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Article

Some Submaximal Elements of the Lattice of Finitary Operations on a Finite Set

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Abstract: The set of Boolean functions has a simple basis formed by the meet and the negation. Is it possible to have a similarly basis for any subset of finitary operations on a finite set with more than three elements? It is known that any set of finitary operations on a finite set can be expressed as a set of operations preserving some relations on this set. A clone on a finite set is a set of finitary operations on this set containing all the projections and stable by composition. It is also known that the set of clones on a finite set is a complete lattice. The maximal elements of this lattice was completely described by Rosenberg in 1965. In this paper, we determine clones of the form $\operatorname{Pol}\sigma \cap \operatorname{Pol}\lambda$ sited directly below $\operatorname{Pol}\lambda$ where λ is a fixed regular relation and σ a binary relation to be characterized.

Keywords: regular relations; binary relation; meet-reducible; submaximal; clones

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1. Preambule

In 1941, Post presented the complete description of the countably many clones on two elements set. It turned out that, all such clones are finitely generated and the lattice of these clones is countable. The structure of the lattice of clones on finitely many (but more than 2) elements is more complex and is of the cardinality $2^{\aleph_0}$. For the k-element case, with $k \geq 3$, not much is known about the structure of the lattice of clones in spite of the efforts made by many researchers in this area. Therefore, every new piece of information is considered valuable. Indeed, it would be very interesting to know the clone lattice on the next level (below the maximal clones) and even a partial description will shed more light onto its structure. The description of this lattice will also help to find a simple basis for any set of operations on a finite set which is useful for the progress of logic and computer science. The complete description of all submaximal clones is known (below certain maximal clones) only for the 2-element case and the 3-element case (see [8–10]), however the result in ([8]) and many result in the literature on clones including those discussed in ([8,9,13,14,16]), require intensive knowledge of submaximal clones on arbitrary finite sets.

Recently, we have determined the eleven types of binary relations σ such that the clones $\operatorname{Pol}\sigma \cap \operatorname{Pol}\rho$ are covered by $\operatorname{Pol}\rho$, where ρ is a fixed h-ary central relation ($h \geq 2$) on a given finite set (see [6]). Clone theory is considered to be very important because of its use to understand universal algebras and the problem of decidability in computer science. In fact, clone theory is also related to constraint satisfaction problem. The main problem is to decide whether a function on a finite set preserving some relations on this set belongs to a fixed clone on that set. For example, given a first order structure $(A; F; R)$ where A is a finite set; F the set of fundamental operations on A and R the set of all relations on A. It is known that $\operatorname{Pol} R$, the set of function on A preserving every relations on R, is a clone. Given a function $g \in \operatorname{Pol} R$, decide whether g belongs to the clone generated by the fundamental operations of $(A; F; R)$ is NP-complete. Clone theory is also useful to determine weak bases on sets of larger size (see [1–3]). Clone theory could also be useful for determining minimal bases to facilitate calculations in logics based on finite residuated lattices, which are tools of artificial intelligence. Our objective is to describe all submaximal clones determined by regular relations on finite set (because these submaximal clones are not studied in the literature for non-trivial regular relations on given finite set) to complete the Burle's theorem (see [9], Page 161) . Since regular relation is difficult to handle, We begin this

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project by the cases of binary relations. A natural way to extend this research would be to consider relations of arity greater than 2.

2. Structure of the Paper

This paper is organized as follows : In Section 3 we give some necessary tools to understand this work and state the main result of this paper. In Section 4 we characterize the non-trivial equivalence relation σ such that the clone $\mathrm{Pol}\,\sigma \cap \mathrm{Pol}\,\lambda$ is maximal below $\mathrm{Pol}\,\lambda$. Section 5 is devoted to the case of bounded partial order. We prove that there is no submaximal clone. Section 6 studies the case of prime permutation. We show that for any prime permutation relation s on E_k, $\mathrm{Pol}\,\lambda \cap \mathrm{Pol}\,s^\circ$ is maximal below $\mathrm{Pol}\,\lambda$ if and only if λ is θ_s-closed. Section 7 is reserved to the case of binary central relations. We prove that $\mathrm{Pol}\,\sigma \cap \mathrm{Pol}\,\lambda$ is maximal below $\mathrm{Pol}\,\lambda$ if and only if $C_\sigma \cap [x]_\epsilon$ is not empty where C_σ is the set of central element of σ and ϵ is the equivalence relation characterizing λ.

3. Preliminaries

In this Section, we give some basic notions necessary to follow this work; for more details you can refer to ([7,9,11–13]). We begin with some generality on clone theory.

Let A be a fixed finite set with k elements, n and h be positive integers. An n-ary *operation* on A is a function $f : A^n \to A$. By $\mathcal{O}_A^{(n)}$ we denote the set of all n-ary operations on A, and by $\mathcal{O}_A$ the set $\bigcup_{n\geq 1} \mathcal{O}_A^{(n)}$ of all finitary operations on A. For $1 \leq i \leq n$, the i-th *projection* on A is the operation $\pi_i^{(n)} : A^n \to A, (a_1, \ldots, a_n) \mapsto a_i$. For arbitrary positive integers m and n, there is a one-to-one correspondence between the functions $f : A^n \to A^m$ and the m-tuples $\boldsymbol{f} = (f_1, \ldots, f_m)$ of functions $f_i : A^n \to A$ (for $i = 1, \ldots, m$) via $f \mapsto \boldsymbol{f} = (f_1, \ldots, f_m)$ with $f_i = \pi_i^{(m)} \circ f$ for all $i = 1, \ldots, m$. In particular, $\pi^{(n)} = (\pi_1^{(n)}, \ldots, \pi_n^{(n)})$ corresponds to the identity function $f : A^n \to A^n$. From now on, we will identify each function $f : A^n \to A^m$ with the corresponding m-tuples $\boldsymbol{f} = (f_1, \ldots, f_m) \in (\mathcal{O}_A^{(n)})^m$ of n-ary operations. Using this convention, the composition of two functions $\boldsymbol{f} = (f_1, \ldots, f_m) : A^n \to A^m$ and $\boldsymbol{g} = (g_1, \ldots, g_p) : A^m \to A^p$ can be described as follows: $\boldsymbol{g} \circ \boldsymbol{f} = (g_1 \circ \boldsymbol{f}, \ldots, g_p \circ \boldsymbol{f}) = (g_1(f_1, \ldots, f_m), \ldots, g_p(f_1, \ldots, f_m))$ where $g_i(f_1, \ldots, f_m)(\boldsymbol{a}) = g_i(f_1(\boldsymbol{a}), \ldots, f_m(\boldsymbol{a}))$ for all $\boldsymbol{a} \in A^n$ and $1 \leq i \leq p$.

A *clone* on A is a subset $\mathcal{C}$ of $\mathcal{O}_A$ that contains all projections and is closed under composition; that is $\pi_i^{(n)} \in \mathcal{C}$ for all $n \geq 1$ and $1 \leq i \leq n$, and $g \circ \boldsymbol{f} \in \mathcal{C}^{(n)}$ whenever $g \in \mathcal{C}^{(m)}$ and $\boldsymbol{f} \in (\mathcal{C}^{(n)})^m$ (for $m, n \geq 1$). The clones on A form a complete lattice $\mathcal{L}_A$ under inclusion. Moreover, for each set $F \subseteq \mathcal{O}_A$ of operations, there exists a smallest clone that contains F, which will be denoted by $\langle F \rangle$ and will be called *clone generated* by F. Clones can also be described via invariant relations. An h-ary relation ρ on A is a subset of A^h. For an n-ary operation $f \in \mathcal{O}_A^{(n)}$ and an h-ary relation ρ on A, we say that f *preserves* ρ (or ρ is *invariant under* f, or f is a *polymorphism* of ρ) if whenever f is applied coordinatewise to h-tuples from ρ, the resulting h-tuple belongs to ρ i.e., for all $(a_{1,i}, \ldots, a_{h,i}) \in \rho, i = 1, \ldots, n$, $(f(a_{1,1}, \ldots, a_{1,n}), f(a_{2,1}, \ldots, a_{2,n}) \ldots, f(a_{h,1}, \ldots, a_{h,n})) \in \rho$. For any family $\mathcal{R}$ of (finitary) relations on A, the set $\mathrm{Pol}\,\mathcal{R}$ of all operations $f \in \mathcal{O}_A$ that preserve each relation in R is easily seen to be a clone on A. Moreover, if A is finite, then it is a well-known fact that every clone on A is of the form $\mathrm{Pol}\,R$ for some family R of relations on A. If $R = \{\rho\}$, we write $\mathrm{Pol}\,\rho$ for $\mathrm{Pol}\{\rho\}$. Let $\rho \subseteq A^h$; for an integer $m > 1$ and $\boldsymbol{a}_i = (a_{1,i}, \ldots, a_{m,i}) \in A^m, 1 \leq i \leq h$, we will write $(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_h) \in \rho$ if for all $j \in \{1, \ldots, m\}, (a_{j,1}, \ldots, a_{j,h}) \in \rho$.

Since A is finite, it is well known that every clone on A other than $\mathcal{O}_A$ is contained in a maximal clone. We say that an h-ary relation ρ on A is *totally reflexive* (*reflexive* for $h = 2$) if ρ contains the h-ary relation ι_A^h defined by

$$\iota_A^h = \{(a_1, \ldots, a_h) \in A^h \mid \exists i, j \in \{1, \ldots, h\} : i \neq j \text{ and } a_i = a_j\},$$

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and is *totally symmetric* (*symmetric* if $h = 2$) if ρ is invariant under any permutation of its coordinates. If ρ is totally reflexive and totally symmetric, we define the *center* of ρ, denoted by C_ρ, as follows:

$$C_\rho = \{a \in A \mid (a, a_2, \ldots, a_h) \in \rho \text{ for all } a_2, \ldots, a_h \in A\}.$$

We say that ρ is a *central relation* if ρ is totally reflexive, totally symmetric and has a nonvoid center which is a proper subset of A. The elements of C_ρ are called *central elements* of ρ. A nonvoid and proper subset of A is called *unary central relation*. Let ρ be a binary relation on A, ρ is an *equivalence relation* if ρ is symmetric, reflexive and transitive; ρ is *non-trivial* if $\rho \neq A^2$ and $\rho \neq \{(a, a) \mid a \in A\}$, ρ is a *bounded partial order* if ρ is reflexive, transitive, antisymmetric and there exist $\perp, \top \in A$ such that for all $x \in A$, $(x, \top) \in \rho$ and $(\perp, x) \in \rho$. ρ is a *prime permutation relation* if $\rho = s^\circ$ the *graph* of the prime permutation s with the same length p (where p is prime).

For an integer $h \geq 3$, a family $T = \{\theta_1, \ldots, \theta_m\}$ ($m \geq 1$) of equivalence relations on A is called *h-regular* if each θ_i ($1 \leq i \leq m$) has exactly h blocks, and for arbitrary blocks B_i of θ_i ($1 \leq i \leq m$), the intersection $B_1 \cap B_2 \cap \ldots \cap B_m$ is nonempty. To each h-regular family $T = \{\theta_1, \ldots, \theta_m\}$ of equivalence relation on A, we associate an h-ary relation λ on A as follows:

$$\lambda = \{(a_1, \ldots, a_h) \in A^h \mid (\forall i)(\exists j)\ j \neq i \text{ such that } (a_i, a_j) \in \theta_i\}.$$

Relations of the form λ are called *h-regular* (or *h-regularly generated*) relations. It is clear from the definition that h-regular relations are totally reflexive and totally symmetric, their arity h satisfies $3 \leq h \leq |A|$, and $h = |A|$ holds if and only if T is the one-element family consisting of the equality relation. The characterization of maximal clones on finite sets given in [15] is well known. Therefore we have that whenever a relation γ is a non-trivial equivalence relation, a bounded partial order, a binary central relation, a prime permutation relation or an h-regular relation with ($3 \leq h \leq |A|$), the clone $\operatorname{Pol}\gamma$ is a maximal clone. The binary diagonal relation on A is $\Delta_A = \{(a, a) \mid a \in A\}$.

We continue with some fundamental results on clone theory. The following result characterizes all maximal clones on finite sets.

Theorem 3.1. *[15] For each finite set A with $|A| \geq 2$, the maximal clones on A are the clones of the form* $\operatorname{Pol}\rho$, *where ρ is a relation of one of the following six types:*

(1) a bounded partial order on A,
(2) a prime permutation on A,
(3) a prime affine relation on A,
(4) a non-trivial equivalence relation on A,
(5) a central relation on A,
(6) an h-regular relation on A.

For $h \in \mathbb{N} \setminus \{0\}$, $R_A^{(h)}$ denotes the set of h-ary relations on A. R_A denotes the set of all finitary relations on A. For a set F of finitary operations on A, $Inv(F)$ denotes the set of relations preserved by every operation in F. A *relational clone* on A is a subset R of R_A such that $Inv(\operatorname{Pol} R) = R$. The set of all relational clones with inclusion is a lattice called lattice of relational clones. For $R \subseteq R_A$, $[R]$ denotes the relational clone generated by R.

For instance, it is nice for us to give the following remark useful to justify some inclusions between clones.

Remark 3.2. *([9], Theorem 2.6.2, 2.6.3 Page 132) Let $R \subseteq R_A$,*

1. *If $f \in \operatorname{Pol} R$, then $f \in \operatorname{Pol}([R])$.*
2. *If σ and σ' are relations such that $\sigma' \in [\{\sigma\}]$, then $\operatorname{Pol}\sigma \subseteq \operatorname{Pol}\sigma'$.*

For two clones $\mathcal{C}$ and $\mathcal{D}$ on A, we say that $\mathcal{C}$ is *maximal* in $\mathcal{D}$ if $\mathcal{D}$ covers $\mathcal{C}$ in $\mathcal{L}_A$, we also say that $\mathcal{C}$ is *submaximal* if $\mathcal{C}$ is maximal in a clone $\mathcal{D}$ and $\mathcal{D}$ is a maximal clone on A. For a maximal clone $\mathcal{D}$, there are two types of clones $\mathcal{C}$ being maximal in $\mathcal{D}$: $\mathcal{C}$ is *meet-reducible* if $\mathcal{C} = \mathcal{D} \cap \mathcal{F}$ for a maximal clone $\mathcal{F}$ distinct from $\mathcal{D}$ (but not necessarily unique) and $\mathcal{C}$ is *meet-irreducible* if it is not meet-reducible.

Definition 3.3. *([9], Page 126) Let $h \in \mathbb{N} \backslash \{0\}$. An h-ary relation $\rho \in R_A^{(h)}$ is called diagonal relation if there exists an equivalence relation ε on $\{1, \ldots, h\}$ such that $\rho := \{(a_1, \ldots, a_h) \in A^h \mid (i, j) \in \varepsilon \Longrightarrow a_i = a_j\}$.*

The set of all diagonal relations on A is denoted by D_A and $D_A = \{\emptyset\} \cup \bigcup_{h \geq 1} D_A^{(h)}$, where $D_A^{(h)}$ is the set of all h-ary diagonal relations on A. In particular, A^h and $\delta^h = \{(x, x, \ldots, x) \in A^h \mid x \in A\}$ are diagonal relations on A. Fore more details, see [9].

From now on, we assume that we are working on the set $E_k = \{0, 1, \ldots, k-1\}$. For any integer $2 \leq h \leq k$, we denote by ι_k^h the set $\iota_{E_k}^h$. It is well known (From Theorem 3.1, relation of type (6)) that the Słupecki clone $\operatorname{Pol} \iota_k^k$ is a maximal clone. Subclones of $\operatorname{Pol} \iota_k^k$ are known (see [9], Page 161, Burle's Theorem). If σ is an equivalence relation, we denote by $[a]_\sigma$ the σ-class of a.

Remark 3.4. *For any h-regular relation on E_k, we have $3 \leq h \leq k$. The case $h = k$ is given by the Słupecki clone $\operatorname{Pol} \iota_k^k$. Submaximal clones of Słupecki clone are known.*

In this paper, we consider the h-ary ($3 \leq h < k$) regular relation λ on E_k defined by the h-ary regular family $T = \{\theta_1, \ldots, \theta_m\}$, where $m \geq 1$ and every non-trivial equivalence relation $\theta_i, 1 \leq i \leq m$ contains h blocks.

Now we define some types of binary relations useful to express our results. Let k and h be two integers such that $h, k \geq 3$. For a prime permutation s of order p on E_k, we denote by θ_s the equivalence relation consisting of pairs $(a, b) \in E_k^2$ with $a = s^i(b)$ for some $0 \leq i < p$.

Definition 3.5. *Let λ be an h-ary regular relation on E_k and σ be a non-trivial equivalence relation on E_k.*

1. *We say that λ is σ-closed if $(a_1, \ldots, a_h) \in \lambda$ whenever $(u_1, \ldots, u_h) \in \lambda$ for some $u_1, \ldots, u_h \in E_k$ with $(a_i, u_i) \in \sigma$ for all $1 \leq i \leq h$.*
2. *There is a transversal T for the σ-classes means that there exist $u_1, \ldots, u_t \in E_k$ such that $(u_i, u_j) \notin \sigma$ for all $1 \leq i < j \leq t, (u_{i_1}, u_{i_2}, \ldots, u_{i_h}) \in \lambda$ for all $1 \leq i_1 \ldots, i_h \leq t$ and $T = \{u_1, \ldots, u_t\}$.*

Let ε be the binary relation defined on E_k by

$$\varepsilon = \{(a, b) \in E_k^2 \mid (a, b, a_3, \ldots, a_h) \in \lambda, \forall a_3, \ldots, a_h \in E_k\}$$

and the h-ary relation γ_h defined by :

$$\gamma_h = \{(a_1, \ldots, a_h) \in E_k^h \mid \exists v_1, \ldots, v_h \in E_k, (v_1, \ldots, v_h) \in \lambda, \text{ and}$$
$$\forall 1 \leq i \leq h, \forall j_1, \ldots, j_{h-2} \in \{1, \ldots, h\} \backslash \{i\}, (v_i, a_i, a_{j_1}, \ldots, a_{j_{h-2}}) \in \lambda\}$$

We will denote by $\underline{h}$, the set $\{1, \ldots, h\}$ and S_h, the set of all permutations on $\{1, \ldots, h\}, h \geq 2$.

Now we state the main result of this paper.

Theorem 3.6. *Let λ be an h-ary regular relation on E_k determined by the h-ary regular family $T = \{\theta_1, \ldots, \theta_m\}$ and σ be a binary relation on E_k such that $\operatorname{Pol} \sigma$ is a maximal clone. Then $\operatorname{Pol} \sigma \cap \operatorname{Pol} \lambda$ is a submaximal clone of $\operatorname{Pol} \lambda$ if and only if one of the following conditions is satisfied:*

I. *σ is a non-trivial equivalence relation and λ is σ-closed;*
II. *σ is a non-trivial equivalence relation, $\lambda = \gamma_h$ and for all $x, y \in E_k$, $[x]_\sigma \cap [y]_\varepsilon \neq \emptyset$;*
III. *σ is a prime permutation and λ is θ_σ-closed;*
IV. *σ is a binary central relation and for all $x \in E_k$, $C_\sigma \cap [x]_\varepsilon \neq \emptyset$.*

Before the Proof of Theorem 3.6, we give some examples to fix some ideas.

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Example 3.7. *Let $E_{10} = \{0,1,2,3,4,5,6,7,8,9\}$. Let θ_1 be the equivalence relation with blocks $\{0,1,2\}$, $\{3,4,5\}, \{6,7,8,9\}$ and θ_2 the equivalence relation with blocks $\{0,3,6\}, \{1,4,7\}, \{2,5,8,9\}$. Then $T = \{\theta_1, \theta_2\}$ is an 3-ary regular family.*

1. *Set $\theta = \theta_1 \cap \theta_2$. We have $\theta = \Delta_{E_4} \cup \{(8,9),(8,9)\}$ and λ is θ-closed. Thus $\operatorname{Pol}\lambda \cap \operatorname{Pol}\theta$ is maximal below $\operatorname{Pol}\lambda$.*
2. *Let $\varepsilon = \{(a,b) \in E_{10}^2 \mid (a,b,c) \in \lambda, \forall c \in E_{10}\}$ and σ be the equivalence relation with blocks $\{0,1,2,3,4,5,6,7\}, \{8,9\}$. It is easy to show that $\varepsilon = \theta_1 \cap \theta_2 = \theta$. Furthermore, $\forall x,y \in E_{10}$, $[x]_\varepsilon \cap [y]_\sigma \neq \emptyset$. Hence $\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma$ is maximal below $\operatorname{Pol}\lambda$.*
3. *Let $\pi = (01)(23)(45)(67)(89)$. π is a prime permutation on E_{10} of order 2 and θ_π is the equivalence relation with blocks $\{0,1\}, \{2,3\}, \{4,5\}, \{6,7\}, \{8,9\}$. Set $T_1 := \{\theta_\pi\}$. Then the regular relation λ_{T_1} is θ_π-closed. So $\operatorname{Pol}\lambda_{T_1} \cap \operatorname{Pol}\pi^\circ$ is maximal below $\operatorname{Pol}\lambda_{T_1}$.*

The Proof of Theorem 3.6 follows from propositions 4.1, 5.1, 6.2 and 7.1.
Firstly we look at the non-trivial equivalence case.

4. Non-Trivial Equivalence Relation

Let λ be an h-ary regular relation $(3 \le h < k)$ on E_k determined by the h-regular family $T = \{\theta_1, \ldots, \theta_m\}$, $m \ge 1$ and σ be a non-trivial equivalence relation on E_k with t classes $(t \ge 2)$.
Here we state the main result of this section.

Proposition 4.1. *Let λ be an h-ary regular relation $(3 \le h < k)$ on E_k determined by the h-regular family $T = \{\theta_1, \ldots, \theta_m\}$, $m \ge 1$ and σ be a non-trivial equivalence relation on E_k with t classes $(t \ge 2)$. Then $\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma$ is maximal below $\operatorname{Pol}\lambda$ if and only if (λ is σ-closed) or ($\lambda = \gamma_h$ and for all $x,y \in E_k$, $[x]_\sigma \cap [y]_\varepsilon \neq \emptyset$).*

The necessary condition of Proposition 4.1 is given by Lemmas 4.2, and 4.4.

Lemma 4.2. *Let λ be an h-ary regular relation $(3 \le h < k)$ on E_k determined by the h-regular family $T = \{\theta_1, \ldots, \theta_m\}$, $m \ge 1$ and σ be a non-trivial equivalence relation on E_k with t classes $(t \ge 2)$ such that λ is σ-closed. Then $\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma$ is maximal below $\operatorname{Pol}\lambda$.*

Before the Proof of Lemma 4.2, we will give some useful properties of σ in Lemma 4.3. For $\boldsymbol{a}, \boldsymbol{b} \in E_k^n$, we write $(\boldsymbol{a}, \boldsymbol{b}) \in \sigma$ if $(a_i, b_i) \in \sigma$ for all $1 \le i \le n$, where $\boldsymbol{a} := (a_1, \ldots, a_n)$ and $\boldsymbol{b} := (b_1, \ldots, b_n)$. Let $g \in \operatorname{Pol}\lambda \setminus \operatorname{Pol}\sigma$ be an n-ary operation, then there exist $(a_1, b_1), \ldots, (a_n, b_n) \in \sigma$ such that $(g(a_1, \ldots, a_n), g(b_1, \ldots, b_n)) \notin \sigma$.

Lemma 4.3. *Let $m \ge 1$ be an integer, λ be an h-ary regular relation $(3 \le h < k)$ on E_k determined by the h-regular family $T = \{\theta_1, \ldots, \theta_m\}$, $m \ge 1$ and σ be a non-trivial equivalence relation on E_k with t classes $(t \ge 2)$, let $g \in \operatorname{Pol}\lambda \setminus \operatorname{Pol}\sigma$ be an n-ary operation . Then for all $\boldsymbol{e}, \boldsymbol{d} \in E_k^m$ such that $\boldsymbol{e} \neq \boldsymbol{d}$ and $(\boldsymbol{e}, \boldsymbol{d}) \in \sigma$, there exists an m-ary operation $f_{\boldsymbol{ed}} \in \langle (\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma) \cup \{g\} \rangle$ such that $(f_{\boldsymbol{ed}}(\boldsymbol{e}), f_{\boldsymbol{ed}}(\boldsymbol{d})) \notin \sigma$.*

Proof. We distinguish two cases: (i) $m = 1$ and (ii) $m > 1$.

Let $g \in \operatorname{Pol}\lambda \setminus \operatorname{Pol}\sigma$ be an n-ary operation. Then there exist $(a_1, b_1), \ldots, (a_n, b_n) \in \sigma$ such that $(g(a_1, \ldots, a_n), g(b_1, \ldots, b_n)) \notin \sigma$. For $m = 1$, let $e, d \in E_k$ such that $e \neq d$ and $(e, d) \in \sigma$, we will construct a unary operation $f_{ed} \in \langle (\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma) \cup \{g\} \rangle$ such that $(f_{ed}(e), f_{ed}(d)) \notin \sigma$. For $1 \le i \le n$, consider the unary operation f_{ed}^i defined on E_k by $f_{ed}^i(x) = a_i$ if $x = e$ and $f_{ed}^i(x) = b_i$ otherwise. Since $(a_i, b_i) \in \sigma$ and $Im f_{ed}^i = \{a_i, b_i\}$, then $f_{ed}^i \in \operatorname{Pol}\sigma$. Furthermore, λ is totally reflexive, thus $f_{ed}^i \in \operatorname{Pol}\lambda$ and $f_{ed}^i \in \operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma$. Let f_{ed} be the unary operation defined on E_k by : $f_{ed}(x) = g(f_{ed}^1(x), \ldots, f_{ed}^n(x))$, $\forall x \in E_k$. We have $f_{ed} \in \langle (\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma) \cup \{g\} \rangle$ (because $f_{ed}^i \in \operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma$) and $(f_{ed}(e), f_{ed}(d)) = (g(a_1, \ldots, a_n), g(b_1, \ldots, b_n)) \notin \sigma$. Hence, f_{ed} is the operation wanted.

Let $m > 1$, $\boldsymbol{e}, \boldsymbol{d} \in E_k^m$ such that $\boldsymbol{e} \neq \boldsymbol{d}$ and $(\boldsymbol{e}, \boldsymbol{d}) \in \sigma$, then there exists $1 \le i \le m$ such that $e_i \neq d_i$ and $(e_i, d_i) \in \sigma$, we can construct $f_{e_i d_i}$ as in **Case** $m = 1$. Set $f_{\boldsymbol{ed}} = f_{e_i d_i} \circ \pi_i^{(m)}$. We have $f_{\boldsymbol{ed}} \in \langle (\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma) \cup \{g\} \rangle$ and $(f_{\boldsymbol{ed}}(\boldsymbol{e}), f_{\boldsymbol{ed}}(\boldsymbol{d}) = (g(a_1, \ldots, a_n), g(b_1, \ldots, b_n)) \notin \sigma$. Hence $f_{\boldsymbol{ed}}$ is the m-ary operation wanted. □

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Proof. (*Proof of Lemma 4.2*). Let $g^n \in \operatorname{Pol}\lambda \setminus \big(\operatorname{Pol}\sigma \cap \operatorname{Pol}\lambda\big)$ be an n-ary operation. Using the operation constructed in Lemma 4.3 we will show that $\langle(\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma) \cup \{g\}\rangle = \operatorname{Pol}\lambda$.

We have $\langle(\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma) \cup \{g\}\rangle \subseteq \operatorname{Pol}\lambda$. It remains to prove that $\operatorname{Pol}\lambda \subseteq \langle(\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma) \cup \{g\}\rangle$. Let $u \in \operatorname{Pol}\lambda$ be an m-ary operation on E_k. Let us show that $u \in \langle(\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma) \cup \{g\}\rangle$. For $\boldsymbol{e}, \boldsymbol{d} \in E_k^m$ such that $\boldsymbol{e} \neq \boldsymbol{d}$ and $(\boldsymbol{e}, \boldsymbol{d}) \in \sigma$, there exists an m-ary operation $f_{\boldsymbol{ed}} \in \langle(\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma) \cup \{g\}\rangle$ such that $(f_{\boldsymbol{ed}}(\boldsymbol{e}), f_{\boldsymbol{ed}}(\boldsymbol{d})) \notin \sigma$ (by Lemma 4.3). Set $S = \{f_{\boldsymbol{ed}} \mid \boldsymbol{e}, \boldsymbol{d} \in E_k^m, (\boldsymbol{e}, \boldsymbol{d}) \in \sigma \text{ and } \boldsymbol{e} \neq \boldsymbol{d}\}$, denoted for reason of simple notation by $S = \{f_i \mid 1 \leq i \leq \ell\}$ ($\operatorname{Card}(S) = \ell$) and define the mapping $\operatorname{ext} : E_k^m \longrightarrow E_k^{m+\ell}$, by $\operatorname{ext}(\boldsymbol{x}) = (\mathbf{x}, f_1(\boldsymbol{x}), \ldots, f_\ell(\boldsymbol{x}))$, for all $\boldsymbol{x} \in E_k^m$.

Let $\boldsymbol{x}, \boldsymbol{y} \in E_k^m$ such that $\boldsymbol{x} \neq \boldsymbol{y}$, we have $(\operatorname{ext}(\boldsymbol{x}), \operatorname{ext}(\boldsymbol{y})) \notin \sigma$. In fact, if $(\operatorname{ext}(\boldsymbol{x}), \operatorname{ext}(\boldsymbol{y})) \in \sigma$, then $(\boldsymbol{x}, \boldsymbol{y}) \in \sigma, \boldsymbol{x} \neq \boldsymbol{y}$ and $(f_{\boldsymbol{xy}}(\boldsymbol{x}), f_{\boldsymbol{xy}}(\boldsymbol{y})) \in \sigma$. Thus, $f_{\boldsymbol{xy}} \in S$; contradiction(due to lemma 4.3).

Now we define the operation H on the range of ext by $H(\operatorname{ext}(\boldsymbol{x})) = u(\boldsymbol{x})$. We choose and fix $T = \{e_1, \ldots, e_t\}$ such that $(e_i, e_j) \notin \sigma$, $\forall 1 \leq i < j \leq t$ (due to σ contains t blocks). Define the unary operation α on E_k by $\alpha(a) = i$ if and only if $(a, e_i) \in \sigma$. Thus for all $a \in E_k$,$(a, e_{\alpha(a)}) \in \sigma$. We have $\operatorname{ext}(E_k^m) \subsetneq E_k^{m+\ell}$, so we can construct an extension $\tilde{H}$ of H on $E_k^{m+\ell}$ as follow:

$$\tilde{H}(\boldsymbol{y}) = \begin{cases} u(\boldsymbol{u}) & \text{if } \exists \boldsymbol{u} \in E_k^m : (\operatorname{ext}(\boldsymbol{u}), \boldsymbol{y}) \in \sigma, \\ u\big(e_{\alpha(y_1)}, \ldots, e_{\alpha(y_m)}\big) & \text{otherwise.} \end{cases}$$

Let us show that $\tilde{H} \in \operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma$. Firstly, we show that $\tilde{H} \in \operatorname{Pol}\sigma$. Let $\boldsymbol{a} = (a_1, \ldots, a_{m+\ell})$ and $\boldsymbol{b} = (b_1, \ldots, b_{m+\ell})$ such that $(\boldsymbol{a}, \boldsymbol{b}) \in \sigma$. We distinguish two cases:

Case 1: There exists $\boldsymbol{u} \in E_k^m$ such that $(\operatorname{ext}(\boldsymbol{u}), \boldsymbol{a}) \in \sigma$. Then $(\operatorname{ext}(\boldsymbol{u}), \boldsymbol{b}) \in \sigma$ (because $(\boldsymbol{a}, \boldsymbol{b}) \in \sigma$ and σ is transitive); hence $(\tilde{H}(\boldsymbol{a}), \tilde{H}(\boldsymbol{b})) = (u(\boldsymbol{u}), u(\boldsymbol{u})) \in \sigma$.

Case 2: For all $\boldsymbol{u} \in E_k^m$, $(ext(\boldsymbol{a}), \boldsymbol{u}) \notin \sigma$, then for all $\boldsymbol{u} \in E_k^m$, $(ext(\boldsymbol{u}), \boldsymbol{b}) \notin \sigma$ (because **(a,b)** $\in \sigma$ and σ is transitive). For all $1 \leq i \leq m + l$, $e_{\alpha(a_i)} = e_{\alpha(b_i)}$ (since $(a_i, b_i) \in \sigma$).

Therefore, $(\tilde{H}(\boldsymbol{a}), \tilde{H}(\boldsymbol{b})) = \Big(u\big(e_{\alpha(a_1)}, \ldots, e_{\alpha(a_{m+\ell})}\big), u\big(e_{\alpha(b_1)}, \ldots, e_{\alpha(b_{m+\ell})}\big)\Big) \in \sigma$. Thus $\tilde{H} \in \operatorname{Pol}\sigma$.

Secondly, we show that $\tilde{H} \in \operatorname{Pol}\lambda$.

Let $\boldsymbol{a}_1 = (a_{1,1}, \ldots, a_{1,h}), \ldots, \boldsymbol{a}_m = (a_{m,1}, \ldots, a_{m,h}), \ldots, \boldsymbol{a}_{m+l} = (a_{m+\ell,\ell}, \ldots, a_{m+\ell,h}) \in \lambda$, then for all $\boldsymbol{y} \in E_k^{m+\ell}$, there exists $\boldsymbol{v} \in E_k^m$ such that $\tilde{H}(\boldsymbol{y}) = u(\boldsymbol{v})$ and $(y_1, \ldots, y_m, \boldsymbol{v}) \in \sigma$ (by the definition of $\tilde{H}$).

For all $1 \leq j \leq h$, set $\mathbf{d}_j = (a_{1,j}, \ldots, a_{m,j}, \ldots, a_{m+l,j})$, $\mathbf{b}_j = (e_{\alpha(a_{1,j})}, \ldots, e_{\alpha(a_{m,j})}, \ldots, e_{\alpha(a_{m+\ell,j})})$, $\mathbf{d}'_j = (a_{1,j}, \ldots, a_{m,j})$ and $\mathbf{b}'_j = \big((e_{\alpha(a_{1,j})}, \ldots, e_{\alpha(a_{m,j})}\big)$. We have $(\mathbf{d}_j, \mathbf{b}_j), (\mathbf{d}'_j, \mathbf{b}'_j) \in \sigma, 1 \leq j \leq h$.

Since $\boldsymbol{d}_j = (a_{1,j}, \ldots, a_{m,j}, \ldots, a_{m+\ell,j}) \in E^{m+l}$, there exists $\boldsymbol{v}_j = (v_{1,j}, \ldots, v_{m,j}) \in E_k^m$ such that $\tilde{H}(\boldsymbol{d}_j) = u(\boldsymbol{v}_j)$ and $\big((a_{1,j}, \ldots, a_{m,j}), (v_{1,j}, \ldots, v_{m,j})\big) \in \sigma$. Thus $(\tilde{H}(\boldsymbol{d}_1), \ldots, \tilde{H}(\boldsymbol{d}_h)) = (u(\boldsymbol{v}_1), \ldots, u(\boldsymbol{v}_h))$.

For all $1 \leq i \leq m$ and $1 \leq j \leq h$, $(v_{i,j}, e_{\alpha(a_{i,j})}) \in \sigma$ and $\big(e_{\alpha(a_{i,1})}, e_{\alpha(a_{i,2})}, \ldots, e_{\alpha(a_{i,h})}\big) \in \lambda$; since λ is σ-closed, we have $(v_{i,1}, \ldots, v_{i,h}) \in \lambda, \forall 1 \leq i \leq m$.

Therefore $\Big(u(v_{1,1}, \ldots, v_{m,1}), u(v_{1,2}, \ldots, v_{m,2}), (u(v_{1,h}, \ldots, v_{m,h})\Big) = (u(\boldsymbol{v}_1), \ldots, u(\boldsymbol{v}_h)) = (\tilde{H}(\mathbf{d}_1), \ldots, \tilde{H}(\mathbf{d}_h)) \in \lambda$ (because $u \in \operatorname{Pol}\lambda$). Thus $\tilde{H} \in \operatorname{Pol}\lambda$. It follows that $\tilde{H} \in \operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma$ and for all $\boldsymbol{x} \in E_k^m$, $\tilde{H}(\boldsymbol{x}, f_{m+\ell}(\boldsymbol{x}), \ldots, f_{m+l}(\boldsymbol{x})) = \tilde{H}(ext(\boldsymbol{x})) = \tilde{H} \circ ext(\boldsymbol{x}) = u(\boldsymbol{x})$. Thus $u = \tilde{H} \circ ext \in \langle(\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma) \cup \{g\}\rangle$. Therefore $\operatorname{Pol}\lambda \subseteq \langle(\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma) \cup \{g\}\rangle$. We conclude that $\langle(\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma) \cup \{g\}\rangle = \operatorname{Pol}\lambda$ and $\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma$ is maximal below $\operatorname{Pol}\lambda$. $\square$

Lemma 4.4. *Let λ be an h-ary regular relation $(3 \leq h < k)$ on E_k determined by the h-regular family $T = \{\theta_1, \ldots, \theta_m\}$, $m \geq 1$ and σ be a non-trivial equivalence relation on E_k with t classes $(t \geq 2)$ such that $\lambda = \gamma_h$ and for all $x, y \in E_k$, $[x]_\sigma \cap [y]_\varepsilon \neq \emptyset$. Then $\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma$ is maximal below $\operatorname{Pol}\lambda$.*

Before the Proof of Lemma 4.4, we recall some notations. The binary relation ε is defined by

$$\varepsilon = \{(a, b) \in E_k^2 \mid (a, b, a_3, \ldots, a_h) \in \lambda, \forall a_3, \ldots, a_h \in E_k\}.$$

Note that $T = \{u_1, \ldots, u_t\}$ is a *transversal* for the σ-classes such that $T^h \subseteq \lambda$ and the h-ary relation γ_h is defined by :

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$\gamma_h = \{(a_1, \ldots, a_h) \in E_k^h \mid \exists v_1, \ldots, v_h \in E_k : (v_1, \ldots, v_h) \in \lambda$, *and* $\forall 1 \le i \le h, \forall j_1, \ldots, j_{h-2} \in \{1, \ldots, h\} \backslash \{i\}, (v_i, a_i, a_{j_1}, \ldots, a_{j_{h-2}}) \in \lambda\}$.

For all $x, y \in E_k$, we choose and fix $\alpha_{xy} \in [x]_\sigma \cap [y]_\varepsilon$ (because $[x]_\sigma \cap [y]_\varepsilon \neq \emptyset$).

Proof. Let $g^n \in \mathrm{Pol}\,\lambda \setminus (\mathrm{Pol}\,\sigma \cap \mathrm{Pol}\,\lambda)$ be an n-ary operation. Using the operation constructed in Lemma 4.3, we will show that $\langle(\mathrm{Pol}\,\lambda \cap \mathrm{Pol}\,\sigma) \cup \{g\}\rangle = \mathrm{Pol}\,\lambda$. We have $\langle(\mathrm{Pol}\,\lambda \cap \mathrm{Pol}\,\sigma) \cup \{g\}\rangle \subseteq \mathrm{Pol}\,\lambda$. It remains to prove that $\mathrm{Pol}\,\lambda \subseteq \langle(\mathrm{Pol}\,\lambda \cap \mathrm{Pol}\,\sigma) \cup \{g\}\rangle$. Let $u \in \mathrm{Pol}\,\lambda$ be an m-ary operation on E_k. We will show that $u \in \langle(\mathrm{Pol}\,\lambda \cap \mathrm{Pol}\,\sigma) \cup \{g\}$. Using the notations given in the proof of Lemma 4.2, we define the operation $\tilde{H}$ as follows:

$$\tilde{H}(\boldsymbol{y}) = \begin{cases} \alpha_{u(\boldsymbol{x})u(y_1,\ldots,y_m)} & \text{if } \exists \boldsymbol{x} \in E_k^m : (\mathrm{ext}(\boldsymbol{x}), \boldsymbol{y}) \in \sigma, \\ \alpha_{u_1 u(y_1,\ldots,y_m)} & \text{otherwise.} \end{cases}$$

Let us show that $\tilde{H} \in \mathrm{Pol}\,\lambda \cap \mathrm{Pol}\,\sigma$. Firstly, we show that $\tilde{H} \in \mathrm{Pol}\,\sigma$. Let $\boldsymbol{a} = (a_1, \ldots, a_{m+\ell})$ and $\boldsymbol{b} = (b_1, \ldots, b_{m+\ell})$ such that **(a,b)** $\in \sigma$. We distinguish two cases:

Case 1: There exits $\boldsymbol{x} \in E_k^m$ such that $(\mathrm{ext}(\boldsymbol{x}), \boldsymbol{a}) \in \sigma$. Then $(\mathrm{ext}(\boldsymbol{x}), \boldsymbol{b}) \in \sigma$ (because **(a,b)** $\in \sigma$ and σ is transitive). We obtain

$(\tilde{H}(\boldsymbol{a}), \tilde{H}(\boldsymbol{b})) = (\alpha_{u(\boldsymbol{x})u(a_1,\ldots,a_m)}, \alpha_{u(\boldsymbol{x})u(b_1,\ldots,b_m)}) \in \sigma$.

Case 2: For all $\boldsymbol{x} \in E_k^m$, $(\mathrm{ext}(\boldsymbol{x}), \boldsymbol{a}) \notin \sigma$, then for all $\boldsymbol{x} \in E_k^m$, $(\mathrm{ext}(\boldsymbol{x}), \boldsymbol{b}) \notin \sigma$ (because $(\boldsymbol{a}, \boldsymbol{b}) \in \sigma$ and σ is transitive).

We have $(\tilde{H}(\boldsymbol{a}), \tilde{H}(\boldsymbol{b})) = (\alpha_{u_1 u(a_1,\ldots,a_m)}, \alpha_{u_1 u(b_1,\ldots,b_m)}) \in \sigma$. Thus $\tilde{H} \in \mathrm{Pol}\,\sigma$.

Secondly, we show that $\tilde{H} \in \mathrm{Pol}\,\lambda$.

Let $\boldsymbol{a}_1 = (a_{1,1}, \ldots, a_{1,h}), \ldots, \boldsymbol{a}_m = (a_{m,1}, \ldots, a_{m,h}), \ldots, \boldsymbol{a}_{m+l} = (a_{m+\ell,\ell}, \ldots, a_{m+\ell,h}) \in \lambda$. For all $1 \le j \le h$, set $\boldsymbol{d}_j = (a_{1,j}, \ldots, a_{m,j}, \ldots, a_{m+l,j})$ and $\boldsymbol{b}_j = (a_{1,j}, \ldots, a_{m,j})$. We will show that $(\tilde{H}(\boldsymbol{d}_1), \ldots, \tilde{H}(\boldsymbol{d}_h)) \in \lambda$. We look at the following two cases:

Case 1: For $1 \le j \le h$, $\exists \boldsymbol{x}_j = (x_{1,j}, \ldots, x_{m,j}) \in E_k^m$ such that $(\mathrm{ext}(\boldsymbol{x}_j), \boldsymbol{d}_j) \in \sigma$. Then, $\tilde{H}(\boldsymbol{d}_j) = \alpha_{u(x_{1,j},\ldots,x_{m,j})u(a_{1,j},\ldots,a_{m,j})} = \alpha_{u(\boldsymbol{x}_j)u(\boldsymbol{b}_j)}$. For all $1 \le j \le h$, set $v_j = u(\boldsymbol{b}_j)$. We have $(v_1, \ldots, v_h) \in \lambda$ (because $u \in \mathrm{Pol}\,\lambda$). Since $\lambda = \gamma_h$, we obtain $(\tilde{H}(\boldsymbol{d}_1), \ldots, \tilde{H}(\boldsymbol{d}_h)) = (\alpha_{u(\boldsymbol{x}_1)u(\boldsymbol{b}_1)}, \ldots, \alpha_{u(\boldsymbol{x}_h)u(\boldsymbol{b}_h)}) \in \lambda$ (due to the fact that for $1 \le i \le h$, $(\alpha_{u(\boldsymbol{x}_i)u(\boldsymbol{b}_i)}, v_i) \in \varepsilon$).

Case 2: There exists $1 \le j \le h$ such that $\forall \boldsymbol{x} \in E_k^m$, $(\mathrm{ext}(\boldsymbol{x}), \boldsymbol{d}_j) \notin \sigma$. Then $\tilde{H}(\boldsymbol{d}_j) = \alpha_{u_1 u(\boldsymbol{b}_j)}$.

We obtain $(\tilde{H}(\boldsymbol{d}_1), \ldots, \tilde{H}(\boldsymbol{d}_h)) = (\alpha_{u(\boldsymbol{x}_1)u(\boldsymbol{b}_1)}, \ldots, \alpha_{u_1 u(\boldsymbol{b}_j)}, \ldots, \alpha_{u(\boldsymbol{x}_h)u(\boldsymbol{b}_h)})$. Since $(u(\boldsymbol{b}_1), \ldots, u(\boldsymbol{b}_h)) \in \lambda$ and $(\alpha_{u(\boldsymbol{x}_1)u(\boldsymbol{b}_1)}, u(\boldsymbol{b}_1)) \in \varepsilon$, we conclude that

$(\alpha_{u(\boldsymbol{x}_1)u(\boldsymbol{b}_1)}, u(\boldsymbol{b}_2), \ldots, u(\boldsymbol{b}_h)) \in \lambda$. Repeating the same argument, we obtain by induction that the resulting h tuple $(\tilde{H}(\boldsymbol{d}_1), \ldots, \tilde{H}(\boldsymbol{d}_h))$ is in λ. Thus, $\tilde{H} \in \mathrm{Pol}\,\lambda$.

For all $\boldsymbol{x} \in E_k^m$, $\tilde{H}(\boldsymbol{x}, f_{m+\ell}(\boldsymbol{x}), \ldots, f_{m+l}(\boldsymbol{x})) = \tilde{H}(ext(\boldsymbol{x})) = \tilde{H} \circ ext(\boldsymbol{x}) = u(\boldsymbol{x})$. Thus, $u = \tilde{H} \circ \mathrm{ext} \in \langle(\mathrm{Pol}\,\lambda \cap \mathrm{Pol}\,\sigma) \cup \{g\}\rangle$. Therefore $\mathrm{Pol}\,\lambda \subseteq \langle(\mathrm{Pol}\,\lambda \cap \mathrm{Pol}\,\sigma) \cup \{g\}\rangle$. We conclude that $\langle(\mathrm{Pol}\,\lambda \cap \mathrm{Pol}\,\sigma) \cup \{g\}\rangle = \mathrm{Pol}\,\lambda$ and $\mathrm{Pol}\,\lambda \cap \mathrm{Pol}\,\sigma$ is maximal below $\mathrm{Pol}\,\lambda$. □

The sufficient condition of Proposition 4.1 is obtained by the following proposition.

Proposition 4.5. *Let λ be an h-ary regular relation on E_k $(3 \le h < k)$ determined by the h-regular family $T = \{\theta_1, \ldots, \theta_m\}$, and σ be a non-trivial equivalence relation on E_k with t classes $(t \ge 2)$such that* $\mathrm{Pol}\,\lambda \cap \mathrm{Pol}\,\sigma$ *is maximal below* $\mathrm{Pol}\,\lambda$. *Then, λ is σ-closed or $\lambda = \gamma_h$ and for all $x, y \in E_k$, $[x]_\sigma \cap [y]_\varepsilon \neq \emptyset$.*

It's proof is obtained from the following Lemmas.

Let λ be an h-ary regular relation $(3 \le h < k)$ on E_k determined by the h-regular family $T = \{\theta_1, \ldots, \theta_m\}$, $m \ge 1$ and σ be a non-trivial equivalence relation on E_k with t classes $(t \ge 2)$. We set :

$\sigma_j = \{(a_1, \ldots, a_h) \in E_k^h \mid \exists u_1 \in [a_1]_\sigma, \ldots u_j \in [a_j]_\sigma, (u_1, \ldots, u_j, a_{j+1}, \ldots, a_h) \in \lambda\}$, $1 \le j \le h$. We have $\lambda \subseteq \sigma_j \subseteq E_k^h$ and $\mathrm{Pol}\,\lambda \cap \mathrm{Pol}\,\sigma \subseteq \mathrm{Pol}\,\lambda \cap \mathrm{Pol}\,\sigma_j$ for $1 \le j \le h$ (due to $\sigma_j \in [\{\lambda, \sigma\}]$).

Let $(a, b) \in E_k^2 \setminus \sigma$ and $(e, d) \in \sigma$ such that $e \neq d$. Consider the unary operation f_1 defined on E_k by

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$$f_1(t) = \begin{cases} a & \text{if } t = e, \\ b & \text{otherwise.} \end{cases}$$

Since $(e,d) \in \sigma$ and $(f_1(e), f_1(d)) = (a,b) \notin \sigma$, then $f_1 \notin \operatorname{Pol}\sigma$; but $f_1 \in \operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma_j$ (because $\operatorname{Im}(f_1) = \{a,b\}, \{a,b\}^h \subseteq \lambda \subseteq \sigma_j$, in fact λ and σ_j are totally reflexive). Therefore, for $1 \leq j \leq h$, $\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma \subsetneq \operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma_j$. Since σ_h is totally reflexive, totally symmetric and $\lambda \subseteq \sigma_h \subseteq E_k^h$, we obtain the following three cases: (1) $\lambda = \sigma_h$, (2) $\lambda \subsetneq \sigma_h \subsetneq E_k^h$ and (3) $\sigma_h = E_k^h$.

We begin with case (1) $\lambda = \sigma_h$. The following lemma shows that λ is σ-closed.

Lemma 4.6. *Under the assumptions of Proposition 4.5 and $\lambda = \sigma_h$, we have λ is σ-closed.*

Proof. Assume that $\lambda = \sigma_h$. It follows from definition that λ is σ-closed. □

We continue with case (2) $\lambda \subsetneq \sigma_h \subsetneq E_k^h$. The next lemma proves that this case can not occur.

Lemma 4.7. *Under the assumptions of Proposition 4.5, the case (2) $\lambda \subsetneq \sigma_h \subsetneq E_k^h$ is impossible.*

Proof. Assume that $\lambda \subsetneq \sigma_h \subsetneq E_k^h$. Let $(a_1, \ldots, a_h) \in E_k^h \setminus \sigma_h$ and $(u_1, \ldots, u_h) \in \sigma_h \setminus \lambda$. There exists an equivalence relation θ_1 in the h-regular family T associated to λ such that for $1 \leq i < j \leq h$, (u_i, u_j) is not an element of θ_1. Consider the unary operation g defined on E_k by $g(x) = a_i$ if and only if $x \in [u_i]_{\theta_1}$. Hence g preserves λ (because g restricted to each θ_1-class is constant) and does not preserve σ_h (because $(u_1, \ldots, u_h) \in \sigma_h$ and $(g(u_1), \ldots, g(u_h)) = (a_1, \ldots, a_h) \notin \sigma_h$). Hence $\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma \subsetneq \operatorname{Pol}\sigma_h \cap \operatorname{Pol}\lambda \subsetneq \operatorname{Pol}\lambda$. Thus $\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma$ is not maximal below $\operatorname{Pol}\lambda$. □

Now we finish our investigation with case (3) $\sigma_h = E_k^h$. We recall that σ has t classes ($t \geq 2$). For $i \geq 2$, we denote by ξ_i the i-ary relation defined on E_k by: $\xi_i = \{(a_1, \ldots, a_i) \in E_k^i \mid \exists a'_1 \in [a_1]_\sigma, \ldots, a'_i \in [a_i]_\sigma, \{a'_1, \ldots, a'_i\}^h \subseteq \lambda\}$.
We have $\sigma_h = \xi_h = E_k^h$ and ξ_t satisfies one of the following two conditions : (3.1) $\xi_t = E_k^t$, (3.2) $\xi_t \neq E_k^t$.

In the case $\xi_t \neq E_k^t$, we denote by n the least integer N such that $\xi_N \neq E_k^N$. Since $\xi_h = E_k^h$, we have $n > h$. The next result shows that only the case (3.1) is possible.

Lemma 4.8. *Under the assumptions of Proposition 4.5 and $\sigma_h = E_k^h$, we have $\xi_t = E_k^t$.*

Proof. Assume that $\xi_t \neq E_k^t$. The minimality of n yields that $\xi_{n-1} = E_k^{n-1}$. It is easy to check that ξ_n is totally reflexive and totally symmetric. Furthermore, we have $\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma \subsetneq \operatorname{Pol}\lambda \cap \operatorname{Pol}\xi_n \subseteq \operatorname{Pol}\lambda$ (due to σ binary, $n > h$, ξ_n totally reflexive and totally symmetric). Let $(a_1, \ldots, a_h) \in E_k^h \setminus \lambda$ and $(u_1, \ldots, u_h) \in \lambda \setminus \iota_k^h$. The unary operation f defined on E_k by

$$f(t) = \begin{cases} a_i & \text{if } t = u_i, 1 < i \leq h, \\ a_1 & \text{otherwise.} \end{cases}$$

preserves ξ_n (because $n > h$, $\operatorname{Im}(f) = \{a_1, \ldots, a_h\}$, ξ_n is totally symmetric and totally reflexive) and does not preserve λ (because $(u_1, \ldots, u_h) \in \lambda$ and $(f(u_1), \ldots, f(u_h)) = (a_1, \ldots, a_h) \notin \lambda$. Thus $Pol\lambda \neq \operatorname{Pol}\xi_n$ and we have $\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma \subsetneq \operatorname{Pol}\lambda \cap \operatorname{Pol}\xi_n \subsetneq \operatorname{Pol}\lambda$; contradicting the maximality of $\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma$ in $\operatorname{Pol}\lambda$. Hence, $\xi_t = E_k^t$. □

Now we assume that $\xi_t = E_k^t$. Therefore there exist $u_1, \ldots, u_t \in E_k$ such that $(u_i, u_j) \notin \sigma$, $1 \leq i < j \leq t$ and $\{u_1, \ldots, u_t\}^h \subseteq \lambda$. We set $W = \{u_1, \ldots, u_t\}$; W is a transversal of σ and λ.

For $j \in \underline{h}$, we set $\delta_j = \bigcap_{s \in S_h} (\sigma_j)_s$. It is easy to check that δ_j is totally symmetric and totally reflexive.
We have $\delta_h = E_k^h$. For $1 \leq j \leq h-1$, $\lambda \subseteq \sigma_j \subseteq E_k^h$ and we distinguish the following cases: (4.1) $\lambda = \delta_j$, (4.2) $\lambda \subsetneq \delta_j \subsetneq E_k^h$ and (4.3) $\delta_j = E_k^h$.

Now, we study the subcase (4.2) $\lambda \subsetneq \delta_j \subsetneq E_k^h$. The following lemma shows that this case can not occur.

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Lemma 4.9. *Under the assumptions of Proposition 4.5 and there is a transversal W of σ-classes such that $W^h \subseteq \lambda$, there is no $1 \leq j \leq h-1$ such that $\lambda \subsetneq \delta_j \subsetneq E_k^h$.*

Proof. Assume that there exists $1 \leq j \leq h-1$ such that $\lambda \subsetneq \delta_j \subsetneq E_k^h$. Then a similar argument as in the Proof of Lemma 4.7 shows that $\mathrm{Pol}\,\lambda \cap \mathrm{Pol}\,\sigma \subsetneq \mathrm{Pol}\,\lambda \cap \mathrm{Pol}\,\delta_j \subsetneq \mathrm{Pol}\,\lambda$ and we obtain a contradiction. □

We continue our discussion with subcase (4.1) $\lambda = \delta_j$ for some $1 \leq j \leq h-1$. We can see in the following lemma that it is also impossible.

Lemma 4.10. *Under the assumptions of Proposition 4.5 and there is a transversal W of σ-classes such that $W^h \subseteq \lambda$, there is no $1 \leq j \leq h-1$ such that $\lambda = \delta_j$.*

Proof. Assume that there is $1 \leq j \leq h-1$ such that $\lambda = \delta_j$. Since $\delta_i \subseteq \delta_j$, for all $1 \leq i \leq j \leq h-1$ and $\lambda \subseteq \delta_i$, for all $1 \leq i \leq h-1$, then $\lambda \subseteq \bigcap_{i=1}^{h-1} \delta_i = \delta_1$.
Thus, $\lambda = \delta_1$ (due to $\lambda \subseteq \delta_1$). Recall that $\delta_1 = \bigcap_{s \in S_h} (\sigma_1)_s$ and

$$\sigma_1 = \{(a_1, \ldots, a_h) \in E_k^h \mid \exists u \in E_k, (a_1, u) \in \sigma \wedge (u, a_2, \ldots, a_h) \in \lambda\}.$$

Thus $\sigma_1 \neq E_k^h$. We have the following possibilities: (i) $\sigma_1 = \lambda$ and (ii) $\lambda \subsetneq \sigma_1 \subsetneq E_k^h$.

Assume that (i) $\lambda = \sigma_1$ holds. Let $(a_1, \ldots, a_h) \in E_k^h \setminus \lambda$; since $W = \{u_1, \ldots, u_t\}$ is a transversal, without loss of generality we can affirm that $(u_1, \ldots, u_h) \in \lambda$ and $(a_1, u_1), \ldots, (a_h, u_h) \in \sigma$ $(\star)$. Thus, $(a_1, u_2, \ldots, u_h) \in \sigma_1 = \lambda$. By induction and the totally symmetry of λ, we obtain $(a_1, a_2, \ldots, a_h) \in \lambda$; contradicting the choice of $(a_1, a_2, \ldots, a_h)$. Therefore (ii) $\lambda \subsetneq \sigma_1 \subsetneq E_k^h$ holds. Since $\lambda = \bigcap_{s \in S_h} (\sigma_1)_s$, we have $\mathrm{Pol}\,\sigma_1 \subseteq \mathrm{Pol}\,\lambda$; in addition $\sigma_1 \in [\{\sigma, \lambda\}]$, so $\mathrm{Pol}\,\lambda \cap \mathrm{Pol}\,\sigma \subseteq \mathrm{Pol}\,\sigma_1$. It follows that $\mathrm{Pol}\,\lambda \cap \mathrm{Pol}\,\sigma \subseteq \mathrm{Pol}\,\sigma_1 \subseteq \mathrm{Pol}\,\lambda$. Let $(a, b) \in E_k^2 \setminus \sigma$ and $(u, v) \in \sigma$ such that $u \neq v$. Let f_1 be defined on E_k by $f(t) = a$ if $t = u$ and $f(t) = b$ otherwise. Then the unary operation f_1 preserves σ_1 (because σ_1 is totally reflexive and $Im(f_1) = \{a, b\}$) and does not preserve σ (due to $(u, v) \in \sigma$ and $(f_1(u), f_1(v)) = (a, b) \notin \sigma$. Therefore $\mathrm{Pol}\,\lambda \cap \mathrm{Pol}\,\sigma \subsetneq \mathrm{Pol}\,\sigma_1$. A similar argument as in the proof of Lemma 4.7 shows that $\mathrm{Pol}\,\lambda \cap \mathrm{Pol}\,\sigma_1 \subsetneq \mathrm{Pol}\,\lambda$. Thus, $\mathrm{Pol}\,\lambda \cap \mathrm{Pol}\,\sigma \subsetneq \mathrm{Pol}\,\sigma_1 \cap \mathrm{Pol}\,\lambda \subsetneq \mathrm{Pol}\,\lambda$; contradicting the maximality of $\mathrm{Pol}\,\lambda \cap \mathrm{Pol}\,\sigma$ in $\mathrm{Pol}\,\lambda$. □

From Lemmas 4.6, 4.7, 4.8, 4.9, and 4.10, we conclude that for all $1 \leq j \leq h-1$, $\delta_j = E_k^h$. Therefore $\delta_1 = E_k^h = \bigcap_{s \in S_h} (\sigma_1)_s$. Hence $E_k^h = \sigma_1 = (\sigma_1)_s$ for all $s \in S_h$.

Now we assume that for all $(a_1, \ldots, a_h) \in E_k^h$, $\exists u \in [a_1]_\sigma$ such that $(u, a_2, \ldots, a_h) \in \lambda$ and there is a transversal $W = \{u_1, \ldots, u_t\}$ for σ and λ such that $W^h \subseteq \lambda$.

For $h \leq l \leq k$, we set $\rho_l = \{(a_1, \ldots, a_l) \in E_k^l \mid \exists u \in E_k, \{u\} \times \{a_1, \ldots, a_l\}^{h-1} \subseteq \lambda\}$.

Clearly $\lambda \subseteq \rho_h \subseteq E_k^h$, and every ρ_l, $h \leq l \leq k$, is totally symmetric. Since σ is a binary relation and $h > 2$, we have $\mathrm{Pol}\,\sigma \cap \mathrm{Pol}\,\lambda \subsetneq \mathrm{Pol}\,\lambda \cap \mathrm{Pol}\,\rho_h$. We distinguish the following three cases : (1) $\lambda = \rho_h$, (2) $\lambda \subsetneq \rho_h \subsetneq E_k^h$ and (3) $\rho_h = E_k^h$.

Now, we study the subcase (2) $\lambda \subsetneq \rho_h \subsetneq E_k^h$. We show in the next lemma that (2) is impossible.

Lemma 4.11. *Under the assumptions of Proposition 4.5, $\sigma_1 = \lambda$ and there exists a transversal $W = \{u_1, \ldots, u_t\}$ of σ-classes such that $W^h \subseteq \lambda$, subcase $\lambda \subsetneq \rho_h \subsetneq E_k^h$ is impossible.*

Proof. Use same argument as in the proof of Lemma 4.7. □

We continue with subcase (3) $\rho_h = E_k^h$. It is also shown that this case is impossible.

Lemma 4.12. *Under the assumptions of Proposition 4.5, $\sigma_1 = \lambda$ and there exists a transversal $W = \{u_1, \ldots, u_t\}$ of σ-classes such that $W^h \subseteq \lambda$, subcase $\rho_h = E_k^h$ is also impossible.*

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Proof. Assume that $\rho_h = E_k^h$. We will show that $\rho_k = E_k^k$.

Suppose that $\rho_k \neq E_k^k$. Let n be the least integer such that $\rho_n \neq E_k^n$. Then $n > h$ (because $\rho_h = E_k^h$). Since $n, h > 2$, we have $\operatorname{Pol}\sigma \cap \operatorname{Pol}\lambda \subsetneq \operatorname{Pol}\rho_h \cap \operatorname{Pol}\lambda$. Furthermore, ρ_n is totally reflexive (due to $\rho_{n-1} = E_k^{n-1}$). Let $(a_1, \dots, a_h) \in E_k^h \setminus \lambda$ and $(u_1, \dots, u_h) \in \lambda \setminus \iota_k^h$. The unary operation f defined on E_k by :

$$f(t) = \begin{cases} a_i & \text{if } t = u_i, 1 < i \leq h, \\ a_1 & \text{otherwise.} \end{cases}$$

preserves ρ_n (because $n > h$ and ρ_n is totally reflexive) and does not preserve λ (because $(u_1, \dots, u_h) \in \lambda$ and $(f(u_1), \dots, f(u_h)) = (a_1, \dots, a_h) \notin \lambda$). Hence $\operatorname{Pol}\sigma \cap \operatorname{Pol}\lambda \subsetneq \operatorname{Pol}\rho_h \cap \operatorname{Pol}\lambda \subsetneq \operatorname{Pol}\lambda$, contradicting the maximality of $\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma$ in $\operatorname{Pol}\lambda$. Thus $\rho_k = E_k^k$. Set $E_k = \{a_1, \dots, a_k\}$. Then $(a_1, \dots, a_k) \in E_k^k = \rho_k$, so there exists $u \in E_k$ such that $\{u\} \times \{a_1, \dots, a_k\}^{h-1} \subseteq \lambda$. Thus u is a central element of λ; contradiction. □

Now we finish our discussion with subcase (3) $\lambda = \rho_h$.
For $h \leq l \leq k$, we set

$\gamma_l = \{(a_1, \dots, a_l) \in E_k^l \mid \exists v_1, \dots, v_l \in E_k : \{v_1, \dots, v_l\}^h \subseteq \lambda, \forall 1 \leq i \leq l, \forall j_1, \dots, j_{h-2} \in \{1, \dots, l\} \setminus \{i\}, (v_i, a_i, a_{j_1}, \dots, a_{j_{h-2}}) \in \lambda\}$.

Clearly $\lambda \subseteq \gamma_h \subseteq E_k^h$, and every γ_l, $h \leq l \leq k$, is totally symmetric. Since σ is a binary relation and $h > 2$, we have $\operatorname{Pol}\sigma \cap \operatorname{Pol}\lambda \subsetneq \operatorname{Pol}\lambda \cap \operatorname{Pol}\gamma_h$. γ_h satisfies one of the following three cases : (1) $\lambda = \gamma_h$, (2) $\lambda \subsetneq \gamma_h \subsetneq E_k^h$ and (3) $\gamma_h = E_k^h$.
Firstly, we show that the subcase (2) $\lambda \subsetneq \gamma_h \subsetneq E_k^h$ is impossible.

Lemma 4.13. *Under the assumptions of Proposition 4.5, $\sigma_1 = \lambda$, there exists a transversal $W = \{u_1, \dots, u_t\}$ of σ-classes such that $W^h \subseteq \lambda$ and $\rho_h = \lambda$, subcase $\lambda \subsetneq \gamma_h \subsetneq E_k^h$ is impossible.*

Proof. Assume that $\lambda \subsetneq \gamma_h \subsetneq E_k^h$. Using similar argument as in the proof of Lemma 4.7 we obtain the conclusion. □

Secondly, we prove also that subcase $\gamma_h = E_k^h$ can not occur.

Lemma 4.14. *Under the assumptions of Proposition 4.5, $\sigma_1 = \lambda$, there exists a transversal $W = \{u_1, \dots, u_t\}$ of σ-classes such that $W^h \subseteq \lambda$ and $\rho_h = \lambda$, subcase $\gamma_h = E_k^h$ is impossible.*

Proof. Assume that $\gamma_h = E_k^h$. We will show that $\gamma_k = E_k^k$.

Suppose that $\gamma_k \neq E_k^k$. Let n be the least integer such that $\gamma_n \neq E_k^n$. Then $n > h$ (because $\gamma_h = E_k^h$) and γ_n is totally reflexive. Using a similar argument as in the proof of Lemma 4.7, we obtain $\operatorname{Pol}\sigma \cap \operatorname{Pol}\lambda \subsetneq \operatorname{Pol}\gamma_n \cap \operatorname{Pol}\lambda \subsetneq \operatorname{Pol}\lambda$, contradicting the maximality of $\operatorname{Pol}\sigma \cap \operatorname{Pol}\lambda$ in $\operatorname{Pol}\lambda$. So, $\gamma_k = E_k^k$.

Now we show that this fact yields $\lambda = E_k^h$. Since $\gamma_k = E_k^k$, then for every $a_0, \dots, a_{h-1} \in E_k$ there exist certain $v_0, \dots, v_{h-1} \in E_k$ with

$$(v_0, v_1, \dots, v_{h-1}) \in \lambda \qquad (4.1)$$

and

$$\forall\ 0 \leq i \leq h-1, \forall j_1, \dots, j_{h-2} \in \{0, \dots, h-1\} \setminus \{i\}, (a_i, v_i, a_{j_1}, \dots, a_{j_{h-2}}) \in \lambda \qquad (4.2).$$

By induction, we will show that

$$\forall l \geq 0 \quad \forall a_0, \dots, a_{l-1} \in E_k \mid (a_0, \dots, a_{l-1}, v_l, v_{t+1}, \dots, v_{h-1}) \in \lambda \qquad (4.3).$$

For $l = 0$, (4.3) follows from (4.1). Assume (4.3) holds for $1 \leq l \leq h-2$. Then $(a_0, \dots, a_{l-1}, v_l, v_{t+1}, \dots, v_{h-1}) \in \lambda$ from (4.3). Choosing $u = v_l$, we can see that $(a_0, \dots, a_{l-1}, a_l, v_{l+1}, \dots, v_{h-1}) \in \rho_h = \lambda$. By (4.3) and $l = h$, we get $\lambda = E_k^h$ which is a contradiction. □

Now we end this discussion with subcase (1) $\lambda = \gamma_h$.

Let ε be the binary relation defined on E_k by

$\varepsilon = \{(a,b) \in E_k^2 \mid (a,b,a_3,\ldots,a_h) \in \lambda, \forall a_3,\ldots,a_h \in E_k\}$. The following proposition show that ε is an equivalence relation and the next lemma gives the link between the equivalence classes of ε and σ.

Proposition 4.15. *([9], page 205-206) Under the assumptions of Proposition 4.5 and $\gamma_h = \lambda$, ε is an equivalence relation on E_k. Furthermore, for all $a, b, a_1, \ldots, a_{h-1} \in E_k$ such that $(a,b) \in \varepsilon$, we have*
$(a, a_1, \ldots, a_{h-1}) \in \lambda \iff (b, a_1, \ldots, a_{h-1}) \in \lambda$.

Lemma 4.16. *Under the assumptions of Proposition 4.5, $\sigma_1 = \lambda$, there exists a transversal $W = \{u_1, \ldots, u_t\}$ of σ-classes such that $W^h \subseteq \lambda$ and $\gamma_h = \lambda$, then for all $x, y \in E_k$, $[x]_\sigma \cap [y]_\varepsilon \neq \emptyset$.*

Proof. Assume that $\gamma_h = \lambda$. We set $\alpha_h = \{(a_1,\ldots,a_h) \in E_k^h \mid (a_1,a_2) \in \sigma\}$.
Assume that $\alpha_h \subseteq \lambda$. Let $(a_1,\ldots,a_h) \in E_k^h$, there exists $v \in E_k$ such that $(a_1, v) \in \sigma$ and $(v, a_2, \ldots, a_h) \in \lambda$. Since $\alpha_h \subseteq \lambda$, we have $\{v\} \times \{a_1,\ldots,a_h\}^{h-1} \subseteq \lambda$. Thus $(a_1,\ldots,a_h) \in \lambda$ (because $\lambda = \rho_h$). So, $E_k^h = \lambda$; contradiction. We conclude that $\alpha_h \nsubseteq \lambda$. Let $(e_1,\ldots,e_h) \in E_k^h \setminus \lambda$ such that $(e_1,e_2) \in \sigma$. We set $\beta_h = \{(a_1,\ldots,a_h) \in E_k^h \mid \exists v \in [a_1]_\sigma, (v, a_2, a_{j_1}, \ldots, a_{j_{h-2}}) \in \lambda \ \forall j_1,\ldots,j_{h-2} \in \{1,\ldots,h\}\}$.

Clearly, we have $\lambda \subseteq \beta_h \subseteq E_k^h$. $(e_1,\ldots,e_h) \in \beta_h \setminus \lambda$, taking $v = e_2$. Hence $\lambda \subsetneq \beta_h \subseteq E_k^h$.

If $\lambda \subsetneq \beta_h \subsetneq E_k^h$, then using a similar argument as in the proof of Lemma 4.7 we obtain a contradiction. So, $\beta_h = E_k^h$. By induction on $l \geq h$ and the fact that β_{h+1} is totally reflexive, we can show that $\beta_k = E_k^k$.

Assume that $E_k = \{a_1,\ldots,a_k\}$. Then $(a_1,\ldots,a_k) \in E_k^k = \beta_k$ and there exists $v \in [a_1]_\sigma$ such that $(v, a_2, a_{j_1}, \ldots, a_{j_{h-2}}) \in \lambda, \forall j_1,\ldots,j_{h-2} \in \{1,\ldots,k\}$. Thus $(v, a_2) \in \varepsilon$ and $[a_1]_\sigma \cap [a_2]_\varepsilon \neq \emptyset$. □

Now we are ready to give the proof of Proposition 4.5 and 4.1.

Proof. (Proof of Proposition 4.5) Combining Lemmas 4.6-4.16 and Proposition 4.15, we obtain the result. □

Proof. (Proof of Proposition 4.1) The necessary condition of Proposition 4.1 is given by Lemmas 4.2 and 4.4 and the sufficient condition is obtained by Proposition 4.5. □

Secondly, we investigate the bounded partial order case.

5. Bounded Partial Order

Let λ be an h-ary regular relation ($3 \leq h < k$) on E_k and σ be a bounded partial order on E_k with greatest element $\top$ and least element $\bot$. In this section, we will show that $\mathrm{Pol}\,\lambda \cap \mathrm{Pol}\,\sigma$ is not maximal below $\mathrm{Pol}\,\lambda$. We recall the relation σ_1 defined below. $\sigma_1 = \{(a_1,\ldots,a_h) \in E_k^h \mid \exists u \in E_k, (a_1,u) \in \sigma \wedge \ (u, a_2, \ldots, a_h) \in \lambda\}$. We have $\lambda \subseteq \sigma_1 \subseteq E_k^h$ and $\mathrm{Pol}\,\lambda \cap \mathrm{Pol}\,\sigma \subsetneq \mathrm{Pol}\,\lambda \cap \mathrm{Pol}\,\sigma_1 \subseteq \mathrm{Pol}\,\lambda$.

Now we state the main result of this section.

Proposition 5.1. *Let λ be an h-ary regular relation on E_k ($3 \leq h < k$). If σ is a bounded partial order on E_k with greatest element $\top$ and least element $\bot$, then $\mathrm{Pol}\,\lambda \cap \mathrm{Pol}\,\sigma$ is not maximal below $\mathrm{Pol}\,\lambda$.*

The proof will be shared in Lemmas 5.2, 5.3 and 5.4. Comparing λ and σ_1, we have the following cases: (a) $\lambda = \sigma_1$, (b) $\lambda \subsetneq \sigma_1 \subsetneq E_k^h$ and (c) $\sigma_1 = E_k^h$. The next lemma shows that the case (a) is impossible.

Lemma 5.2. *If $\mathrm{Pol}\lambda \cap \mathrm{Pol}\,\sigma$ is maximal below $\mathrm{Pol}\,\lambda$, then the case $\lambda = \sigma_1$ is impossible.*

Proof. Assume that $\lambda = \sigma_1$. Let $a_2,\ldots,a_h \in E_k$, we have $(a_2, \top) \in \sigma$ and $(\top, \top, \ldots, a_h) \in \lambda$ (due to $\top$ greatest element of σ and λ totally reflexive). Thus $(a_2, \top, \ldots, a_h) \in \sigma_1 = \lambda$. Hence $\top$ is a central element of λ; contradiction.
□

Now we look at the case (c). The following lemma shows that it is impossible.

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Lemma 5.3. *If Pol$\lambda \cap$ Pol σ is maximal below* Pol λ, *then the case* $\sigma_1 = E_k^h$ *is impossible.*

Proof. Assume that $\sigma_1 = E_k^h$. Let $a_2, \ldots, a_h \in E_k$; we have $(\top, a_2, \ldots, a_h) \in E_k^h = \sigma_1$, there exists $u \in E_k$ such that $(\top, u) \in \sigma$ and $(u, a_2, \ldots, a_h) \in \lambda$. Hence $u = \top$ (due to $\top$ is the greatest element of σ) and $(\top, a_2, \ldots, a_h) \in \lambda$. Therefore $\top$ is a central element of λ; contradiction.
□

Finally we show that the case (b) is also impossible in the next lemma.

Lemma 5.4. *If Pol$\lambda \cap$ Pol σ is maximal below* Pol λ, *then the case* $\lambda \subsetneq \sigma_1 \subsetneq E_k^h$ *is impossible.*

Proof. Assume that $\lambda \subsetneq \sigma_1 \subsetneq E_k^h$. Let $(a_1, \ldots, a_h) \in E_k^h \setminus \sigma_1$ and $(u_1, \ldots, u_h) \in \sigma_1 \setminus \lambda$. Consider again the unary operation g defined on E_k by $g(x) = a_i$ if and only if $x \in [u_i]_{\theta_1}$ where $\theta_1 \in T$ and T is the h-regular family associated to λ. g preserves λ and does not preserve σ_1, since $(u_1, \ldots, u_h) \in \sigma_1$ and $(g(u_1), \ldots, g(u_h)) = (a_1, \ldots, a_h) \notin \sigma_1$ and g restricted to each θ_1-class is constant. Hence Pol $\lambda \cap$ Pol $\sigma \subsetneq$ Pol $\sigma_1 \cap$ Pol $\lambda \subsetneq$ Pol λ, contradicting the maximality of Pol $\lambda \cap$ Pol σ below Pol λ. □

Proof. (Proof of **Proposition** 5.1) Using Lemma 5.2, Lemma 5.3 and Lemma 5.4, we observe that Pol $\sigma \cap$ Pol λ is not maximal below Pol λ. □

Thirdly, we investigate the case of prime permutation relation.

6. Prime Permutation Relation

In this section, s is a fixed point free permutation on E_k with $s^p = id$ (p prime). We denote by s°, the graph of s. In other words, $s^\circ = \{(a, b) \in E_k^2 \mid b = s(a)\}$. Let θ_s be the equivalence relation on E_k consisting of all pairs $(a, b) \in E_k^2$ with $a = s^i(b)$ for some $0 \leq i < p$.

Definition 6.1. *Let* $k \geq 3$, λ *be an h-ary regular relation* $(3 \leq h < k)$ *and s be a fixed point free permutation on* E_k *with* $s^p = id$ *(p prime). An equivalence relation* θ *is said transversal to s if* $s \in$ Pol θ *and* $\theta_s \cap \theta = \Delta_{E_k}$ *where* Δ_{E_k} *is the equality relation on* E_k.

Here we state the main result of this section.

Proposition 6.2. *Let* λ *be an h-ary regular relation on* E_k *and s be a fixed point free permutation on* E_k *with* $s^p = id$ *(p prime). Then* Pol $\lambda \cap$ Pol s° *is maximal below* Pol λ *if and only if* λ *is* θ_s*-closed.*

Before the proof of this proposition, we recall the following characterization of the relational clone generated by the prime permutation s and the regular relation λ.

Proposition 6.3. *([16], Proposition 3.14) Let* $k \geq 3$, λ *be an h- ary regular relation* $(3 \leq h < k)$ *and s be a fixed point free permutation on* E_k *with* $s^p = id$ *(p prime). The relational algebra* $[\{s^\circ, \lambda\}]$ *contains one of the following relations:*

- *(1)* *a non-trivial equivalence relation which is either* θ_s*-closed or transversal to s;*
- *(2)* *an affine relation determined by an elementary abelian p-group* $(\mathbf{k}; +)$ *such that there exists an element* $c \in \mathbf{k}$ *with* $s(x) = x + c$ *for all* $x \in \mathbf{k}$*;*
- *(3)* *a central relation;*
- *(4)* *a* θ_s*-closed regular relation.*

The next lemma gives the sufficient condition in Proposition 6.2.

Lemma 6.4. *Let* $k \geq 3$, λ *be an h-ary regular relation* $(3 \leq h < k)$ *and s be a fixed point free permutation on* E_k *with* $s^p = id$ *(p prime). If* Pol $\lambda \cap$ Pol s° *is maximal below* Pol λ, *then* λ *is* θ_s*-closed.*

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Proof. Assume that $\operatorname{Pol}\lambda \cap \operatorname{Pol} s^\circ$ is maximal below $\operatorname{Pol}\lambda$. From Proposition 6.3, the relational algebra $[\{s^\circ, \lambda\}]$ contains a relation γ satisfying (1), (2), (3) or (4). Thus $\operatorname{Pol}\lambda \cap \operatorname{Pol} s^\circ \subseteq \operatorname{Pol}\lambda \cap \operatorname{Pol}\gamma \subseteq \operatorname{Pol}\lambda$. From Rosenberg's classification theorem (see Theorem 3.1) $\operatorname{Pol}\gamma$ is a maximal clone. Assume that $\gamma \neq \lambda$, then $\operatorname{Pol}\lambda \cap \operatorname{Pol}\gamma \subsetneq \operatorname{Pol}\lambda$ (because $\operatorname{Pol}\lambda$ and $\operatorname{Pol}\gamma$ are two different maximal clones). Thus $\operatorname{Pol}\lambda \cap \operatorname{Pol} s^\circ \subseteq \operatorname{Pol}\lambda \cap \operatorname{Pol}\gamma \subsetneq \operatorname{Pol}\lambda$. Let $a \in E_k$. Consider the constant unary operation C_a with value a. It is easy to see that C_a preserves λ and γ and C_a does not preserve s°. Hence $\operatorname{Pol}\lambda \cap \operatorname{Pol} s^\circ \subsetneq \operatorname{Pol}\lambda \cap \operatorname{Pol}\gamma \subsetneq \operatorname{Pol}\lambda$, contradicting the choice of s. Therefore $\gamma = \lambda$ and we conclude that λ is θ_s-closed. □

The converse of the equivalence in Proposition 6.2 follows from the following result due to Rosenberg and Szendrei.

Proposition 6.5. *([16], Proposition 4.3) Let $k \geq 3$, s be a fixed point free permutation on E_k with $s^p = id$ (p prime) and λ be an h-ary θ_s-closed regular relation ($3 \leq h < k$). The relational subalgebras of $[\{s^\circ, \lambda\}]$ form a 4-element boolean lattice consisting of $[\{s^\circ, \lambda\}]$, $[\{s^\circ\}]$, $[\{\lambda\}]$ and $[\{\Delta_{E_k}\}]$.*

The next Corollary gives the maximality of $\operatorname{Pol}\lambda \cap \operatorname{Pol} s^\circ$ in $\operatorname{Pol}\lambda$.

Corollary 6.6. *Let λ be an h-ary regular relation on E_k ($3 \leq h < k$) and s be a prime permutation relation on E_k such that λ is θ_s-closed. Then $\operatorname{Pol} s^\circ \cap \operatorname{Pol}\lambda$ is maximal below $\operatorname{Pol}\lambda$.*

Proof. It follows from Proposition 6.5. □

Proof. (Proof of Proposition 6.2) It follows from Lemma 6.4 and Corollary 6.6. □

Fourthly, we focus our attention on binary central relation case.

7. Binary Central Relation

In this section, we give a necessary and sufficient condition on a binary central relation σ such that the clone $\operatorname{Pol}\sigma \cap \operatorname{Pol}\lambda$ is covered by $\operatorname{Pol}\lambda$, where λ is an h-ary regular relation on E_k ($3 \leq h < k$).

Let σ be a binary central relation on E_k. For $l \geq 2$, we set

$\omega_l = \{(a_1, \ldots, a_l) \in E_k^l \mid \exists u \in E_k, (a_1, u), \ldots, (a_l, u) \in \sigma \text{ and } \{u\} \times \{a_1, \ldots, a_l\}^{h-1} \subseteq \lambda\}$. From definition ω_l is totally symmetric for $2 \leq l$. We consider again the equivalence relation ε defined on E_k by

$$\varepsilon = \{(a, b) \in E_k^2 \mid (a, b, a_3, \ldots, a_h) \in \lambda, \forall\ a_3, \ldots, a_h \in E_k\}.$$

Here we state the main result of this section.

Proposition 7.1. *Let $k \geq 3$, λ be an h-ary regular relation on E_k ($3 \leq h < k$) and σ be a binary central relation on E_k. Then $\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma$ is maximal below $\operatorname{Pol}\lambda$ if and only if for all $x \in E_k$, $C_\sigma \cap [x]_\varepsilon \neq \emptyset$.*

The necessary condition of Proposition 7.1 is stated in the following Lemma.

Lemma 7.2. *Let λ be an h-ary regular relation on E_k ($3 \leq h < k$) determined by the h-regular family $T = \{\theta_1, \ldots, \theta_m\}$, $m \geq 1$ and σ be a binary central relation on E_k such that $\forall x \in E_k$, $C_\sigma \cap [x]_\varepsilon \neq \emptyset$. Then $\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma$ is a submaximal clone of $\operatorname{Pol}\lambda$.*

Proof. Assume that $\forall x \in E_k$, $C_\sigma \cap [x]_\varepsilon \neq \emptyset$. For all $x \in E_k$, we choose and fix $c_x \in C_\sigma \cap [x]_\varepsilon$. Let $g^n \in \operatorname{Pol}\lambda \setminus (\operatorname{Pol}\sigma \cap \operatorname{Pol}\lambda)$ be an n-ary operation. Using the operation H constructed in the proof of Lemma 4.3, we will show that $\langle(\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma) \cup \{g\}\rangle = \operatorname{Pol}\lambda$.

Clearly, we have $\langle(\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma) \cup \{g\}\rangle \subseteq \operatorname{Pol}\lambda$. It remains to prove that $\operatorname{Pol}\lambda \subseteq \langle(\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma) \cup \{g\}\rangle$. Let $u \in \operatorname{Pol}\lambda$ be an m-ary operation on E_k. We will show that $u \in \langle(\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma) \cup \{g\}$. We extend H to $\tilde{H}$ defined by:

$$\tilde{H}(\boldsymbol{y}) = \begin{cases} u(\boldsymbol{x}) & \text{if } \boldsymbol{y} = \operatorname{ext}(\boldsymbol{x}), \\ c_{u(y_1, \ldots, y_m)} & \text{otherwise.} \end{cases}$$

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Let's show that $\tilde{H} \in \operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma$. Firstly, we show that $\tilde{H} \in \operatorname{Pol}\sigma$.

Let $\boldsymbol{a} = (a_1, \ldots, a_{m+\ell})$ and $\boldsymbol{b} = (b_1, \ldots, b_{m+\ell})$ such that $(\boldsymbol{a}, \boldsymbol{b}) \in \sigma$. We distinguish two cases:

Case 1: If $\boldsymbol{a} = \text{ext}(\boldsymbol{x})$ and $\boldsymbol{b} = \text{ext}(\boldsymbol{y})$, then $\boldsymbol{x} = \boldsymbol{y}$ (From Lemma 4.3). Hence $(\tilde{H}(\boldsymbol{a}), \tilde{H}(\boldsymbol{b})) = (u(\boldsymbol{x}), u(\boldsymbol{x})) \in \sigma$.

Case 2: For all $\boldsymbol{x} \in E_k^m, \boldsymbol{a} \neq ext(\boldsymbol{x})$ or For all $\boldsymbol{x} \in E_k^m, \boldsymbol{b} \neq ext(\boldsymbol{x})$; without loss of generality, we suppose that For all $\boldsymbol{x} \in E_k^m, \boldsymbol{a} \neq ext(\boldsymbol{x})$. Then $(\tilde{H}(\boldsymbol{a}), \tilde{H}(\boldsymbol{b})) = (c_{u(a_1,\ldots,a_m)}, \tilde{H}(\boldsymbol{b})) \in \sigma$ (because $c_{u(a_1,\ldots,a_m)} \in C_\sigma$). Thus $\tilde{H} \in \operatorname{Pol}\sigma$.

Secondly, we show that $\tilde{H} \in \operatorname{Pol}\lambda$.

Let $\boldsymbol{a}_1 = (a_{1,1}, \ldots, a_{1,h}), \ldots, \boldsymbol{a}_m = (a_{m,1}, \ldots, a_{m,h}), \ldots, \boldsymbol{a}_{m+l} = (a_{m+\ell,\ell}, \ldots, a_{m+\ell,h}) \in \lambda$. For all $1 \leq j \leq h$, set $\boldsymbol{d}_j = (a_{1,j}, \ldots, a_{m,j}, \ldots, a_{m+l,j})$ and $\boldsymbol{b}_j = (a_{1,j}, \ldots, a_{m,j})$. We will show that $(\tilde{H}(\boldsymbol{d}_1), \ldots, \tilde{H}(\boldsymbol{d}_h)) \in \lambda$. We distinguish again two cases:

Case 1: $\forall 1 \leq j \leq h$, $\boldsymbol{d}_j = \text{ext}(\boldsymbol{b}_j)$, then we obtain $(\tilde{H}(\boldsymbol{d}_1), \ldots, \tilde{H}(\boldsymbol{d}_h)) = (u(\boldsymbol{b}_1), \ldots, u(\boldsymbol{b}_h)) \in \lambda$ (due to $(\boldsymbol{b}_1, \ldots, \boldsymbol{b}_h) \in \lambda$ and $u \in \operatorname{Pol}\lambda$).

Case 2: There exists $1 \leq j \leq h$ such that $\tilde{H}(\boldsymbol{d}_j) = c_{u(\boldsymbol{b}_j)}$, then we have $(\tilde{H}(\boldsymbol{d}_1), \ldots, \tilde{H}(\boldsymbol{d}_h)) = (\tilde{H}(\boldsymbol{d}_1), \ldots, c_{u(\boldsymbol{b}_j)}, \ldots, \tilde{H}(\boldsymbol{d}_h)) \in \lambda$ because $(c_{u(\boldsymbol{b}_j)}, u(\boldsymbol{b}_j)) \in \varepsilon$, λ is totally symmetric, $(u(\boldsymbol{b}_1), \ldots, u(\boldsymbol{b}_h)) \in \lambda$ and we can replace gradually $u(\boldsymbol{b}_k)$ by $c_{u(\boldsymbol{b}_k)}$ until obtained the desire tuple. Thus $\tilde{H} \in \operatorname{Pol}\lambda$. For all $\boldsymbol{x} \in E_k^m$, $\tilde{H}(\boldsymbol{x}, f_{m+\ell}(\boldsymbol{x}), \ldots, f_{m+l}(\boldsymbol{x})) = \tilde{H}(ext(\boldsymbol{x})) = \tilde{H} \circ \text{ext}(\boldsymbol{x}) = u(\boldsymbol{x})$. Thus, $u = \tilde{H} \circ \text{ext} \in \langle (\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma) \cup \{g\} \rangle$. Therefore $\operatorname{Pol}\lambda \subseteq \langle (\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma) \cup \{g\} \rangle$. We conclude that $\langle (\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma) \cup \{g\} \rangle = \operatorname{Pol}\lambda$. Thus $\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma$ is maximal below $\operatorname{Pol}\lambda$. □

The sufficient condition in Proposition 7.1 is given by the following statement.

Proposition 7.3. *Let λ be an h-ary regular relation $(3 \leq h < k)$ on E_k and σ be a binary central relation on E_k. If $\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma$ is maximal below $\operatorname{Pol}\lambda$, then $\forall x \in E_k$, $C_\sigma \cap [x]_\varepsilon \neq \emptyset$.*

The proof of Proposition 7.3 is investigated in the following Lemmas.

Let λ be an h-ary regular relation $(3 \leq h < k)$ on E_k and σ be a binary central relation on E_k such that $\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma$ is maximal below $\operatorname{Pol}\lambda$. Recall that $\omega_2 = \{(a_1, a_2) \in E_k^2 \mid \exists u \in E_k, (a_1, u), (a_2, u) \in \sigma \text{ and } \{u\} \times \{a_1, a_2\}^{h-1} \subseteq \lambda\}$.

Clearly $\sigma \subseteq \omega_2 \subseteq E_k^2$ and $\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma \subseteq \operatorname{Pol}\lambda \cap \operatorname{Pol}\omega_2 \subseteq \operatorname{Pol}\lambda$ (due to λ totally reflexive and $\omega_2 \in [\lambda, \sigma]$). We look at the following three subcases: (1) $\sigma \subsetneq \omega_2 \subsetneq E_k^2$, (2) $\omega_2 = \sigma$ and (3) $\omega_2 = E_k^2$.

Firstly, we show in the next lemma that the subcase (1) $\sigma \subsetneq \omega_2 \subsetneq E_k^2$ can not occur.

Lemma 7.4. *Under the assumptions of Proposition 7.3, the subcase $\sigma \subsetneq \omega_2 \subsetneq E_k^2$ is impossible.*

Proof. Let $c \in C_\sigma$. Since $\sigma \subsetneq \omega_2 \subsetneq E_k^2$, then ω_2 is a binary central relation with $c \in C_{\beta_2}$. Therefore σ and ω_2 are two different central relations. Thus $\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma_2 \subsetneq \operatorname{Pol}\lambda \cap \operatorname{Pol}\omega_2 \subsetneq \operatorname{Pol}\lambda$ (because $\operatorname{Pol}\lambda$ and $\operatorname{Pol}\omega_2$ are two different maximal clones and $\sigma \subsetneq \omega_2$) , contradicting the maximality of $\operatorname{Pol}\sigma \cap \operatorname{Pol}\lambda$ below $\operatorname{Pol}\lambda$. □

Secondly, we show that subcase $\omega_2 = \sigma$ is also impossible.

Lemma 7.5. *Under the assumptions of Proposition7.3, the subcase $\omega_2 = \sigma$ is also impossible.*

Proof. Assume that $\omega_2 = \sigma$. In this case we have $h = 3$ (because λ is totally reflexive). For $l \geq 3$, we set

$\theta'_l = \{(a_1, \ldots, a_l) \in E_k^l \mid \exists u \in E_k, (a_1, u) \in \sigma \text{ and } \forall j_1, \ldots, j_{h-2} \in \{1, \ldots, l\}, (u, a_2, a_{j_1}, \ldots, a_{jh-2}) \in \lambda\}$.

Clearly $\lambda \subseteq \theta'_3 \subseteq E_k^3$, hence θ'_3 is totally reflexive and $\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma \subsetneq \operatorname{Pol}\lambda \cap \operatorname{Pol}\theta'_3 \subseteq \operatorname{Pol}\lambda$(due to $\theta'_3 \in [\lambda, \sigma]$).

Let $c \in C_\sigma$. Since c is not a central element of λ, there exist $a_2, a_3 \in E_k$ such that $(c, a_2, a_3) \notin \lambda$. So $(c, a_2) \in \sigma$, $(a_2, a_2, c), (a_2, a_2, a_3) \in \lambda$. Hence $(c, a_2, a_3) \in \theta'_3 \setminus \lambda$. Thus $\lambda \subsetneq \theta'_3 \subseteq E_k^3$.

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If $\lambda \subsetneq \theta'_3 \subsetneq E_k^3$, then using a similar argument as in the proof of Lemma 4.7, we obtain $\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma \subsetneq \operatorname{Pol}\lambda \cap \operatorname{Pol}\theta'_3 \subsetneq \operatorname{Pol}\lambda$; contradicting the maximality of $\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma$ below $\operatorname{Pol}\lambda$. Hence $\theta'_3 = E_k^3$.

Now we will show that $\theta'_k = E_k^k$. We assume that $\theta'_k \neq E_k^k$. Let n be the least integer N such that $\theta'_N \neq E_k^N$. Then $3 < n \leq k$ and θ'_n is totally reflexive (due to $\theta'_{n-1} = E_k^{n-1}$). A similar argument as in the proof of Lemma 4.8 yields that $\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma \subsetneq \operatorname{Pol}\lambda \cap \operatorname{Pol}\theta'_n \subsetneq \operatorname{Pol}\lambda$(due to θ'_n is not a diagonal relation); contradicting the maximality of $\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma$ below $\operatorname{Pol}\lambda$. Thus $\theta'_k = E_k^k$.

Furthermore, We show that $\varepsilon \subseteq \sigma$. Let $(a,b) \in \varepsilon$. Since $\theta'_k = E_k^k$, there exists $u \in E_k$ such that $(u,a) \in \varepsilon$ and $(b,u) \in \sigma$. Let $c \in C_\sigma$. We have $(a,c),(b,c) \in \sigma$ and $(a,b,c) \in \lambda$ (because $(a,b) \in \varepsilon$). Hence $(a,b) \in \omega_2 = \sigma$. Therefore $\varepsilon \subseteq \sigma$.

Let $(x,y) \in E_k^2$. We assume that $E_k = \{x,y,a_3,\ldots,a_k\}$. Therefore $(x,y,a_3,\ldots,a_k) \in E_k^k = \theta'_k$. So there exists $u \in E_k$ such that $(x,u) \in \sigma$ and $(u,y,v_1,\ldots,v_{h-2}) \in \lambda$ for all $v_1,\ldots,v_{h-2} \in E_k$. Thus $(y,u) \in \varepsilon$. Hence $(y,u) \in \sigma$ (because $\varepsilon \subseteq \sigma$). Thus $(x,u),(y,u) \in \sigma$ and $(x,y,u) \in \lambda$ (because $(y,u) \in \varepsilon$). Therefore $(x,y) \in \omega_2$, and $\sigma = \omega_2 = E_k^2$; contradiction. □

We finish with subcase (3) $\omega_2 = E_k^2$. The following lemma shows that every ε-block contains a central element of σ.

Lemma 7.6. *Under the assumptions of Proposition 7.3 and $\omega_2 = E_k^2$, then for all $x \in E_k$, $C_\sigma \cap [x]_\varepsilon \neq \emptyset$.*

Proof. Assume that $\omega_2 = E_k^2$. We will show that $\omega_{h-1} = E_k^{h-1}$. If $\omega_{h-1} \neq E_k^{h-1}$, then we denote by n the least integer N such that $\omega_N \neq E_k^N$. Then $2 < n < h-1$. Since ω_n is totally reflexive and $n > 2$, for $(a,b) \in \sigma$ such that $a \neq b$ and $(u,v) \notin \sigma$, the unary operation f defined by $f(a) = u$ and $f(x) = v$ if $x \neq a$, preserves w_n and λ; and does not preserve σ. Hence $\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma \subsetneq \operatorname{Pol}\lambda \cap \operatorname{Pol}\omega_n$; it remains the next inequality to conclude. Let $a,b \in \sigma$ such that $a \neq b$, we fix $\boldsymbol{a_1} = (a,a,\ldots,a)$, $\boldsymbol{a_2} = (a,b,\ldots,a),\ldots,\boldsymbol{a_n} = (a,a,\ldots,b)$, n tuples of E_k^n. Let $(u_1,\ldots,u_n) \in E_k^n$. The n-ary operation g defined on E_k by $g(\boldsymbol{a_i}) = u_i$ for $1 \leq i \leq n$ and $g(\boldsymbol{x}) = u_1$ otherwise, preserves λ (due to λ totally reflexive and $h > n$) and does not preserve ω_n (due to $\boldsymbol{a_i} \in \omega_n$, $1 \leq i \leq n$ and $g(\boldsymbol{a_1},\ldots,\boldsymbol{a_n}) = (u_1,\ldots,u_n) \notin E_k^n$); therefore $\operatorname{Pol}\lambda \cap \operatorname{Pol}\omega_n \subsetneq \operatorname{Pol}\lambda$; contradicting the maximality of $\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma$ in $\operatorname{Pol}\lambda$. Thus $\omega_{h-1} = E_k^{h-1}$.

Further we show that $\omega_h = E_k^h$. It is easy to see that $\omega_h \cap \lambda$ is an h-ary totally reflexive and totally symmetric relation contains in λ and $\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma \subseteq \operatorname{Pol}\lambda \cap \operatorname{Pol}(\omega_h \cap \lambda)$. We discuss two subcases: (i) $\omega_h \cap \lambda = \lambda$, (ii) $\omega_h \cap \lambda \subsetneq \lambda$. We proceed first with case (ii). Since $\lambda \cap \omega_h$ is totally reflexive, we have $\operatorname{Pol}\lambda \cap \operatorname{Pol}\sigma \subsetneq \operatorname{Pol}\lambda \cap \operatorname{Pol}(\omega_h \cap \lambda)$. Choosing $(u_1,\ldots,u_h) \in \lambda \setminus \omega_h$ and using an operation similar to g below, we obtain that $\operatorname{Pol}\lambda \cap \operatorname{Pol}(\omega_h \cap \lambda) \subsetneq Pol\lambda$, contradicting the maximality. Hence case (ii) can not occur. Now we suppose that (i) holds. It means that λ is a subset of ω_h. Thus three subcases can occur: (i) $\lambda = \omega_h$, (ii) $\lambda \subsetneq \omega_h \subsetneq E_k^h$ and (iii) $\omega_h = E_k^h$.

We begin with subcase (iii) $\omega_h = E_k^h$. Using a similar argument as above in the proof of the fact that $\omega_{h-1} = E_k^{h-1}$, we can observe that $\omega_k = E_k^k$. Therefore λ has a central element which is a contradiction.

Now we continue with subcase (ii) $\lambda \subsetneq \omega_h \subsetneq E_k^h$. A similar argument as in the proof of Lemma 4.7 shows that this case is also impossible.

We finish with subcase (i) $\lambda = \omega_h$.

We consider the relation

$\varphi_l = \{(a_1,\ldots,a_l) \in E_k^l \mid \exists u \in E_k, (a_1,u),\ldots,(a_l,u) \in \sigma$ and $\forall j_1,\ldots,j_{h-2} \in \{1,\ldots,l\}$, $(u,a_2,a_{j_1},\ldots,a_{jh-2}) \in \lambda\}$ for $l \geq 2$.

We observe that λ is a subset of φ_h. Hence, we obtain two possibilities: (i) $\lambda \subseteq \varphi_h \subsetneq E_k^h$ and (ii) $E_k^h = \varphi_h$.

Firstly, we discuss case (i) $\lambda \subseteq \varphi_h \subsetneq E_k^h$. Let $c \in C_\sigma$. Since c is not a central element of λ, there exist $a_2,\ldots,a_h \in E_k$ such that $(c,a_2,\ldots,a_h) \notin \lambda$. Hence, $(a_2,c,a_3,\ldots,a_h) \notin \lambda$ (because λ is totally symmetric). We have $(a_i,c) \in \sigma$ for $2 \leq i \leq h$ and $(c,c,a_{j_1},\ldots,a_{j_{h-2}}) \in \lambda$ for all $j_1,\ldots,j_{h-2} \in \{1,\ldots,h\}$.

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So $(a_2, c, \ldots, a_h) \in \varphi_h \setminus \lambda$. Using the h tuple $(a_2, c, \ldots, a_h)$ and the operation g defined in the proof of Lemma 4.7, we obtain $\mathrm{Pol}\,\lambda \cap \mathrm{Pol}\,\sigma \subsetneq \mathrm{Pol}\,\lambda \cap \mathrm{Pol}\,\varphi_h \subsetneq \mathrm{Pol}\,\lambda$, which is a contradiction.

Secondly we investigate case $\varphi_h = E_k^h$. A similar argument as in the proof of Lemma 4.7 shows that $\varphi_k = E_k^k$.

We assume that $E_k = \{a_1, \ldots, a_k\}$. Then $(a_1, \ldots, a_k) \in E_k^k = \varphi_k$, so there exists $u \in E_k$ such that $(a_1, u), \ldots, (a_k, u) \in \sigma$ and $(u, a_2, a_{j_1}, \ldots, a_{j_{h-2}}) \in \lambda$ for all $j_1, \ldots, j_{h-2} \in \{1, \ldots, k\}$. Hence $u \in C_\sigma$ and $(u, a_2) \in \varepsilon$. Thus $\forall x \in E_k, C_\sigma \cap [x]_\varepsilon \neq \emptyset$. □

Proof. (Proof of Proposition 7.3) It follows from the combination of Lemma 7.4, Lemma 7.5 and Lemma 7.6. □

Proof. (Proof of Proposition 7.1) It follows from the combination of Proposition 7.1, Lemma 7.2 and Proposition 7.3. □

8. Conclusion and Further Research

In this work, we have characterized all binary relations σ such that the clone $\mathrm{Pol}\,\lambda \cap \mathrm{Pol}\,\sigma$ is a submaximal clone of $\mathrm{Pol}\,\lambda$ for a fixed regular relation λ on a k-element set ($k > 2$). The complete characterization of submaximal clone in the lattice of clone on finite set is still opened and we will continue to explore it in our future work.

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References

1. Beherisch M.: Weak bases for Boolean relational clones revisited, in IEEE 52nd ISMVL 2022, Dallas, TX, May 18–20, 2022, 68–73 (2022). https://doi.org/10.1109/ISMVL52857.2022.00017.
2. Behrisch M.: On weak bases for Boolean relational clones and reductions for computational problems, To appear in Journal of Applied Logics– IfCoLog Journal (2023).
3. Behrisch M.: Weak bases for maximal clones, in IEEE 53rd ISMVL 2023, Matsue, Shimane, Japan, May 22–24, 2023, 128–133 (2023). Available from https://doi.org/10.1109/ISMVL57333.2023.00034.
4. Diekouam L.E.F., Jeufack Y.L.T.,Temgoua E.R.A., Tonga, M.: Meet-reducible submaximal clones determined by non-trivial equivalence relations, Journal of Algebra, Number theory; advances and Applications. Vol 17, Number 1, 29–93 (2017).
5. Jeufack Y.L.T., Diekouam L.E.F., Temgoua E.R.A.: Meet-reducible submaximal clones determined by two central relations.Asian-Eropean Journal of Mathematics, Vol.12, N_0.1,1950064. (26pages),(2018).
6. Jeufack Y.L.T., Djoumessi J., Temgoua E.R.A.: Binary relations and submaximal clones determined by central relations. Discussiones Mathematicae-General Algebra and Applications 43. (38 pages),263-300,(2023).
7. Kerkoff S., Pöschel R., Friedrich M. S.: A Short Introduction to Clones, Published in WACT 1 March 2014 Mathematics.
8. Lau D.: Submaximale Klassen von P_3. J. Inf. Process. Cybern. EIK 18, 4/5, 227-243 (1982).
9. Lau D.: Function Algebras on finite sets. A basic course on many-valued logic and clone theory. Springer-Verlag, New York, Heidelberg, Berlin (2006).
10. Post E.L.: Introduction to a general theory of elementary propositions. Amer. J. Math. 43, 163-185 (1921).
11. Post E.L.: The two-valued iterative systems of mathematical logic. Ann. Math. Studies, Vol. 5. Princeton University Press (1941).
12. Szendrei Á.: Clones in universal Algebra. Séminaire de mathématiques supérieures,Vol. 99. Les presses de l'université de Montréal (1986).
13. Temgoua E.R.A., Rosenberg I.G.: Binary central relations and submaximal clones determined by non-trivial equivalence relations. Algebra universalis 67, 299-311 (2012).
14. Temgoua E.R.A.: Meet-irreducible submaximal clones determined by non-trivial equivalence relations. Algebra universalis 70, 175-196 (2013).

doi:10.20944/preprints202412.0377.v1

15. Rosenberg I.G.: La structure des fonctions de plusieurs variables sur un ensemble fini. C. R. Acad. Sci. Paris, Ser.A–B 260, 3817-3819 (1965).
16. Rosenberg I.G., Szendrei Á.: Submaximal clones with a prime order automorphism. Acta Sci. Math. 49, 29-48 (1985).