{(\mathbb{R}^d\times\mathbb{T})^2} e^{4\Re\mathbf{B}(t)}|u_2(x_2)|^{\beta+2}|u_1(x_1)|^2\Delta_{x_2}a(x_1,x_2)\,dx_1dx_2dy_1dy_2 \\ &= \frac{4\beta}{\beta+1}\int_{(\mathbb{R}^d\times\mathbb{T})^2} e^{4\Re\mathbf{B}(t)}|u_1(x_1)|^{\beta+2}|u_2(x_2)|^2\Delta_{x_1}a(x_1,x_2)\,dx_1dx_2dy_1dy_2 \end{aligned} \tag{32}$$

We will consider now the linear terms, that are the ones associated to $\Delta_{x,y}$. The approach displayed in [7] and [11] brings to

$$\begin{aligned} &2\Re\int_{(\mathbb{R}^d\times\mathbb{T})^2}\Delta_{x_1,x_2}z(x_1,x_2)][(\Delta_{x_1,x_2}a(x_1,x_2)\bar{z}(x_1,x_2)]\,dx_1dx_2dy_1dy_2 \\ &+4\Re\int_{(\mathbb{R}^d\times\mathbb{T})^2}[\Delta_{x_1,x_2}z(x_1,x_2)][(\nabla_{x_1},\nabla_{x_2})a(x_1,x_2)\cdot(\nabla_{x_1},\nabla_{x_2})\bar{z}(x_1,x_2)]\,dx_1dx_2dy_1dy_2 \\ &=\int_{(\mathbb{R}^d\times\mathbb{T})^2}\Delta^2_{x_1}a(x_1,x_2)|e^{\mathbf{B}(t)}u(x_1)|^2|e^{\mathbf{B}(t)}u(x_2)|^2\,dx_1dx_2dy_1dy_2 \\ &+4e^{4\Re\mathbf{B}(t)}\Re\int_{(\mathbb{R}^d\times\mathbb{T})^2}\nabla_{x_1}u_1(x_1)D^2_{x_1x_1}a(x_1,x_2)\nabla_{x_1}\bar{u}_1(x_1)|u_2(x_2)|^2\,dx_1dx_2dy_1dy_2 \\ &+4e^{4\Re\mathbf{B}(t)}\Re\int_{(\mathbb{R}^d\times\mathbb{T})^2}\nabla_{x_2}u_2(x_2)D^2_{x_2x_2}a(x_1,x_2)\nabla_{x_2}\bar{u}_2(x_2)|u_1(x_1)|^2\,dx_1dx_2dy_1dy_2. \end{aligned} \tag{33}$$

Notice that, by using again the symmetry of $a(x_1,x_2)$ and integration by parts we get

$$\begin{aligned} &\int_{(\mathbb{R}^d\times\mathbb{T})^2}\Delta^2_{x_1}a(x_1,x_2)|e^{\mathbf{B}(t)}u(x_1)|^2|e^{\mathbf{B}(t)}u(x_2)|^2\,dx_1dx_2dy_1dy_2 \\ &=\int_{(\mathbb{R}^d\times\mathbb{T})^2}\Delta_{x_1}\Delta_{x_2}a(x_1,x_2)|e^{\mathbf{B}(t)}u(x_1)|^2|e^{\mathbf{B}(t)}u(x_2)|^2\,dx_1dx_2dy_1dy_2 \\ &=\int_{(\mathbb{R}^d\times\mathbb{T})^2}D^2_{x_1x_2}a(x_1,x_2)\nabla_{x_1}|e^{\mathbf{B}(t)}u(x_1)|^2\nabla_{x_2}|e^{\mathbf{B}(t)}u(x_2)|^2\,dx_1dx_2dy_1dy_2. \end{aligned} \tag{34}$$

By merging (32) and (33) one earns finally the idenity (29). □

3.1. A Localized Morawetz Inequality

We start this section with an outcome that is a consequence of Lemma 3.1 above. More precisely

Lemma 3.3. *Assume $d\geq 1$ and let $u\in\mathcal{C}([0,\infty);H^1_{x,y})$ be a global solution to (1) with initial data $f\in H^1_{x,y}$ such that (2) and (3) are satisfied. Then it holds that*

$$\begin{aligned} &\int_{(\mathbb{R}^d\times\mathbb{T})^2}\frac{e^{4\Re\mathbf{B}(t)}}{|x_1-x_2|}|u_1(t,x_1,y_1)|^{\beta+2}|u_2(t,x_2,y_1)|^2\,dx_1dx_2dy_1dy_2 \\ &\lesssim\Im\int_{(\mathbb{R}^d\times\mathbb{T})^2}e^{4\Re\mathbf{B}(t)}\nabla_{x_1}\bar{u}_1(t,x_1,y_1)\frac{(x_1-x_2)}{|x_1-x_2|}\cdot\nabla_{x_1}u_1(t,x_1,y_1)|u_2(t,x_2,y_2)|^2\,dx_1dx_2dy_1dy_2, \end{aligned} \tag{35}$$

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for $d \geq 2$ and

$$\int_{(\mathbb{R}^d\times\mathbb{T})^2} \frac{e^{4\Re\mathbf{B}(t)}}{\langle x_1-x_2\rangle^3}|u_1(t,x_1,y_1)|^{\beta+2}|u_2(t,x_2,y_1)|^2\,dx_1dx_2dy_1dy_2$$
$$\lesssim \Im\int_{(\mathbb{R}^d\times\mathbb{T})^2} e^{4\Re\mathbf{B}(t)}\nabla_{x_1}\bar{u}_1(t,x_1,y_1)\frac{(x_1-x_2)}{\langle x_1-x_2\rangle}\cdot\nabla_{x_1}u_1(t,x_1,y_2)|u_2(t,x_2,y_2)|^2\,dx_1dx_2dy_1dy_2, \tag{36}$$

for $d = 1$, where $\langle x_1 - x_2\rangle := (1 + |x_1 - x_2|^2)^{\frac{1}{2}}$.

Proof. We first notice, by Fubini's Theorem, that

$$\mathcal{M}(t)$$
$$= \Im\int_{(\mathbb{R}^d\times\mathbb{T})^2} e^{4\Re\mathbf{B}(t)}\nabla_{x_1}\bar{u}_1(t,x_1,y_1)\nabla_{x_1}a(x_1,x_2)\cdot\nabla_{x_1}u_1(t,x_1,y_2)|u_2(t,x_2,y_2)|^2\,dx_1dx_2dy_1dy_2. \tag{37}$$

We choose now $\psi = \psi(x_1, x_2) = |x_1 - x_2|$ if $d \geq 2$ and $\psi = \langle x_1 - x_2\rangle$ for $d = 1$. Elementary computations bring to

$$\begin{aligned}\nabla_{x_1}|x_1-x_2| &= \frac{x_1-x_2}{|x_1-x_2|},\quad \Delta_{x_1}|x_1-x_2| = \frac{d-1}{|x_1-x_2|},\\ D^2_{x_1x_1}|x_1-x_2| &= D^2_{x_2x_2}|x_1-x_2| = -D^2_{x_1x_2}|x_1-x_2|\\ &= \frac{1}{|x_1-x_2|}\left[\delta_{\ell k} - \frac{(x_1-x_2)_\ell}{|x_1-x_2|}\frac{(x_1-x_2)_k}{|x_1-x_2|}\right],\end{aligned} \tag{38}$$

for $\ell, k = 1, \ldots, d$, and

$$\begin{aligned}\nabla_{x_1}\langle x_1-x_2\rangle &= \frac{x_1-x_2}{\langle x_1-x_2\rangle},\quad \Delta_{x_1}\langle x_1-x_2\rangle = \frac{1}{\langle x_1-x_2\rangle^3},\\ D^2_{x_1x_1}\langle x_1-x_2\rangle &= D^2_{x_2x_2}\langle x_1-x_2\rangle = -D^2_{x_1x_2}\langle x_1-x_2\rangle\\ &= \frac{1}{\langle x_1-x_2\rangle}\left[\delta_{\ell k} - \frac{(x_1-x_2)_\ell}{\langle x_1-x_2\rangle}\frac{(x_1-x_2)_k}{\langle x_1-x_2\rangle}\right].\end{aligned} \tag{39}$$

We will look now at the identity (29) in the case $d \geq 2$ only, since the case $d = 1$ can be handled in a similar way. We have then

$$\begin{aligned}\Re\int_{(\mathbb{R}^d\times\mathbb{T})^2} & e^{4\Re\mathbf{B}(t)}\nabla_{x_1}u(x_1)D^2_{x_1x_1}a(x_1,x_2)\nabla_{x_1}\overline{u}(x_1)|u(x_2)|^2\,dx_1dx_2dy_1dy_2\\ &= \Re\int_{(\mathbb{R}^d\times\mathbb{T})^2} e^{4\Re\mathbf{B}(t)}\frac{|\widetilde{\nabla}_{x_2}u(x_1)|^2}{|x-y|}|u(x_2)|^2\,dx_1dx_2dy_1dy_2,\end{aligned} \tag{40}$$

with

$$\widetilde{\nabla}_{x_i}u(x_j) = \nabla u(x_j) - \nabla u(x_i)\cdot\frac{x_i-x_j}{|x_i-x_j|}\frac{x_i-x_j}{|x_i-y_j|},$$

for $i, j = 1, 2$, $i \neq j$. Analogously, we have

$$\begin{aligned}\Re\int_{(\mathbb{R}^d\times\mathbb{T})^2} & e^{4\Re\mathbf{B}(t)}\nabla_{x_2}u(x_2)D^2_{x_2x_2}a(x_1,x_2)\nabla_{x_2}\overline{u}(x_2)|u(x_1)|^2\,dx_1dx_2dy_1dy_2\\ &= \Re\int_{(\mathbb{R}^d\times\mathbb{T})^2} e^{4\Re\mathbf{B}(t)}\frac{|\widetilde{\nabla}_{x_1}u(x_2)|^2}{|x-y|}|u(x_1)|^2\,dx_1dx_2dy_1dy_2.\end{aligned} \tag{41}$$

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Moreover, by Cauchy-Schwartz inequality, one achieve also

$$
\begin{aligned}
&2\int_{(\mathbb{R}^d\times\mathbb{T})^2} D^2_{x_1x_2}a(x_1,x_2)\nabla_{x_1}|e^{\mathbf{B}(t)}u(x_1)|^2\nabla_{x_2}|e^{\mathbf{B}(t)}u(x_2)|^2\,dx_1dx_2dy_1dy_2\\
&= 8\int_{(\mathbb{R}^d\times\mathbb{T})^2} e^{4\Re\mathbf{B}(t)}D^2_{x_1x_2}a(x_1,x_2)\Re(\bar u(x_1)\nabla_{x_1}u(x_1))\Re(\bar u(x_2)\nabla_{x_2}u(x_2))\,dx_1dx_2dy_1dy_2\\
&= 8\int_{(\mathbb{R}^d\times\mathbb{T})^2}\frac{e^{4\Re\mathbf{B}(t)}}{|x_1-x_2|}\Re(\bar u(x_1)\tilde\nabla_{x_1}u(x_1))\cdot\Re(\bar u(x_2)\tilde\nabla_{x_2}u(x_2))\,dx_1dx_2dy_1dy_2\\
&\leq 4\int_{(\mathbb{R}^d\times\mathbb{T})^2} e^{4\Re\mathbf{B}(t)}\left(\frac{|\tilde\nabla_{x_2}u(x_1)|^2}{|x-y|}|u(x_2)|^2+\frac{|\tilde\nabla_{x_1}u(x_2)|^2}{|x-y|}|u(x_1)|^2\right)dx_1dx_2dy_1dy_2.
\end{aligned}
\tag{42}
$$

The above inequality in combination with (40) and (41) yields

$$
\begin{aligned}
&2\int_{(\mathbb{R}^d\times\mathbb{T})^2} D^2_{x_1x_2}a(x_1,x_2)\nabla_{x_1}|e^{\mathbf{B}(t)}u_1(t,x_1)|^2\nabla_{x_2}|e^{\mathbf{B}(t)}u_2(t,x_2)|^2\,dx_1dx_2dy_1dy_2\\
&+4\Re\int_{(\mathbb{R}^d\times\mathbb{T})^2} e^{4\Re\mathbf{B}(t)}\nabla_{x_1}u(x_1)D^2_{x_1x_1}a(x_1,x_2)\nabla_{x_1}\bar u(x_1)|u(x_2)|^2\,dx_1dx_2dy_1dy_2\\
&+4\Re\int_{(\mathbb{R}^d\times\mathbb{T})^2} e^{4\Re\mathbf{B}(t)}\nabla_{x_2}u(x_2)D^2_{x_2x_2}a(x_1,x_2)\nabla_{x_2}\bar u(x_2)|u(x_1)|^2\,dx_1dx_2dy_1dy_2\geq 0.
\end{aligned}
$$

By an use of the previous bound in the equality (29), one arrives at (35). □

We obtain the following corollary, that is a direct outcome of the previous lemma,

Corollary 3.1. *Let $u\in\mathcal{C}([0,\infty);H^1_{x,y})$ be a global solution to (1) with initial data $f\in H^1_{x,y}$ such that (2) and (3) are satisfied. Then, far any $\mathcal{Q}^d_{\tilde x}(r)=\tilde x+[-r,r]^d$, with $r>0$ and $\tilde x\in\mathbb{R}^d$ one has,*

$$
\int_0^\infty\int_{(\mathcal{Q}^d_{\tilde x}(r)\times\mathbb{T})^2} e^{4\Re\mathbf{B}(t)}|u(t,x_1,y_1)|^{\beta+2}|u(t,x_2,y_1)|^2\,dx_1dx_2dy_1dy_2<\infty. \tag{43}
$$

Proof. By integrating (35) with $a(x_1,x_2)$ as in (38) w.r.t. the time variable on the interval $J=[t_1,t_2]$, with $t_1,t_2\in[0,\infty)$, one obtains

$$
\begin{aligned}
&\left[\Im\int_{(\mathbb{R}^d\times\mathbb{T})^2} e^{4\Re\mathbf{B}(t)}\nabla_{x_1}\bar u_1(t,x_1,y_1)\frac{(x_1-x_2)}{|x_1-x_2|}\cdot\nabla_{x_1}u_1(t,x_1,y_1)|u_2(t,x_2,y_2)|^2\,dx_1dx_2dy_1dy_2\right]_{t=t_1}^{t=t_2}\\
&\gtrsim\int_{(\mathbb{R}^d\times\mathbb{T})^2}\frac{e^{4\Re\mathbf{B}(t)}}{|x_1-x_2|}|u_1(t,x_1,y_1)|^{\beta+2}|u_2(t,x_2,y_1)|^2\,dx_1dx_2dy_1dy_2\\
&\gtrsim\int_0^\infty\int_{(\mathcal{Q}^d_{\tilde x}(r)\times\mathbb{T})^2} e^{4\Re\mathbf{B}(t)}|u(t,x_1,y_1)|^{\beta+2}|u(t,x_2,y_1)|^2\,dx_1dx_2dy_1dy_2,
\end{aligned}
$$

where in the last line of the previous chain of inequalities we used that, for any $\tilde x\in\mathbb{R}^d$,

$$
\inf_{x_1,x_2\in\mathcal{Q}^d_{\tilde x}(r)}\frac{1}{|x_1-x_2|}=\inf_{x_1,x_2\in\mathcal{Q}^d_0(r)}\frac{1}{|x_1-x_2|}>0.
$$

Applying again the Cauchy-Schwartz inequality and by the conservation laws (19), one infers also

$$
\begin{aligned}
&\left[\Im\int_{(\mathbb{R}^d\times\mathbb{T})^2} e^{4\Re\mathbf{B}(t)}\nabla_{x_1}\bar u_1(t,x_1,y_1)\frac{(x_1-x_2)}{|x_1-x_2|}\cdot\nabla_{x_1}u_1(t,x_1,y_1)|u_2(t,x_2,y_2)|^2\,dx_1dx_2dy_1dy_2\right]_{t=t_1}^{t=t_2}\\
&\lesssim\|f\|^4_{H^1_{x,y}}<\infty.
\end{aligned}
\tag{44}
$$

Finally, we get (43) when $t_1\to 0, t_2\to\infty$. □

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4. The Decay of Solutions

This section focuses on the demonstration of the first part of Theorem 1.1. Namely one has:

Proof of (4). It is enough to prove the decay in (4) for a suitable $2 < q < \frac{2d+2}{d-1}$, because the general case follows by the conservation laws (19) and interpolation. More precisely it is enough to show that

$$\lim_{t\to\pm\infty} \|e^{\mathbf{B}(t)}u(t,x,y)\|_{L^{\frac{2d+6}{d+1}}_{x,y}} = 0. \tag{45}$$

Then the property (4) follows for all $2 < q < \frac{2d+2}{d-1}$ by (45) and the fact that

$$\sup_{t\in\mathbb{R}} \|e^{\mathbf{B}(t)}u(t,x,y)\|_{H^1_{x,y}} < \infty. \tag{46}$$

We recall the following localized Gagliardo-Nirenberg inequality (see [12] and [13])

$$\|\chi\|_{L^{\frac{2d+6}{d+1}}_{x,y}} \le C \sup_{\tilde{x}\in\mathbb{R}^d}\left(\|\chi\|_{L^2_{(\mathcal{Q}^d_{\tilde{x}}(1)\times\mathbb{T})}}\right)^{\frac{2}{d+3}} \|\chi\|^{\frac{d+1}{d+3}}_{H^1_{x,y}}, \tag{47}$$

where $\mathcal{Q}^d_{\tilde{x}}(1) = \tilde{x} + [-1,1]^d$. Assume now by contradiction that (45) is not fulfilled, then by (46) and by (47) we deduce the existence of a sequence $(t_n, x_n) \in \mathbb{R}\times\mathbb{R}^d$ with $t_n \to \infty$ and $\epsilon_0 > 0$ such that

$$\inf_n \|e^{\mathbf{B}(t_n)}u(t_n,x)\|^2_{L^2(\mathcal{Q}^d_{x_n}(1)\times\mathbb{T})} = \epsilon_0^2. \tag{48}$$

Note that by (26) and (46) one attains

$$\sup_{n,t}\left|\frac{d}{dt}\int_{\mathbb{R}^d\times\mathbb{T}} \xi(x-x_n)|e^{\mathbf{B}(t)}u(t,x,y)|^2\,dx\right| < \infty,$$

where $\xi(x)$ is a smooth, non-negative cut-off function, so that $\xi(x) = 1$ for $x \in \mathcal{Q}^d_0(1) = [-1,1]^d$ and $\xi(x) = 0$ for $x \notin \mathcal{Q}^d_0(2) = [-2,2]^d$. Therefore, by an application of the Fundamental Theorem of calculus we establish the inequality

$$\left|\int_{\mathbb{R}^d\times\mathbb{T}} \xi(x-x_n)|e^{\mathbf{B}(\sigma)}u(\sigma,x)|^2dx - \int_{\mathbb{R}^d\times\mathbb{T}} \xi(x-x_n)|e^{\mathbf{B}(t)}u(t,x,y)|^2dx\right| \le \widetilde{C}|t-\sigma|, \tag{49}$$

for a $\widetilde{C} > 0$ which does not depend on n. We choose $t = t_n$ and have

$$\int_{\mathbb{R}^d\times\mathbb{T}} \xi(x-x_n)|e^{\mathbf{B}(\sigma)}u(\sigma,x)|^2dx \ge \int_{\mathbb{R}^d\times\mathbb{T}} \xi(x-x_n)|e^{\mathbf{B}(t_n)}u(t_n,x)|^2dx - \widetilde{C}|t_n-\sigma|, \tag{50}$$

which results, having in mind the support property of the function $\xi(x)$, in

$$\int_{\mathcal{Q}^d_{x_n}(2)\times\mathbb{T}} |e^{\mathbf{B}(\sigma)}u(\sigma,x)|^2dx \ge \int_{\mathcal{Q}^d_{x_n}(1)\times\mathbb{T}} |e^{\mathbf{B}(t_n)}u(t_n,x)|^2dx - \widetilde{C}|t_n-\sigma|. \tag{51}$$

By combining this result with (48), it follows that there exists $T > 0$ so that

$$\inf_n \Big(\inf_{t\in(t_n,t_n+T)} \|e^{\mathbf{B}(t)}u(t,x)\|^2_{L^2(\mathcal{Q}_{x_n}(2)\times\mathbb{T})}\Big) \gtrsim \epsilon_1^2, \tag{52}$$

and, by Hölder inequality,

$$\inf_n \Big(\inf_{t\in(t_n,t_n+T)} \|e^{\mathbf{B}(t)}u(t,x)\|^{\beta+2}_{L^{\beta+2}(\mathcal{Q}_{x_n}(2)\times\mathbb{T})}\Big) \gtrsim \epsilon_1^{\beta+2}, \tag{53}$$

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for some $\epsilon_1 > 0$. Observe also that, because $t_n \to \infty$, it is possible to assume, up to a subsequence, that the intervals $(t_n, t_n + T)$ are mutually disjoint. In particular we come by

$$\begin{aligned}\sum_n T\epsilon_1^{\beta+4} &\lesssim \sum_n \int_{t_n}^{t_n+T} \int_{(\mathcal{Q}^d_{x_n}(2)\times\mathbb{T})^2} |e^{\mathbf{B}(t)}u(t,x_1,y_1)|^{\beta+2}|e^{\mathbf{B}(t)}u(t,x_2,y_2)|^2\,dx_1dx_2dy_1dy_2 \\ \lesssim \int_0^\infty \sup_{\tilde{x}\in\mathbb{R}^d}\int_{(\mathcal{Q}^d_{\tilde{x}}(2)\times\mathbb{T})^2} & e^{-\beta\|\Re\mathbf{B}(t)\|_{L^\infty([0,\infty))}}|e^{\mathbf{B}(t)}u(t,x_1,y_1)|^{\beta+2}|e^{\mathbf{B}(t)}u(t,x_2,y_2)|^2\,dx_1dx_2dy_1dy_2 \\ &\lesssim \int_0^\infty \sup_{\tilde{x}\in\mathbb{R}^d}\int_{(\mathcal{Q}^d_{\tilde{x}}(2)\times\mathbb{T})^2} e^{4\Re\mathbf{B}(t)}|u(t,x_1,y_1)|^{\beta+2}|u(t,x_2,y_2)|^2\,dx_1dx_2dy_1dy_2.\end{aligned} \tag{54}$$

Thus we get a contradiction since the right hand side of (54) is bounded, as it can be seen form (43). □

Remark 4.1. *The condition (3) is required to ensure not only an effective use of the Morawetz technique as one can see from Lemmas 3.1 and 3.2. We do not need to impose further lower bounds to $e^{\Re\mathbf{B}(t)}$ when $4/d < \beta < 4/(d-1)$. Notice also that, for $0 < \beta \le 4/d$ one has $e^{-\beta\Re\mathbf{B}(t)} < (1+t)^{-1}$, this bound guarantees that $e^{\Delta_{x,y}+\mathbf{B}(t)}u$ has a strong limit in $L^2_{x,y}$ when $\Re b(t) \ge 0$, as underlined in [19], avoiding the nonexistence of scattering solutions.*

5. Analysis of the Solutions in the Strichartz Spaces and Scattering

Here, we present some results associated to the control of the solution of (1) in the Strichartz norms. They are pivotal for the proof of the second part of the Theorems 1.1.

Proposition 5.1. *Let $d \ge 1$, $\lambda > 0$ and let $u \in \mathcal{C}([0,\infty); H^1(\mathbb{R}^d))$ be a unique global solution to (1) with initial data $f \in H^1(\mathbb{R}^d \times \mathbb{T})$ such that (2) is satisfied. One has:*

a) if $\frac{4}{d} < \beta < \frac{4}{d-1}$, and assuming (3) is also fulfilled, then

$$e^{\mathbf{B}(t)}u(t,x,y) \in L_t^{q_\varrho}L_x^{r_\varrho}H_y^{\tilde{s}}, \tag{55}$$

with (q_ϱ, r_ϱ) as in Lemma 2.1;

b) if $0 < \beta \le \frac{4}{d}$, and assuming (2) is fulfilled with strict inequality, then

$$e^{\mathbf{B}(t)}u(t,x,y) \in L_t^{q}L_x^{r}H_y^{\tilde{s}}, \tag{56}$$

with $(q,r) = \left(\frac{4(\beta+2)}{d\beta}, \beta+2\right)$.

Proof. Case $\frac{4}{d} < \beta < \frac{4}{d-1}$. Bear in mind the integral operator associated to (22) be defined, for any $f \in H^1_{x,y}$ as

$$\mathcal{T}_f(e^{\mathbf{B}(t)}u) = e^{it\Delta_{x,y}}f + k\int_0^t e^{-\beta\Re\mathbf{B}(\tau)}e^{i(t-\tau)\Delta_{x,y}}\left(|e^{\mathbf{B}(\tau)}u(\tau)|^\beta e^{\mathbf{B}(\tau)}u(\tau)\right)d\tau. \tag{57}$$

Then, in view of (2), one gets by a combination of Proposition 2.1, Lemmas 2.1, 2.3 and the Hölder inequality

$$\begin{aligned}\|e^{\mathbf{B}(\cdot)}u\|_{L^{q_\varrho}_{(t_0,\infty)}L^{r_\varrho}_xH^{\tilde{s}}_y} &\lesssim \|u(t_0)\|_{H^\sigma_xH^{\tilde{s}}_y} + \|e^{-\beta\Re\mathbf{B}(\cdot)}|e^{\mathbf{B}(\cdot)}u|^\beta e^{\mathbf{B}(\cdot)}u\|_{L^{\tilde{q}_\varrho'}_{(t_0,\infty)}L^{\tilde{r}_\varrho'}_xH^{\tilde{s}}_y} \\ \lesssim \|u(t_0)\|_{H^1_{x,y}} + \||e^{\mathbf{B}(\cdot)}u|^\beta e^{\mathbf{B}(\cdot)}u\|_{L^{\tilde{q}_\varrho'}_{(t_0,\infty)}L^{\tilde{r}_\varrho'}_xH^{\tilde{s}}_y} &\lesssim \|u(t_0)\|_{H^1_{x,y}} + \|e^{\mathbf{B}(\cdot)}u\|^{1+\beta}_{L^{(1+\beta)\tilde{q}_\varrho'}_{(t_0,\infty)}L^{(1+\beta)\tilde{r}_\varrho'}_xH^{\tilde{s}}_y} \\ &\lesssim \|u(t_0)\|_{H^1_{x,y}} + \|e^{\mathbf{B}(\cdot)}u\|^{(1-\varrho)(1+\beta)}_{L^\infty_{(t_0,\infty)}L^{\beta d/2}_xH^{\tilde{s}}_y}\|e^{\mathbf{B}(\cdot)}u\|^{\varrho(1+\beta)}_{L^{q_\varrho}_{(t_0,\infty)}L^{r_\varrho}_xH^{\tilde{s}}_y}.\end{aligned} \tag{58}$$

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An application of (4) and Lemma 2.5 in [13] gives, for a suitable big $t_0 > 0$, that $e^{\mathbf{B}(t)}u(t,x,y) \in L^{q_\varrho}_{(t_0,\infty)} L^{r_\varrho}_x H^{\tilde{s}}_y$. Finally, by a continuity argument we deduce (55).

Case $0 < \beta \leq \frac{4}{d}$**.** Within this framework, the previous approach can be followed with slight adjustments. Having in mind the Sobolev embedding $H^{\tilde{s}}_y \subset L^\infty_y$, which is valid for $\tilde{s} > \frac{1}{2}$, then one can acquire the chain of inequalities

$$\begin{aligned} \|e^{\mathbf{B}(t)}u\|_{L^q_{(t_0,\infty)} L^r_x H^{\tilde{s}}_y} &\lesssim \|u(t_0)\|_{H^\sigma_x H^{\tilde{s}}_y} + \|e^{-\beta\Re\mathbf{B}(\cdot)}|e^{\mathbf{B}(\cdot)}u|^\beta e^{\mathbf{B}(\cdot)}u\|_{L^{q'}_{(t_0,\infty)} L^{r'}_x H^{\tilde{s}}_y} \\ &\lesssim \|u(t_0)\|_{H^1_{x,y}} + \|e^{-\mathbf{B}(\cdot)}e^{\mathbf{B}(\cdot)}u\|^\beta_{L^\eta_{(t_0,\infty)} L^r_x H^{\tilde{s}}_y} \|e^{\mathbf{B}(\cdot)}u\|_{L^q_{(t_0,\infty)} L^r_x H^{\tilde{s}}_y} \\ &\lesssim \|u(t_0)\|_{H^1_{x,y}} + \|e^{-\mathbf{B}(\cdot)}\|^\beta_{L^\eta_{(t_0,\infty)}} \|e^{\mathbf{B}(\cdot)}u\|^\beta_{L^\infty_{(t_0,\infty)} L^r_x H^{\tilde{s}}_y} \|e^{\mathbf{B}(\cdot)}u\|_{L^q_{(t_0,\infty)} L^r_x H^{\tilde{s}}_y} \\ &\lesssim \|u(t_0)\|_{H^1_{x,y}} + \|e^{\mathbf{B}(\cdot)}u\|^\beta_{L^\infty_{(t_0,\infty)} L^r_x H^{\tilde{s}}_y} \|e^{\mathbf{B}(\cdot)}u\|_{L^q_{(t_0,\infty)} L^r_x H^{\tilde{s}}_y}, \end{aligned} \tag{59}$$

with

$$\eta = \frac{2\beta(\beta+2)}{4-(d-2)\beta}$$

and because the strict inequality in (2). Again by (4), Lemma 2.5 in [13] and a continuity argument we conclude that (56) holds true. □

Corollary 5.1. *Assume (ℓ, p) are given as in Lemma 2.2 and let $u(t,x,y)$ be the unique solution to (1) with $0 < \beta < \frac{4}{d-1}$ and such that (2) is satisfied. Then one gets*

$$\|e^{\mathbf{B}(t)}u(t,x,y)\|_{L^\ell_t L^p_x L^2_y} + \|e^{\mathbf{B}(t)}u(t,x,y)\nabla_{x,y}u(t,x,y)\|_{L^\ell_t L^p_x L^2_y} < \infty, \tag{60}$$

a) if $\frac{4}{d} < \beta < \frac{4}{d-1}$, and assuming (3) is also fulfilled;
b) if $0 < \beta \leq \frac{4}{d}$, and assuming (2) holds with strict inequality.

Proof. We will concentrate first on the case $\frac{4}{d} < \beta < \frac{4}{d-1}$, the proof of case $0 < \beta \leq \frac{4}{d}$ follows from (59). We display $\|e^{\mathbf{B}(t)}u(t,x,y)\|_{L^\ell_t L^p_x L^2_y} < \infty$, the other estimate can be handled in a similar way. By (9), Lemma 2.2 and Hölder inequality one achieves

$$\begin{aligned} \|e^{\mathbf{B}(t)}u(t,x,y)\|_{L^\ell_{(t_0,\infty)} L^p_x L^2_y} &\lesssim \|u(t_0)\|_{H^1_{x,y}} + \|e^{-\beta\Re\mathbf{B}(t)}|e^{\mathbf{B}(\cdot)}u|^\beta e^{\mathbf{B}(\cdot)}u\|_{L^{\ell'}_{(t_0,\infty)} L^{p'}_x L^2_y} \\ &\lesssim \|u(t_0)\|_{H^1_{x,y}} + \|e^{\mathbf{B}(\cdot)}u\|^\beta_{L^{q_\varrho}_{(t_0,\infty)} L^{r_\varrho}_x H^{\tilde{s}}_y} \|e^{\mathbf{B}(\cdot)}u\|_{L^\ell_{(t_0,\infty)} L^p_x L^2_y}. \end{aligned}$$

We conclude by choosing a suitable big $t_0 > 0$ and utilizing the previous Proposition 5.1. □

We now turn back to the proof of the scattering for the solution to (1).

Proof of (5). By using the integral operator (57) and the Strichartz estimates (7), (9) we obtain

$$\begin{aligned} \left\|e^{\mathbf{B}(t)}e^{-it\Delta_{x,y}}u(t) - e^{\mathbf{B}(t')}e^{-it\Delta_{x,y}}u(t')\right\|_{H^1_{x,y}} &= \left\|\int_{t'}^t e^{-is\Delta_{x,y}}\left(e^{-\beta\Re\mathbf{B}(\cdot)}|e^{\mathbf{B}(\cdot)}u|^\beta e^{\mathbf{B}(\cdot)}u\right)(s)ds\right\|_{H^1_{x,y}} \\ &\lesssim \|e^{-\beta\Re\mathbf{B}(\cdot)}|e^{\mathbf{B}(\cdot)}u|^\beta e^{\mathbf{B}(\cdot)}u\|_{L^\ell_{(t,t')} L^p_x L^2_y} + \|e^{-\beta\Re\mathbf{B}(\cdot)}\nabla_{x,y}\left(|e^{\mathbf{B}(\cdot)}u|^\beta e^{\mathbf{B}(\cdot)}u\right)\|_{L^\ell_{(t,t')} L^p_x L^2_y}. \end{aligned}$$

Hence

$$\lim_{t_1,t_2\to\infty}\left\|e^{\mathbf{B}(t)}e^{-it\Delta_{x,y}}u(t) - e^{\mathbf{B}(t')}e^{-it\Delta_{x,y}}u(t')\right\|_{H^1_{x,y}} = 0 \tag{61}$$

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if

$$\lim_{t,t'\to\infty}\left(\|e^{-\beta\Re\mathbf{B}(\cdot)}|e^{\mathbf{B}(\cdot)}u|^{\beta}e^{\mathbf{B}(\cdot)}u\|_{L^{\ell}_{(t,t')}L^{p}_{x}L^{2}_{y}}+\|e^{-\beta\Re\mathbf{B}(\cdot)}\nabla_{x,y}\left(|e^{\mathbf{B}(\cdot)}u|^{\beta}e^{\mathbf{B}(\cdot)}u\right)\|_{L^{\ell}_{(t,t')}L^{p}_{x}L^{2}_{y}}\right)=0.$$

This limit can be provided arguing as in the proof of Proposition 5.1, by exploiting (55), (56), (60) together with Lemma 2.2. So, (5) follows thus from (61) above. □

Remark 5.1. *We also emphasize that exponential decay can be obtained directly through interpolation between the conservation of mass stated in* (19) *and the estimates derived from the Sobolev embedding combined with* (25)*. More explicitly, one obtains*

$$\|e^{\mathbf{B}(t)}u(t)\|_{L^{r}_{x,y}}\lesssim 1.$$

Nevertheless, this inequality is insufficient to establish a decay behavior as strong as the one described in (4) *of Theorem* 1.1*, which characterizes a more restrictive property of solutions to* (1)*. Additionally, the analysis above does not take into account the scenario where* $\Re\mathbf{B}(t)=0$*, that is when* $ib(t)$ *is a real-valued function.*

Remark 5.2. *It is important to emphasize that our results apply broadly to a wide class of damped nonlinearities satisfying the condition* (2)*. In particular, they cover situations where the damping term behaves asymptotically as*

$$\Re\mathbf{b}(t)=\frac{\tilde{a}}{(1+t)},\quad \alpha>0,\quad \tilde{a}>0,\quad t\ge 0,$$

as well as cases of the form

$$\Re\mathbf{b}(t)=\frac{\tilde{a}}{(1+t)^{\alpha}},\quad 0\le\alpha<1,\quad \tilde{a}>0,\quad t\ge 0,$$

which were examined in [19]*. Moreover, the techniques and results developed herein extend naturally to even more general forms of damping terms, thereby enabling the analysis of nonlinear Schrödinger equations with complex time-dependent damping structures, such as*

$$i\partial_t u+\Delta_{x,y}u+\frac{i\tilde{a}t^{\alpha_1}\ln t}{(1+t)^{\alpha_2}}u+\tilde{b}(t)u-\lambda u|u|^{\beta}=0,\quad t\ge\delta,$$

with $\delta\ge 1$*, where* $\tilde{b}(t)$ *is a real-valued continuous function, and parameters* $\tilde{a}$*,* α_1 *and* α_2 *that characterize the precise rate and structure of the damping effect. These types of equations were previously considered in the literature, for example in* [17]*, highlighting the flexibility and applicability of our approach.*

6. Conclusions

We extend the outcomes obtained in [16], [17], [18] and [19] to the partially periodic framework. In our work, the assumptions imposed on the function $ib(t)$ are close to the more general ones stated in [19], with the added novelty that our approach is capable of treating the one-dimensional case $d=1$ as well. Moreover, we incorporate an oscillatory component within the perturbed propagator $e^{\mathbf{B}(t)+it\Delta_{x,y}}$, a feature that has not been addressed in the aforementioned works. In addition, our strategy significantly simplifies the proof of scattering in the energy space for the damped Schrödinger equation, even when considered on the flat Euclidean geometry $\mathbb{R}^d$ for $0<\beta<\frac{4}{d-2}$. The implementation of bilinear Morawetz inequalities provides insight into the decay behavior of the $L^{p}_{x,y}$norm of the solutions of (1): specifically, we show that the decay rate is faster than $e^{-\Re\mathbf{B}(t)}$ when the Hamiltonian (20) is positive. This enhanced decay property broadens the class of admissible perturbations, even allowing the limit case $\mathbf{b}=0$. It is important to emphasize that we have opted for a conservative set of assumptions regarding the nonlinear terms in (1). Indeed, we are convinced that our methodology extends naturally to perturbations $b(t)$ satisfying

$$e^{-\beta\Re\mathbf{B}(t)}\le\frac{\tilde{a}\,t^{\alpha_1}}{(1+t)^{\alpha_2-\alpha_1}},\quad 0\le\alpha_1<1,\quad \alpha_1<\alpha_2,\quad \tilde{a}>0,\quad t\ge 0,$$

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with $\mathbf{B}(t)$ defined as in (2), as well as to the case $\Re\mathbf{B}(t) < 0$. These extensions form the basis of our future investigations.

7. Open Problems and Further Developments

The theoretical framework developed in this paper is general and robust, allowing for a direct analysis of energy decay in solutions to damped Schrödinger equations with partial periodic local nonlinearities. The versatility of our approach suggests its applicability to several significant open problems, particularly:

- A detailed analysis of scattering phenomena in energy spaces for solutions to the equation (1) within the focusing regime, characterized by $\lambda < 0$. Such an investigation would deepen the understanding of the interplay between damping and focusing nonlinearities.
- An exploration of decay and scattering behavior of solutions to fourth-order nonlinear Schrödinger equations, such as
$$\partial_t u - (\Delta_{x,y})^2 u + ib(t)u - \lambda u|u|^\beta = 0,$$
where $(\Delta_{x,y})^2 = \Delta_{x,y}(\Delta_{x,y})$ is the bilaplacian operator. This would help clarify the long-term dynamics and stability of higher-order dispersive models under nonlinear damping effects.
- A thorough investigation into the decay rates and scattering properties of solutions to other related nonlinear dispersive equations, including the nonlinear Beam equation such as
$$\partial_{tt} u + \Delta_x^2 u + b(t)u + u + \lambda u|u|^\beta = 0,$$
with $(\Delta_x)^2 = \Delta_x(\Delta_x)$. This can be done also in the partially periodic setting.
- A comprehensive study of the scattering dynamics for nonlinear Klein–Gordon equations of the form
$$\partial_{tt} u - \Delta_{x,y} u + b(t)u + u + \lambda u|u|^\beta = 0,$$
including the the partially periodic case.

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