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Automorphisms and Definability for Upward Complete Structures

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Abstract: The Svenonius theorem establishes the correspondence between definability of relations in a countable structure and automorphism groups of these relations in the extension of the structure. This correspondence may help in finding the description of the definability lattice constituted by all definability spaces (reducts) of the original structure. However, the major difficulty here is the necessity to consider the extensions which generally are obscure and hardly amenable to classification. Because of that results on definability lattices were obtained only for ω-categorical structures (i. e. those in which all elementary extensions are isomorphic to the structure itself) with finite signature. In this work we introduce the concept of upward complete structure as such in which all its extensions are isomorphic. Further we define upward completion of structure. For upward complete structures Galois correspondence between definability lattice and the lattice of closed supergroups of automorphism group of the structure is an anti-isomorphism. These lattices could be infinite in general. We describe the natural class of structures which have upward completion, we call them discretely homogeneous graphs, present the explicit construction of their completion and automorphism groups of completions. We establish the general *localness* property of discretely homogeneous graphs and present the examples of completable structures and their completions.

Keywords: definability; definability lattice; automorphism group; reduct; Svenonius theorem

0. Introduction

The definitions of concepts of definability space and definability lattice (reducts lattice) can be found in [1].

The original idea of Alessandro Padoa[2] as well as general philosophy of Alfred Tarski [3] assuming "Erlangen Program" of "geometrization" and "algebraization" of logic have found their refinement in the Svenonius Theorem ("Completeness Theorem for definability") published in 1959 [4]. This remarkable result by Lars Svenonius however have not found applications for a long time. One of the reasons, perhaps, is historical and subjective – the paper was published in a little known to mathematicians Swedish philosophical journal. Another, and likely main, reason is that the Svenonius theorem reduces the problem of description of definability lattice of a structure to the consideration of supergroups of automorphism groups of all countable elementary extensions of that structure; and these extensions generally appear to be obscure and hardly amenable to classification.

Starting from 1965 (see [5]) definability lattices have been constructed for several homogeneous structures with finite relational signature. Since such structures are ω-categorical, the studying of closed supergroups were required only for the structures themselves. Furthermore, homogeneity of the structures, which is essentially a transformational and geometric property, did help to simplify greatly the problem. All found latices proved to be finite. The hypothesis of finitness of definability lattices of all homogenenous structures with finite basis (finite relational signature) was proposed by Simon Thomas (see [6]) in 1991 and remains open.

Continuing the direction of [1] in the present work we consider the structures for which the Svenonious theorem "works" by restricting the class of elementary extensions where studying of authomorphism groups is required. We believe that it may become an important step in the description of corresponding definability lattices. Such an approach was used in [1], where a complete description of the infinite definability lattice for the successor structure on integers had been obtained.

The Svenonius Theorem is formulated as follows [7]:

Svenonius Theorem. *Let $M = \langle A, \Sigma \rangle$ be a countable structure, and $\Sigma' \subset \Sigma$. Then for any relation $P \in \Sigma$ the following two conditions are equivalent:*

i) P is not definable in Σ';

ii) There exists an elementary extension $M' = \langle A', \Sigma \rangle$ of the structure M and a permutation φ of the domain of A', preserving all relations in Σ' and not preserving P.

In the present paper we discuss a class of structures, where the construction of their definability lattices is reduced to the analysis of closed supergroups for one structure and its automorphism group, that is the class of *upward complete structures*. We present several examples of structures from this class.

In [1], for the description of definability lattice of $\langle \mathbb{Z}, +1 \rangle$ upward completeness of an extension of this structure was used. (We used a different terminology there.)

For other structures we consider in the present work (excluding $\langle \mathbb{Z}, +1 \rangle$) full descriptions of their definability lattices are unknown. We hope, however that the existence of upwards complete elementary extensions will help to obtain such descriptions.

1. Upward complete structures and completions

Definition.

A countable structure $\mathcal{M}$ is called upward complete *, if $\mathcal{M} \cong \mathcal{M}'$ for every countable elementary extension of $\mathcal{M}' \succ \mathcal{M}$. An upward complete structure elementary equivalent to a structure is called a completion of the latter.*

It immediately follows from Svenonius theorem that definability lattice of every upward complete structure $\mathcal{M}$ (and of each structure elementary equivalent to that) is determined completely by the lattice of all closed supergroups of its automorphism group $Aut(\mathcal{M}) \subset Sym(\mathcal{M})$ (the group of all permutations of the domain of $\mathcal{M}$).

It is obvious that any ω-categorical structure is upward complete. For example, the structure of non-negative rational numbers with the standard order is ω-categorical. It provides an example of upward complete but not homogeneous structure. The simplest example of not $\omega-$categorical upward complete structure is $\mathbb{Z}^\omega$, the union of countably many disjoint copies of $\langle \mathbb{Z}, +1 \rangle$. Following tradition we call such copies (and similar components in other cases) *galaxies*. It is easy to see that $\mathbb{Z}^\omega$ is a completion of $\langle \mathbb{Z}, +1 \rangle$.

For every natural number n we denote:

the relation $|x_1 - x_2| = n$ by $A_{0,n}$,

the relation $x_1 - x_2 = x_3 - x_4 = n \vee x_1 - x_2 = x_3 - x_4 = -n$ by $A_{1,n}$,

the relation $x_1 - x_2 = n$ by $A_{2,n}$.

It is shown in [1] that the lattice of closed supergroups of the group $Aut(\mathbb{Z}^\omega)$ consists of groups $\Gamma_{A_{0,d}}, \Gamma_{A_{1,d}}, \Gamma_{A_{2,d}}$ for all $d \in \mathbb{N}$ corresponding to the relations above.

For any d we denote by $C(d)$ the set of all residue classes modulo d taken from all galaxies.

Then:

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– the group $\Gamma_{A_{2,d}}$ consists of all permutations γ, such that $\gamma(a+d) = \gamma(a) + d$ for all a. It is possible to have an explicit description of the elements of $\Gamma_{A_{2,d}}$ — these are mappings arbitrarily permuting the elements of $C(d)$ and preserving the order on each element.

– the group $\Gamma_{A_{1,d}}$ is a proper supergroup of $\Gamma_{A_{2,d}}$, which consists of all permutations γ such that either $\gamma(a+d) = \gamma(a) + d$ for all a, or $\gamma(a+d) = \gamma(a) - d$ for all a. The explicit description of the elements of $\Gamma_{A_{1,d}}$ is similar to the previous case of $\Gamma_{A_{2,d}}$ The difference here is that two variants are possible: the residues classes in $C(d)$ are permuted either all preserving or all reversing the order.

– the group $\Gamma_{A_{0,d}}$ is a proper supergroup of $\Gamma_{A_{1,d}}$, which consists of all permutations γ such that $|\gamma(a+d) - \gamma(a)| = d$ for all $a \in \mathbb{Z}^{\omega}$. The explicit description of the elements of the group is similar to the previous cases. A permutation from $\Gamma_{A_{0,d}}$ sends each residue class from $C(d)$ to another residue class with either preserving or reversing the order. The choice of the variant for each class is done independently.

In [1] it is proved that these groups exhaust the lattice of closed supergroups for successor structure on the domain of integers, and generators for the corresponding definability spaces are presented as given above. Notice that all these groups with the exception of $\Gamma_{A_{0,1}}$ are realizable as supergorups of the automorphism groups of the successor structure over integers. So, we need extension for $\Gamma_{A_{0,1}}$ only.

Here are some more examples Every countable infinite vector space over a finite field is an upward complete structure. The countably dimensional vector space over the field of rationals is a completion of the additive group of rationals.

Further we present more examples of upward complete structures.

The set of all integers with successor operation can be naturally generalized to a countable set with several successor operations (the definitions will be given later).

In this paper we consider two examples of such structures and show that for each of them an upward complete elementary extension does exist.

2. Graphs

Consider a countable infinite oriented graph $\langle G, R \rangle$ with its vertices and edges labelled by the elements of a finite set and denote the resulting signature by Σ.

We will use the concepts of *connectedness, distances between vertices, diameter, neighbourhoods, neighbourhood centre, radius* etc, ignoring the orientations and labels.

Further on we assume that our graphs have infinite diameter. It is true if degrees of all vertices are finite.

Non-standard extension of graphs

Let us enrich the signature by adding the relation $d_i(x, y)$ for every $i \in \mathbb{N}$, meaning that the distance between vertices x, y is greater than i, and the constant symbol $\odot$ denoting an arbitrary fixed element of G. We denote the resulting structure by G^+.

Let us now add to the theory of G^+ the statements on the existence of infinitely many non-standard and not connected to each other elements, that is elements c_j, such that for all k, i, j both $d_k(c_i, \odot)$ and $d_k(c_i, c_j), i \neq j$ hold true. By compactness argument there is a countable model of the theory. Let us denote it by G^{+*}, and let G^* be a reduct of the structure G^{+*} to the structure in the original signature Σ.

Proposition 1. *The structure G^* as well as any of its elementary extensions contains apart of the original structure G countably many components not connected to each other and to G.*

Proof. It follows from the existence of the countably many non-standard not connected elements. □

We will call the components of every structure obtained by the procedure above and of all its elementary extensions *galaxies* of the structure (graph) G.

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Proposition 2. *Let all galaxies of G be isomorphic. Then the structure G^* is an upward complete extension of G.*

Proof. The proof is straightforward.

We need to demonstrate that

(i) G^* is an elementary extension of G,

(ii) any countable elementary extension G^* is isomorphic to G^*.

The condition (i) is obvious since G^* is obtained by just signature reduction.

As we have noticed in Proposition 1 G^* is the union of the original structure G and countably many non connected galaxies. The same is true for any countable elementary extension of G. Then (ii) follows from Proposition 1. □

Discretely homogeneous graphs

We will call a graph G *discretely homogeneous* if it is connected and for each radius r, all neighbourhoods, with the exception of finitely many, of radius r are isomorphic (as oriented labelled graphs with a pointed neighbourhood centre of it). We will call such (non exceptional) neighbourhoods *typical*.

Proposition 3. *Let graph G be discretely homogeneous. Then each of its galaxies is isomorphic to a countable union of increasing typical neighbourhoods with a common centre.*

Proof. For every natural n and vertex $a \in G$ we denote by $D_n(a)$ the neighbourhood of radius n with the centre a. Thus $D_n(a) = \{x|\neg d_n(a,x)\}$. Define the equivalence relation $E_n(x,y)$ on the vertices of the graph G as

$$E_n(a,b) \leftrightharpoons D_n(a) \cong D_n(b)$$

.

The relation $E_n(x,y)$ partitions the set of graph vertices into finitely many equivalence classes. The condition of discrete homogeneity means that for each fixed n only one of the equivalence classes is infinite and it is definable. We denote such a class by A_n.

Let us consider now an elementary extension of G and an element a of a galaxy of it. The element a equals to no element of the original graph G. So, the condition $a \in A_n$ is satisfied for every n. Furthermore, since each connected component G' of a graph coincides with the union $\bigcup_{i\in\mathbb{N}} D_i(a)$ for some vertex $a \in G'$, every galaxy coincides with $\bigcup_{i\in\mathbb{N}} D_i(a)$ for every (non-standard) element a from that galaxy.

□

Automorphisms

For every discretely homogeneous G the automorphism group of its upward complete extension $Aut(G^*)$ has very simple structure.

Proposition 4. *Let $\Gamma(G)$ consists of all galaxies of G plus G itself, if G is isomorphic to a galaxy.*

All automorphisms from the $Aut(G^*)$ can be constructed as follows:

– take an arbitrary permutation a of $\Gamma(G)$,

– for every g from $\Gamma(G)$ take an arbitrary isomorphism β_g from g to a(g),

– make the union of all β_g.

if G is not isomorphic to a galaxy multiply directly the union by $Aut(G)$.

The proposition does not give us an immediate description for the lattice of closed supergroups for the automorphism group. But it provides a possible base for further investigation.

The situation becomes more interesting and productive when the galaxies are not isomorphic to the original structure. They can have richer groups of authomorphisms

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representing 'imaginary' elements of the geometry of the original space G. The group $Aut(G^*)$ in this case is the direct product of $Aut(G)$ and the group acting on galaxies as described above.

Localness

From intuitive viewpoint discretely homogeneous structures should satisfy *localness* property: the value of any formula on them should not depend on the "distant" values of its variables. This is made precise by the following proposition.

Proposition 5. *For every formula $F(x_1, \ldots, x_n, y)$ in the signature of a discretely homogeneous graph G there exists a natural number w such that for every tuple $\bar{a} \in G$ and for every $b_1, b_2 \in G$ if $d(\odot, b_j) > w$ and $d(a_i, b_j) > w, j = 1, 2; 1 \leqslant i \leqslant n$ then $G \vDash (F(\bar{a}, b_1) \equiv F(\bar{a}, b_2))$*

Proof. Consider a structure GN with a domain which is the union of the graph G and the set of natural numbers $\mathbb{N}$. The signature of GN consists of the signature of G applied to elements of G only (false outside), as well as:

– two unary relations $G(x), N(x)$ defining the set of vertices of G and the set of natural numbers, respectively;

– ternary relation $D(a, b, k)$ meaning that a, b are vertices of the graph G with the distance between them greater than k and $NG \vDash N(k)$.

Let $F(x_1, \ldots, x_n, y)$ be a formula in the signature of graph G. Let

$$P(x_1, \ldots, x_n, y, w) \leftrightharpoons (G(y) \wedge N(w) \wedge D(\odot, y, w) \wedge \bigwedge_{i=1}^{n} (G(x_i) \wedge D(x_i, y, w)))$$

The statement to prove is

$$NG \vDash (\exists w)(\forall \bar{a}, b_1, b_2)(P(\bar{a}, b_1, w) \wedge P(\bar{a}, b_2, w)) \rightarrow (F(\bar{a}, b_1) \equiv F(\bar{a}, b_2)))$$

Consider a structure NG^* being an arbitrary non-standard extension of NG, in which the set $\mathbb{N}^* = \{x | NG^* \vDash N(x)\}$ contains non-standard elements and the structure $G^* = \{x | NG^* \vDash G(x)\}$ is a "non-standard" elementary extension of G.

Let $w_0 \in \mathbb{N}^* \setminus \mathbb{N}$ be an arbitrary non-standard element. If $P(\bar{a}, b_1, w_0)$ holds true then

– $b \in G^* \setminus G$ is a non-standard element, and

– the galaxies containing the elements a_i and b, respectively, do not intersect.

Furthermore, if

$$(P(\bar{a}, b_1, w_0) \wedge P(\bar{a}, b_2, w_0))$$

holds true then there exists an automorphism of G^* mapping b_1 to b_2 which is an identity on G and an identity on all galaxies not containing b_1, b_2. It means the formula

$$(\forall \bar{a}, b_1, b_2)((P(\bar{a}, b_1, w_0) \wedge P(\bar{a}, b_2, w_0)) \rightarrow (F(\bar{a}, b_1) \equiv F(\bar{a}, b_2)))$$

is satisfied in the structure NG^*, that is

$$NG^* \vDash (\forall \bar{a}, b_1, b_2)((P(\bar{a}, b_1, w) \wedge P(\bar{a}, b_2, w)) \rightarrow (F(\bar{a}, b_1) \equiv F(\bar{a}, b_2)))$$

for some $w \in \mathbb{N}$ and hence it is satisfied in NG too. □

3. Examples of discretely homogeneous structures

The important examples of homogeneous structures (alongside of rational numbers) are provided by random graphs. Further homogeneous structures were studied, whose homogeneity resembles homogeneity (universality) of random graphs [8]. The graphs we have studied are also homogeneous, but in a different sense — they are the same "almost everywhere"; (and in particular could be the same "everywhere").

In the following paragraphs we present corresponding examples. In each case it is not difficult to establish the property of discrete homogeneity. In some cases the property of 'strong' discrete homogeneity holds, where all neighbourhoods of a given radius are isomorphic.

Acyclic (free) non-oriented n-graph (that is a graph, in which the degree of every vertex is n). Discrete homogeneity holds for both graphs with labelled and not labelled edges.

Oriented n-tree with a root, in which for every vertex there are n outgoing edges, and for every vertex except the root there is one incoming edge. The variants with labelled and not labelled edges are possible. In the case of labelled edges we get an unoid, an algebraic structure. One can also consider trees without root.

The points of the plane with integer coordinates and **two successor** relations: the domain is $\mathbb{Z} \times \mathbb{Z}$; the signature with two binary relations:

$\langle\langle x, y\rangle, \langle x+1, y\rangle\rangle; \langle\langle x, y\rangle, \langle x, y+1\rangle\rangle$

The description of the authomorphisms group $Aut(\mathcal{M}^*)$ for these examples is given by the Proposition 4. For example in the case of $G = \mathbb{Z} \times \mathbb{Z}$ the group $Aut(G)$ is generated by shits along the axes and the full authomorphisms group is constructed similar to one-dimensional case.

The modifications of the proposed construction can be used to construct completions of the structure of rational numbers with binary relations for multiplication by rational constants, and plausibly, also to construct completions of the following structures:

– **positive rational numbers** with binary relations for **multiplication by rational constants**;

– the same for **non-negative rational numbers**.

4. Discussion, open problems and hypotheses

In the proposed approach the problem of construction of definability lattice for a structure can be split into two problems:

– finding an elementary extension of the structure, such that its lattice of closed supergroups is isomorphic to the definability lattice;

– finding the structure of closed supergroups of automorphisms for the elementary extension.

In present work we made a progress towards a solution of the first problem, which was trivial for considered previously homogeneous structures.

As we already noticed, definability lattices for all mentioned above structures (with the exception of $\langle\mathbb{Z}, +1\rangle$) are unknown to us.

As the first problem concerned, the structures considered in this work can be further extended by several unary relations. Among the classes of such relations one can identify the simplest: *periodic* (further generalizable to *almost periodic* [9,10]) and *sparse* with respect to successor relation — meaning that the distance between neighbouring elements within the union of unary relations tends to infinity.

Further direction is to extend the class of discretely homogeneous structures to the class which have not one but uniformly limited number of neighbourhoods that occurs infinitely.

We think that a version of 'quantifire elimination' can be conducted for our structures and the generalisations. The infinite signature can consist of atomic formulas with several vertices pointed and "distance is bigger then".

It looks plausible that the technique used in [1] could help to describe the lattices for both $\langle\mathbb{N}, +1\rangle$ and upward complete n-trees, considered above. In particular, the hypothesis for $\langle\mathbb{N}, +1\rangle$ can be formulated as follows:

– Every element of the lattice can be generated by a relation from the list $A_{0,d}$, $A_{1,d}$, $A_{2,d}$ generating an element of the lattice for $\langle\mathbb{Z}, +1\rangle$ as mentioned in the first section of the paper with a parameter d, possibly extended by several constants (unary one element relations) 0,..., $d-1$.

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At the moment it is not clear to which extent this technique can be transferred to the structures of higher dimensions: example $\mathbb{Z} \times \mathbb{Z}$ above, etc. The hypothesis, however, is that the lattice generators in this case will be the pairs/tuples of the elements corresponding to axes. New situation will occur if we add, for example, binary predicate of equality for the components of two-dimensional vectors.

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References

1. Semenov, A. L.; Soprunov, S. F. Lattice of definability (of reducts) for integers with successor. *Izvestiya: Mathematics* **2021**, *85:6*, 1257–1269.
2. Padoa, A. Essai d'une theorie algebraique des nombre entiers, precede d'une introduction logique a une theorie deductive quelconque. In *Bibliotheque du Congres international de philosophie*; Paris, France, 1901; pp. 309–365.
3. Tarski, A. What are logical notions? *History and philosophy of logic* **1986**, *7.2*, 143–154.
4. Svenonius, L. A theorem on permutations in models *Theoria* **1959**, *25.3*, 173–178.
5. Frasnay, C. Quelques problemes combinatoires concernant les ordres totaux et les relations monomorphes *Annales de l' institut Fourier* **1965**, *15:2*, 415–524.
6. Thomas, S. Reducts of the random graph *Journal of Symbolic Logic* **1991**, *56(1)*, 176–181.
7. Hodges, W. Model theory. In *Encyclopedia of Mathematics and its Applications*, vol. 42; Cambridge University Press; Cambridge, UK, 1993; pp. 309–365.
8. Macpherson, D. A survey of homogeneous structures *Discrete Mathematics* **2011**, *311.15*, 1599–1634.
9. Semenov, A. L. On certain extensions of the arithmetic of addition of natural numbers *Izvestiya: Mathematics* **1980**, *15.2*, 401–418.
10. Muchnik, A. A.; Pritykin, Y. L.; Semenov, A. L. Sequences Close to Periodic *Russian Mathematical Surveys* **2009**, *64(5)*, 805–871.