Article

Not peer-reviewed version

General Fractional Calculus Operators of Distributed Order

Mohammed Al-Refai and Yuri Luchko *

Posted Date: 7 November 2023

doi: 10.20944/preprints202311.0384.v1

Keywords: Sonin kernels; Sonin condition; general fractional derivatives; general fractional derivatives of distributed order; fractional integrals of distributed type.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 7 November 2023 doi:10.20944/preprints202311.0384.v1

Article

General Fractional Calculus Operators of Distributed Order

Mohammed Al-Refai [1] and Yuri Luchko [2,*]

[1] Dept. of Mathematics, Yarmouk University, Irbid, Jordan; m_alrefai@yu.edu.jo
[2] Dept. of Mathematics, Physics, and Chemistry, Berlin University of Applied Sciences and Technology, Berlin, Germany; luchko@bht-berlin.de
* Correspondence: luchko@bht-berlin.de

Abstract: In this paper, two types of the general fractional derivatives of distributed order as well as a corresponding fractional integral of distributed type are defined and their basic properties are investigated. The general fractional derivatives of distributed order are constructed for a special class of the one-parametric Sonin kernels with a power law singularities at the origin. The conventional fractional derivatives of distributed order based on the Riemann-Liouville and the Caputo fractional derivatives are particular cases of the general fractional derivatives of distributed order introduced in this paper.

Keywords: Sonin kernels; Sonin condition; general fractional derivatives; general fractional derivatives of distributed order; fractional integrals of distributed type

MSC: 26A33; 33E12; 35S10; 45K05

---

1. Introduction

The formal constructions of the operators nowadays referred to as the general fractional integrals (GFIs) and the general fractional derivatives (GFDs) were suggested for the first time by Sonin in [27]. In this paper, Sonin extended Abel's method for solving the Abel integral equation presented in [1,2] to a class of the integral equations with the so-called Sonin kernels. He recognized that the basic ingredient of Abel's method for solving the Abel integral equation (in the modern notations)

$$f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\tau)\, d\tau,\ t > 0 \tag{1}$$

is nothing else than a simple formula for the power law kernels

$$(h_\alpha * h_{1-\alpha})(t) \equiv 1,\ t > 0,\ h_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)},\ \alpha > 0, \tag{2}$$

and suggested its generalisation in the form

$$(\kappa * k)(t) \equiv 1,\ t > 0, \tag{3}$$

where $*$ stands for the Laplace convolution. Nowadays the condition (3) is referred to as the Sonin condition and the functions that satisfy this conditions are called the Sonin kernels.

The simplest pair of the Sonin kernels are the power law kernels h_α and $h_{1-\alpha}$ that was known already to Abel, see the relation (2). In [27], Sonin introduced an important class of the Sonin kernels that can be represented in the following form:

$$\kappa(t) = h_\alpha(t) \cdot \kappa_1(t),\ \kappa_1(t) = \sum_{k=0}^{+\infty} a_k t^k,\ a_0 \neq 0,\ 0 < \alpha < 1, \tag{4}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 7 November 2023 doi:10.20944/preprints202311.0384.v1

$$k(t) = h_{1-\alpha}(t) \cdot k_1(t),\; k_1(t) = \sum_{k=0}^{+\infty} b_k t^k, \tag{5}$$

where the functions $\kappa_1 = \kappa_1(t)$ and $k_1 = k_1(t)$ are analytical on $\mathbb{R}$ and their coefficients satisfy the following infinite system of linear equations with a triangular matrix:

$$a_0 b_0 = 1,\; \sum_{k=0}^{n} \Gamma(k+1-\alpha)\Gamma(\alpha+n-k) a_{n-k} b_k = 0,\; n = 1,2,3,\ldots.$$

In particular, he derived the famous pair of the Sonin kernels

$$\kappa(t) = (\sqrt{t})^{\alpha-1} J_{\alpha-1}(2\sqrt{t}),\; k(t) = (\sqrt{t})^{-\alpha} I_{-\alpha}(2\sqrt{t}),\; 0 < \alpha < 1$$

in terms of the Bessel function J_ν and the modified Bessel function I_ν.

Following Abel, Sonin formally solved the integral equation with the Sonin kernel κ

$$f(t) = (\kappa * u)(t) \;=\; \int_0^t \kappa(t-\tau) u(\tau)\, d\tau$$

and represented its solution in the form

$$u(t) = \frac{d}{dt}(k * f)(t) \;=\; \frac{d}{dt} \int_0^t k(t-\tau) u(\tau)\, d\tau,$$

where the kernel k is the Sonin kernel associated to the kernel κ through the Sonin condition (3). In particular, for the Abel integral equation (1), the solution takes the well-known form:

$$u(t) = \frac{d}{dt}(h_{1-\alpha} * f)(t) = \frac{d}{dt} \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} f(\tau)\, d\tau,\; t > 0. \tag{6}$$

However, Sonin did not interpreted his results in terms of the Fractional Calculus. This was done by Kochubei much later in his paper [13], where among other things he introduced and investigated the regularized GFD or the GFD of the Caputo type in the form

$$({}_*\mathbb{D}_{(k)}\, u)(t) = \frac{d}{dt}(k * u)(t) - u(0)k(t),\; t > 0 \tag{7}$$

with the Sonin kernel k from a special class $\mathcal{K}$ of kernels described in terms of their Laplace transforms (see [13] for details). Kochubei also considered the GFD of the Riemann-Liouville type

$$(\mathbb{D}_{(k)}\, u)(t) = \frac{d}{dt}\, (k * u)(t)\, d\tau,\; t > 0. \tag{8}$$

Evidently, the regularized GFD (7) and the GFD (8) are connected each to other by the relation

$$({}_*\mathbb{D}_{(k)}\, u)(t) = (\mathbb{D}_{(k)}\, [u(\cdot) - u(0)])(t) = (\mathbb{D}_{(k)}\, u)(t) - u(0)k(t),\; t > 0. \tag{9}$$

On the space of absolutely continuous functions, another useful representation of the regularized GFD is valid in the following form:

$$({}_*\mathbb{D}_{(k)}\, u)(t) = (k * u')(t),\; t > 0. \tag{10}$$

For the Sonin kernels k from the class $\mathcal{K}$, Kochubei showed existence of the associated completely monotonic Sonin kernels κ and introduced the corresponding GFI as follows:

$$(\mathbb{I}_{(\kappa)}\, u)(t) = (\kappa * u)(t) = \int_0^t \kappa(t-\tau) u(\tau)\, d\tau. \tag{11}$$

It is worth mentioning that the constructions of the GFI and GFDs with the Sonin kernels were first formally introduced by Sonin in [27]. However, their theory essentially depends on the kernels or on the classes of kernels and on the spaces of functions where these operators are studied. Say, even in the case of the power law Sonin kernels h_α and $h_{1-\alpha}$ that generate the Riemann-Liouville fractional integral and the Riemann-Liouville and Caputo fractional derivatives, the properties of these Fractional Calculus (FC) operators are very different on different spaces of functions. In [13], Kochubei introduced and investigated a very important special case of the GFI and GFDs with the Sonin kernels from the class $\mathcal{K}$. However, one of the conditions posed on the kernels from $\mathcal{K}$ (the Laplace transform of the kernel k has to be a Stieltjes function) is very restrictive. As a consequence, not all of the known Sonin kernels belong to this class.

In the subsequent publications devoted to the GFI and GFDs, essentially larger classes of the Sonin kernels compared to those suggested in [27] and [13] were introduced and investigated. In [15,16], the GFI and GFDs with the Sonin kernels from the class $\mathcal{S}_{-1}$ as well as fractional differential equations with these GFDs were studied. The Sonine kernels from $\mathcal{S}_{-1}$ are the functions continuous on $\mathbb{R}_+$ that have an integrable singularity of the power law type at the origin (see [15,16] for details). The class $\mathcal{S}_{-1}$ is a very general one and contains both the kernels (4), (5) introduced by Sonin and the Kochubei class $\mathcal{K}$ of the Sonin kernels.

In [17], the Sonin condition (3) was generalized and the GFI and GFDs of arbitrary order were introduced (the operators (7), (8), and (11) with the Sonin kernels have a "generalized order" less than one). In [18], the so-called 1st level GFD was suggested. This derivative is constructed as a composition of two GFIs with different kernels and the first order derivative and contains both the GFD (8) and the regularized GFD (10) as its particular cases. In the recent publications by Tarasov, the multi-kernel approach for definition of the GFI and GFDs, the general vector calculus based on the GFI and GFDs, and the Riesz form of the general FC operators in the multi-dimensional space were suggested, see [28,29,35], respectively. We also refer to the paper [5], where the left- and right-hand sided GFIs and GFDs on a finite interval were introduced and investigated. For an overview of the recent publications devoted to the fractional differential equations, both ordinary and partial, with the GFDs we refer to the recent survey paper [19].

It is worth mentioning that the GFI and GFDs introduced in the publications mentioned above have already found there first applications. In the papers [30–34], Tarasov suggested several non-local physical theories based on the GFI and GFDs with the Sonin kernels including the general fractional dynamics, the general non-Markovian quantum dynamics, the general non-local electrodynamics, the non-local classical theory of gravity, and the non-local statistical mechanics. Furthermore, the GFI and GFDs with the Sonin kernels were used in some mathematical models for anomalous diffusion and in the linear viscoelasticity, see, e.g., [6,7,11,21,22,25].

Another active direction of research in modern FC concerns an object that does not possess a counterpart in the world of integer order derivatives, namely, the so-called fractional derivatives of distributed order. This interest has a clear physical background. Recently, a lot of attention in modelling of the anomalous diffusion processes was attracted by the so-called ultra-slow diffusion that is characterized by the logarithmic behaviour of the mean squared displacement of the diffusing particles, see, e.g., [8,9,23,26], and the references therein. One of the most promising approaches for description of such processes is by means of the time-fractional diffusion equations with the fractional derivatives of distributed order. From the mathematical viewpoint, the fractional derivatives of distributed order and the fractional differential equations with these derivatives were studied, e.g., in [4,10,12,14,20,36].

Until now, the definitions of the fractional derivatives of distributed order were based on the conventional fractional derivatives and especially on the Riemann-Liouville and the Caputo derivatives. The main subject of this paper is in introducing a concept of the GFDs of distributed order and in investigation of their basic properties. These operators are a generalization of both the GFDs and the fractional derivatives of distributed order introduced so far. Moreover, in this paper, we also define the

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 7 November 2023 doi:10.20944/preprints202311.0384.v1

corresponding fractional integrals of distributed type and prove the fundamental theorems of FC for these integrals and the GFDs distributed order.

The rest of the paper is organized as follows: In the 2nd Section, a one-parametric class of the Sonin kernels is introduced and the GFDs of distributed order with the kernels from this class are defined and investigated. The 3rd Section is devoted to fractional integral of distributed type and some connections between the GFDs and the fractional integral of distributed type in form of two fundamental theorems of FC. In the final 4th Section, some examples of the GFDs of distributed order and the corresponding fractional integrals of distributed type are presented.

2. General fractional derivatives of distributed order

First we remind the readers on the definitions of the Riemann-Liouville and the Caputo fractional derivatives of distributed order on the interval $(a,\, b), 0 \le a < b \le 1$, respectively:

$$(\mathbb{D}^{w} u)(t) = \int_a^b (D_{0+}^{\alpha} u)(t)\, w(\alpha)\, d\alpha, \tag{12}$$

$$({}_*\mathbb{D}^{w} u)(t) = \int_a^b ({}_*D_{0+}^{\alpha} u)(t)\, w(\alpha)\, d\alpha, \tag{13}$$

where the weight function w satisfies the properties $w \in C([a,b]),\, w(\alpha) \ge 0\ \forall \alpha \in [a,b]$ and $w(\alpha) \not\equiv 0$, the Riemann-Liouville fractional derivative of the order $\alpha,\ 0 < \alpha < 1$ is defined by

$$(D_{0+}^{\alpha} u)(t) = \frac{d}{dt}\,(h_{1-\alpha} * u)(t),\ t > 0, \tag{14}$$

and the Caputo fractional derivative of the order $\alpha,\ 0 < \alpha < 1$ is given by

$$({}_*D_{0+}^{\alpha} u)(t) = (h_{1-\alpha} * u')(t),\ t > 0. \tag{15}$$

Please note that in some FC publications, the distributed order derivatives (12) and (13) are defined on the interval $(a,\, b)$ with $0 \le a < b \le 2$. However, in this paper, we restrict ourselves to the case $0 \le a < b \le 1$.

For a generalization of the distributed order fractional derivatives (12) and (13) to the case of the GFDs of distributed order, we need a class of the Sonin kernels that explicitly depend on a parameter that can be interpreted as a "generalized order" of the corresponding GFDs.

In what follows, we deal with the Sonin kernels $k_\alpha(t)$ and their associated kernels $\kappa_\alpha(t)$ that satisfy the following two constraints:

C1) The Sonin condition

$$(k_\alpha * \kappa_\alpha)(t) \equiv 1,\ t > 0 \tag{16}$$

holds valid for all $\alpha \in (a,\, b)$, where $0 \le a < b \le 1$.

C2) The kernels k and κ can be represented as follows:

$$k_\alpha(t) = h_{1-\alpha}(t) k_1(t,\alpha),\ \kappa_\alpha(t) = h_\alpha(t)\kappa_1(t,\alpha), \tag{17}$$

where $k_1,\, \kappa_1 \in C([0,+\infty) \times [a,b])$ and $k_1(0,\alpha) \neq 0,\ \kappa_1(0,\alpha) \neq 0,\ t > 0,\ \alpha \in (a,\, b)$.

The class of the Sonin kernels that satisfy the conditions C1) and C2) will be denoted by ${}_*S_\alpha$. Evidently, any associated kernel κ_α to a kernel $k_\alpha \in {}_*S_\alpha$ also belongs to the class ${}_*S_\alpha$ and k_α is its associated kernel.

Please note that the kernels from the class ${}_*S_\alpha$ are functions of two variables: $t\ (t \in \mathbb{R}_+)$ and $\alpha\ (\alpha \in (a,\, b))$. The simplest and very important example of this kind are the Sonin kernels $k_\alpha(t) = h_{1-\alpha}(t),\ \alpha \in (0,1)$ and $\kappa_\alpha(t) = h_\alpha(t),\ \alpha \in (0,1)$ of the Riemann-Liouville fractional derivative and the Riemann-Liouville fractional integral, respectively.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 7 November 2023 doi:10.20944/preprints202311.0384.v1

In general, the Sonin kernels from the class ${}_*S_\alpha$ and their associated kernels are the power law functions $h_{1-\alpha}(t)$ and $h_\alpha(t)$ that are disturbed by (multiplied with) some continuous functions (compare to the kernels (4) and (5) introduced by Sonin). These continuous functions can depend on α and/or other parameters or not. Most of the known Sonin kernels (see, e.g., [15,16]) belong to the class ${}_*S_\alpha$ and thus our theory will cover many known particular cases including those presented in the examples from our paper, see the 4th Section. However, as mentioned in [24], the Sonin kernels can also possess other kinds of singularities at the origin, say, the ones of the power-logarithmic type. Such kernels are not covered by our theory.

It is also worth mentioning that for any fixed $\alpha \in (a, b)$, the Sonin kernels from the class ${}_*S_\alpha$ are functions of the t-variable that belong to the space $C_{-1}(0, +\infty)$ defined as follows:

$$C_{-1}(0,+\infty) = \{u : u = t^p v(t),\ t > 0,\ p > -1,\ v \in C[0,\infty)\}. \tag{18}$$

The space $C_{-1}(0, +\infty)$ was employed in several publications devoted to the GFIs and GFDs with the Sonin kernels from the class $\mathcal{S}_{-1}$ (see, e.g., [15,16]):

$$(\kappa,\ k \in \mathcal{S}_{-1}) \ \Leftrightarrow\ (\kappa,\ k \in C_{-1}(0,+\infty)) \wedge ((\kappa * k)(t) \ =\ 1,\ t > 0). \tag{19}$$

For a fixed $\alpha \in (a, b)$, the Sonin kernels from the class ${}_*S_\alpha$ introduced above evidently belong to the class $\mathcal{S}_{-1}$. Thus, we can use the results derived in these publications for analysis of the GFDs and GFIs with the kernels from the class ${}_*S_\alpha$.

In what follows, we also employ another useful subspace of the space $C_{-1}(0, +\infty)$ defined by

$$C^n_{-1}(0,+\infty) = \{u : u^{(n)} \in C_{-1}(0,+\infty)\},\ n \in \mathbb{N}. \tag{20}$$

For a kernel $k_\alpha \in {}_*S_\alpha$, the GFD (8) and the regularized GFD (10) are defined for all values of α from the interval (a, b) as follows:

$$(\mathbb{D}_{k_\alpha} u)(t) = \frac{d}{dt}\int_0^t k_\alpha(t-\tau)u(\tau)\,d\tau,\ t > 0, \tag{21}$$

$$({}_*\mathbb{D}_{k_\alpha} u)(t) = \int_0^t k_\alpha(t-\tau)u'(\tau)\,d\tau,\ t > 0. \tag{22}$$

Now we proceed with the definitions of the corresponding general fractional derivatives of distributed order.

Definition 2.1. *Let a kernel k_α belong to the class ${}_*S_\alpha$ and a weight function w satisfy the conditions $w \in C([a,b])$, $w(\alpha) \geq 0$, $\alpha \in [a,b]$, and $w(\alpha) \not\equiv 0$, $\alpha \in [a,b]$.*

The general distributed order fractional derivative (GDOFD) of the Riemann-Liouville type and the regularized or Caputo type GDOFD are defined as follows, respectively:

$$(\mathbb{D}^w_{k_\alpha} u)(t) = \int_a^b (\mathbb{D}_{k_\alpha} u)(t) w(\alpha) d\alpha, \tag{23}$$

$$({}_*\mathbb{D}^w_{k_\alpha} u)(t) = \int_a^b ({}_*\mathbb{D}_{k_\alpha} u)(t) w(\alpha) d\alpha. \tag{24}$$

Remark 2.2. *The GDOFDs (23) and (24) are well defined in particular for the functions from the space $C^1_{-1}(0, +\infty)$. Indeed, let the inclusion $u \in C^1_{-1}(0, +\infty)$ hold true. Then*

$$u' = t^\beta u_1(t),\ t > 0,\ \beta > -1,\ u_1 \in C[0,\infty).$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 7 November 2023 doi:10.20944/preprints202311.0384.v1

The regularized GFD with a kernel $k_\alpha \in {}_*S_\alpha$ takes the form

$$({}_*\mathbb{D}_{k_\alpha} u)(t) = \int_0^t h_{1-\alpha}(t-\tau) k_1(t-\tau,\alpha)\tau^\beta u_1(\tau)d\tau.$$

Because $h_{1-\alpha}(t-\tau)\tau^\beta$ is integrable and of one sign and $k_1(t-\tau,\alpha)u_1(\tau)$ is continuous, applying the mean value theorem for the last integral yields

$$\begin{aligned}({}_*\mathbb{D}_{k_\alpha} u)(t) &= u_1(\tau_0)k_1(t-\tau_0,\alpha)\int_0^t h_{1-\alpha}(t-\tau)\tau^\beta d\tau \\ &= u_1(t-\tau_0)k_1(\tau_0,\alpha)\frac{\Gamma(\beta+1)}{\Gamma(\beta+2-\alpha)}t^{\beta+1-\alpha}. \end{aligned} \tag{25}$$

Due to the inequalities $\beta > -1$ and $0 < \alpha < 1$, we thus arrive at the inclusion

$$({}_*\mathbb{D}_{k_\alpha} u)(t) = u_1(t-\tau_0)k_1(\tau_0,\alpha)\frac{\Gamma(\beta+1)}{\Gamma(\beta+2-\alpha)}t^{\beta+1-\alpha} \in C((0,\infty)\times[a,b]),$$

which proves that the regularized GDOFD (24) is well-defined on the space $C^1_{-1}(0,+\infty)$.

To prove the same statement for the GDOFD (23), we employ the result proved above and the relation (9) between the regularized GFD (7) and the GFD (8) that is valid on the space $C^1_{-1}(0,+\infty)$.

Now we derive a useful representation of the regularized GDOFD (24) that by definition is the following iterated integral

$$\begin{aligned}({}_*\mathbb{D}^w_{k_\alpha} u)(t) &= \int_a^b ({}_*\mathbb{D}_{k_\alpha} u)(t)w(\alpha)d\alpha \\ &= \int_a^b \int_0^t k_\alpha(t-\tau)u'(\tau)d\tau\, w(\alpha)d\alpha. \end{aligned}$$

The representation (25) ensures that the corresponding double integral

$$\int_a^b \int_0^t k_\alpha(t-\tau)u'(\tau)w(\alpha)d\tau d\alpha$$

is absolutely integrable. Thus, by using Fubini's theorem we can interchange the order of integration in the iterated integral and get the following formula:

$$\begin{aligned}({}_*\mathbb{D}^w_{k_\alpha} u)(t) &= \int_0^t u'(\tau)\int_a^b k_\alpha(t-\tau)w(\alpha)d\alpha\, d\tau \\ &= \int_0^t k^w(t-\tau)u'(\tau)d\tau, \qquad (26) \\ &= (k^w * u')(t), \qquad (27) \end{aligned}$$

where

$$k^w(t) = \int_a^b k_\alpha(t)w(\alpha)d\alpha, \tag{28}$$

and $*$ denotes the Laplace convolution.

Remark 2.3. *The last formula looks like the representation (10) of the regularized GFD with the kernel k^w. Because the kernel k_α belongs to the space $C_{-1}(0,+\infty)$ for all $\alpha \in (a,b)$, we have the inclusion $k^w(t) = \int_a^b k_\alpha(t)w(\alpha)d\alpha \in C_{-1}(0,+\infty)$. Moreover, as we see in the next section, under some additional conditions, the function k^w is a Sonin kernel from the class $\mathcal{S}_{-1}$ and thus the GDOFD (24) can be interpreted as a regularized GFD (10) with the kernel $k^w \in \mathcal{S}_{-1}$. In its turn, this means that the GDOFDs with such kernels and weight functions are a special subclass of the regularized GFDs with the Sonin kernels from the class $\mathcal{S}_{-1}$. Thus, one*

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 7 November 2023 doi:10.20944/preprints202311.0384.v1

can employ the results derived in the publications devoted to the GFDs (see, e.g., [5,15–17]) for investigation of the GDOFDs.

Applying the same procedure to the GDOFD (23), we arrive at the analogous representation

$$(\mathbb{D}^w_{k_\alpha} u)(t) = \frac{d}{dt}(k^w * u)(t), \tag{29}$$

where the kernel k^w is defined as in (28).

As in the case of the GDOFD (24), under some additional conditions (see the next section), the GDOFD (23) can be interpreted as a GFD of the Riemann-Liouville type with the kernel $k^w \in \mathcal{S}_{-1}$.

Now we discuss an important relation between the GDOFD (23) and the regularized GDOFD (24). To derive it, we employ the relation

$$(\mathbb{D}_{k_\alpha} u)(t) = ({}_*\mathbb{D}_{k_\alpha} u)(t) + u(0)k_\alpha(t),\ \alpha \in (a,\, b) \tag{30}$$

that holds true for any $u \in C^1_{-1}(0,\infty)$ and for any kernel from the class $\mathcal{S}_{-1}$, see [15,16]. As already mentioned, for a fixed $\alpha \in (a, b)$, the kernels from the class ${}_*\mathcal{S}_\alpha$ belong to the class $\mathcal{S}_{-1}$. Using the formula (30) we thus arrive at the relation

$$\begin{aligned}
(\mathbb{D}^w_{k_\alpha} u)(t) &= \int_a^b (\mathbb{D}_{k_\alpha} u)(t) w(\alpha) d\alpha \\
&= \int_a^b \Big(({}_*\mathbb{D}_{k_\alpha} u)(t) + u(0)k_\alpha(t) \Big) w(\alpha) d\alpha \\
&= ({}_*\mathbb{D}^w_{k_\alpha} u)(t) + u(0) \int_a^b k_\alpha(t) w(\alpha) d\alpha \\
&= ({}_*\mathbb{D}^w_{k_\alpha} u)(t) + u(0) k^w(t).
\end{aligned} \tag{31}$$

Some examples of the GDOFDs introduced above are provided in Section 4.

3. General fractional integrals of distributed type

To introduce the general fractional integrals of distributed type (GFIDs), in what follows, we impose some additional conditions on the kernels $k_\alpha(t)$ from the class ${}_*\mathcal{S}_\alpha$:

C3) The Laplace transform

$$K_\alpha(\rho) = (\mathcal{L}\, k_\alpha)(\rho) := \int_0^\infty e^{-\rho t} k_\alpha(t) dt$$

exists for all $\rho > 0$,

C4) The Laplace transform $K_\alpha(\rho)$ satisfies the following standard conditions:

$$\rho K_\alpha(\rho) \to \infty, \text{ as } \rho \to \infty, \tag{32}$$

$$\rho K_\alpha(\rho) \to 0, \text{ as } \rho \to 0. \tag{33}$$

The class of the kernels from ${}_*\mathcal{S}_\alpha$ that satisfy the conditions C3) and C4) will be denoted by ${}_*\hat{\mathcal{S}}_\alpha$. In what follows, we always consider the operators with the kernels from the class ${}_*\hat{\mathcal{S}}_\alpha$, i.e., the kernels that satisfy the conditions C1)-C4).

For a definition of the GFIDs, we need some auxiliary results. First, we mention an evident relation

$$K^w(\rho) := (\mathcal{L}\, k^w)(\rho) = \int_a^b K_\alpha(\rho) w(\alpha) d\alpha \tag{34}$$

between the Laplace transform $K^w(\rho)$ of the kernel k_w given by (28) and the Laplace transform $K_\alpha(\rho)$ of the kernel k_α. Moreover, for $k_\alpha \in {}_*\hat{\mathcal{S}}_\alpha$, the Laplace transform $K^w(\rho)$ satisfy the conditions (32) and (33).

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 7 November 2023 doi:10.20944/preprints202311.0384.v1

The last formula, the Laplace convolution theorem, and the known Laplace transform formula for the first order derivative result in the following useful representation of the Laplace transform for the regularized GDOFD (24):

$$(\mathcal{L}\,{}_*\mathbb{D}^w_{k_\alpha}\,u)(\rho) = K^w(\rho)(\rho(\mathcal{L}\,u)(\rho) - u(0)). \tag{35}$$

Now let us consider the following initial-value problem for the fractional differential equation with the regularized GDOFD (24):

$$({}_*\mathbb{D}^w_{k_\alpha}u)(t) = g(t),\ u(0) = 0. \tag{36}$$

Assuming existence of the Laplace transform of the function g for all $\rho > 0$, we apply the Laplace transform to the above equation and using the formula (35), we get the relation

$$K^w(\rho)\rho(\mathcal{L}\,u)(\rho) = (\mathcal{L}\,g)(\rho).$$

The solution u to the problem (36) can be formally represented as follows:

$$u(t) = \left(\mathcal{L}^{-1}\frac{(\mathcal{L}\,g)(\rho)}{\rho K^w(\rho)}\right)(t).$$

The convolution theorem for the Laplace transform leads then to the representation

$$u(t) = (\psi_w * g)(t) \tag{37}$$

of the solution u in the time-domain, where the kernel ψ_w is defined in terms of the inverse Laplace transform:

$$\psi_w(t) = \left(\mathcal{L}^{-1}\frac{1}{\rho K^w(\rho)}\right)(t). \tag{38}$$

The function ψ_w will play the role of a kernel of the GFID. In the following theorem, we provide some important characteristics of this function.

Theorem 3.1. *Let the kernel k_α be from the class ${}_*\hat{S}_\alpha$.*

Then the function ψ_w defined by (38) belongs to the space $C_{-1}(0,\infty)$. Moreover, the functions k^w and ψ_w form a pair of the Sonine kernels from the class $\mathcal{S}_{-1}$.

Proof. By definition, any kernel $k_\alpha \in {}_*\hat{S}_\alpha$ can be represented in the form

$$k_\alpha(t) = h_{1-\alpha}(t)k_1(t,\alpha),\ k_1(t,\alpha) \in C([0,\infty)\times[a,b]), k_1(0,\alpha) \neq 0,\ a < \alpha < b.$$

For the Laplace transform $K_\alpha(\rho)$, we have the expression

$$K_\alpha(\rho) = (\mathcal{L}\,k_\alpha)(\rho) = \int_0^\infty e^{-\rho t}k_\alpha(t)dt = \int_0^\infty e^{-\rho t}h_{1-\alpha}(t)k_1(t,\alpha)dt.$$

Because $k_1(t,\alpha) \in C([0,\infty)\times[a,b])$ and $e^{-\rho t}h_{1-\alpha}(t) \in L^1(0,\infty)$, applying the mean value theorem to the last integral yields the representation

$$K_\alpha(\rho) = k_1(t_0,\alpha)\int_0^\infty e^{-\rho t}h_{1-\alpha}(t)dt = k_1(t_0,\alpha)\int_0^\infty \frac{e^{-\rho t}\,t^{-\alpha}}{\Gamma(1-\alpha)}dt = \frac{k_1(t_0,\alpha)}{\rho^{1-\alpha}},$$

where $t_0 > 0$ and $k_1(t_0,\alpha) \neq 0$.

 doi:10.20944/preprints202311.0384.v1

Taking into account the last formula and the relation (34), the Laplace transform $K^w(\rho)$ of the kernel k_w takes the form

$$K^w(\rho) = \int_a^b K_\alpha(\rho) w(\alpha) d\alpha = \int_a^b \frac{k_1(t_0,\alpha)}{\rho^{1-\alpha}} w(\alpha) d\alpha = \frac{1}{\rho} \int_a^b k_1(t_0,\alpha) \rho^\alpha w(\alpha) d\alpha.$$

Applying the mean value theorem to the last integral, we arrive at the representation

$$K^w(\rho) = \rho^{\alpha_0 - 1} W k_1(t_0, \alpha_0)$$

with $\alpha_0 \in (a,b)$ and $W = \int_a^b w(\alpha) d\alpha > 0$. Thus, the relation

$$\frac{1}{\rho K^w(\rho)} = \frac{1}{k_1(t_0,\alpha_0) W} \frac{1}{\rho^{\alpha_0}}$$

holds true. The inverse Laplace transform of the right-hand side of the last formula is well-known and we arrive at the representation

$$\psi_w(t) = \left(\mathcal{L}^{-1} \frac{1}{\rho K^w(\rho)} \right)(t) = \frac{1}{k_1(t_0,\alpha_0) W} \frac{t^{\alpha_0 - 1}}{\Gamma(\alpha_0)}$$

that immediately implicates the inclusion $\psi_w \in C_{-1}(0,\infty)$.

By definition $(\mathcal{L}\,\psi_w)(\rho) = \frac{1}{\rho K^w(\rho)}$. Then we get the relation

$$(\mathcal{L}\,\psi_w)(\rho)(\mathcal{L}\,k^w)(\rho) = \frac{1}{\rho}$$

that in time-domain can be rewritten as

$$(k^w * \psi_w)(t) = 1, \; t > 0.$$

Thus, the functions ψ_w and k^w belong to the space $C_{-1}(0,\infty)$ and form a pair of the Sonine kernels, i.e., $\psi_w, k^w \in \mathcal{S}_{-1}$. □

Motivated by the form (37) of the solution to the fractional differential equation (36) and by Theorem 3.1, we now proceed with defining the GFIDs.

Definition 3.2. *Let the kernel k_α be from the class ${}_*\hat{S}_\alpha$.*

The general fractional integral operator of distributed type (GFID) is defined by

$$(\mathbb{I}_{\psi_w} u)(t) = (\psi_w * u)(t) = \int_0^t \psi_w(t-\tau) u(\tau) d\tau, \tag{39}$$

where the function ψ_w is as in (38).

Remark 3.3. *For a kernel $k_\alpha \in {}_*\hat{S}_\alpha$, let κ_α be its associated Sonin kernel. The Sonin condition in Laplace domain takes the form*

$$(\mathcal{L}\,k_\alpha)(\rho) \cdot (\mathcal{L}\,\kappa_\alpha)(\rho) = \frac{1}{\rho}.$$

Then we get the formulas

$$K^w(\rho) = \int_a^b K_\alpha(\rho) w(\alpha) d\alpha = \int_a^b \frac{1}{\rho(\mathcal{L}\,\kappa_\alpha)(\rho)} w(\alpha) d\alpha$$

and

$$\frac{1}{\rho K^w(\rho)} = \left(\int_a^b \frac{1}{(\mathcal{L}\,\kappa_\alpha)(\rho)} w(\alpha) d\alpha \right)^{-1}.$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 7 November 2023 doi:10.20944/preprints202311.0384.v1

Thus, we arrive at another representation of the kernel ψ_w in terms of the kernel κ_α of the corresponding GFI:

$$\psi_w(t) = \left(\mathcal{L}^{-1} \left(\int_a^b \frac{1}{(\mathcal{L}\,\kappa_\alpha)(\rho)} w(\alpha) d\alpha \right)^{-1} \right)(t).$$

As shown in Theorem 3.1, the kernel ψ_w of the GFID (39) is from the class $\mathcal{S}_{-1}$ of the Sonin kernels. Thus, the operator (39) is a special case of the GFIs with the kernels from $\mathcal{S}_{-1}$ and we can employ the results already derived for the GFIs on the space $C_{-1}(0,\infty)$ (see, e.g., [15–17] and subsequent publications). In particular, the following properties are worth mentioning:

$$\begin{aligned} \mathbb{I}_{\psi_w} &: C_{-1}(0,\infty) \to C_{-1}(0,\infty) \text{ (mapping property)}, \\ \mathbb{I}_{\psi_{w_1}} \mathbb{I}_{\psi_{w_2}} &= \mathbb{I}_{\psi_{w_2}} \mathbb{I}_{\psi_{w_1}} \text{ (commutativity law)} \\ \mathbb{I}_{\psi_{w_1}} \mathbb{I}_{\psi_{w_2}} &= \mathbb{I}_{\psi_{w_1} * \psi_{w_2}} \text{ (index law).} \end{aligned} \quad (40)$$

According to Theorem 3.1, the kernel k^w of the GDOFD (23) and of the regularized GDOFD (24) is an associated Sonin kernel to the kernel ψ_w of the GFID (39). Thus, we can apply the first and the second fundamental theorems for the GFDs and the GFIs with the Sonin kernels from the class $\mathcal{S}_{-1}$ derived in [15] and arrive at the following important results:

Theorem 3.4 (1st Fundamental theorem for the distributed order fractional operators)**.** *Let the kernel k_α be from the class $_*\hat{S}_\alpha$.*

Then the GDOFD (23) and the regularized GDOFD (24) are the left-inverse operators to the GFID (39):

$$(\mathbb{D}^w_{k_\alpha} \mathbb{I}_{\psi_w} u)(t) = u(t),\ u \in C_{-1}(0,\infty),\ t > 0 \quad (41)$$

$$({}_*\mathbb{D}^w_{k_\alpha} \mathbb{I}_{\psi_w} u)(t) = u(t),\ u \in C^1_{-1}(0,\infty),\ t > 0. \quad (42)$$

Theorem 3.5 (2nd Fundamental theorem for the distributed order fractional operators)**.** *Let the kernel k_α be from the class $_*\hat{S}_\alpha$ and $u \in C^1_{-1}(0,\infty)$.*

Then the relations

$$(\mathbb{I}_{\psi_w}\, {}_*\mathbb{D}^w_{k_\alpha} u)(t) = u(t) - u(0),\ t > 0 \quad (43)$$

$$((\mathbb{I}_{\psi_w}\, \mathbb{D}^w_{k_\alpha} u)(t) = u(t),\ t > 0 \quad (44)$$

hold valid.

For the proofs of the fundamental theorems for the GFDs and the GFIs with the Sonin kernels from the class $\mathcal{S}_{-1}$ we refer the interested readers to [15].

4. Examples of the general fractional operators of distributed order

In this section, we discuss three particular examples of the Sonin kernels from the class $_*\hat{S}_\alpha$ and the corresponding general fractional operators of distributed order.

1st Example: We start with the power law kernel $k_\alpha(t) = h_{1-\alpha}(t)$ of the Riemann-Liouville and the Caputo fractional derivatives with the associated kernel $\kappa_\alpha(t) = h_\alpha(t)$ of the Riemann-Liouville fractional integral.

In this case, the GDOFDs introduced in this paper are nothing else than the Riemann-Liouville and the Caputo fractional derivatives of distributed order on the interval (0, 1) defined as in (12) and (13), respectively. As mentioned in Introduction, the distributed order fractional derivatives of the Riemann-Liouville and the Caputo types are well-studied (see, e.g., [4,10,12,14,20], and [36]) and have many applications. In this example, we look at these operators from the viewpoint of our general theory.

For the kernels $k_\alpha(t) = h_{1-\alpha}(t)$ and $\kappa_\alpha(t) = h_\alpha(t)$, we set $k_1(t,\alpha) = \kappa_1(t,\alpha) = 1$ and $a = 0,\ b = 1$ and show now that they belong to the class ${}_*\hat{S}_\alpha$ of the Sonin kernels introduced in Section 2. Indeed, they evidently satisfy the conditions C1) and C2).

Moreover, the Laplace transform $K_\alpha(\rho)$ of the kernel $k_\alpha(t) = h_{1-\alpha}(t)$ does exist for $\rho > 0$ and can be written down in explicit form:

$$K_\alpha(\rho) = (\mathcal{L}\, k_\alpha)(\rho) = \rho^{\alpha-1},\ \ \rho > 0.$$

For the function $K_\alpha(\rho) = \rho^{\alpha-1}$, the conditions (32) and (33) are evidently satisfied and thus the kernel $k_\alpha(t) = h_{1-\alpha}(t)$ belongs to the class ${}_*\hat{S}_\alpha$ of the Sonin kernels.

This means that all of the results that were presented in the previous sections including the properties of the corresponding GFID and the fundamental theorems of FC for the GDOFDs and the GFID hold true. However, we found instructive to do some independent calculations and derivations and to establish some explicit formulas that are not possible in the general case.

For the power law kernels, the formula (34) takes the form

$$K^w(\rho) = (\mathcal{L}\, k^w)(\rho) = \int_0^1 \rho^{\alpha-1} w(\alpha)\, d\alpha.$$

Because $w \in C[0,1]$ and $\rho^{\alpha-1}$ is integrabale and of one sign for $0 < \alpha < 1$, applying the mean value theorem for the last integral yields the relation

$$K^w(\rho) = w(\alpha_0) \int_0^1 \rho^{\alpha-1} d\alpha = w(\alpha_0) \frac{\rho - 1}{\rho \ln(\rho)}, \tag{45}$$

for some α_0 $(0 < \alpha_0 < 1)$. We also mention that $w(\alpha_0) \neq 0$ because of the evident inequality $K^w(\rho) > 0$ for any $\rho > 0$.

Now we get the representation

$$\frac{1}{\rho K^w(\rho)} = \frac{1}{w(\alpha_0)} \frac{\ln(\rho)}{\rho - 1}.$$

Because the function at the right-hand of the last formula has finitely many singular point and tends to 0 as $\rho \to \infty$, its inverse Laplace transform

$$\psi_w(t) = \frac{1}{w(\alpha_0)} \left(\mathcal{L}^{-1} \frac{\ln(\rho)}{\rho - 1} \right)(t)$$

is well-defined and can be represented in explicit form ([3], p. 1027):

$$\psi_w(t) = \frac{1}{w(\alpha_0)} \left(\mathcal{L}^{-1} \frac{\ln(\rho)}{\rho - 1} \right)(t) = \frac{1}{w(\alpha_0)} e^t E_1[t],$$

where E_1 is the exponential integral $E_1[t] := \int_1^\infty \frac{e^{-tx}}{x} dx$.

2nd Example: In this example, we consider the kernels

$$k_\alpha(t) = h_{1-\alpha}(t)\, exp(-\mu t),\ \kappa_\alpha(t) = h_\alpha(t)\, exp(-\mu t) + \mu \int_0^t h_\alpha(s)\, exp(-\mu s) ds$$

with $\mu \geq 0$. It is well-known (see, e.g. [15]) that for any $\alpha \in (0,\, 1)$ the function $k_\alpha(t)$ is a Sonin kernel and $\kappa_\alpha(t)$ is its associated Sonin kernel. Furthermore, direct calculations show that these kernels are from the class ${}_*S_\alpha$ of the Sonin kernels with $a = b = 1$.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 7 November 2023 doi:10.20944/preprints202311.0384.v1

The Laplace transform $K_\alpha(\rho)$ of the kernel k_α can be explicitly evaluated:

$$K_\alpha(\rho) = (\mathcal{L}\, k_\alpha)(\rho) = \frac{1}{(\rho+\mu)^{1-\alpha}},\ \rho+\mu > 0.$$

For this function, the condition C4) is also satisfied and thus the kernels k_α and κ_α belong to the class $_*\hat{S}_\alpha$ of the Sonin kernels.

Then we proceed with the formula (34) that takes the form

$$K^w(\rho) = (\mathcal{L}\, k^w)(\rho) = \int_0^1 \frac{1}{(\rho+\mu)^{1-\alpha}} w(\alpha)d\alpha = \frac{1}{(\rho+\mu)} \int_0^1 (\rho+\mu)^\alpha w(\alpha)d\alpha.$$

Because $(\rho+\mu)^\alpha \in C[0,1]$ and the weight function w is integrabale and of one sign for $0<\alpha<1$, applying the mean value theorem for the last integral yields the relation

$$K^w(\rho) = \frac{1}{(\rho+\mu)}(\rho+\mu)^{\alpha_0} W = (\rho+\mu)^{\alpha_0-1}W, \tag{46}$$

for some α_0 $(0<\alpha_0<1)$ and with $W = \int_0^1 w(\alpha)d\alpha$.

Then we get the formula

$$\frac{1}{\rho K^w(\rho)} = \frac{1}{W}\frac{1}{\rho}\frac{\rho+\mu}{(\rho+\mu)^{\alpha_0}} = \frac{1}{W}\left(\frac{1}{(\rho+\mu)^{\alpha_0}} + \frac{\mu}{\rho(\rho+\mu)^{\alpha_0}}\right).$$

By definition, the kernel function $\psi_w(t)$ of the GFID is the inverse Laplace transform of the last expression. Thus we arrive at the representation

$$\psi_w(t) = \frac{1}{W}\left(\frac{t^{\alpha_0-1}}{\Gamma(\alpha_0)} exp(-\mu t) + \mu\int_0^t \frac{s^{\alpha_0-1}}{\Gamma(\alpha_0)} exp(-\mu s)ds)\right).$$

As we see, the kernel ψ_w has the from of the kernel κ_α with a certain $\alpha_0 \in (0,1)$ that depend on μ and the weight function w. Thus, it is well-defined and belongs to the class $\mathcal{S}_{-1}$ of the Sonin kernels as stated in Theorem 3.1.

3rd Example: In the last example, we consider the Sonin kernels (see, e.g. [15])

$$k_\alpha(t) = t^{-\alpha}E_{\gamma,1-\alpha}(-t^\gamma),\ \kappa_\alpha(t) = h_\alpha(t) + h_{\alpha+\gamma}(t),\ t>0,\ 0<\alpha,\ \gamma<1.$$

First we represent the kernel k_α in the form

$$k_\alpha(t) = h_{1-\alpha}(t)\Gamma(1-\alpha)E_{\gamma,1-\alpha}(-t^\gamma).$$

It is easy to verify that $k_\alpha(t) \in {}_*S_\alpha$ provided that $0 \le a < b < 1$. Similarly, $\kappa_\alpha(t) \in {}_*S_\alpha$, provided that $0 < a < b \le 1$. Thus, $k_\alpha, \kappa_\alpha \in {}_*S_\alpha$, provided that $0<a<b<1$ (in the previous examples, we had the case $a=0$ and $b=1$). In the further derivations, we assume that this condition holds valid.

The Laplace transform $K_\alpha(\rho)$ of the kernel k_α can be explicitly evaluated:

$$K_\alpha(\rho) = (\mathcal{L}\, k_\alpha)(\rho) = \frac{\rho^{\gamma+\alpha-1}}{\rho^\gamma+1},\ \rho>0.$$

For $0<\alpha,\ \gamma<1$, this function satisfies the condition C4) and thus the kernels k_α and κ_α belong to the class $_*\hat{S}_\alpha$ of the Sonin kernels.

 doi:10.20944/preprints202311.0384.v1

The formula (34) takes now the form

$$K^w(\rho) = (\mathcal{L}\, k^w)(\rho) = \int_a^b \frac{\rho^{\gamma+\alpha-1}}{\rho^\gamma+1} w(\alpha)d\alpha = \frac{\rho^{\gamma-1}}{\rho^\gamma+1}\int_a^b \rho^\alpha w(\alpha)d\alpha.$$

Because $\rho^\alpha \in C[a,b]$ and w is integrabale and of one sign for $a < \alpha < b$, applying the mean value theorem for the last integral yields the relation

$$K^w(\rho) = \frac{\rho^{\gamma-1}}{\rho^\gamma+1}\rho^{\alpha_0}\int_a^b w(\alpha)d\alpha = \frac{\rho^{\gamma+\alpha_0-1}}{\rho^\gamma+1}W \tag{47}$$

for some α_0 $(a < \alpha_0 < b)$ and with $W = \int_a^b w(\alpha)d\alpha$.

From the last formula, we get

$$\frac{1}{\rho K^w(\rho)} = \frac{1}{W}\frac{\rho^\gamma+1}{\rho^{\gamma+\alpha_0}} = \frac{1}{W}(\rho^{-\alpha_0}+\rho^{-\gamma-\alpha_0}).$$

Applying the inverse Laplace transform to the right-hand side of the last formula, we arrive at the following representation for the kernel ψ_w of the corresponding GFID:

$$\psi_w(t) = \frac{1}{W}\left(\frac{t^{\alpha_0-1}}{\Gamma(\alpha_0)} + \frac{t^{\gamma+\alpha_0-1}}{\Gamma(\gamma+\alpha_0)}\right).$$

This function is well-defined for $t > 0$, belongs to the space $C_{-1}(0,\infty)$ and is a Sonin kernel from the class $\mathcal{S}_{-1}$ as predicted by Theorem 3.1.

Author Contributions: Conceptualization, M.A.-R. and Y.L.; methodology, M.A.-R. and Y.L.; validation, M.A.-R. and Y.L.; formal analysis, M.A.-R.; investigation, Y.L.; writing—original draft preparation, M.A.-R.; writing—review and editing, Y.L.; visualization, M.A.-R. and Y.L. All authors have read and agreed to the published version of the manuscript.

Funding: There are no external funding resources of this research.

Data Availability Statement: Not Applicable.

Acknowledgments: The first author would like to express his sincere appreciation to the Research Affairs at Yarmouk University for their support.

Conflicts of Interest: The authors have no conflict of interest to disclose.

References

1. N.H. Abel, Oplösning af et par opgaver ved hjelp af bestemte integraler. *Mag. Naturvidenskaberne* **1823**, *2*, 2.
2. N.H. Abel, Auflösung einer mechanischen Aufgabe. *J. Die Reine Angew. Math.* **1826**, *1*, 153–157.
3. M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, National Bureau of Standards Applied Mathematics, Series 55, 10th Edition, (1972).
4. Al-Refai, M., Yu. Luchko, Analysis of fractional diffusion equations of distributed order: Maximum principles and their applications. Analysis 36 (2016), 123–133.
5. Al-Refai, M., Yu. Luchko, The general fractional integrals and derivatives on a finite interval. *Mathematics* 2023, **11**(4), 1031.
6. Atanackovic, T.M..; Pilipovic, S. Zener Model with General Fractional Calculus: Thermodynamical Restrictions. *Fractal Fract.* **2022**, 6, 617.
7. Bazhlekova, E.; Pshenichnov, S. Two Integral Representations for the Relaxation Modulus of the Generalized Fractional Zener Model. *Fractal Fract.* **2023**, 7, 636.
8. M. Caputo, M. Fabrizio, The kernel of the distributed order fractional derivatives with an application to complex materials, Fractal and Fractional, 1(13) (2017), 1–11.
9. A.V. Chechkin, R. Gorenflo, and I.M. Sokolov, Retarding subdiffusion and accelerating superdiffusion governed by distributed order fractional diffusion equations. *Phys. Rev. E* **66** (2002), 1-7.

10. R. Gorenflo, Yu. Luchko, and M. Stojanovic, Fundamental solution of a distributed order time-fractional diffusion-wave equation as probability density. Fract. Calc. Appl. Anal. 16(2013), 297–316.
11. Gorska, K.; Horzela, A. Subordination and memory dependent kinetics in diffusion and relaxation phenomena. *Fract. Calc. Appl. Anal.* 26 (**2023**), 480-512.
12. A.N. Kochubei, Distributed order calculus and equations of ultraslow diffusion, Journal of Mathematical Analysis, 340 (2008), 252–281.
13. A.N. Kochubei, General fractional calculus, evolution equations, and renewal processes, Integral Equations and Operator Theory, 71 (2011), 583-600.
14. Yu. Luchko, Boundary value problems for the generalized time-fractional diffusion equation of distributed order. Fract. Calc. Appl. Anal. 12(2009), 409–422.
15. Yu. Luchko, General Fractional Integrals and Derivatives with the Sonine Kernels. *Mathematics* **2021**, *9(6)*, 594.
16. Yu. Luchko, Operational Calculus for the general fractional derivatives with the Sonine kernels. *Fract. Calc. Appl. Anal.* **2021**, *24*, 338–375.
17. Yu. Luchko, General Fractional Integrals and Derivatives of Arbitrary Order. *Symmetry* **2021**, *13(5)*, 755.
18. Yu. Luchko, The 1st Level General Fractional Derivatives and some of their Properties. J Math Sci, 266 (**2022**), 709-722.
19. Luchko, Yu. and Yamamoto, M. The General Fractional Derivative and Related Fractional Differential Equations. *Mathematics* **2020**, *8(12)*, 2115.
20. M.M. Meerschaert, H.-P. Scheffler, Stochastic model for ultraslow diffusion. *Stochastic Process. Appl.* **116** (2006) 1215-1235.
21. Miskovic-Stankovic, V.; Atanackovic, T.M. On a System of Equations with General Fractional Derivatives Arising in Diffusion Theory. *Fractal Fract.* **2023**, 7, 518.
22. Miskovic-Stankovic, V., Janev, M., Atanackovic, T.M. Two compartmental fractional derivative model with general fractional derivative. *J Pharmacokinet Pharmacodyn* 50 (**2023**), 79-87.
23. M. Naber, Distributed order fractional subdiffusion, *Fractals* **12** (2004), 23-32.
24. S.G. Samko, R.P. Cardoso, Integral equations of the first kind of Sonine type, International Journal of Mathematics and Mathematical Sciences, Volume 2003 | Article ID 238394 | https://doi.org/10.1155/S0161171203211455.
25. V.M. Stankovic, T.M. Atanackovic, On a system of equations with general fractional derivatives arising in diffusion theory, Fractal and Fractional, 7(5) (2023), 1–13.
26. I.M. Sokolov, A.V. Chechkin, and J. Klafter, Distributed-order fractional kinetics. *Acta Phys. Polon. B* **35** (2004), 1323-1341.
27. N. Sonine, Sur la généralisation d'une formule d'Abel. *Acta Math.* **1884**, *4*, 171–176,
28. V.E. Tarasov, General Fractional Calculus: Multi-Kernel Approach. *Mathematics* **2021**, *9*, 1501.
29. V.E. Tarasov, General fractional vector calculus. *Mathematics* 2021, 9, 2816.
30. V.E. Tarasov, General Fractional Dynamics. *Mathematics* **2021**, *9*, 1464.
31. V.E. Tarasov, General non-Markovian quantum dynamics. *Entropy* **2021**, *23*, 1006.
32. V.E. Tarasov, General non-local electrodynamics: Equations and non-local effects. *Annals of Physics*, 445 (**2022**), 169082.
33. V.E. Tarasov, Nonlocal classical theory of gravity: massiveness of nonlocality and mass shielding by nonlocality. *Eur. Phys. J. Plus* 137 (**2022**), 1336.
34. V.E. Tarasov, Nonlocal statistical mechanics: general fractional Liouville equations and their solutions. *Physica A: Statistical Mechanics and its Applications* 609 (**2023**), 128366.
35. V.E. Tarasov, General Fractional Calculus in Multi-Dimensional Space: Riesz Form. *Mathematics* 2023, 11, 1651.
36. S. Umarov and R. Gorenflo, Cauchy and nonlocal multi-point problems for distributed order pseudo-differential equations. *Z. Anal. Anwend.* **24** (2005), 449-466.