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On Generalized Tetranacci Quaternions

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Abstract. In this paper, we introduce the generalized Tetranacci quaternions and present some properties of these quaternions.

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1. Introduction

In this paper, we define generalized Tetranacci quaternions in the next section and give some properties of them and Tetranacci and Tetranacci-Lucas quaternions as special cases. First, we present some background about quaternions and generalized Tetranacci numbers.

A quaternion is a hyper-complex number and is defined by

$$q = a_0 + ia_1 + ja_2 + ka_3 = (a_0, a_1, a_2, a_3)$$

where a_0, a_1, a_2 and a_3 are real numbers or scalers and $1, i, j, k$ are the standard orthonormal basis in $\mathbb{R}^4$. The set of all quaternions are denoted by $\mathbb{H}$. Note that we can write

$$q = a_0 + p$$

where $p = ia_1 + ja_2 + ka_3$. a_0 and p are called the scalar part and the vector part of the quaternion q, respectively. The a_0, a_1, a_2, a_3 are called the components of the quaternion q.

Addition of quaternions is defined as componentwise and the quaternion multiplication is defined as follows:

$$i^2 = j^2 = k^2 = ijk = -1. \tag{1.1}$$

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Note that from (1.1), we have

$$ij = k = -ji, \; jk = i = -kj, \; ki = j = -ik. \tag{1.2}$$

So, multiplication on $\mathbb{H}$ is not commutative.

The product of two quaternions $q = a_0 + ia_1 + ja_2 + ka_3$ and $p = b_0 + ib_1 + jb_2 + kb_3$ is

$$\begin{aligned} qp &= (a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3) + i(a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2) \\ &\quad + j(a_0b_2 - a_1b_3 + a_2b_0 + a_3b_1) + k(a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0). \end{aligned}$$

The conjugate of the quaternion q is defined by

$$q^* = (a_0 + ia_1 + ja_2 + ka_3)^* = a_0 - ia_1 - ja_2 - ka_3.$$

For two quaternions p, q we have

$$(q^*)^* = q, \; (p+q)^* = p^* + q^*, \; (pq)^* = q^*p^* \text{ and } (p^*q)^* = q^*p.$$

The norm of a quaternion q is defined by

$$N(q) = \|q\| := qq^* = a_0^2 + a_1^2 + a_2^2 + a_3^2.$$

The norm is multiplicative:

$$N(pq) = N(p)N(q).$$

Division is uniquely defined (except by zero), thus quaternions form a division algebra. For two quaternions $p, q \in \mathbb{H}$, we have

$$(pq)^{-1} = q^{-1}p^{-1}.$$

The inverse (reciprocal) of a nonzero quaternion q is given by

$$q^{-1} = \frac{q^*}{N(q)}.$$

A generalized Tetranacci sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2, V_3)\}_{n\geq 0}$ is defined by the fourth-order recurrence relations

$$V_n = V_{n-1} + V_{n-2} + V_{n-3} + V_{n-4} \tag{1.3}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2, V_3 = c_3$ not all being zero.

This sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example [11], [16], [17], [19], [27], [28].

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -V_{-(n-1)} - V_{-(n-2)} - V_{-(n-3)} + V_{-(n-4)}$$

for $n = 1, 2, 3, \ldots.$ Therefore, recurrence (1.3) holds for all integer n.

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The first few generalized Tetranacci numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized Tetranacci numbers

n	V_n	V_{-n}
0	c_0	c_0
1	c_1	$c_3 - c_2 - c_1 - c_0$
2	c_2	$2c_2 - c_3$
3	c_3	$2c_1 - c_2$
4	$c_0 + c_1 + c_2 + c_3$	$2c_0 - c_1$
5	$c_0 + 2c_1 + 2c_2 + 2c_3$	$2c_3 - 2c_2 - 2c_1 - 3c_0$
6	$2c_0 + 3c_1 + 4c_2 + 4c_3$	$c_0 + c_1 + 5c_2 - 3c_3$
7	$4c_0 + 6c_1 + 7c_2 + 8c_3$	$4c_1 - 4c_2 + c_3$
8	$8c_0 + 12c_1 + 14c_2 + 15c_3$	$4c_0 - 4c_1 + c_2$

If we set $V_0 = 0, V_1 = 1, V_2 = 1, V_3 = 2$, then $\{V_n\}$ is the well-known Tetranacci sequence and if we set $V_0 = 4, V_1 = 1, V_2 = 3, V_3 = 7$ then $\{V_n\}$ is the well-known Tetranacci-Lucas sequence. In other words, Tetranacci sequence $\{M_n\}_{n\geq 0}$ and Tetranacci-Lucas sequence $\{R_n\}_{n\geq 0}$ are defined by the fourth-order recurrence relations

$$M_n = M_{n-1} + M_{n-2} + M_{n-3} + M_{n-4}, \qquad M_0 = 0, M_1 = 1, M_2 = 1, M_3 = 2 \tag{1.4}$$

and

$$R_n = R_{n-1} + R_{n-2} + R_{n-3} + R_{n-4}, \qquad R_0 = 4, R_1 = 1, R_2 = 3, R_3 = 7. \tag{1.5}$$

The sequences $\{M_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$M_{-n} = -M_{-(n-1)} - M_{-(n-2)} - M_{-(n-3)} + M_{-(n-4)}$$

and

$$R_{-n} = -R_{-(n-1)} - R_{-(n-2)} - R_{-(n-3)} + R_{-(n-4)}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.4) and (1.5) hold for all integer n. Next, we present the first few values of the Tetranacci and Tetranacci-Lucas numbers with positive and negative subscripts:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	...
M_n	0	1	1	2	4	8	15	29	56	108	208	401	773	1490	...
M_{-n}	0	0	0	1	-1	0	0	2	-3	1	0	4	-8	5	...
R_n	4	1	3	7	15	26	51	99	191	367	708	1365	2631	5071	...
R_{-n}	4	-1	-1	-1	7	-6	-1	-1	15	-19	4	-1	31	-53	...

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For all integers n, usual Tetranaci and Tetranacci-Lucas numbers can be expressed using Binet's formulas

$$M_n = \frac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{\delta^{n+2}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}$$

(see for example [11] or [30])

or

$$M_n = \frac{\alpha-1}{5\alpha-8}\alpha^{n-1} + \frac{\beta-1}{5\beta-8}\beta^{n-1} + \frac{\gamma-1}{5\gamma-8}\gamma^{n-1} + \frac{\delta-1}{5\delta-8}\delta^{n-1} \tag{1.6}$$

(see for example [7])

and

$$R_n = \alpha^n + \beta^n + \gamma^n + \delta^n$$

respectively, where α, β, γ and δ are the roots of the cubic equation $x^4 - x^3 - x^2 - x - 1 = 0$. Moreover,

$$\begin{aligned}
\alpha &= \frac{1}{4} + \frac{1}{2}\omega + \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 + \frac{13}{4}\omega^{-1}},\\
\beta &= \frac{1}{4} + \frac{1}{2}\omega - \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 + \frac{13}{4}\omega^{-1}},\\
\gamma &= \frac{1}{4} - \frac{1}{2}\omega + \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 - \frac{13}{4}\omega^{-1}},\\
\delta &= \frac{1}{4} - \frac{1}{2}\omega - \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 - \frac{13}{4}\omega^{-1}},
\end{aligned}$$

where

$$\omega = \sqrt{\frac{11}{12} + \left(\frac{-65}{54} + \sqrt{\frac{563}{108}}\right)^{1/3} + \left(\frac{-65}{54} - \sqrt{\frac{563}{108}}\right)^{1/3}}.$$

We present Binet's formula of the generalized Tetranacci sequence.

Corollary 1. *The Binet's formula of the generalized Tetranacci sequence* $\{V_n\}$ *is given as*

$$V_n = A\alpha^{n-6} + B\beta^{n-6} + C\gamma^{n-6} + D\delta^{n-6}$$

where

$$\begin{aligned}
A &= \frac{\alpha-1}{5\alpha-8}(V_3\alpha^3 + (V_0+V_1+V_2)\alpha^2 + (V_1+V_2)\alpha + V_2)\\
B &= \frac{\beta-1}{5\beta-8}(V_3\beta^3 + (V_0+V_1+V_2)\beta^2 + (V_1+V_2)\beta + V_2)\\
C &= \frac{\gamma-1}{5\gamma-8}(V_3\gamma^3 + (V_0+V_1+V_2)\gamma^2 + (V_1+V_2)\gamma + V_2)\\
D &= \frac{\delta-1}{5\delta-8}(V_3\delta^3 + (V_0+V_1+V_2)\delta^2 + (V_1+V_2)\delta + V_2)
\end{aligned}$$

Proof. For a proof see [21, Corollary 1.3.].

In fact, Corollary 1 is a special case of a result in [1, Remark 2.3.].

Note that the Binet form of a sequence satisfying (1.3) for non-negative integers is valid for all integers n, for a proof of this result see [13]. This result of Howard and Saidak [13] is even true in the case of higher-order recurrence relations.

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} a_n x^n$ of the sequence V_n.

Lemma 2. *Suppose that* $f_{V_n}(x) = \sum_{n=0}^{\infty} a_n x^n$ *is the ordinary generating function of the generalized Tetranacci sequence* $\{V_n\}_{n\geq 0}$. *Then,* $f_{V_n}(x)$ *is given by*

$$f_{V_n}(x) = \frac{V_0 + (V_1 - V_0)x + (V_2 - V_1 - V_0)x^2 + (V_3 - V_2 - V_1 - V_0)x^3}{1 - x - x^2 - x^3 - x^4}. \tag{1.7}$$

Proof. Using (1.3) and some calculation, we obtain

$$f_{V_n}(x) - xf_{V_n}(x) - x^2 f_{V_n}(x) - x^3 f_{V_n}(x) - x^4 f_{V_n}(x) = V_0 + (V_1 - V_0)x + (V_2 - V_1 - V_0)x^2 + (V_3 - V_2 - V_1 - V_0)x^3$$

which gives (1.7).

The previous Lemma gives the following results as particular examples: generating function of the Tetranacci sequence M_n is

$$f_{M_n}(x) = \sum_{n=0}^{\infty} M_n x^n = \frac{x}{1 - x - x^2 - x^3 - x^4}$$

and generating function of the Tetranacci-Lucas sequence R_n is

$$f_{R_n}(x) = \sum_{n=0}^{\infty} R_n x^n = \frac{4 - 3x - 2x^2 - x^3}{1 - x - x^2 - x^3 - x^4}.$$

2. Generalized Tetranacci Quaternions and their Generating Functions and Binet's Formulas

In this section, we define generalized Tetranacci quaternions and give generating functions and Binet formulas for them. First, we give some information about quaternion sequences from the literature.

There are various types of quaternion sequences which have been studied by many researchers. Horadam [12] introduced nth Fibonacci and nth Lucas quaternions as

$$Q_n = F_n + F_{n+1}e_1 + F_{n+2}e_2 + F_{n+3}e_3 = \sum_{s=0}^{3} F_{n+s}e_s$$

and

$$R_n = L_n + L_{n+1}e_1 + L_{n+2}e_2 + L_{n+3}e_3 = \sum_{s=0}^{3} L_{n+s}e_s$$

respectively, where F_n and L_n are the nth Fibonacci and Lucas numbers respectively. He also defined generalized Fibonacci quaternion as

$$P_n = H_n + H_{n+1}e_1 + H_{n+2}e_2 + H_{n+3}e_3 = \sum_{s=0}^{3} H_{n+s}e_s$$

where H_n is the nth generalized Fibonacci number (which is now called Horadam number) by the recursive relation $H_1 = p$, $H_2 = p + q$, $H_n = H_{n-1} + H_{n-2}$ (p and q are arbitrary integers). Halici [8] gave the generating functions and Binet formulas for the Fibonacci and Lucas quaternions. We can list a few references of quaternion sequences. We list the references of a few second order quaternion sequences as

[4],[9],[18],[23],[25]

and we list the references of a few third order quaternion sequences as

[3],[6],[24],[26].

Soykan [22] introduced the Tetranacci and Tetranacci-Lucas quaternions as fourth order quaternion sequences.

We now define generalized Tetranacci quaternions over the quaternion algebra $\mathbb{H}$. The nth generalized Tetranacci quaternion is

$$\widehat{V}_n = V_n + iV_{n+1} + jV_{n+2} + kV_{n+3}. \tag{2.1}$$

As special cases, the nth Tetranacci quaternion and the nth Tetranacci-Lucas quaternion are given as

$$\widehat{M}_n = M_n + iM_{n+1} + jM_{n+2} + kM_{n+3}$$

and

$$\widehat{R}_n = R_n + iR_{n+1} + jR_{n+2} + kR_{n+3}$$

respectively. It can be easily shown that

$$\widehat{V}_n = \widehat{V}_{n-1} + \widehat{V}_{n-2} + \widehat{V}_{n-3} + \widehat{V}_{n-4}. \tag{2.2}$$

The sequence $\{\mathbb{BC}V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\widehat{V}_{-n} = -\widehat{V}_{-(n-1)} - \widehat{V}_{-(n-2)} - \widehat{V}_{-(n-3)} + \widehat{V}_{-(n-4)}.$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrence (2.2) holds for all integer n.

The conjugate of $\widehat{V}_n$ is defined by

$$\overline{\widehat{V}_n} = V_n - iV_{n+1} - jV_{n+2} - kV_{n+3}.$$

Now, we will state Binet's formula for the generalized Tetranacci quaternions and in the rest of the paper, we fix the following notations:

$$\begin{aligned}
\widehat{\alpha} &= 1 + i\alpha + j\alpha^2 + k\alpha^3, \\
\widehat{\beta} &= 1 + i\beta + j\beta^2 + k\beta^3, \\
\widehat{\gamma} &= 1 + i\gamma + j\gamma^2 + k\gamma^3, \\
\widehat{\delta} &= 1 + i\delta + j\delta^2 + k\delta^3.
\end{aligned}$$

THEOREM 3. *(Binet's Formula) For any integer n, the nth generalized Tetranacci quaternion is*

$$\widehat{V}_n = A\widehat{\alpha}\alpha^{n-6} + B\widehat{\beta}\beta^{n-6} + C\widehat{\gamma}\gamma^{n-6} + D\widehat{\delta}\delta^{n-6} \tag{2.3}$$

where A, B, C and D are as in Corollary 1.

Proof. Using Binet's formula of the generalized Tetranacci numbers, we obtain

$$\begin{aligned} \widehat{V}_n &= V_n + iV_{n+1} + jV_{n+2} + kV_{n+3} \\ &= A\alpha^{n-6} + B\beta^{n-6} + C\gamma^{n-6} + D\delta^{n-6} + i(A\alpha^{n-5} + B\beta^{n-5} + C\gamma^{n-5} + D\delta^{n-5}) \\ &\quad +j(A\alpha^{n-4} + B\beta^{n-4} + C\gamma^{n-4} + D\delta^{n-4}) + k(A\alpha^{n-3} + B\beta^{n-3} + C\gamma^{n-3} + D\delta^{n-3}) \\ &= A\widehat{\alpha}\alpha^{n-6} + B\widehat{\beta}\beta^{n-6} + C\widehat{\gamma}\gamma^{n-6} + D\widehat{\delta}\delta^{n-6}. \end{aligned}$$

This proves (2.3).

As special cases, for any integer n, the Binet's Formula of nth Tetranacci quaternion is

$$\widehat{M}_n = \frac{\alpha - 1}{5\alpha - 8}\widehat{\alpha}\alpha^{n-1} + \frac{\beta - 1}{5\beta - 8}\widehat{\beta}\beta^{n-1} + \frac{\gamma - 1}{5\gamma - 8}\widehat{\gamma}\gamma^{n-1} + \frac{\delta - 1}{5\delta - 8}\widehat{\delta}\delta^{n-1} \tag{2.4}$$

and the Binet's Formula of nth Tetranacci-Lucas quaternion is

$$\widehat{R}_n = \widehat{\alpha}\alpha^n + \widehat{\beta}\beta^n + \widehat{\gamma}\gamma^n + \widehat{\delta}\delta^n. \tag{2.5}$$

Next, we present generating function.

THEOREM 4. *The generating function for the generalized Tetranacci quaternions is*

$$\sum_{n=0}^{\infty} \widehat{V}_n x^n = \frac{\widehat{V}_0 + (\widehat{V}_1 - \widehat{V}_0)x + (\widehat{V}_2 - \widehat{V}_1 - \widehat{V}_0)x^2 + \widehat{V}_{-1}x^3}{1 - x - x^2 - x^3 - x^4} \tag{2.6}$$

Proof. Let

$$g(x) = \sum_{n=0}^{\infty} \widehat{V}_n x^n$$

be generating function of generalized Tetranacci quaternions. Then, using the definition of the generalized Tetranacci quaternions, and substracting $xg(x)$, $x^2g(x)$, $x^3g(x)$ and $x^4g(x)$ from $g(x)$, we obtain (note the

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shift in the index n in the third line)

$$\begin{aligned}
&(1-x-x^2-x^3-x^4)g(x)\\
=&\sum_{n=0}^{\infty}\widehat{V}_nx^n-x\sum_{n=0}^{\infty}\widehat{V}_nx^n-x^2\sum_{n=0}^{\infty}\widehat{V}_nx^n-x^3\sum_{n=0}^{\infty}\widehat{V}_nx^n-x^4\sum_{n=0}^{\infty}\widehat{V}_nx^n\\
=&\sum_{n=0}^{\infty}\widehat{V}_nx^n-\sum_{n=0}^{\infty}\widehat{V}_nx^{n+1}-\sum_{n=0}^{\infty}\widehat{V}_nx^{n+2}-\sum_{n=0}^{\infty}\widehat{V}_nx^{n+3}-\sum_{n=0}^{\infty}\widehat{V}_nx^{n+4}\\
=&\sum_{n=0}^{\infty}\widehat{V}_nx^n-\sum_{n=1}^{\infty}\widehat{V}_{n-1}x^n-\sum_{n=2}^{\infty}\widehat{V}_{n-2}x^n-\sum_{n=3}^{\infty}\widehat{V}_{n-3}x^n-\sum_{n=4}^{\infty}\widehat{V}_{n-4}x^n\\
=&(\widehat{V}_0+\widehat{V}_1x+\widehat{V}_2x^2+\widehat{V}_3x^3)-(\widehat{V}_0x+\widehat{V}_1x^2+\widehat{V}_2x^3)-(\widehat{V}_0x^2+\widehat{V}_1x^3)-\widehat{V}_0x^3\\
&+\sum_{n=4}^{\infty}(\widehat{V}_n-\widehat{V}_{n-1}-\widehat{V}_{n-2}-\widehat{V}_{n-3}-\widehat{V}_{n-4})x^n\\
=&\widehat{V}_0+(\widehat{V}_1-\widehat{V}_0)x+(\widehat{V}_2-\widehat{V}_1-\widehat{V}_0)x^2+(\widehat{V}_3-\widehat{V}_2-\widehat{V}_1-\widehat{V}_0)x^3.
\end{aligned}$$

Note that we used the recurrence relation $\widehat{V}_n=\widehat{V}_{n-1}+\widehat{V}_{n-2}+\widehat{V}_{n-3}+\widehat{V}_{n-4}$. Rearranging above equation, we get

$$g(x)=\frac{\widehat{V}_0+(\widehat{V}_1-\widehat{V}_0)x+(\widehat{V}_2-\widehat{V}_1-\widehat{V}_0)x^2+(\widehat{V}_3-\widehat{V}_2-\widehat{V}_1-\widehat{V}_0)x^3}{1-x-x^2-x^3-x^4},$$

or

$$g(x)=\frac{\widehat{V}_0+(\widehat{V}_1-\widehat{V}_0)x+(\widehat{V}_2-\widehat{V}_1-\widehat{V}_0)x^2+\widehat{V}_{-1}x^3}{1-x-x^2-x^3-x^4},$$

since $\widehat{V}_3=\widehat{V}_2+\widehat{V}_1+\widehat{V}_0+\widehat{V}_{-1}$.

As special cases, the generating functions for the Tetranacci and Tetranacci-Lucas quaternions are

$$\sum_{n=0}^{\infty}\widehat{M}_nx^n=\frac{(i+j+2k)+(1+j+2k)x+(j+2k)x^2+(j+k)x^3}{1-x-x^2-x^3-x^4} \tag{2.7}$$

and

$$\sum_{n=0}^{\infty}\widehat{R}_nx^n=\frac{(4+i+3j+7k)+(-3+2i+4j+8k)x+(-2+3i+5j+4k)x^2+(-1+4i+j+3k)x^3}{1-x-x^2-x^3-x^4} \tag{2.8}$$

respectively.

Now, we present the formula which give the summation of the first n generalized Tetranacci numbers.

THEOREM 5. *For $n\geq 1$, we have*

$$\sum_{p=1}^{n}V_p=\frac{1}{3}(V_{n+2}+2V_n+V_{n-1}-V_0+V_1-V_3). \tag{2.9}$$

Proof. This is a Theorem in Soykan [21, Theorem 2.6].

Note that from above theorem, we have

$$\begin{aligned}
\sum_{p=0}^{n}V_p &= V_0+\sum_{p=1}^{n}V_p=V_0+\frac{1}{3}(V_{n+2}+2V_n+V_{n-1}-V_0+V_1-V_3)\\
&=\frac{1}{3}(V_{n+2}+2V_n+V_{n-1}+2V_0+V_1-V_3). \qquad (2.10)
\end{aligned}$$

Note that, as special cases, for every integer $n \geq 0$, we have

$$\sum_{p=0}^{n} M_p = \frac{1}{3}(M_{n+2} + 2M_n + M_{n-1} - 1)$$

and

$$\sum_{p=0}^{n} R_p = \frac{1}{3}(R_{n+2} + 2R_n + R_{n-1} + 2).$$

Next, we present the formulas which give the summation of the first n generalized Tetranacci quaternions.

Theorem 6. *The summation formula for generalized Tetranacci quaternions is*

$$\sum_{p=0}^{n} \widehat{V}_p = \frac{1}{3}(\widehat{V}_{n+2} + 2\widehat{V}_n + \widehat{V}_{n-1} + c)$$

where

$$c = 2V_0 + V_1 - V_3 + i(-V_0 + V_1 - V_3) + j(-V_0 - 2V_1 - V_3) + k(-V_0 - 2V_1 - 3V_2 - V_3).$$

Proof. Using (2.10), we obtain

$$\begin{aligned} \sum_{p=0}^{n} \widehat{V}_p &= \sum_{p=0}^{n} V_p + i\sum_{p=0}^{n} V_{p+1} + j\sum_{p=0}^{n} V_{p+2} + k\sum_{p=0}^{n} V_{p+3} \\ &= (V_0 + ... + V_n) + i(V_1 + ... + V_{n+1}) \\ &\quad +j(V_2 + ... + V_{n+2}) + k(V_3 + ... + V_{n+3}). \end{aligned}$$

and so,

$$\begin{aligned} 3\sum_{p=0}^{n} \widehat{V}_p &= (V_{n+2} + 2V_n + V_{n-1} + 2V_0 + V_1 - V_3) \\ &\quad +i(V_{n+3} + 2V_{n+1} + V_n + 2V_0 + V_1 - V_3 - 3V_0) \\ &\quad +j(V_{n+4} + 2V_{n+2} + V_{n+1} + 2V_0 + V_1 - V_3 - 3(V_0 + V_1)) \\ &\quad +k(V_{n+5} + 2V_{n+3} + V_{n+2} + 2V_0 + V_1 - V_3 - 3(V_0 + V_1 + V_2)) \\ &= \widehat{V}_{n+2} + 2\widehat{V}_n + \widehat{V}_{n-1} + c \end{aligned}$$

where

$$\begin{aligned} c &= 2V_0 + V_1 - V_3 + i(2V_0 + V_1 - V_3 - 3V_0) + j(2V_0 + V_1 - V_3 - 3(V_0 + V_1)) \\ &\quad +k(2V_0 + V_1 - V_3 - 3(V_0 + V_1 + V_2)) \\ &= 2V_0 + V_1 - V_3 + i(-V_0 + V_1 - V_3) + j(-V_0 - 2V_1 - V_3) + k(-V_0 - 2V_1 - 3V_2 - V_3). \end{aligned}$$

Hence

$$\sum_{p=0}^{n} \widehat{V}_p = \frac{1}{3}(\widehat{V}_{n+2} + 2\widehat{V}_n + \widehat{V}_{n-1} + c).$$

This proves (2.9).

As special cases we have the following summation formula for Tetranacci and Tetranacci-Lucas quaternions:

$$\sum_{p=0}^{n}\widehat{M}_p=\frac{1}{3}(\widehat{M}_{n+2}+2\widehat{M}_n+\widehat{M}_{n-1}-(1+i+4j+7k)) \tag{2.11}$$

and

$$\sum_{p=0}^{n}\widehat{R}_p=\frac{1}{3}(\widehat{R}_{n+2}+2\widehat{R}_n+\widehat{R}_{n-1}+(2-10i-13j-22k)). \tag{2.12}$$

respectively.

Now, we present the formulas which give the summation of odd and even generalized Tetranacci numbers.

THEOREM 7. *For $n\geq 1$, we have the following formulas:*

(a): $\sum_{p=1}^{n}V_{2p+1}=\frac{1}{3}(2V_{2n+2}+V_{2n}-V_{2n-1}-2V_0-V_1-3V_2+V_3)$

(b): $\sum_{p=1}^{n}V_{2p}=\frac{1}{3}(2V_{2n+1}+V_{2n-1}-V_{2n-2}+V_0-V_1+3V_2-2V_3)$.

Proof. This is a Theorem in Soykan [21, Theorem 2.6].

Note that from above theorem, we have

$$\begin{aligned}\sum_{p=0}^{n}V_{2p+1} &= V_1+\sum_{p=1}^{n}V_{2p+1}\\ &= V_1+\frac{1}{3}(2V_{2n+2}+V_{2n}-V_{2n-1}-2V_0-V_1-3V_2+V_3)\\ &= \frac{1}{3}(2V_{2n+2}+V_{2n}-V_{2n-1}-2V_0+2V_1-3V_2+V_3) \end{aligned}\tag{2.13}$$

and

$$\begin{aligned}\sum_{p=0}^{n}V_{2p} &= V_0+\sum_{p=1}^{n}V_{2p}\\ &= V_0+\frac{1}{3}(2V_{2n+1}+V_{2n-1}-V_{2n-2}+V_0-V_1+3V_2-2V_3)\\ &= \frac{1}{3}(2V_{2n+1}+V_{2n-1}-V_{2n-2}+4V_0-V_1+3V_2-2V_3). \end{aligned}\tag{2.14}$$

Note that, as special cases, for every integer $n\geq 0$, we have

$$\begin{aligned}\sum_{p=0}^{n}M_{2p+1} &= \frac{1}{3}(2M_{2n+2}+M_{2n}-M_{2n-1}-2M_0+2M_1-3M_2+M_3),\\ \sum_{p=0}^{n}M_{2p} &= \frac{1}{3}(2M_{2n+1}+M_{2n-1}-M_{2n-2}+4M_0-M_1+3M_2-2M_3)\end{aligned}$$

and

$$\begin{aligned}\sum_{p=0}^{n}R_{2p+1} &= \frac{1}{3}(2R_{2n+2}+R_{2n}-R_{2n-1}-2R_0+2R_1-3R_2+R_3),\\ \sum_{p=0}^{n}R_{2p} &= \frac{1}{3}(2R_{2n+1}+R_{2n-1}-R_{2n-2}+4R_0-R_1+3R_2-2R_3).\end{aligned}$$

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THEOREM 8. *For $n \geq 0$, we have the following formulas:*

(a):

$$\sum_{p=0}^{n} \widehat{V}_{2p+1} = \frac{1}{3}(2\widehat{V}_{2n+2} + \widehat{V}_{2n} - \widehat{V}_{2n-1} + d)$$

where

$$d = (-2V_0 + 2V_1 - 3V_2 + V_3) + i(V_0 - V_1 + 3V_2 - 2V_3) + j(-2V_0 - V_1 - 3V_2 + V_3) + k(V_0 - V_1 - 2V_3),$$

(b):

$$\sum_{p=0}^{n} \widehat{V}_{2p} = \frac{1}{3}(2\widehat{V}_{2n+1} + \widehat{V}_{2n-1} - \widehat{V}_{2n-2} + e)$$

where

$$e = (4V_0 - V_1 + 3V_2 - 2V_3) + (-2V_0 + 2V_1 - 3V_2 + V_3)i + (V_0 - V_1 + 3V_2 - 2V_3)j + (-2V_0 - V_1 - 3V_2 + V_3)k.$$

Proof. *The proof can be easily obtained by using (2.13) and (2.14), so we omit it.*

As special cases, we have the following two corollaries.

COROLLARY 9. *For $n \geq 0$, we have the following formulas:*

(a): $\sum_{p=0}^{n} \widehat{M}_{2p+1} = \frac{1}{3}(2\widehat{M}_{2n+2} + \widehat{M}_{2n} - \widehat{M}_{2n-1} + (1 - 2i - 2j - 5k))$

(b): $\sum_{p=0}^{n} \widehat{M}_{2p} = \frac{1}{3}(2\widehat{M}_{2n+1} + \widehat{M}_{2n-1} - \widehat{M}_{2n-2} - (2 - i + 2j + 2k))$

COROLLARY 10. *For $n \geq 0$, we have the following formulas:*

(a): $\sum_{p=0}^{n} \widehat{R}_{2p+1} = \frac{1}{3}(2\widehat{R}_{2n+2} + \widehat{R}_{2n} - \widehat{R}_{2n-1} - (8 + 2i + 11j + 11k))$

(b): $\sum_{p=0}^{n} \widehat{R}_{2p} = \frac{1}{3}(2\widehat{R}_{2n+1} + \widehat{R}_{2n-1} - \widehat{R}_{2n-2} + (10 - 8i - 2j - 11k)).$

3. Matrices related with Generalized Tetranacci Quaternions

We define the square matrix B of order 4 as:

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $\det B = -1$.

Consider the sequence $\{U_n\}$ which is defined by the fourth-order recurrence relation

$$U_n = U_{n-1} + U_{n-2} + U_{n-3} + U_{n-4}, \qquad U_0 = U_1 = 0, U_2 = U_3 = 1.$$

The numbers U_n can be expressed using Binet's formula

$$U_n = \frac{\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}.$$

Induction proof may be used to establish

$$B^n = \begin{pmatrix} U_{n+2} & U_{n+1}+U_n+U_{n-1} & U_{n+1}+U_n & U_{n+1} \\ U_{n+1} & U_n+U_{n-1}+U_{n-2} & U_n+U_{n-1} & U_n \\ U_n & U_{n-1}+U_{n-2}+U_{n-3} & U_{n-1}+U_{n-2} & U_{n-1} \\ U_{n-1} & U_{n-2}+U_{n-3}+U_{n-4} & U_{n-2}+U_{n-3} & U_{n-2} \end{pmatrix} \tag{3.1}$$

and (the matrix formulation of V_n)

$$\begin{pmatrix} V_{n+3} \\ V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} V_3 \\ V_2 \\ V_1 \\ V_0 \end{pmatrix}. \tag{3.2}$$

Now, we define the matrices B_V as

$$B_V = \begin{pmatrix} \widehat{V}_5 & \widehat{V}_4+\widehat{V}_3+\widehat{V}_2 & \widehat{V}_4+\widehat{V}_3 & \widehat{V}_4 \\ \widehat{V}_4 & \widehat{V}_3+\widehat{V}_2+\widehat{V}_1 & \widehat{V}_3+\widehat{V}_2 & \widehat{V}_3 \\ \widehat{V}_3 & \widehat{V}_2+\widehat{V}_1+\widehat{V}_0 & \widehat{V}_2+\widehat{V}_1 & \widehat{V}_2 \\ \widehat{V}_2 & \widehat{V}_1+\widehat{V}_0+\widehat{V}_{-1} & \widehat{V}_1+\widehat{V}_0 & \widehat{V}_1 \end{pmatrix}.$$

This matrice B_V is called generalized Tetranacci quaternion matrix. As special cases, Tetranacci quaternion matrix and Tetranacci-Lucas quaternion matrix are

$$B_M = \begin{pmatrix} \widehat{M}_5 & \widehat{M}_4+\widehat{M}_3+\widehat{M}_2 & \widehat{M}_4+\widehat{M}_3 & \widehat{M}_4 \\ \widehat{M}_4 & \widehat{M}_3+\widehat{M}_2+\widehat{M}_1 & \widehat{M}_3+\widehat{M}_2 & \widehat{M}_3 \\ \widehat{M}_3 & \widehat{M}_2+\widehat{M}_1+\widehat{M}_0 & \widehat{M}_2+\widehat{M}_1 & \widehat{M}_2 \\ \widehat{M}_2 & \widehat{M}_1+\widehat{M}_0+\widehat{M}_{-1} & \widehat{M}_1+\widehat{M}_0 & \widehat{M}_1 \end{pmatrix} \text{ and } B_R = \begin{pmatrix} \widehat{R}_5 & \widehat{R}_4+\widehat{R}_3+\widehat{R}_2 & \widehat{R}_4+\widehat{R}_3 & \widehat{R}_4 \\ \widehat{R}_4 & \widehat{R}_3+\widehat{R}_2+\widehat{R}_1 & \widehat{R}_3+\widehat{R}_2 & \widehat{R}_3 \\ \widehat{R}_3 & \widehat{R}_2+\widehat{R}_1+\widehat{R}_0 & \widehat{R}_2+\widehat{R}_1 & \widehat{R}_2 \\ \widehat{R}_2 & \widehat{R}_1+\widehat{R}_0+\widehat{R}_{-1} & \widehat{R}_1+\widehat{R}_0 & \widehat{R}_1 \end{pmatrix},$$

respectively.

Theorem 11. *For $n \geq 0$, the following is valid:*

$$B_V \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} \widehat{V}_{n+5} & \widehat{V}_{n+4}+\widehat{V}_{n+3}+\widehat{V}_{n+2} & \widehat{V}_{n+4}+\widehat{V}_{n+3} & \widehat{V}_{n+4} \\ \widehat{V}_{n+4} & \widehat{V}_{n+3}+\widehat{V}_{n+2}+\widehat{V}_{n+1} & \widehat{V}_{n+3}+\widehat{V}_{n+2} & \widehat{V}_{n+3} \\ \widehat{V}_{n+3} & \widehat{V}_{n+2}+\widehat{V}_{n+1}+\widehat{V}_n & \widehat{V}_{n+2}+\widehat{V}_{n+1} & \widehat{V}_{n+2} \\ \widehat{V}_{n+2} & \widehat{V}_{n+1}+\widehat{V}_n+\widehat{V}_{n-1} & \widehat{V}_{n+1}+\widehat{V}_n & \widehat{V}_{n+1} \end{pmatrix}. \tag{3.3}$$

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Proof. We prove by mathematical induction on n. If $n=0$, then the result is clear. Now, we assume it is true for $n=k$, that is

$$B_V B^k = \begin{pmatrix} \widehat{V}_{k+5} & \widehat{V}_{k+4}+\widehat{V}_{k+3}+\widehat{V}_{k+2} & \widehat{V}_{k+4}+\widehat{V}_{k+3} & \widehat{V}_{k+4} \\ \widehat{V}_{k+4} & \widehat{V}_{k+3}+\widehat{V}_{k+2}+\widehat{V}_{k+1} & \widehat{V}_{k+3}+\widehat{V}_{k+2} & \widehat{V}_{k+3} \\ \widehat{V}_{k+3} & \widehat{V}_{k+2}+\widehat{V}_{k+1}+\widehat{V}_{k} & \widehat{V}_{k+2}+\widehat{V}_{k+1} & \widehat{V}_{k+2} \\ \widehat{V}_{k+2} & \widehat{V}_{k+1}+\widehat{V}_{k}+\widehat{V}_{k-1} & \widehat{V}_{k+1}+\widehat{V}_{k} & \widehat{V}_{k+1} \end{pmatrix}.$$

If we use (2.1), then we have $\widehat{V}_{k+4}=\widehat{V}_{k+3}+\widehat{V}_{k+2}+\widehat{V}_{k+1}+\widehat{V}_{k}$. Then, by induction hypothesis, we obtain

$$\begin{aligned} B_V B^{k+1} &= (B_V B^k)B \\ &= \begin{pmatrix} \widehat{V}_{k+5} & \widehat{V}_{k+4}+\widehat{V}_{k+3}+\widehat{V}_{k+2} & \widehat{V}_{k+4}+\widehat{V}_{k+3} & \widehat{V}_{k+4} \\ \widehat{V}_{k+4} & \widehat{V}_{k+3}+\widehat{V}_{k+2}+\widehat{V}_{k+1} & \widehat{V}_{k+3}+\widehat{V}_{k+2} & \widehat{V}_{k+3} \\ \widehat{V}_{k+3} & \widehat{V}_{k+2}+\widehat{V}_{k+1}+\widehat{V}_{k} & \widehat{V}_{k+2}+\widehat{V}_{k+1} & \widehat{V}_{k+2} \\ \widehat{V}_{k+2} & \widehat{V}_{k+1}+\widehat{V}_{k}+\widehat{V}_{k-1} & \widehat{V}_{k+1}+\widehat{V}_{k} & \widehat{V}_{k+1} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \widehat{V}_{k+5}+\widehat{V}_{k+4}+\widehat{V}_{k+3}+\widehat{V}_{k+2} & \widehat{V}_{k+5}+\widehat{V}_{k+4}+\widehat{V}_{k+3} & \widehat{V}_{k+5}+\widehat{V}_{k+4} & \widehat{V}_{k+5} \\ \widehat{V}_{k+4}+\widehat{V}_{k+3}+\widehat{V}_{k+2}+\widehat{V}_{k+1} & \widehat{V}_{k+4}+\widehat{V}_{k+3}+\widehat{V}_{k+2} & \widehat{V}_{k+4}+\widehat{V}_{k+3} & \widehat{V}_{k+4} \\ \widehat{V}_{k+3}+\widehat{V}_{k+2}+\widehat{V}_{k+1}+\widehat{V}_{k} & \widehat{V}_{k+3}+\widehat{V}_{k+2}+\widehat{V}_{k+1} & \widehat{V}_{k+3}+\widehat{V}_{k+2} & \widehat{V}_{k+3} \\ \widehat{V}_{k+2}+\widehat{V}_{k+1}+\widehat{V}_{k}+\widehat{V}_{k-1} & \widehat{V}_{k+2}+\widehat{V}_{k+1}+\widehat{V}_{k} & \widehat{V}_{k+2}+\widehat{V}_{k+1} & \widehat{V}_{k+2} \end{pmatrix} \\ &= \begin{pmatrix} \widehat{V}_{k+6} & \widehat{V}_{k+5}+\widehat{V}_{k+4}+\widehat{V}_{k+3} & \widehat{V}_{k+5}+\widehat{V}_{k+4} & \widehat{V}_{k+5} \\ \widehat{V}_{k+5} & \widehat{V}_{k+4}+\widehat{V}_{k+3}+\widehat{V}_{k+2} & \widehat{V}_{k+4}+\widehat{V}_{k+3} & \widehat{V}_{k+4} \\ \widehat{V}_{k+4} & \widehat{V}_{k+3}+\widehat{V}_{k+2}+\widehat{V}_{k+1} & \widehat{V}_{k+3}+\widehat{V}_{k+2} & \widehat{V}_{k+3} \\ \widehat{V}_{k+3} & \widehat{V}_{k+2}+\widehat{V}_{k+1}+\widehat{V}_{k} & \widehat{V}_{k+2}+\widehat{V}_{k+1} & \widehat{V}_{k+2} \end{pmatrix}. \end{aligned}$$

Thus, (3.3) holds for all non-negative integers n.

COROLLARY 12. *For $n \geq 0$, the following holds:*

$$\widehat{V}_{n+3} = \widehat{V}_3 U_{n+2} + (\widehat{V}_2+\widehat{V}_1+\widehat{V}_0)U_{n+1} + (\widehat{V}_1+\widehat{V}_2)U_n + \widehat{V}_2 U_{n-1}.$$

Proof. The proof can be seen by the coefficient of the matrix B_V and (3.1).

References

[1] Bacani, J. B., Rabago, J. F. T., On Generalized Fibonacci Numbers, Applied Mathematical Sciences, 9 (25), 3611-3622, 2015.

[2] Baez, J., The octonions, Bull. Amer. Math. Soc. 39 (2), 145-205, 2002.

[3] Akkus, I., and Kızılaslan, G., On Some Properties of Tribonacci Quaternions, arXiv:1708.05367v1 [math.CO] 17 Aug 2017.

[4] Catarino, P., The Modified Pell and Modified k-Pell Quaternions and Octonions. Advances in Applied Clifford Algebras 26, 577-590, 2016.

[5] Cerda-Morales, G., On a Generalization for Tribonacci Quaternions, Mediterranean Journal of Mathematics, 14 (239), 1–12, 2017.

[6] Cerda-Morales, G., Identities for Third Order Jacobsthal Quaternions, Adv. Appl. Clifford Algebras 27, 1043-1053, 2017.

[7] Dresden, G. P., Du, Z., A Simplified Binet Formula for k-Generalized Fibonacci Numbers, J. Integer Seq. 17, art. 14.4.7, 9 pp., 2014.

[8] Halici, S., On Fibonacci Quaternions, Adv. Appl. Cliff ord Algebras 22, 321–327, 2012.

[9] Halici, S., Karataş, A., On a Generalization for Fibonacci Quaternions. Chaos Solitons and Fractals 98, 178–182, 2017.

[10] Hankins, T. L., Sir William Rowan Hamilton, Johns Hopkins University Press, Baltimore, 1980.

[11] Hathiwala, G. S., Shah, D. V., Binet–Type Formula For The Sequence of Tetranacci Numbers by Alternate Methods, Mathematical Journal of Interdisciplinary Sciences 6 (1), 37–48, 2017.

[12] Horadam, A. F., Complex Fibonacci Numbers and Fibonacci quaternions, Amer. Math. Monthly 70, 289–291, 1963.

[13] Howard, F.T., Saidak, F., Zhou's Theory of Constructing Identities, Congress Numer. 200, 225-237, 2010.

[14] Lewis, D. W., Quaternion Algebras and the Algebraic Legacy of Hamilton's Quaternions, Irish Math. Soc. Bulletin 57, 41–64, 2006.

[15] Melham, R. S., Shannon, A.G., A Generalization of a Result of D'Ocagne, The Fibonacci Quarterly, 33 (2), 135-138, 1995.

[16] Melham, R. S., Some Analogs of the Identity $F_n^2 + F_{n+1}^2 = F_{2n+1}^2$, Fibonacci Quarterly, 305-311, 1999.

[17] Natividad, L. R., On Solving Fibonacci-Like Sequences of Fourth, Fifth and Sixth Order, International Journal of Mathematics and Computing, 3 (2), 2013.

[18] Polatlı, E., A Generalization of Fibonacci and Lucas Quaternions, Advances in Applied Clifford Algebras, 26 (2), 719-730, 2016.

[19] Singh, B., Bhadouria, P., Sikhwal, O., Sisodiya, K., A Formula for Tetranacci-Like Sequence, Gen. Math. Notes, 20 (2), 136-141, 2014.

[20] Sloane, N.J.A., The on-line encyclopedia of integer sequences, http://oeis.org/

[21] Soykan, Y., Gaussian Generalized Tetranacci Numbers,. arXiv:1902.03936 [math.NT], 2019.

[22] Soykan Y, Tetranacci and Tetranacci-Lucas Quaternions, arXiv:1902.05868v1 [math.RA], 2019.

[23] Szynal-Liana, A., and Wloch, I., The Pell quaternions and the Pell octonions. Advances in Applied Clifford Algebras 26.1, 435-440, 2016.

[24] Szynal-Liana, A., and Wloch, I., Some Properties of Generalized Tribonacci Quaternions, Scientfic Issues, Jan Dlugosz University in Czestochowa, Mathematics XXII, 73-81, 2017.

[25] Tasci, D., On k-Jacobsthal and k-Jacobsthal-Lucas Quaternions, Journal of Science and Arts, year 17, No. 3(40), pp. 469-476, 2017.

[26] Tasci, D., Padovan and Pell-Padovan Quaternions, Journal of Science and Arts, year 18, No. 1(42), pp. 125-132, 2018.

[27] Waddill, M. E., Another Generalized Fibonacci Sequence, Fibonacci Quarterly., 5 (3), 209-227, 1967.

[28] Waddill, M. E., The Tetranacci Sequence and Generalizations, The Fibonacci Quarterly, 9-20, 1992.

[29] Ward, J. P., Quaternions and Cayley Numbers: Algebra and Applications, Kluwer Academic Publishers, London, 1997.

[30] Zaveri, M. N., Patel, J. K., Binet's Formula for the Tetranacci Sequence, International Journal of Science and Research (IJSR), 5 (12), 1911-1914, 2016.