Article

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Article

Edgeworth Coefficients for Standard Multivariate Estimates

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Abstract

We give for the first time explicit formulas for the coefficients needed for the 4th order Edgeworth expansions of a multivariate standard estimate. We call these *the Edgeworth coefficients*. They are Bell polynomials in *the cumulant coefficients*. Standard estimates include most estimates of interest, including smooth functions of sample means and other empirical estimates. We also give applications to ellipsoidal and hyperrectangular sets.

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1. Introduction and Summary

Suppose that $\hat{w}$ is a standard estimate, as defined in §2, of an unknown parameter $w \in R^q$ of a statistical model, based on a sample of size n. For example $\hat{w}$ may be a smooth function of a sample mean, or a smooth functional of an empirical distribution. A smooth function of a standard estimate is also a standard estimate: see Withers (2024). §2 summarises the multivariate Edgeworth expansions of Withers and Nadarajah (2010) for the distribution and density of $X_n = n^{1/2}(\hat{w} - w)$ in powers of $n^{-1/2}$ about the multivariate normal in terms of the *Edgeworth coefficients*, the P_r-coefficients of (11). For $r \geq 1$, these P_r are needed for the $r + 1$st term of the multivariate Edgeworth expansions. They are Bell polynomials in the *cumulant coefficients* of (5). They are given for $r = 1$ by (12) and for $r = 2, 3$ by (12)–(14) in terms of the symmetrizing operator $\mathcal{S}$, and explicitly in Appendix A. §3 derives expansions on ellipsoidal and hyperrectangular sets.

When $q = 2$, §4 simplifies these Edgeworth coefficients using dual notation. Examples include the distribution and density of a sample mean, and of bivariate *entangled* gamma random variables.

§5 and §6 give conclusions and discussion, and suggests some future directions. Appendix B gives explicitly the bivariate Hermite polynomials needed for bivariate Edgeworth expansions to $O(n^{-2})$.

Univariate estimates. Suppose that $\hat{w}$ is a *standard estimate* of $w \in R$ with respect to n, (typically the sample size). That is, $\hat{w}$ is non-lattice,
$E\ \hat{w} \to w$ as $n \to \infty$, and its rth cumulant can be expanded as

$$\kappa_r(\hat{w}) \approx \sum_{j=r-1}^{\infty} n^{-j}\, a_{rj} \text{ for } r \geq 1, \tag{1}$$

where the *cumulant coefficients* a_{rj} may depend on n but are bounded as $n \to \infty$, and a_{21} is bounded away from 0. Here and below $\approx$ indicates an asymptotic expansion that need not converge. So (1) holds in the sense that

$$\kappa_r(\hat{w}) = \sum_{j=r-1}^{I-1} n^{-j}\, a_{rj} + O(n^{-I}) \text{ for } I \geq r \geq 1,$$

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where $y_n = O(x_n)$ means that y_n/x_n is bounded in n. Withers (1984) replaced the artifical assumptions of Cornish and Fisher (1937) and Fisher and Cornish (1960) by (1), and gave the distribution, density and quantiles of

$$U_n = (n/a_{21})^{1/2}(\hat{w} - w)$$

as asymptotic expansions in powers of $n^{-1/2}$:

$$P_n(u) = Prob.(U_n \leq u) \approx \Phi(u) - \phi(u) \sum_{r=1}^{\infty} n^{-r/2}\, h_r(u), \tag{2}$$

$$p_n(u) = dP_n(u)/du \approx \phi(u)\ [1 + \sum_{r=1}^{\infty} n^{-r/2}\, \overline{h}_r(u)], \tag{3}$$

$$\Phi^{-1}(P_n(u)) \approx u - \sum_{r=1}^{\infty} n^{-r/2}\, f_r(u),\ P_n^{-1}(\Phi(u)) \approx u + \sum_{r=1}^{\infty} n^{-r/2}\, g_r(u), \tag{4}$$

where $Prob.(A)$ is the probability that A is true,

$$\Phi(u) = Prob.(N \leq u) = \int_{-\infty}^{u} \phi(u)du,\ \phi(u) = (2\pi)^{-1/2}e^{-u^2/2},\ N \sim \mathcal{N}(0,1)$$

is a unit normal random variable with even moments $E\ N^{2k} = 1.3\ldots(2k-1)$, and $h_r(u), \overline{h}_r(u), f_r(u), g_r(u)$ are polynomials in u and also in *the standardized cumulant coefficients* $A_{ri} = a_{ri}/a_{21}^{r/2}$. For example

$$h_1(u) = f_1(u) = g_1(u) = A_{11} + A_{32}H_2/6,\ \overline{h}_1(u) = A_{11}H_1 + A_{32}H_3/6, \tag{5}$$
$$h_2(u) = (A_{11}^2 + A_{22})H_1/2 + (A_{11}A_{32} + A_{43}/4)H_3/6 + A_{32}^2 H_5/72,$$

where H_k is the kth Hermite polynomial. By Withers (2000), for $I = \sqrt{-1}$,

$$H_k = H_k(u) = \phi(u)^{-1}(-d/du)^k\phi(u) = E\ (u + IN)^k \text{ for } k \geq 0: \tag{6}$$
$$H_0 = 1,\ H_1 = u,\ H_2 = u^2 - 1,\ H_3 = u^3 - 3u,\ H_4 = u^4 - 6u^2 + 3.$$

(A1) gives H_k for $k \leq 9$. Since $h_r(u)$ is even/odd for r odd/even,

$$Prob.(|U_n| \leq u) \approx \Phi(u) - 2\phi(u) \sum_{r=1}^{\infty} n^{-r}\, h_{2r}(u) \text{ for } u > 0. \tag{7}$$

From (4), it follows that

$$U_n \approx u + \sum_{r=1}^{\infty} n^{-r/2}\, g_r(N) \text{ where } N \sim \mathcal{N}(0,1). \tag{8}$$

For a discussion of when to truncate (2)–(4), see Withers and Nadarajah (2012a).

NOTE 1.1. *Edgeworth's expansion was for $\hat{w}$ the mean of n independent identically distributed random variables on R from a distribution with rth cumulant κ_r. So (1) holds with $a_{r,r-1} = \kappa_r$, and other $a_{ri} = 0$. An explicit formula for its general term was given in Withers and Nadarajah (2009).*

For examples of the many applications of Cornish-Fisher expansions and the extensions of Hill and Davis (1968), see Withers and Nadarajah (2011, 2014, 2015). Applications to finance include Simonato (2011) and Zhang et al. (2011). For an application to Rayleigh fading amplitudes, see Withers and Nadarajah (2008). TianLi et al. (2009) used them for GPS accuracy. Winterbottom (1974, 1980, 1984) and Abdel-Wahed and Winterbottom (1983) successfully used them for system reliability, even though binomial random variables fall on a lattice.

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Now suppose that $\hat{w}_*$ is another standard estimate of w with the same asymptotic variance a_{21}/n. Denote its standardized cumulant coefficients by A_{ri*}. Suppose that

$$U_{n*} = (n/a_{21})^{1/2}(\hat{w}_* - w) \sim \Phi_n(u) = \int_{-\infty}^{u} \phi_n(u)du \to \Phi(u) \text{ as } n \to \infty.$$

Then one can expand $P_n(u)$ about $\Phi_n(u)$ rather than $\Phi(u)$. The above expressions for $P_n(u)$, $h_k(u)$ become

$$P_n(u) = Prob.(U_n \leq u) \approx \Phi_n(u) - \phi_n(u)\sum_{r=1}^{\infty} n^{-r/2}\, h_{rn}(u), \text{ where}$$
$$h_{1n}(u) = A_{110} + A_{320}H_{2n}/6,$$
$$h_{2n}(u) = (A_{110}^2 + A_{220})H_{1n}/2 + (A_{110}A_{320} + A_{430}/4)H_{3n}/6 + A_{320}^2 H_{5n}/72,$$
$$A_{rj0} = A_{rj} - A_{rj*}, \text{ and } H_{kn} = H_{kn}(u) = \phi_n(u)^{-1}(-d/du)^k\phi_n(u).$$

So if for example we choose $\hat{w}_*$ so that $A_{110} = A_{320} = 0$, then

$$h_{1n}(u) = 0,\ h_{2n}(u) = A_{220}H_{1n}/2 + A_{430}H_{3n}/24,$$

and the number of terms in the analogues of h_r, f_r, g_r are greatly reduced. The disadvantage is that $H_{kn}(u)$ is more complicated than $H_k(u)$. See Withers and Nadarajah (2010). For expansions about a matching Student's, gamma, or F distribution, see Withers and Nadarajah (2011, 2012b, 2014).

2. Multivariate Edgeworth Expansions

Ordinary Bell polynomials. For a sequence $e = (e_1, e_2, \ldots)$ from R, the *partial ordinary Bell polynomial* $\tilde{B}_{rs} = \tilde{B}_{rs}(e)$, is defined by the identity

$$\text{for } s = 0,1,2,\ldots, \text{ and } z \in R,\ S^s = \sum_{r=s}^{\infty} z^r \tilde{B}_{rs}(e) \text{ where } S = \sum_{r=1}^{\infty} z^r e_r. \tag{1}$$
$$\text{So, } \tilde{B}_{r0} = \delta_{r0},\ \tilde{B}_{r1} = e_r,\ \tilde{B}_{rr} = e_1^r,\ \tilde{B}_{32} = 2e_1e_2, \tag{2}$$

where $\delta_{00} = 1$, $\delta_{r0} = 0$ for $r \neq 0$. They are tabled on p309 of Comtet (1974). (The partial *exponential* Bell polynomials are not used in this paper.) The *complete ordinary Bell polynomial*, $\tilde{B}_r(e)$ is defined in terms of S of (1) by

$$e^S = \sum_{r=0}^{\infty} z^r \tilde{B}_r(e). \text{ So } \tilde{B}_0(e) = 1 \text{ and for } r \geq 1,\ \tilde{B}_r(e) = \sum_{s=1}^{r} \tilde{B}_{rs}(e)/s!\ : \tag{3}$$
$$\tilde{B}_1(e) = e_1,\ \tilde{B}_2(e) = e_2 + e_1^2/2,\ \tilde{B}_3(e) = e_3 + e_1e_2 + e_1^3/6. \tag{4}$$

Multivariate estimates. Suppose that $\hat{w}$ is a *standard estimate* of $w \in R^q$ with respect to n. That is, $\hat{w}$ is non-lattice, $E\,\hat{w} \to w$ as $n \to \infty$, and for $r \geq 1, 1 \leq i_1, \ldots, i_r \leq q$, the rth order cumulants of $\hat{w}$ can be expanded as

$$\bar{k}^{1-r} = k^{i_1\ldots i_r} = \kappa(\hat{w}^{i_1}, \ldots, \hat{w}^{i_r}) \approx \sum_{d=r-1}^{\infty} n^{-d}\, \bar{k}_d^{1-r} \text{ where } \bar{k}_d^{1-r} = k_d^{i_1\ldots i_r}, \tag{5}$$

and the *cumulant coefficients* $\bar{k}_d^{1-r}$ may depend on n but are bounded as $n \to \infty$. So *the bar* replaces each i_k by k. *The use of* i_k *is reserved for this purpose.* For example $\bar{k}_0^1 = w^{i_1}$ and $\bar{k}_1^{12} = k_1^{i_1i_2}$. We use this *bar notation* repeatedly to avoid double subscripts i_k. As $n \to \infty$,

$$X_n = n^{1/2}(\hat{w} - w) \xrightarrow{\mathcal{L}} X = \mathcal{N}_q(0, V) \text{ for } V = (\bar{k}_1^{12}),\ q \times q, \tag{6}$$

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the multivariate normal on R^q, with density and distribution

$$\phi_V(x) = (2\pi)^{-q/2}(det\ V)^{-1/2}\exp(-x'V^{-1}x/2),\ \Phi_V(x) = \int_{-\infty}^{x}\phi_V(x)dx. \tag{7}$$

V may depend on n, but we assume that $det\ V$ is bounded away from 0. Set

$$Y = V^{-1}X \sim \mathcal{N}_q(0, V^{-1}),\ \bar{Y}_j = Y_{i_j},\ \bar{\mu}^{1-k} = E\ \bar{Y}_1 \dots \bar{Y}_k \text{ of (A2)}, \tag{8}$$

$$\bar{b}_{2d}^{1-r} = \bar{k}_d^{1-r},\ \bar{b}_{2d+1}^{1-r} = 0,\ \bar{t}_k = t_{i_k},\ e_j(t) = \sum_{r=1}^{j+2} \bar{b}_{r+j}^{1-r}\ \bar{t}_1 \dots \bar{t}_r/r!,\ t \in R^q. \tag{9}$$

So $E\ \bar{Y}_1\bar{Y}_2 = \bar{V}^{12}$ where $\bar{V}^{12} = V^{i_1 i_2}$ is the (i_1, i_2) element of V^{-1},
$e_1(t) = \bar{k}_1^1\bar{t}_1 + \bar{k}_2^{1-3}\bar{t}_1\bar{t}_2\bar{t}_3/6,\ e_2(t) = \bar{k}_2^{12}\bar{t}_1\bar{t}_2/2 + \bar{k}_3^{1-4}\bar{t}_1 \dots \bar{t}_4/24,$

where here and below, we use *the tensor summation convention* of implicitly summing each i_k over its range $1, \dots, q$. For example

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$$\bar{b}_2^1\bar{t}_1 = \bar{k}_1^1\bar{t}_1 = \sum_{i_1=1}^{q} k_1^{i_1} t_{i_1} \text{ and } \bar{b}_2^{12}\bar{t}_1\bar{t}_2 = \bar{k}_1^{12}\bar{t}_1\bar{t}_2 = \sum_{i_1,i_2=1}^{q} k_1^{i_1 i_2} t_{i_1} t_{i_2}. \tag{10}$$

$$\text{Then for } r \geq 1,\ \tilde{B}_r(e(t)) = \sum_{k=1}^{3r} [\bar{P}_r^{1-k}\ \bar{t}_1 \ldots \bar{t}_k :\ k-r \text{ even}], \tag{11}$$

where the rth *Edgeworth coefficient*, $\bar{P}_r^{1-k} = P_r^{i_1 \ldots i_k}$, is a function of the $\bar{k}_d^{1-r}$. One could use unsymmetrized $\bar{P}_r^{1-k}$. For example by (4) $\tilde{B}_2(e(t))$ needs the cross term in $e_1(t)^2/2$, $\bar{k}_1^4\bar{k}_2^{1-3}\bar{t}_1\bar{t}_2\bar{t}_3\bar{t}_4/6$. So we could use $\bar{k}_1^4\bar{k}_2^{1-3}/6$ in $\bar{P}_2^{1-4}$. But as (3) below illustrates, there is a big advantage in making $\bar{P}_r^{1-k}$ symmetric in $i_1, \ldots, i_k$ using the operator $\mathcal{S}$ that symmetrizes over $i_1, \ldots, i_k$. So the $\bar{P}_r^{1-k}$ for $r = 1,2,3$ that we need are

$$\bar{P}_1^1 = \bar{k}_1^1,\ \bar{P}_1^{1-3} = \bar{k}_2^{1-3}/3!,\ \bar{P}_2^{12} = \bar{k}_2^{12}/2 + k_1^1\bar{k}_1^2/2, \tag{12}$$

$$\bar{P}_2^{1-4} = \bar{k}_3^{1-4}/4! + \mathcal{S}\ \bar{k}_1^4\bar{k}_2^{1-3}/6,\ \bar{P}_2^{1-6} = \mathcal{S}\ \bar{k}_2^{1-3}\bar{k}_2^{4-6}/72, \tag{13}$$

$$\begin{aligned}
&\bar{P}_3^1 = \bar{k}_2^1,\ \bar{P}_3^{1-3} = \bar{k}_3^{1-3}/6 + \mathcal{S}\ \bar{k}_2^{12}\bar{k}_1^3/2 + \bar{k}_1^1\bar{k}_1^2\bar{k}_1^3/6,\\
&\bar{P}_3^{1-5} = \bar{k}_4^{1-5}/5! + \mathcal{S}\ \bar{k}_3^{1-4}\bar{k}_1^5/24 + \mathcal{S}\ \bar{k}_2^{12}\bar{k}_2^{3-5}/12 + \mathcal{S}\ \bar{k}_2^{123}\bar{k}_1^4\bar{k}_1^5/12,\\
&\bar{P}_3^{1-7} = \mathcal{S}\ \bar{k}_2^{123}\bar{k}_3^{4-7}/144 + \mathcal{S}\ \bar{k}_2^{123}\bar{k}_2^{4-6}\bar{k}_1^7/72,\\
&\bar{P}_3^{1-9} = \mathcal{S}\ \bar{k}_2^{1-3}\bar{k}_2^{4-6}\bar{k}_2^{7-9}/6^3.
\end{aligned} \tag{14}$$

These formulas are mostly new. The terms involving $\mathcal{S}$ are given explicitly for the first time in Appendix A. By Withers and Nadarajah (2010), the distribution and density of X_n of (6), can be expanded as

$$Prob.(X_n \leq x) \approx \sum_{r=0}^{\infty} n^{-r/2} P_r(x),\ p_{X_n}(x) \approx \sum_{r=0}^{\infty} n^{-r/2} p_r(x),\ x \in R^q, \tag{15}$$

where $P_0(x) = \Phi_V(x),\ p_0(x) = \phi_V(x)$, and

$P_r(x) = \tilde{B}_r(e(-\partial/\partial x))\ \Phi_V(x),\ p_r(x) = \tilde{B}_r(e(-\partial/\partial x))\ \phi_V(x)$ for $r \geq 1$.

Set $\partial_i = \partial/\partial x_i,\ \bar{\partial}_k = \partial_{i_k}$ and $\bar{O}^{1-k} = O^{i_1 \ldots i_k} = (-\bar{\partial}_1) \ldots (-\bar{\partial}_k)$.

$$\text{So by (11), for } r \geq 1, P_r(x) = \sum_{k=1}^{3r} [P_{rk}(x) :\ k-r \text{ even}], \text{ and} \tag{16}$$

$$p_r(x)/\phi_V(x) = \sum_{k=1}^{3r} [\tilde{p}_{rk} :\ k-r \text{ even}] = \tilde{p}_r(x) \text{ say}, \tag{17}$$

$$\text{where } P_{rk}(x) = \bar{P}_r^{1-k}\ \bar{H}_*^{1-k},\ \tilde{p}_{rk} = \bar{P}_r^{1-k}\bar{H}^{1-k}, \tag{18}$$

$$\bar{H}_*^{1-k} = \bar{H}_*^{1-k}(x,V) = \bar{O}^{1-k}\ \Phi_V(x) = \int_{-\infty}^{x} \bar{H}^{1-k}\phi_V(x)dx, \tag{19}$$

$$\text{and } \bar{H}^{1-k} = H^{i_1 \ldots i_k} = \bar{H}^{1-k}(x,V) = \phi_V(x)^{-1}\bar{O}^{1-k}\ \phi_V(x) \tag{20}$$

is the *multivariate Hermite polynomial.* By Withers (2020), for $I = \sqrt{-1}$,

$$\bar{H}^{1-k} = E\,\Pi_{j=1}^{k}(\bar{y}_j + I\bar{Y}_j) \text{ for } Y \text{ of (8) where } \bar{y}_j = y_{i_j},\ y = V^{-1}x, \tag{21}$$

$$\text{So, } \bar{H}^1 = \bar{y}_1,\ \bar{H}^{12} = \bar{y}_1\bar{y}_2 - \bar{V}^{12}, \tag{22}$$

$$\bar{H}^{1-3} = \bar{y}_1\bar{y}_2\bar{y}_3 - \sum^3 \bar{y}_1\bar{V}^{23},\ \sum^3 \bar{y}_1\bar{V}^{23} = \bar{y}_1\bar{V}^{23} + \bar{y}_2\bar{V}^{13} + \bar{y}_3\bar{V}^{12}, \tag{23}$$

$$H^{1-4} = y_1 \dots y_4 - \sum^6 V^{12}y_3y_4 + \mu^{1-4}, \tag{24}$$

$$H^{1-5} = y_1 \dots y_5 - \sum^{10} V^{12}y_3 \dots y_5 + \sum^5 y_5\mu^{1-4}, \tag{25}$$

$$H^{1-6} = \bar{y}_1 \dots \bar{y}_6 - \sum^{15} \bar{y}_1 \dots \bar{y}_4\bar{V}^{56} + \sum^{15} \bar{y}_1\bar{y}_2\bar{\mu}^{3-6} - \bar{\mu}^{1-6}, \tag{26}$$

for $\bar{\mu}^{1-2k}$ of (8). These give $p_1(x)$ and $p_2(x)$, and so $p_{X_n}(x)$ of (15) to $O(n^{-3/2})$. For $\bar{H}^{1-k}$ for $7 \le k \le 9$ needed for $p_3(x)$ when $q = 2$, see Appendix B. $P_{rk}(x)$ *is just* $\tilde{p}_{rk}$ *with* $\bar{H}^{1-k}$ *replaced by* $\bar{H}_*^{1-k}$ *of* (19). For example,

$$\begin{aligned}
&\bar{H}_*^1 = \bar{J}^1,\ \bar{H}_*^{12} = \bar{J}^{12} - \bar{V}^{12}\Phi_V(x),\ \bar{H}_*^{1-3} = \bar{J}^{123} - \sum^3 \bar{J}^1\bar{V}^{23} \text{ where}\\
&\bar{J}^{1-k} = \bar{J}^{1-k}(x,V) = E\,\bar{Y}_1 \dots \bar{Y}_k\, I(X \le x) = \bar{V}^{1,k+1} \dots \bar{V}^{k,2k}\, \bar{M}^{k+1-2k},\\
&\text{and } \bar{M}^{a-b} = \bar{M}^{a-b}(x,V) = E\,\bar{X}_a \dots \bar{X}_b\, I(X \le x) = \int_{-\infty}^{x} \bar{x}_a \dots \bar{x}_b\, \phi_V(x)dx.
\end{aligned}$$

We call $\{\bar{M}^{a-b}(x,V)\}$ *the partial moments of* $\Phi_V(x)$. So by (15), the Edgeworth expansions to $O(n^{-2})$ for the distribution and density of X_n of (6) about those of $X = \mathcal{N}_q(0,V)$, are given by

$$\begin{aligned}
&P_1(x) = e_1(-\partial/\partial x)\,\Phi_V(x) = \sum_{r=1}^{3} \bar{b}_{r+1}^{1-r}\,\bar{O}^{1-r}\,\Phi_V(x)/r! = \sum_{k=1,3} P_{1k}(x),\\
&\text{where } P_{11}(x) = \bar{k}_1^1\,(-\bar{\partial}_1)\,\Phi_V(x),\ P_{13}(x) = \bar{k}_2^{1-3}\,\bar{O}^{1-3}\,\Phi_V(x)/6,\\
&p_1(x) = \bar{k}_1^1\,(-\bar{\partial}_1)\,\phi_V(x) + \bar{k}_2^{1-3}\,\bar{O}^{1-3}\,\phi_V(x)/6,\\
&\tilde{p}_1(x) = p_1(x)/\phi_V(x) = \sum_{k=1,3} \tilde{p}_{1k},\ \tilde{p}_{11} = \bar{k}_1^1\,\bar{H}^1,\ \tilde{p}_{13} = \bar{k}_2^{1-3}\,\bar{H}^{1-3}/6,
\end{aligned} \tag{27}$$

$$P_2(x) = \sum_{k=2,4,6} P_{2k}(x),\ \tilde{p}_2(x) = \sum_{k=2,4,6} \tilde{p}_{2k}, \tag{28}$$

$$P_3(x) = \sum_{k=1,3,5,7,9} P_{3k}(x),\ \tilde{p}_3(x) = \sum_{k=1,3,5,7,9} \tilde{p}_{3k}, \tag{29}$$

for $\tilde{p}_{rk}$ and P_{rk} of (18). Each has q^k terms, but many are duplicates, as it is of great advantage to make $\bar{P}_r^{1-k}$ symmetric in $i_1, \dots, i_k$. The density of X_n relative to its asymptotic value is

$$p_{X_n}(x)/\phi_V(x) \approx 1 + \sum_{r=1}^{\infty} n^{-r/2}\tilde{p}_r(x) = 1 + n^{-1/2}\tilde{p}_1(x) + O(n^{-1}) \text{ for } x \in R^q,$$

for $\tilde{p}_r(x)$ of (17). So $n^{-1/2}\tilde{p}_1(x)$ is a simple measure of the accuracy of the Central Limit Theorem (CLT) approximation. An exception is

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Example 1. *If the distribution of $\hat{w}$ is symmetric about w, then for r odd, $p_r(x) = \tilde{p}_r(x) = P_r(x) = 0$ and by (13), $\tilde{p}_{26}(x) = 0$. So,*

$$p_{X_n}(x)/\phi_V(x) = 1 + n^{-1}\tilde{p}_2(x) + O(n^{-2}) \tag{30}$$
$$\text{where } \tilde{p}_2(x) = \tilde{p}_{22} + \tilde{p}_{24},\ \tilde{p}_{22} = \bar{k}_2^{12}\bar{H}^{12}/2,\ \tilde{p}_{24} = \bar{k}_3^{1-4}\bar{H}^{1-4}/24. \tag{31}$$
$$\text{Also, } Prob.(X_n \leq x) = \Phi_V(x) + n^{-1}P_2(x) + O(n^{-2}) \text{ where } P_2 = P_{22} + P_{24},$$

since $\bar{P}_2^{12} = \bar{k}_2^{12}/2,\ \bar{P}_2^{1-4} = \bar{k}_3^{1-4}/24$. In this case $n^{-1}\tilde{p}_2(x)$ is a measure of the accuracy of the CLT approximation. Also,

$$Prob.(X_n \leq x) = \Phi_V(x) + n^{-1}P_2(x) + O(n^{-2}) \text{ where } P_2 = P_{22} + P_{24}. \tag{32}$$

Example 2. *Let $\hat{w}$ be the sample mean from a distribution with finite cross cumulants $\bar{\kappa}^{1-r}, r \geq 1$. Then $E\,\hat{w} = w$, and only the leading coefficient in (5) is non-zero. $\bar{k}_1^1 = \bar{k}_2^{12} = \bar{k}_2^1 = \bar{k}_3^{123} = 0 \Rightarrow$ $\tilde{p}_{11} = \tilde{p}_{22} = \tilde{p}_{31} = \tilde{p}_{33} = 0$. For $r = 1,2,3$, the non-zero P_r^{1-k} are*

$$\bar{P}_1^{1-3} = \bar{\kappa}^{1-3}/3!,\ \bar{P}_2^{1-4} = \bar{\kappa}^{1-4}/4!,\ \bar{P}_2^{1-6} = \mathcal{S}\,\bar{\kappa}^{1-3}\bar{\kappa}^{4-6}/72,,$$
$$\bar{P}_3^{1-5} = \bar{\kappa}^{1-5}/5!,\ \bar{P}_3^{1-7} = \mathcal{S}\,\bar{\kappa}^{123}\bar{\kappa}^{4-7}/144,\ \bar{P}_3^{1-9} = \mathcal{S}\,\bar{\kappa}^{1-3}\bar{\kappa}^{4-6}\bar{\kappa}^{7-9}/6^3.$$

By (27)–(29), for $1 \leq r \leq 3$, $P_r(x)$ and $\tilde{p}_r(x)$ have only r terms.

NOTE 2.1. *For $H_k(x)$ the univariate Hermite polynomial of (6), and $1 \leq j \leq q$, we define the jth* marginal *Hermite polynomial as*

$$H^{j^k} = E\,(y_j + IY_j)^k = \tau_j^k H_k(\tau_j^{-1} y_j) \text{ where } I = \sqrt{-1},\ \tau_j = (V^{jj})^{1/2}: \tag{33}$$

$$H^j = y_j,\ H^{j^2} = y_j^2 - V^{jj},\ H^{j^3} = y_j^3 - 3V^{jj} y_j,\ H^{j^4} = y_j^4 - 6V^{jj} y_j^2 + 3(V^{jj})^2.$$

$$\text{So if } q = 1,\ H^{1^k}(x, V) = \sigma^{-k} H_k(\sigma x) = H_k(x, V) \text{ say, where } \sigma^2 = V. \tag{34}$$

Takemura and Takeuchi (1988) showed how to extend (4) to the multivariate case by replacing (8) by

$$X_n \approx X + \sum_{r=1}^{\infty} n^{-r/2}\, g_r(X). \tag{35}$$

It would very useful to obtain these multivariate $g_r(X)$ explicitly. How does one extend the univariate formula for $g_r(x)$ in terms of the $h_r(x)$? Their §5 shows that the Cornish-Fisher expansions (4) are valid under the usual conditions for validity of the Edgeworth expansion (2).

NOTE 2.2. *Standard estimates have a natural extension to* Type b estimates. *These are estimates for which the cumulant expansion (5) is replaced by*

$$\bar{k}^{1-r} = k^{i_1 \ldots i_r} = \kappa(\hat{w}^{i_1}, \ldots, \hat{w}^{i_r}) \approx \sum_{j=2r-2}^{\infty} n^{-j/2}\, \bar{b}_j^{1-r} \text{ where } \bar{b}_j^{1-r} = b_j^{i_1 \ldots i_r}.$$

Examples are 1-sided confidence interval limits. These have the form

$$\hat{w} \approx \sum_{j=0}^{\infty} n^{-j/2}\, t_j(\hat{\theta}),$$

where $\hat{\theta}$ is a standard estimate and t_j are smooth functions. See Withers and Nadarajah (2010) for more details.

3. Secondary or Derived Expansions

Let V have Hermitian form $H'\Lambda H$ where $H'H = I_q$.

$$\text{Set } S = V^{-1/2} = H'\Lambda^{-1/2}H \text{ and } T = V^{1/2} = H'\Lambda^{1/2}H.$$

As in (18)–(20), we use tensor summation and the bar notation, and

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$$N \sim \mathcal{N}_q(0, I_q),\ X = TN \sim \mathcal{N}_q(0, V),\ Y = V^{-1}X = SN \sim \mathcal{N}_q(0, V^{-1}).$$

From the 2nd Edgeworth expansion in (15), it follows that for $C \subset R^q$,

$$Prob.(X_n \in C) \approx \sum_{r=0}^{\infty} n^{-r/2} p_{rC} \text{ where} \tag{1}$$

$$p_{rC} = E\ p_r(X)\ I(X \in C) = \int_C p_r(x)\phi_V(x)dx.$$

$$\text{So, } p_{0C} = Prob.(X \in C) = \int_C d\Phi_V(x) = \Phi_V(C), \tag{2}$$

$$\text{and for } r \geq 1,\ p_{rC} = \sum_{k=1}^{3r} [\tilde{p}_{rk}(C) :\ k - r \text{ even}] \text{ where} \tag{3}$$

$$\tilde{p}_{rk}(C) = E\ \tilde{p}_{rk}(X)I(X \in C) = \int_C \tilde{p}_{rk}(x)\phi_V(x)dx = \bar{P}_r^{1-k}\bar{H}^{1-k}(C), \tag{4}$$

$$\text{and } \bar{H}^{1-k}(C) = E\ \bar{H}^{1-k}(X, V)\ I(X \in C) = \int_C \bar{H}^{1-k}\phi_V(x)dx : \tag{5}$$

$$p_{1C} = \tilde{p}_{11}(C) + \tilde{p}_{13}(C),\ \tilde{p}_{r1}(C) = \bar{P}_r^1\bar{H}^1(C),\ \tilde{p}_{r3}(C) = \bar{P}_r^{1-3}\bar{H}^{1-3}(C), \tag{6}$$

$$p_{2C} = \sum_{k=2,4,6} \tilde{p}_{2k}(C) \text{ where } \tilde{p}_{r2}(C) = \bar{P}_r^{12}\bar{H}^{12}(C), \tag{7}$$

$$\tilde{p}_{r4}(C) = \bar{P}_r^{1-4}\bar{H}^{1-4}(C),\ \tilde{p}_{r6}(C) = \bar{P}_r^{1-6}\bar{H}^{1-6}(C).$$

As $\bar{H}^{1-k}$ is a linear combination of $\bar{y}_1 \ldots \bar{y}_s$, we can write $\bar{H}^{1-k}(C)$ in terms of

$$\bar{Q}^{1-s} = E\ \bar{Y}_1 \ldots \bar{Y}_s\ I(VY \in C) = \int_C \bar{y}_1 \ldots \bar{y}_s \phi_V(x)dx : \tag{8}$$

$$\bar{H}^1(C) = \bar{Q}^1,\ \bar{H}^{12}(C) = \bar{Q}^{12} - p_{0C}\ \bar{V}^{12},\ \bar{H}^{1-3}(C) = \bar{Q}^{1-3} - \sum^{3} \bar{Q}^1\bar{V}^{23},$$

$$\bar{H}^{1-4}(C) = \bar{Q}^{1-4} - \sum^{6} \bar{Q}^{12}\ \bar{V}^{34} + p_{0C}\ \bar{\mu}^{1-4},$$

$$\bar{H}^{1-6}(C) = \bar{Q}^{1-6} - \sum^{15} \bar{Q}^{1-4}\ \bar{V}^{56} + \sum^{15} \bar{Q}^{12}\ \bar{\mu}^{3-6} - p_{0C}\ \bar{\mu}^{1-6}.$$

This gives p_{1C} and p_{2C} of (1). So it gives $Prob.(X_n \in C)$ to $O(n^{-3/2})$. For $\hat{w}$ a sample mean, $\tilde{p}_{22}(C) = 0$ and for $k = 4, 6$, $\bar{P}_2^{1-k}$ needed for $\tilde{p}_{2k}(C)$, are given by Example 2.2. If $-C = C$, then for r odd, $\bar{Q}^{1-r} = \tilde{p}_{rk}(C) = p_{rC} = 0$, so that

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$$Prob.(X_n \in C) \approx \sum_{r=0}^{\infty} n^{-r} p_{2rC} = \Phi_V(C) + n^{-1} p_{2C} + O(n^{-2}). \tag{9}$$

We now consider three such C. Ellipsoidal C. Take $C = \{x : x'V^{-1}x \leq u\}$ for some $u > 0$.

$$\text{Then } -C = C,\ p_{0C} = Prob.(N'N < u) = Prob.(\chi_q^2 < u).$$

(We might choose u such that $p_{0C} = .5$ or $.9$.) $\bar{Q}^{1-s}$ of (8) is given by

$$\bar{Q}^{1-s} = E\ \bar{Y}_1 \ldots \bar{Y}_s\ I(N'N < u) = \bar{S}_{1,k+1} \ldots \bar{S}_{k,2k}\ \bar{R}^{k+1-2k} \tag{10}$$
$$\text{where } \bar{R}^{1-k} = E\ \bar{N}_1 \ldots \bar{N}_k\ I(N'N < u). \tag{11}$$
$$\text{So } R^{12} = 0,\ R^{11} = E\ Z_1\ I(Z_1 + Z_2 < u) = R_2 \text{ say, where} \tag{12}$$
$$Z_1 = N_1^2 \sim \chi_1^2,\ Z_2 = N'N - N_1^2 \sim \chi_{q-1}^2 \text{ are independent.} \tag{13}$$
$$\text{So, } \bar{Q}^{12} = R^{11} \bar{S}_1^{12} \text{ where}$$
$$\bar{S}_1^{12} = \sum_{i_3,i_4} \bar{S}_{13}\bar{S}_{24}\ I(i_3 = i_4) = \sum_{i_3=1}^{q} \bar{S}_{13}\bar{S}_{23} = \bar{V}^{12}. \tag{14}$$
$$\text{So by (7), } \tilde{p}_{r2}(C) = (R_2 - p_{0C})\ \bar{P}_r^{12} \bar{V}^{12}. \tag{15}$$

Similarly $\bar{R}^{1-k} = 0$ unless k is even and $\{i_1, \ldots, i_k\} = \{1^{J_1}, \ldots, q^{J_q}\}$ where $J_1, \ldots, J_q$ are even integers and j^k is a string of js of length k.
For Z_1, Z_2 of (13), set

$$R_{2k} = E\ N_1^{2k}\ I(N'N < u) = E\ Z_1^k\ I(Z_1 + Z_2 < u).$$
$$\text{Set } R_{2k_1,2k_2} = E\ N_1^{2k_1} N_2^{2k_2}\ I(N'N < u) = E\ Z_1^{k_1} Z_2^{k_2}\ I(Z_1 + Z_2 + Z_3 < u),$$

where now for $Z_1 = N_1^2$, $Z_2 = N_2^2$ and $Z_3 = N'N - Z_1 - Z_2 \sim \chi_{q-2}^2$, so that Z_1, Z_2, Z_3 are independent.
Set

$$I_1 = I(i_5 = i_6 \neq i_7 = i_8) + I(i_5 = i_7 \neq i_6 = i_8) + I(i_5 = i_8 \neq i_6 = i_7).$$
$$\text{Then } \bar{Q}^{1-4} = \sum_{i_5,\ldots,i_8} \bar{S}_{15} \ldots \bar{S}_{48}\ [R_4\ I(i_5 = \cdots = i_8) + R_{22}\ I_1] = R_4\ \bar{S}_1^{1-4} + R_{22}\bar{S}_2^{1-4}$$
$$\text{where } \bar{S}_1^{1-k} = \sum_{j=1}^{q} S_{i_1 j} \ldots S_{i_k j},\ \bar{S}_2^{1-4} = \bar{S}^{12:34} + \bar{S}^{13:24} + \bar{S}^{14:23},$$
$$\text{and } \bar{S}^{12:34} = \sum_{j \neq k} S_{i_1 j} S_{i_2 j} S_{i_3 k} S_{i_4 k} = \bar{V}^{12}\bar{V}^{34} - \bar{S}_1^{1-4} \text{ by (14).}$$

$$\text{So, } \bar{S}_2^{1-4} = \bar{\mu}^{1-4} - 3\bar{S}_1^{1-4},$$
$$\bar{Q}^{1-4} = \bar{S}_1^{1-4}\ r_4 + \bar{\mu}^{1-4}\ R_{22},\ \text{where } r_4 = R_4 - 3R_{22}, \tag{16}$$
$$\tilde{p}_{r4}(C) = \bar{P}_r^{1-4}\ [\bar{S}_1^{1-4}\ r_4 + 3\bar{V}^{12}\bar{V}^{34}\ r_5] \text{ where } r_5 = R_{22} - 2R_2 + p_{0C}. \tag{17}$$

Similarly, using $\sum'_{j,k,l}$ to mean a sum over distinct j,k,l,

$$\begin{aligned}
&\bar{Q}^{1-6} = \bar{S}_{17}\dots\bar{S}_{6,12}\,[R_6 I(i_7 = \dots = i_{12}) + R_{42}I_2 + R_{222}I_3]\\
&\text{where } R_{222} = E\ N_1^2N_2^2N_3^2\ I(N'N < u),\ I_2 = \sum^{15}\bar{S}^{1-4:56},\ I_3 = \sum^{90}\bar{S}^{12:34:56},\\
&\bar{S}^{1-4:56} = \sum_{j\neq k} S_{i_1j}\dots S_{i_4j}S_{i_5k}S_{i_6k},\ \bar{S}^{12:34:56} = \sum_{j,k,l}{}' S_{i_1j}S_{i_2j}S_{i_3k}S_{i_4k}S_{i_5l}S_{i_6l}.\\
&\text{So, } \bar{P}_r^{1-6}\bar{Q}^{1-6} = \bar{P}_r^{1-6}\,[\bar{S}_1^{1-6}R_6 + 15\bar{S}^{1-4:56}R_{42} + 90\bar{S}^{12:34:56}R_{222}],\\
&\tilde{p}_{r6}(C) = \bar{P}_r^{1-6}\,[\bar{Q}^{1-6} - 15\bar{Q}^{1-4}\bar{V}^{56} + 15\bar{Q}^{12}\bar{\mu}^{3-6} - p_{0C}\bar{\mu}^{1-6}]\\
&= \bar{P}_r^{1-6}\,[\bar{S}_1^{1-6}R_6 + 15\bar{S}^{1-4:56}R_{42} + 90\bar{S}^{12:34:56}R_{222} - 15\bar{S}_1^{1-4}\bar{V}^{56}r_4\\
&+ 3\bar{V}^{12}\bar{V}^{34}\bar{V}^{56}r_6] \text{ where } r_6 = -15R_{42} + 15R_2 - p_{0C}.
\end{aligned} \tag{18}$$

p_{2C} of (9) is now given by (15), (17), (18). Note that

$$\bar{V}^{12}\bar{V}^{34}\bar{V}^{56} = \bar{S}_1^{1-6} + \sum^{3}\bar{S}_1^{1-4:56} + \bar{S}_1^{12:34:56}.$$

We can write $R_{2k_1,\dots}$ in terms of $F_k(u) = Prob.(\chi_k^2 \le u)$:

$$\begin{aligned}
R_{2k} &= E\ Z_1^k\ F_{q-1}(u - Z_1) = \int_0^u z_1^k F_{q-1}(u - z_1)dF_1(z_1) = G_{q-1}(u:k) \text{ say},\\
R_{2k_1,2k_2} &= E\ Z_2^{k_2}\ G_{q-2}(u - Z_2 : k_1) = \int_0^u z_2^{k_2} G_{q-2}(u - z_2 : k_1)dF_1(z_2)\\
&= G_{q-2}(u : k_1k_2) \text{ say},\\
R_{222} &= E\ Z_3\ G_{q-3}(u - Z_3 : 11) = \int_0^u z_3 G_{q-3}(u - z_3 : 11)dF_1(z_3).
\end{aligned}$$

For X, X_n of (6), set

$$Q = X'V^{-1}X \sim \chi_q^2,\ Q_n = X_n'V^{-1}X_n,\ \hat{Q}_n = X_n'\hat{V}^{-1}X_n = n(\hat{w} - w)'\hat{V}^{-1}(\hat{w} - w),$$

where $\hat{V}$ is an estimate $\hat{V}$, (empirical, semi-parametric, or parametric depending on the model used). So

$$Prob.(Q_n \le u) = Prob.(\chi_q^2 \le u) + n^{-1}p_{2C} + O(n^{-2}) = 1 - \alpha + O(n^{-1}) \text{ if } u = \chi^2_{q,1-\alpha},$$

the $1-\alpha$ quantile of χ_q^2 . It is often true that

$$Prob.(\hat{Q}_n \le u) = Prob.(\chi_q^2 \le u) + O(n^{-1}).$$

(An exact confidence region for w is only possible if $\hat{w}$ is normal: see 5.3.2 of Anderson (1958).) $\chi_q^2/q = G_\gamma/\gamma$ where $\gamma = q/2$ and G_γ is a gamma random variable with mean γ. So if $q = 2$,

$$p_{0C} = Prob.(G_1 < u/2) = 1 - e^{-u/2} = 1 - \alpha \text{ if } 0 < u = -2\ln\alpha,$$

$Q_n < -2\ln\alpha$ with probability $1-\alpha+O(n^{-1})$, and typically,

$$\hat{Q}_n \le -2\ln\alpha \text{ with probability } 1-\alpha+O(n^{-1}).$$

By way of illustration, Figure 3.1 plots the elliptical contours $x = X_n$ when $Q_n = -2\ln\alpha$, for $1-\alpha = .5, .9, .99$, when

$$V = \begin{pmatrix} 2 & 1\\ 1 & 2\end{pmatrix},\ V^{-1} = \begin{pmatrix} 2 & -1\\ -1 & 2\end{pmatrix}/3. \tag{19}$$

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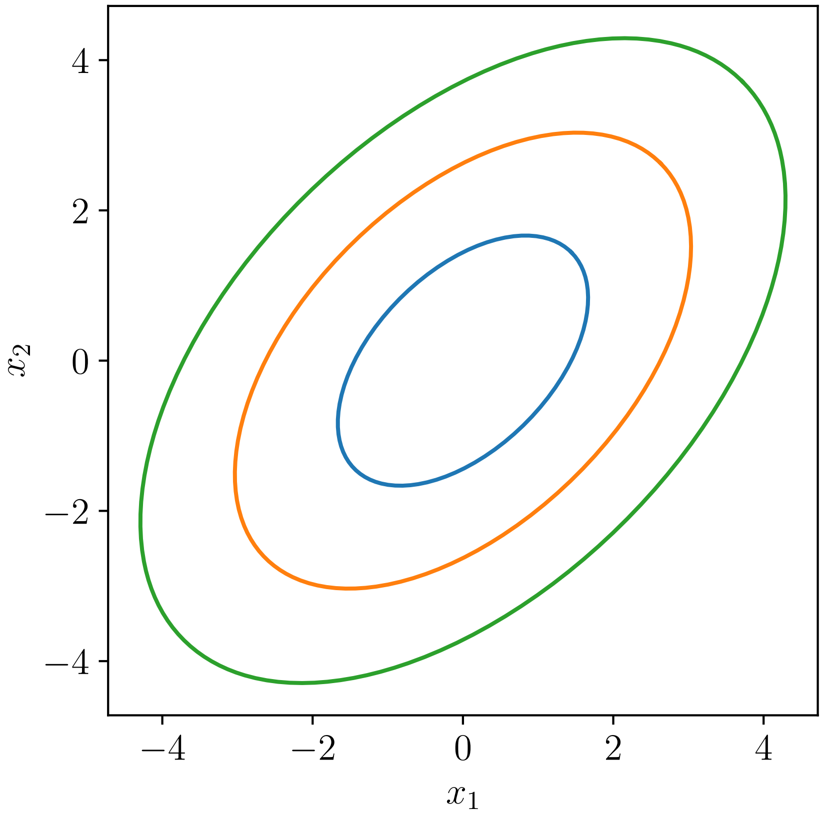
So the asymptotic correlation of $\hat{w}_1$ and $\hat{w}_2$ is $1/2$.

Figure 3.1: $x = X_n$ when $Q_n = -2\ln\alpha$ for $1-\alpha = .5, .9, .99$



courtesy of Dr Paul Teal:

One can do similarly for $q > 2$, using $\chi^2_{q,1-\alpha}$ and 2-dimensional slices of these ellipsoids. For related references on confidence regions, see Withers and Nadarajah (2012a).

The R functions. If $q = 1$, then $R_{2k_1,2k_2} = R_{222} = 0$ for $k_2 > 0$, and

$$R_{2k} = 2g_{2k}(u^{1/2}) \text{ where } g_k(u) = E\,N^k\,I(0 < N < u) = \int_0^u n^k\phi(n)dn$$

is given in Example 3.2 using a recurrence formula. (A simpler way is just to use (7).)

Now take $q = 2$**.** Then

$$\begin{aligned} R_{2k_1,2k_2} &= 4E\,N_1^{2k_1}N_2^{2k_2}I(N_1^2 + N_2^2 < u,\ N_1 > 0, N_2 > 0) \\ &= 4g_{2k_1,2k_2}(u^{1/2}) \text{ where } g_{k_1k_2}(v) = \int_0^v dn_1\phi(n_1)n_1^{k_1}g_{k_2}((v^2 - n_1^2)^{1/2}). \end{aligned} \tag{20}$$

To get a recurrence formula for $g_{k_1k_2} = g_{k_1k_2}(v)$, set

$$
\begin{aligned}
& f_k(u) = g_k(u^{1/2}),\ A = \phi(n_1)n_1^{k_1-1} f_{k_2}(v^2 - n_1^2). \\
& \text{Then } g_{k_1k_2} = -\int_0^v A\, d\phi(n_1) = [A]_v^0 + \bar{g}_{k_12k_2} \text{ where} \\
& \bar{g}_{k_12k_2} = \int_0^v \phi(n_1)\, dA = (k_1-1)A_1 + A_2, \\
& A_1 = \int_0^v dn_1\, \phi(n_1)n_1^{k_1-2} f_{k_2}(v^2 - n_1^2) = g_{k_1-2,k_2}, \\
& A_2 = \int_0^v dn_1\, \phi(n_1)n_1^{k_1-1}(-2n_1)\dot{f}_{k_2}(v^2 - n_1^2),\ \dot{f}_k(u) = df_k(u)/du. \\
& f_k(v^2) = g_k(v) \Rightarrow 2v\dot{f}_{k_2}(v^2) = \dot{g}_k(v) = v^k\phi(v). \\
& \text{So } \dot{f}_{k_2}(v^2 - n_1^2) = [x^{k-1}\phi(x)]_{x=v^2-n_1^2} = (v^2 - n_1^2)^{(k-1)/2}\phi(v)e^{n_1^2/2}. \\
& \text{So } A_2 = -\phi(0)\phi(v)A_3 \text{ where for } B_{12} = B(k_1+1)/2, k_2+1)/2), \\
& A_3 = \int_0^v dn_1\, n_1^{k_1} v^2 - n_1^{2(k_2-1)/2} = v^{k_1+k_2}B_{12}/2. \\
& \text{So } A_2 = -\phi(0)\phi(v)v^{k_1+k_2}B_{12}/2 = -A_{k_1k_2} \text{ say.} \\
& \text{So for } k_1 \geq 1,\ g_{k_1k_2} = (k_1-1)g_{k_1-2,k_2} + G_{k_1k_2} \\
& \text{ where } G_{k_1k_2} = I(k_1 = 1)g_{k_2}(v) + A_{k_1k_2}.
\end{aligned}
\tag{21}
$$

This recurrence formula for $g_{k_1k_2}$ of (20) gives

$$\begin{aligned}
&R_{22} = 4g_{22}(u^{1/2}) \text{ where } g_{22}(v) = g_{22} = g_{02} + A_{22},\\
&\text{and } A_{22} = v^4 e^{-v^2}/32 \text{ as } B_{12} = \pi/8,\\
&R_{42} = 4g_{42}(u^{1/2}) \text{ where } g_{42}(v) = g_{42} = g_{02} + A_{42},\\
&\text{and } A_{42} = v^6 e^{-v^2}/64 \text{ as } B_{12} = \pi/16.
\end{aligned}$$

So we need g_{02}. By Example 3.2,

$$g_2(u) = \Phi_0(u) - u\phi(u) \text{ where } \Phi_0(u) = \Phi(u) - 1/2.$$

Transforming to $u^2 = v^2 - n_1^2$,

$$\begin{aligned}
&g_{02} = \int_0^v dn_1 \phi(n_1) g_2(u) = e^{v^2/2}(A - B),\\
&\text{where } A = \int_0^v (v^2 - u^2)^{1/2} \Phi_0(u) u\phi(u) du,\\
&B = \phi(0) \int_0^v u^2 (v^2 - u^2)^{1/2} \phi(u) du = (2\pi)^{-1} v^2 I(-v^2),\\
&\text{and } I(t) = \int_0^1 x^{1/2}(1-x)^{-1/2} e^{tx} dx = \sum_{i=0}^{\infty} B(i+3/2, 1/2) t^i / i!.\\
&B(i+3/2, 1/2) = \pi^{1/2}\Gamma(i+3/2)/i!.\\
&\Gamma(i+3/2) = \Gamma(3/2)\ [3/2]_i \text{ where } [t]_i = t(t+1)\ldots(t+i-1).\\
&\text{So } B = (v^2/8)\ {}_1F_1(3/2, 1 : -v^2)
\end{aligned}$$

where ${}_1F_1(3/2, 1 : t)$ is the confluent or degenerate hypergeometric function: see 9.21 of Gradshteyn and Ryzhik (2000) and p504 of Abramowitz and Stegun (1964). **Now suppose that** $q > 2$**.** Then

$$\begin{aligned}
&R_{2k_1,2k_2} = 4E\ N_1^{2k_1} N_2^{2k_2} I(N_1^2 + N_2^2 < u - Z, N_1 > 0, N_2 > 0)\\
&= 4E\ g_{2k_1,2k_2}((u-Z)^{1/2}) \text{ where } Z = \sum_{i=3}^{q} \sim \chi^2_{q-2},\\
&\text{and } R_{222} = 8E\ N_1^2 N_2^2 N_3^2\ I(N'N < u, N_1 > 0, N_2 > 0, N_3 > 0)\\
&= 8E\ N_3^2 R_{22}(u - N_3^2 - Z) I(N_3 > 0)
\end{aligned}$$

where $R_{22}(u) = R_{22}$ of (20) and $Z = \sum_{i=4}^{q} N_i^2 \sim \chi^2_{q-3}$ is independent of N_3.

By Theorem 2.2 of Withers and Nadarajah (2012a), for $r \geq 1$,

$p_{rC} = \sum_{j=1}^{3r} b_{2r,j}(U-1)^j F_q(x)$ where $F_q(x) = Prob.(\chi_q^2 < x),\ U^j F_q(x) = F_{q+2j}(x),$
$b_{2r,j} = \bar{b}_{2r}^{1-2j}\ \bar{\mu}^{1-2j},\ \bar{\mu}^{1-2j} = E\ \bar{X}_1 \dots \bar{X}_{2j}.$
For example, p_{rC} is given by $b_{21} = (\bar{k}_2^{12} + \bar{k}_1^1\bar{k}_1^2)\bar{V}^{12},$
$b_{22} = \bar{k}_3^{1-4}\bar{V}^{12}\bar{V}^{34}/4 + \bar{k}_1^1\bar{k}_2^{34}\bar{V}^{12}\bar{V}^{34},$
$b_{23} = \bar{k}_2^{123}\bar{k}_2^{456}(\bar{V}^{12}\bar{V}^{34}\bar{V}^{56}/4 + \bar{V}^{14}\bar{V}^{25}\bar{V}^{36}/6).$

Its extension to $\hat{Q}_n$ is given in §3 there for parametric and non-parametric models. Take $C = \{x : |(V^{-1/2}x)_j| \le u_j,\ j = 1,\dots,q\}$. So $-C = C$. For $Y = V^{-1/2}N$ and $N \sim \mathcal{N}_q(0, I_q)$,

$$p_{0C} = \Pi_{j=1}^q G_0(u_j) \text{ where } G_0(u) = Prob.(|N_1| < u) = 2\Phi(u) - 1.$$

(We might choose $u_j \equiv u$ such that $p_{0C} = .5$ or $.9$.) For $S = V^{-1/2}$,

$\bar{Q}^{1-k} = E\ \bar{Y}_1 \dots \bar{Y}_k\ I(|N|_j < u_j,\ j = 1,\dots,q) = \bar{S}_{1,k+1} \dots \bar{S}_{k,2k}\ \bar{R}^{k+1-2k}$
where now $\bar{R}^{1-k} = E\ \bar{N}_1 \dots \bar{N}_k\ I(|N|_j < u_j,\ j = 1,\dots,q) = \bar{F}^{1-k}(u_1 \dots u_q)$ say.
Set $R_{2k_1,\dots,2k_s} = E\ N_1^{2k_1} \dots N_s^{2k_s}\ I(|N|_j < u_j,\ j = 1,\dots,q)$
$= [\Pi_{j=1}^s G_{k_j}(u_j)]\ [\Pi_{j=s+1}^q G_0(u_j)]$ where
$G_k(u) = E\ N_1^{2k}\ I(|N_1| < u) = 2g_{2k}(u)$ where
$g_k(u) = g_k = \int_0^u n^k\phi(n)dn = -\int_0^u n^{k-1}d\phi(n) = [n^{k-1}\phi(n)]_u^0 + (k-1)g_{k-2}:$
$g_0 = \Phi(u) - 1/2,\ g_1 = \phi(0) - \phi(u),\ g_2 = g_0 - u\phi(u),$
$g_3 = 2\phi(0) - (u^2+2)\phi(u),\ g_4 = 3g_0 - (u^3+3u)\phi(u),$
$g_5 = 2.4\phi(0) - (u^4+4u^2+4.2)\phi(u),\ g_6 = 3.5g_0 - (u^5+5u^3+5.3u)\phi(u),$
$g_7 = 2.4.6\phi(0) - (u^6+6u^4+66.4u^2+6.4.2)\phi(u),$
$g_8 = 3.5.7g_0 - (u^7+7u^5+7.5u^3+7.5.3u)\phi(u).$
As $R^{12} = 0,\ \bar{Q}^{12} = \bar{V}^{12}R_2$ by (14) where $R_{2k} = 2^q g_{2k}(u_1)\ \Pi_{j=2}^q g_0(u_j).$
Set $R_{2k_1,2k_2} = 2^q[\Pi_{j=1}^2 g_{2k_j}(u_j)]\ [\Pi_{j=3}^q g_0(u_j)],$
$R_{2k_1,2k_2,2k_3} = 2^q[\Pi_{j=1}^3 g_{2k_j}(u_j)]\ [\Pi_{j=4}^q g_0(u_j)],$

So now $\bar{Q}^{1-4}$ is given by (16) and (17) in terms of these new $R_{2k_1,\dots}$. Similarly $\bar{Q}^{1-6}$ and $\tilde{p}_{rk}(C)$ are given by Example 3.1 with these new $R_{2k_1,\dots}$. So now we have p_{2C} of (9). If we choose $u_j \equiv u$, then

$$R_{2k} = 2^q g_{2k}\ g_0^{q-1},\ R_{2k_1,2k_2} = 2^q[\Pi_{j=1}^2 g_{2k_j}]\ g_0^{q-2},$$
$$R_{2k_1,2k_2,2k_3} = 2^q[\Pi_{j=1}^3 g_{2k_j}]\ g_0^{q-3}.$$

Take $C = \{x : |x_j| \le u_j,\ j = 1,\dots,q\}$. So $-C = C$. This choice gives a set of q simultaneous intervals. For $Y = SN$ and $T = V^{1/2}$

$p_{0C} = Q = Prob.(|(VY)_j| < u_j,\ j = 1,\dots,q) = Prob.(|(TN)_j| < u_j,\ j = 1,\dots,q),$
$\bar{Q}^{1-k} = E\ \bar{Y}_1 \dots \bar{Y}_k\ I(|(VY)_j| < u_j,\ j = 1,\dots,q) = \bar{S}_{1,k+1} \dots \bar{S}_{k,2k}\ \bar{R}^{k+1-2k}$
where now $\bar{R}^{1-k} = E\ \bar{N}_1 \dots \bar{N}_k\ I(|(TN)_j| < u_j,\ j = 1,\dots,q).$

So $\bar{R}^{1\dots k} = 0$ for k odd. (R^{12} is no longer 0.) But each of those non-zero require q numerical integrations. Consider the case $q = 2$.

$$
\begin{aligned}
& I(|X_j| < u_j,\ j = 1,2) = I(X_j < u_j,\ j = 1,2) - I(X_1 < -u_1, X_2 < u_2) \\
& - I(X_1 < u_1, X_2 < -u_2) + I(X_1 < -u_1, X_2 < -u_2) \Rightarrow \\
& R^{jk} = F_{jk}(u_1, u_2) - F_{jk}(u_1, -u_2) - F_{jk}(-u_1, u_2) + F_{jk}(-u_1, -u_2) \\
& \text{where } F_{jk}(u_1, u_2) = E\ N_j N_k\ I(X_1 < u_1, X_2 < u_2) \\
& = \int dn_1 \int dn_2 \phi(n_1)\phi(n_2)\ I(\sum_{k=1}^{2} T_{jk} n_k < u_j, j = 1,2).
\end{aligned}
$$

For more on these types of expansions and their inverses, see Hill and Davis (1968) and their simplifications given in §4–§5 of Withers and Nadarajah (2015). However in these papers the need to express results in terms of $S = V^{-1/2}$ did not arise there. For the case of a sample mean with estimated covariance, it is possible to avoid the use of $S = V^{-1/2}$: see Xu and Gupta (2006).

4. The Distribution of $X_n = n^{1/2}(w - w)$ for $q = 2$

As above, we set $y = V^{-1}x,\ Y = V^{-1}x$. We now switch notation to

$$\mu_{ab} = \mu^{1^a 2^b} = E\ Y_1^a\ Y_2^b, \text{ so that } \mu_{20} = V^{11},\ \mu_{11} = V^{12}, \tag{1}$$

$$H_{ab} = H_{ab}(x) = H^{1^a 2^b} = E\ (y_1 + IY_1)^a\ (y_2 + IY_2)^b \text{ for } I = \sqrt{-1}. \tag{2}$$

So H_{k0} and H_{0k} are given by (33) with $j = 1$ and 2: $H_{10} = y_1,\ H_{01} = y_2,$

$$H_{11} = y_1 y_2 - \mu_{11}, H_{21} = y_1^2 y_2 - \mu_{20} y_2 - 2\mu_{11} y_1, H_{12} = y_1 y_2^2 - \mu_{02} y_1 - 2\mu_{11} y_2,$$

and so on. The μ_{ab} and H_{ab} needed here are given in Appendix B. Let us write the cumulant expansion (5) as

$$\kappa_{ab}(\hat{w}_1, \hat{w}_2) \approx \sum_{d=a+b-1}^{\infty} n^{-d} k_{abd} \text{ for } a + b \geq 1 \text{ where } k_{abd} = k_d^{1^a 2^b}.$$

So $k_{100} = w_1,\ k_{010} = w_2$. The density of $X_n = n^{1/2}(\hat{w} - w)$, $p_{X_n}(x)$, is given by (15) and (18) in terms of $\tilde{p}_r(x)$ and $\tilde{p}_{rk}$ of (18). As $barP_r^{1-k}$ is symmetric,

$$\tilde{p}_{rk} = \sum_{b=0}^{k} P_r(k - b, b)\ H_{k-b,b} \text{ where } P_r(ab) = \binom{a+b}{a} P_r^{1^a 2^b}, \tag{3}$$

for $\bar{P}_r^{1-k}$ of (12)–(14). So $P_r(ba)$ is just $P_r(ab)$ with 1 and 2 reversed.

$$\text{Set } \sum^{2} P_r(ab) H_{ab} = P_r(ab) H_{ab} + P_r(ba) H_{ba}.$$

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Then $\tilde{p}_r(x)$ is given for $r = 1,2,3$ by (27)–(29) in terms of

$$\begin{aligned}
\tilde{p}_{r1} &= \sum^{2} P_r(10)H_{10},\ \tilde{p}_{r3} = \sum^{2}[P_r(30)H_{30} + P_r(21)H_{21}],\ r \text{ odd},\\
\tilde{p}_{22} &= \sum^{2} P_2(20)H_{20} + P_2(11)H_{11},\\
\tilde{p}_{24} &= \sum^{2}[P_2(40)H_{40} + P_2(31)H_{31}] + P_2(22)H_{22},\\
\tilde{p}_{26} &= \sum^{2}[P_2(60)H_{60} + P_2(51)H_{51} + P_2(42)H_{42}] + P_2(33)H_{33},\\
\tilde{p}_{35} &= \sum^{2}[P_3(50)H_{50} + P_3(41)H_{41} + P_3(32)H_{32}],\\
\tilde{p}_{37} &= \sum^{2}[P_3(70)H_{70} + P_3(61)H_{61} + P_3(52)H_{52} + P_3(43)H_{43}],\\
\tilde{p}_{39} &= \sum^{2}[P_3(90)H_{90} + P_3(81)H_{81} + P_3(72)H_{72} + P_3(63)H_{63} + P_3(54)H_{54}].
\end{aligned}$$

The $P_r(ab)$ needed in (3) for $\tilde{p}_{rk},\ r \leq 3$, are as follows.

$$\begin{aligned}
&\text{For } \tilde{p}_{11},\ P_1(10) = k_1^1, \text{ so } P_1(01) = k_1^2.\\
&\text{For } \tilde{p}_{13},\ P_1(30) = k_2^{111}/6,\ \text{ so } P_1(03) = k_2^{222}/6,\ P_1(21) = k_2^{112}/2.
\end{aligned}$$

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$$
\begin{aligned}
&\text{For } \tilde{p}_{22},\ P_2(20) = k_2^{11}/2 + (k_1^1)^2/2,\ P_2(11) = k_2^{12} + k_1^1 k_1^2. \qquad (4)\\
&\text{For } \tilde{p}_{24},\ P_2(40) = k_3^{1^4}/24 + k_1^1 k_2^{1^3}/6,\ P_2(31) = k_3^{1112}/6 + k_1^1 k_2^{112}/3 + k_1^2 k_2^{111}/3,\\
&\ P_2(22) = k_3^{1122}/4 + k_1^1 k_2^{112}/2 + k_1^2 k_2^{122}/2,\\
&\text{For } \tilde{p}_{26},\ \ P_2(60) = (k_2^{111})^2/72,\ P_2(51) = k_2^{111} k_2^{112}/12,\\
&\ P_2(42) = k_2^{111} k_2^{122}/12 + (k_2^{112})^2/8,\\
&P_2(33) = k_2^{111} k_2^{222}/36 + k_2^{112} k_2^{122}/4,\\
&\text{For } \tilde{p}_{31},\ P_3(10) = k_2^1. \text{ For } \tilde{p}_{33},\ P_3(30) = k_3^{111}/6 + k_2^{11} k_1^1/2 + (k_1^1)^3/6,\\
&\ P_3(21) = [k_3^{112} + 2k_1^1 k_2^{12} + k_1^2 k_2^{11} + (k_1^1)^2 k_1^2]/6.\\
&\text{For } \tilde{p}_{35},\ P_3(50) = k_4^{1^5}/120 + k_3^{1^4} k_1^1/24 + k_2^{111}[k_2^{11} + (k_1^1)^2]/12,\\
&\ P_3(41)/5 = P_3^{1^4 2} = k_4^{1^4 2}/120 + S_1/24 + S_2/12 + S_3/12 \text{ where}\\
&\ S_1 = (4k_1^1 k_3^{1112} + k_1^2 k_3^{1111})/5,\ S_2 = (3k_2^{11} k_2^{112} + 2k_2^{12} k_2^{111})/5,\\
&\ S_3 = [2k_1^1 k_1^2 k_2^{111} + 3(k_1^1)^2 k_2^{112}]/5,\\
&P_3(32) = 10P_3^{1112} = k_4^{1112^2}/12 + k_3^{1112} k_1^2/3 + k_3^{1122} k_1^1/4 + k_3^{111}(k_1^2)^2/12\\
&\ + k_3^{112} k_1^2 k_1^2/2 + k_3^{122}(k_1^1)^2/4.\\
&\text{For } \tilde{p}_{37},\ P_3(70) = k_3^{1111} k_2^{111}/144 + (k_2^{1^3})^2 k_1^1/72,\\
&\ P_3(61) = k_2^{112} k_3^{1111}/48 + k_2^{111} k_3^{1112}/36 + k_1^2 (k_2^{111})^2/72 + k_1^1 k_2^{111} k_2^{112}/12.\\
&\ P_3(52) = k_3^{1111} k_2^{122}/48 + k_3^{1^3 2} k_2^{112}/12 + k_3^{1122} k_2^{111}/24 + 5k_2^{111} k_2^{112} k_1^2/24\\
&\ + k_2^{111} k_2^{122} k_1^1/12 + (k_2^{112})^2 k_1^1/12,\\
&\ P_3(43) = 35P_3^{1^4 2^3} = A/144 + B/72 \text{ for } A = k_3^{1^4} k_2^{222} + 13k_3^{1^3 2} k_2^{122} + 17k_3^{1122} k_2^{112}\\
&\ + k_3^{1222} k_2^{111},\ B = (k_2^{111} k_2^{222} + 15k_2^{112} k_2^{122}) k_1^1 + (9k_2^{111} k_2^{122} + 10k_2^{112} k_2^{112}) k_1^2.\\
&\text{For } \tilde{p}_{39},\ P_3(90) = (k_2^{1^3}/6)^3,\ P_3(81) = 9(k_2^{111})^2 k_2^{112}/6^3,\\
&\text{and for } \bar{a}^{123} = \bar{k}_2^{123},\ \ P_3(72) = [9(a^{111})^2 a^{122} + 3a^{111}(a^{112})^2]/6^3,\\
&\ P_3(63) = 3[(a^{111})^2 a^{222} + 18a^{111} a^{112} a^{122} + 9(a^{112})^3]/6^3,\\
&\ P_3(54) = 9[a^{111} a^{112} a^{222} + 15a^{111}(a^{122})^2 + 27(a^{112})^2 a^{122}]/6^3.
\end{aligned}
$$

(15) and (18) now give the density of $X_n = n^{1/2}(\hat{w} - w)$ to $O(n^{-2})$. Set

$$
\begin{aligned}
&H_{ab}^* = (-\partial_1)^a (-\partial_2)^b\ \Phi_V(x) = \int_{-\infty}^x H_{ab} \phi_V(x) dx. \qquad (5)\\
&\text{So } H_{ab}^* = H_{a-1,b-1}\ \phi_V(x) \text{ if } a \geq 2, b \geq 1,
\end{aligned}
$$

$$
\begin{aligned}
H_{10}^* &= \int_{-\infty}^{x_2} \phi_V(x) dx_2 = \partial_1 \Phi_V(x),\ H_{a0}^* = (-\partial_1)^{a-1} H_{10}^* \text{ if } a \geq 2,\\
H_{01}^* &= \int_{-\infty}^{x_1} \phi_V(x) dx_1 = \partial_2 \Phi_V(x),\ H_{0b}^* = (-\partial_2)^{b-1} H_{01}^* \text{ if } b \geq 1.
\end{aligned}
$$

The distribution of $X_n = n^{1/2}(\hat{w} - w)$ is given by (15) and (19) in terms of $P_{rk}(x)$. *$P_{rk}(x)$ is just $\tilde{p}_{rk}$ above with H_{ab} replaced by H_{ab}^* of (5).* That is,

$$
P_{rk}(x) = \sum_{b=0}^{k} P_r(k-b, b)\ H_{k-b,b}^*. \qquad (6)
$$

(15) and (19) now give $Prob.(X_n \leq x)$ to $O(n^{-2})$ for $X_n = n^{1/2}(\hat{w} - w)$. For example for $x = (1,1)'$ and $x = (2,2)'$, the values of H_{ab} are given in Example B.1. For $r = 1,2,3$, p_{rC} of (1) and (9) is given by replacing $\tilde{p}_{rk}, H_{ab}$ above by $\tilde{p}_{rk}(C)$ of (4) and $H_{abC} = H^{1^a 2^b}(C)$ of (5). By §3,

$$\begin{aligned}
&H_{10C} = Q_{10},\ H_{20C} = Q_{20} - p_{0C}\mu_{20},\ H_{11C} = Q_{11} - p_{0C}\mu_{11},\\
&H_{30C} = Q_{30} - 3Q_{10}\mu_{20},\ H_{21C} = Q_{21} - 2Q_{10}\mu_{11} - Q_{01}\mu_{20},\\
&H_{40C} = Q_{40} - 6Q_{20}\mu_{20} + p_{0C}\mu_{40},\ H_{31C} = Q_{31} - 3Q_{20}\mu_{11} - 3Q_{11}\mu_{20} + p_{0C}\mu_{31},\\
&H_{22C} = Q_{22} - Q_{20}\mu_{02} - Q_{02}\mu_{20} - 4Q_{11}\mu_{11} + p_{0C}\mu_{22},\\
&H_{60C} = Q_{60} - 15Q_{40}\mu_{20} + 15Q_{20}\mu_{40} - p_{0C}\mu_{60},\\
&H_{51C} = Q_{51} - 10Q_{31}\mu_{20} + 5Q_{11}\mu_{40} - 5Q_{40}\mu_{11} + 10Q_{20}\mu_{31}p_{0C}\mu_{51},\\
&H_{42C} = Q_{42} - 6Q_{22}\mu_{20} + Q_{02}\mu_{40} - 8Q_{31}\mu_{11} + 8Q_{11}\mu_{31} - Q_{40}\mu_{02} + 6Q_{20}\mu_{22}\\
&\quad - p_{0C}\mu_{42},\\
&H_{33C} = Q_{33} + 3\sum^{2}(Q_{20}\mu_{13} - Q_{13}\mu_{20}) - 9Q_{22}\mu_{11} + 9Q_{11}\mu_{22} - p_{0C}\mu_{33},\\
&\text{where } Q_{ab} = E\ Y_1^a Y_2^b\ I(VY \in C).
\end{aligned}$$

These expressions can be read off those for H_{ab} in Appendix B. If $-C = C$, then $0 = Q_{ab} = H_{abC}$ for $a+b$ odd. For Examples 3.1 and 3.2, $Q_{11} = \mu_{11}R_2$, and $\tilde{p}_{r2}(C)$ of (4) is given by (15) in terms of $\bar{P}_r^{12}\bar{V}^{12} = \sum^2 P_r(20)\mu_{20} + 2P_r(11)\mu_{11}$. For $r = 2$ this is given by (4). Let $\hat{w}$ be a sample mean of a distribution with cumulants κ_{ab}. By Example 2.2, for $(rk) = (11), (22), (31), (33)$, $\tilde{p}_{rk} = 0$, and we need the following Edgeworth coefficients.

$$\begin{aligned}
&\text{For } \tilde{p}_{13} \text{ and } P_{13}(x):\ P_1(30) = \kappa_{30}/3!,\ P_1(21) = \kappa_{21}/2,\ P_1(12) = \kappa_{12}/2.\\
&\text{For } \tilde{p}_{24},\ P_{24}(x):\ P_2(40) = \kappa_{40}/4!,\ P_2(31) = \kappa_{31}/6,\ P_2(22) = \kappa_{22}/4.\\
&\text{For } \tilde{p}_{26},\ P_{26}(x):\ P_2(60) = \kappa_{30}^2/72,\ P_2(51) = \kappa_{30}\kappa_{21}/12,\\
&P_2(42) = (2\kappa_{30}\kappa_{12} + 3\kappa_{21}^2)/24,\ P_2(33) = (\kappa_{30}\kappa_{03} + 9\kappa_{21}\kappa_{12})/6.\\
&\text{For } \tilde{p}_{35},\ P_{35}(x):\ P_3(50) = \kappa_{50}/5!,\ P_3(41)/5 = \kappa_{41}/5!,\ P_3(32) = \kappa_{32}/12.\\
&\text{For } \tilde{p}_{37},\ P_{37}(x):\ P_3(70) = \kappa_{40}\kappa_{30}/144,\ P_3(61) = \kappa_{21}\kappa_{40}/48 + \kappa_{30}\kappa_{31}/36,\\
&P_3(52) = \kappa_{40}\kappa_{12}/48 + \kappa_{31}\kappa_{21}/12 + \kappa_{22}\kappa_{30}/24,\\
&P_3(43) = A/144 \text{ for } A = \kappa_{40}\kappa_{03} + 13\kappa_{31}\kappa_{12} + 17\kappa_{22}\kappa_{21} + \kappa_{13}\kappa_{30}.\\
&\text{For } \tilde{p}_{39},\ P_{39}(x):\ P_3(90) = \kappa_{30}^3/6^3,\ P_3(81) = \kappa_{30}^2\kappa_{21}/6^3,\\
&P_3(72) = (\kappa_{30}^2\kappa_{12} + 3\kappa_{30}\kappa_{21}^2)/4.6^3,\ P_3(63) = (\kappa_{30}^2\kappa_{03} + 12\kappa_{30}\kappa_{21}\kappa_{12}\\
&\quad + 15\kappa_{21}^3)/28.6^3,\ P_3(54) = (5\kappa_{30}\kappa_{21}\kappa_{03} + 9\kappa_{30}\kappa_{12}^2 + 21\kappa_{21}^2\kappa_{12})/35.6^3.
\end{aligned}$$

For $a < b$, $P_r(ab)$ is $P_r(ba)$ with superscripts 1 and 2 reversed. The $P_2(ab)$ needed for p_{2C} of (1) and (9) are those given above for $\tilde{p}_{24}, \tilde{p}_{26}$. For a specific case of a bivariate sample mean, consider An **entangled gamma** model. Let G_0, G_1, G_2 be independent gamma random variables with means $\gamma = \gamma_0, \gamma_1, \gamma_2$. For $i = 1,2$, set $X_i = G_0 + G_i$, $w_i = E\ X_i = \gamma + \gamma_i$, and let $\hat{w}$ be the mean of a random sample of size n distributed as (X_1, X_2). So, $E\hat{w} = w$, and $n\hat{w} \stackrel{\mathcal{L}}{=} (G_{n0} + G_{n1}, G_{n0} + G_{n2})'$ where G_{n0}, G_{n1}, G_{n2} are independent gamma random variables with means $n\gamma, n\gamma_1, n\gamma_2$. The rth order cumulants of $X = (X_1, X_2)'$ are $\kappa^{i^r} = (r-1)!w_i$, and otherwise $(r-1)!\gamma$. For example $\kappa_{20} = \kappa^{11} = w_1$, $\kappa_{02} = \kappa^{22} = w_2$, $\kappa_{11} = \kappa^{12} = \gamma$, $\kappa_{21} = \kappa^{112} = \kappa_{12} = \kappa^{122} = 2!\gamma$, and

$$V = \begin{pmatrix} w_1 & \gamma \\ \gamma & w_2 \end{pmatrix},\ V^{-1} = \begin{pmatrix} w_2 & -\gamma \\ -\gamma & w_1 \end{pmatrix}/D \text{ where } D = det\ V = w_1 w_2 - \gamma^2.$$

So, $y = V^{-1}x = (w_2 x_1 - \gamma x_2, -\gamma x_1 + w_1 x_2)'/D$. Set $\nu_i = w_i/\gamma$. Then V has correlation $(\nu_1\nu_2)^{-1/2}$. This ranges from 0 at $\gamma = 0$ to 1 at $\gamma = \infty$. So $\hat{w}_1$ and $\hat{w}_2$ are positively entangled. (For a negatively entangled example, replace $G_{n0}+G_{n2}$ by $-G_{n0}+G_{n2}$: the correlation is then $-(\nu_1\nu_2)^{-1/2}$. An extension to R^q is $n\hat{w}_i = \lambda_i G_{n0}+G_{ni},\ i=1,\dots q$.) For $c=0,1,\dots,$ set

$$\sum^2 w_1^c\, H_{ab} = w_1^c\, H_{ab} + w_2^c\, H_{ba}.$$

By Example 3.1, $P_1(30)=w_1/3$, (so $P_1(03)=w_2/3$,) $P_1(22)=2\gamma$,

$$\begin{aligned}
&\tilde{p}_1(x) = \tilde{p}_{13} = \sum^2 [w_1H_{30}/3+\gamma H_{21}],\\
&\tilde{p}_2(x) = \tilde{p}_{24}+\tilde{p}_{26} \text{ where } \tilde{p}_{24} = \sum^2 [w_1H_{40}/4+\gamma H_{31}]+\gamma H_{22},\\
&\tilde{p}_{26} = \sum^2 [w_1^2H_{60}/18 + w_1\gamma(H_{51}+H_{42})/3] + (w_1w_2/9+\gamma^2)H_{33},\\
&\tilde{p}_3(x) = \sum_{k=5,7,9} \tilde{p}_{3k} \text{ where } 5\tilde{p}_{35} = \sum^2 [w_1H_{50}+\gamma(H_{41}+H_{32})],\\
&12\tilde{p}_{37} = \sum^2 [w_1^2H_{70}+7w_1\gamma H_{61}+6(w_1\gamma+2\gamma^2)H_{52}+(w_1w_2+w_1\gamma+30\gamma^2)H_{43}],\\
&27\tilde{p}_{39} = \sum^2 [w_1^3H_{90}+9w_1^2\gamma H_{81}+(9w_1^2\gamma+3w_1\gamma^2)H_{72}\\
&+3(w_1^2w_2+18w_1\gamma^2+9\gamma^3)H_{63}+9(w_1w_2\gamma+15w_1\gamma^2+27\gamma^3)H_{54}].
\end{aligned}$$

Now consider the **entangled exponential** case $\gamma_i \equiv 1$. So $w_i \equiv 2$, V and V^{-1} are given by (19),

$$\begin{aligned}
&\tilde{p}_1(x) = \tilde{p}_{13} = \sum^2 [2H_{30}/3+H_{21}]\\
&\tilde{p}_{24} = \sum^2 [H_{40}/2+H_{31}]+H_{22},\ 9\tilde{p}_{26} = 2\sum^2 [H_{60}+3H_{51}+3H_{42}]+13H_{33},\\
&5\tilde{p}_{35} = \sum^2 [2H_{50}+H_{41}+H_{32}],\ 6\tilde{p}_{37} = \sum^2 [2H_{70}+7H_{61}+12H_{52}+18H_{43}],\\
&27\tilde{p}_{39} = \sum^2 [8H_{90}+36H_{81}+42H_{72}+159H_{63}+549H_{54}]. \qquad (7)
\end{aligned}$$

For $x=(1,1)'$ and $x=(2,2)'$, the values of H_{ab} are given in Example B.1.

So if $x=(1,1)'$, then $y=(1,1)'/3,\ \tilde{p}_1(x) = -62/81 \approx -.765$, so that our measure of the accuracy of the CLT is $n^{-1/2}\tilde{p}_1(x) = -31/81 \approx -.383$ for $n=4$ and $-62/243 \approx -.255$

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$$\begin{aligned}
&\text{For } x=(1,1)',\ \tilde{p}_{24}=4/81\approx .049,\ \tilde{p}_{26}=11552/3^8\approx 1.761,\\
&\tilde{p}_2(x)=11876/3^8\approx 1.810,\ \tilde{p}_{35}=352/3^5\approx 1.449,\ \tilde{p}_{37}=65139/3^8\approx 9.928,\\
&\tilde{p}_{39}=5530972/3^{12}\approx 10.408,\ \tilde{p}_3(x)=11577055/3^{12}\approx 21.784,\\
&p_{X_n}(x)/\phi_V(x)\approx 1-.765\,n^{-1/2}+1.810\,n^{-1}+21.784\,n^{-3/2}+O(n^{-2})\\
&\approx 1-.191+.113+.340 \text{ if } n=16 \text{ so that only 3 terms can be used.}
\end{aligned}$$

If $x=(2,2)'$, then $y=(2,2)'/3$, $\tilde{p}_1(x)=-64/81\approx -.790$, and our measure of the accuracy of the CLT is $n^{-1/2}\tilde{p}_1(x)=-32/81\approx -.395$ for $n=4$, and $-16/81\approx -.197$ for $n=16$.

$$\begin{aligned}
&\text{For } x=(2,2)',\ \tilde{p}_{24}=-58/81\approx -.716,\ \tilde{p}_{26}=22910/3^8\approx 3.492,\ \tilde{p}_2(x)\approx 2.776,\\
&\tilde{p}_{35}=392/3^5\approx 1.613,\ \tilde{p}_{37}=8000/3^7\approx 3.658,\ \tilde{p}_{39}=2528960/3^{12}\approx 4.759,\\
&\tilde{p}_3(x)=5330264/3^{12}\approx 10.030,\\
&p_{X_n}(x)/\phi_V(x)\approx 1-.790\,n^{-1/2}+2.776\,n^{-1}+10.030\,n^{-3/2}+O(n^{-2})\\
&\approx 1-.198+.174+.157 \text{ if } n=16.
\end{aligned}$$

As in Example 2.1, suppose that the distribution of $\hat{w}$ is symmetric about w. By (31),

$$\begin{aligned}
2\tilde{p}_{22}&=k_2^{11}H_{20}+2k_2^{12}H_{11}+k_2^{22}H_{02},\\
24\tilde{p}_{24}&=k_3^{1111}H_{40}+4k_3^{1112}H_{31}+6k_3^{1122}H_{22}+4k_3^{1222}H_{13}+k_3^{2222}H_{04},
\end{aligned}$$

and $P_{2k}(x)$ needed for (32) is $\tilde{p}_{2k}$ with H_{ab} replaced by H^*_{ab} of (5). Now suppose that $\hat{w}=\hat{W}_1-\hat{W}_2$ where $\hat{W}_1$ and $\hat{W}_2$ are independent copies of a random vector $\hat{W}$. Then the cumulants of $\hat{w}$ of odd order are zero, and the cumulants of even order are twice those of $\hat{W}$.

Consider the case $\hat{W}=\hat{w}$, the bivariate entangled gamma of Example 5.2. So $w=(0,0)'$, the odd cumulants of $\hat{w}$ are zero, and the odd $\tilde{p}_r(x), P_r(x)$ are zero. Denote V,x,y,X,Y of Example 3.2 as V_0,x_0,y_0,X_0,Y_0. Then

$$\begin{aligned}
&V=2V_0,\ X=2^{1/2}X_0,\ Y=2^{-1/2}Y_0,\\
&H_{ab}=2^{-(a+b)/2}H_{ab}(x_0,V_0) \text{ where } x=2^{1/2}x_0,\ y=2^{-1/2}y_0.
\end{aligned}$$

By Example 2.2, $\tilde{p}_{22}=0,\ P_2(40)=k_3^{14}/24,\ P_2(31)=\kappa_{40}/6,$
$P_2(22)=\kappa_{22}/4$, where $\kappa_{40}=12(\gamma_1+\gamma_3),\ \kappa_{31}=\kappa_{22}=12\gamma_3.$

$$\text{So, } \tilde{p}_2(x)=\sum^2[(\gamma_1+\gamma_3)H_{40}+4\gamma_3H_{31}]+6\gamma_3H_{22}.$$

Now consider the exponential case $\gamma_i\equiv 1$. So

$$\begin{aligned}
\tilde{p}_2(x)=\tilde{p}_{24}&=\sum^2[H_{40}+2H_{31}]+2H_{22}\\
&=7/9\approx .778 \text{ if } x=(1,1) \text{ or } -14/9\approx -1.556 \text{ if } x=(2,2).
\end{aligned}$$

So for $n=16$, $n^{-1}\tilde{p}_2(x)\approx .049$ if $x=(1,1)$, or $-.194$ if $x=(2,2)$. Here we used the values of H_{ab} given in the example of Teal (2024). Takeuchi (1978) gave explicit expressions for the Cornish-Fisher expansion when $q=2$.

5. Conclusions

Most estimates of interest are standard estimates, including functions of sample moments, like the sample correlation, and any smooth multivariate function of Fisher's k-statistics: see Stuart and Ord

(1991) for these. In §2 we gave the density and distribution of $X_n = n^{1/2}(\hat{w} - w)$ to $O(n^{-2})$, for $\hat{w}$ any standard estimate, in terms of functions of the cumulants coefficients $\bar{k}_j^{1-r}$ of (5), the coefficients $\bar{P}_r^{1-k}$ of (11)–(14). §3 gave explicit detail when $q = 2$ using the dual notation $P_r(ab)$. §3 gave as examples Edgeworth expansions for the probability of X_n lying in an ellipsoidal or hyperrectangular set.

6. Discussion

A good approximation for the distribution of an estimate, is vital for accurate inference. It enables one to explore the distribution's dependence on underlying parameters. The analytic results given here avoid the need for simulation or jack-knife or bootstrap methods while providing greater accuracy than them. Hall (1992) used the Edgeworth expansion to show that the bootstrap gives accuracy to $O(n^{-1})$. Hall (1988) said that "2nd order correctness usually cannot be bettered". But this is not true using these analytic results. Simulation, while popular, can at best shine a light on behaviour when there is only a small number of parameters.

Estimates based on a sample of independent but not identically distributed random vectors, are also generally standard estimates. For example for a univariate sample mean $\bar{w} = n^{-1}\sum_{j=1}^n X_{jn}$ where X_{jn} has rth cumulant κ_{rjn}, then $\kappa_r(\bar{w}) = n^{1-r}\kappa_r$ where $\kappa_r = n^{-1}\sum_{j=1}^n \kappa_{rjn}$ is the average rth cumulant. For some examples, see Skovgaard (1981a, 1981b) and Withers and Nadarajah (2010, 2020). The last is for a function of a weighted mean of complex random matrices. It gives an application in electrical engineering to channel capacity for multiple arrays.

For conditions for the validity of multivariate Edgeworth expansions, see Skovgaard (1986) and its references.

Cornish and Fisher (1937) and Fisher and Cornish (1960) did not deal with the question of when these expansions diverge. We showed how to confront this in the numerical examples in Example 4.2 and Withers (1984).

Lastly we discuss numerical computation. We have used Teal (2024) for the numerical calculations in Example 4.2. One could also download R-4.4.1 for Windows and google for the routines needed. dmvnorm computes the density function of the multivariate normal specified by mean and co-variance matrix. Use mvtnorm for the multivariate normal. See also qmvnorm for quantiles. rmvnorm generates multivariate normal variables. Googling 'bivariate hermite polynomials r' gives
https://cran.r-project.org/web/packages/calculus/vignettes/hermite.html
One can then use install.packages("calculus"). However we have not used this route.

Some future directions.

1. Hill and Davis (1968) showed how to generalise the expansions of Cornish and Fisher about $N(0,1)$ to expansions about an *arbitrary* continuous distribution. Their results are cumbersome as they involve partition theory. In Withers and Nadarajah (2015) we overcame this using Bell polynomials. It would be straightforward to apply these to expansions about χ_q^2 in Example 3.1, to obtain the percentiles of $n(\hat{w} - w)'V^{-1}(\hat{w} - w)$ and $n(\hat{w} - w)'\hat{V}^{-1}(\hat{w} - w)$. However in the latter case we first need to derive the cumulant coefficients of $\hat{V}^{-1/2}(\hat{w} - w)$ from those of $\hat{w}$. This can be done by applying Withers (2024).
2. It would very useful to obtain the multivariate $g_r(X)$ of (35) explicitly.
3. The multivariate expansions considered here have been about the multivariate normal. However as noted at the end of §1, expansions about other distributions can greatly reduce the number of terms in each $P_r(x)$, $\tilde{p}_r(x)$ and p_{rC} by matching bias and/or skewness. While this has been done for $q = 1$ by Withers and Nadarajah (2011, 2012a, 2014) about Student's distribution , the F-distribution, and the gamma, to date this has yet to be done for multivariate expansions about, for example, a multivariate gamma distribution.
4. The results given here are the first step for constructing confidence intervals and confidence regions. See Withers (1989).
5. The results here can be extended to tilted (saddlepoint) expansions by applying the results of Withers and Nadarajah (2010). These are very useful where convergence fails, that is, where the CLT

cannot be improved upon, typically due to $\hat{w}$ being in a tail. The tilted version of the multivariate distribution and density of a standard estimate are given by Corollaries 3, 4 there. Tilting was 1st used in statistics by Daniels (1954). He gave an approximation to the density of a sample mean. Jensen (1988) gave a univariate extension to S_N where S_n is the sum of i.i.d. observations and N is Poisson. The extension of the present results from $\hat{w}_n = \hat{w}$ to $\hat{w}_N$ would be useful for both univariate and multivariate observations. For a review of references on tilting, see Withers and Nadarajah (2010).
6. Teal (2024) wrote a python program to obtain both analytic and numerical values of multivariate normal moments and multivariate Hermite polynomials when $q = 2$. It would be useful to have these extended to $q = 3$ and 4. (The dual notation for $\bar{\mu}^{1-k}$ and $\bar{H}^{1-k}$ when $q = 3$ or 4 is straightforward.)

Appendix A: The Edgeworth coefficients $\bar{P}_r^{1-k}$ needed for (11)

Here we give for the first time the symmetric form of the coefficients $\bar{P}_r^{1-k}$ needed for (11) for $r \leq 3$, that is, for the Edgeworth expansions (15) to $O(n^{-2})$, using the symmetrising operator $\mathcal{S}$. They are given for $r = 1$ by (12), and for $r = 2, 3$ by (13)–(14) and the following.

$$\bar{P}_2^{1-4} \text{ needs } \mathcal{S}\,\bar{k}_1^1\bar{k}_2^{234} \text{ where } \mathcal{S}\,a^1b^{234} = (a^1b^{234} + a^2b^{341} + a^3b^{412} + a^4b^{123})/4.$$

$$\bar{P}_2^{1-6} \text{ needs } \mathcal{S}\,\bar{k}_2^{1-3}\bar{k}_2^{4-6} \text{ where}$$

$$\mathcal{S}\,a^{1-3}a^{4-6} = \mathcal{S}\,123.4-6 = (a^{1-3}a^{4-6} + a^{124}a^{356} + a^{125}a^{346} + a^{126}a^{346}$$
$$+ a^{134}a^{256} + a^{135}a^{246} + a^{136}a^{245} + a^{145}a^{236} + a^{146}a^{235} + a^{156}a^{234})/10. \quad \text{(A1)}$$

$$\bar{P}_3^{1-3} \text{ needs } \mathcal{S}\,\bar{k}_2^{12}\bar{k}_1^3 \text{ where } \mathcal{S}\,a^{12}b^3 = (a^{12}b^3 + a^{13}b^2 + a^{23}b^1)/3.$$

$$\bar{P}_3^{1-5} \text{ needs } \mathcal{S}\,\bar{k}_3^{1-4}\bar{k}_1^5,\ \mathcal{S}\,\bar{k}_2^{12}\bar{k}_2^{3-5},\ \mathcal{S}\,\bar{k}_2^{123}\bar{k}_1^4\bar{k}_1^5, \quad \text{(A2)}$$

$$\text{where } \mathcal{S}\,a^1b^{2-5} = (a^1b^{2-5} + \cdots + a^5b^{1-4})/5, \quad \text{(A3)}$$

$$\mathcal{S}\,a^{12}b^{345} = (12.345 + 13.245 + 14.235 + 15.234 + 23.145 + 24.135 + 25.134$$
$$+ 34.125 + 35.124 + 45.123)/10 \text{ for } 12.345 = a^{12}b^{345}, \quad \text{(A4)}$$

$$\text{and for } 1.2. = 1.2.345 = a^1a^2b^{345},\ \mathcal{S}\,a^1a^2b^{345}$$
$$= (1.2. + 1.3. + 1.4. + 1.5. + 2.3. + 2.4. + 2.5. + 3.4. + 3.5. + 4.5.)/10. \quad \text{(A5)}$$

$$\bar{P}_3^{1-7} \text{ needs } \mathcal{S}\,\bar{k}_2^{123}\bar{k}_3^{4-7},\ \mathcal{S}\,\bar{k}_2^{123}\bar{k}_2^{456}\bar{k}_1^7, \text{ where} \quad \text{(A6)}$$

$$\text{for } 123. = a^{123}b^{4-7},\ \mathcal{S}\,a^{123}b^{4-7} = (123. + 124. + \cdots + 567.)/\binom{7}{3}, \quad \text{(A7)}$$

$$\mathcal{S}\,a^{123}a^{456}b^7 = \mathcal{S}\,123.456.7 = (b^7\mathcal{S}\,123.456 + \cdots + b^1\mathcal{S}\,234.567)/7 \quad \text{(A8)}$$

$$\text{where } \mathcal{S}\,123.456 = \mathcal{S}\,a^{1-3}a^{4-6} \text{ of (A1)},$$

$$\bar{P}_3^{1-9} = \mathcal{S}\,\bar{k}_2^{1-3}\bar{k}_2^{4-6}\bar{k}_2^{7-9}/6^3, \text{ where for } 123.456.789 = a^{1-3}a^{4-6}a^{7-9}$$

$$\text{and } 123- = 123.[\mathcal{S}\,456.789 \text{ of (A1)}],$$

$$\mathcal{S}\,a^{1-3}a^{4-6}a^{7-9} = [123- + 124- + 125- + 126- + 127- + 128- + 129-$$
$$+ 134- + 135- + 136- + 137- + 138- + 139- + 145- + 146- + 147-$$
$$+ 148- + 149- + 156- + 157- + 158- + 159- + 167- + 168- + 169-$$
$$+ 178- + 179- + 189-]/28.$$

Appendix B: μ_{ab} and H_{ab} of (2) for $a + b \leq 9$.

Here we give the bivariate normal moments μ_{ab} of (1) and the bivariate Hermite polynomials H_{ab} of (2) for $a + b \leq 9$. These are needed for the Edgeworth expansions to $O(n^{-2})$.

The 1st 9 univariate Hermite polynomials are

$$\begin{aligned}
&H_0 = 1,\ H_1 = u,\ H_2 = u^2 - 1,\ H_3 = u^3 - 3u,\ H_4 = u^4 - 6u^2 + 3,\\
&H_5 = u^5 - 10u^3 + 15u, H_6 = u^6 - 15u^4 + 45u^2 - 15,\\
&H_7 = u^7 - 21u^5 + 105u^3 - 105u,\ H_8 = u^8 - 28u^6 + 210u^4 - 420u^2 + 105,\\
&H_9 = u^9 - 36u^7 + 378u^5 - 1260u^3 + 945u. \qquad \text{(A1)}
\end{aligned}$$

These are needed for $h_r(u),\ r = 1,2,3$ of (2), (3), that is, for the univariate Edgeworth expansions to $O(n^{-2})$. See Stuart and Ord (1991).

Let $X \sim \mathcal{N}_q(0, V)$ be a q-variate normal random variable with mean $0 \in R^q$, positive-definite covariance V, with density and distribution $\phi_V(x),\ \Phi_V(x)$ of (7). Set $V^{ab} = E\ Y_a Y_b$, as in (8), so that $V^{-1} = (V^{ab})$. Y has odd moments 0 and even moments

$$\mu^{1-2k} = E\ Y_1 \ldots Y_{2k} = \sum^{2k-1} V^{12}\mu^{3-3k} = \sum^{1.3\ldots(2k-1)} V^{12} \ldots V^{2k-1,2k}. \qquad \text{(A2)}$$

$$\text{For example } \mu^{1-4} = \sum^{3} V^{12}V^{34} = V^{12}V^{34} + V^{13}V^{24} + V^{14}V^{32}, \qquad \text{(A3)}$$

$$\mu^{1-6} = \sum^{5} V^{12}\mu^{3-6} = \sum^{15} V^{12}V^{34}V^{56} = V^{12}V^{34}V^{56} + \cdots + V^{16}V^{25}V^{34}, \qquad \text{(A4)}$$

where $\sum_r^m$ sums over the m permutations of $1, 2, \ldots, r$ giving distinct terms. For example,

$$\sum_{3}^{3} V^{12}y_3 = V^{12}y_3 + V^{13}y_2 + V^{23}y_1.$$

Teal (2024) has written a python program to obtain the bivariate moments using (A2). Let $H^{1-k} = \bar{H}^{1-k}(x, V)$ be the multivariate Hermite polynomial is defined by of (20), (21). For $k \leq 6,\ H^{1-k}$ is given by (8)–(26). In these expressions $1, 2, \ldots, k$ can be replaced by any integers in $1, 2, \ldots, q$. For example

$$H^{11} = y_1^2 - V^{11},\ H^{112} = y_1^2 y_2 - 2V^{12}y_1 - V^{11}y_2.$$

Now consider the bivariate case, $q = 2$ and denote the moments of Y by

$$\mu_{ab} = E\ Y_1^a Y_2^b.$$

Two special cases are

$$\begin{aligned}
&\mu_{2k,0} = 1.3\ldots(2k-1)\mu_{20}^k,\ k \geq 0 \text{ where } \mu_{20} = E\ Y_1^2 = V^{11}, \qquad \text{(A5)}\\
&\mu_{2k-1,1} = (2k-1)\mu_{11}\mu_{2k-2,0} = 1.3\ldots(2k-1)\mu_{20}^{k-1}\mu_{11},\ k \geq 1.
\end{aligned}$$

The μ_{ab} needed here are those up to order $a + b = 8$:

$$\begin{aligned}
&\mu_{20} = EY_1^2 = V^{11},\ \mu_{11} = EY_1Y_2 = V^{12},\ \mu_{40} = 3\mu_{20}^2,\ \mu_{31} = 3\mu_{20}\mu_{11},\\
&\mu_{22} = \mu_{20}\mu_{02} + 2\mu_{11}^2,\ \mu_{60} = 15\mu_{20}^3,\ \mu_{51} = 15\mu_{20}^2\mu_{11},\\
&\mu_{42} = \mu_{40}\mu_{02} + 4\mu_{11}\mu_{31} = 3\mu_{20}(\mu_{20}\mu_{02} + 4\mu_{11}^2),\\
&\mu_{33} = 2\mu_{02}\mu_{31} + 3\mu_{11}\mu_{22} = 3\mu_{11}(3\mu_{20}\mu_{02} + 2\mu_{11}^2),\\
&\mu_{80} = 105\mu_{20}^4,\ \mu_{71} = 105\mu_{20}^3\mu_{11},\ \mu_{62} = \mu_{02}\mu_{60} + 6\mu_{11}\mu_{51} = 15\mu_{20}^2(\mu_{02} + 6\mu_{11}^2),\\
&\mu_{53} = 2\mu_{02}\mu_{51} + 5\mu_{11}\mu_{42} = 15\mu_{20}\mu_{11}(3\mu_{20}\mu_{02} + 4\mu_{11}^2),\\
&\mu_{44} = 3\mu_{02}\mu_{42} + 4\mu_{11}\mu_{33} = 3(3c^2 + 24cd + 8d^2) \text{ for } c = \mu_{20}\mu_{02},\ d = \mu_{11}^2.
\end{aligned}$$

For example to derive μ_{53} from μ^{1-8} of (A2), replace $k \leq 5$ by 1 and $k = 6, 7, 8$ by 2. μ_{ba} can be read off μ_{ab} using symmetry. For example $\mu_{02} = V^{22}$. Formulas for the *general* μ_{ab} were given by Isserlis (1918) and Withers (2025). By (2),

$$H_{ab} = \sum_{j=0}^{a} \binom{a}{j} y_1^{a-j} \sum_{k=0}^{b} \binom{b}{k} y_1^{b-k} i^{j+k} \mu_{jk}. \tag{A6}$$

This is the formula used in Teal (2024) to obtain bivariate Hermite polynomials. H_{ab} is said to be *of order* $a+b$. Most of those of order up to $a+b=9$ are needed to expand the density of $n^{1/2}(\hat{w}-w)$ to $O(n^{-2})$, but not those of order 8, (nor those of order 6 if the distribution of $\hat{w}$ is symmetric about w). But we include them here for completeness.
We can write H_{k0} in terms of μ_{a0} of (A5) and $y = V^{-1}x$:

$$\begin{aligned}
&H_{10} = y_1,\ H_{20} = y_1^2 - \mu_{20},\ H_{30} = y_1^3 - 3y_1\mu_{20},\\
&H_{40} = y_1^4 - 6y_1^2\mu_{20} + \mu_{40},\ H_{50} = y_1^5 - 10y_1^3\mu_{20} + 5y_1\mu_{40},\\
&H_{60} = y_1^6 - 15y_1^4\mu_{20} + 15y_1^2\mu_{40} - \mu_{60},\\
&H_{70} = y_1^7 - 21y_1^5\mu_{20} + 35y_1^3\mu_{40} - 7y_1\mu_{60},\\
&H_{80} = y_1^8 - 28y_1^6\mu_{20} + 70y_1^4\mu_{40} - 28y_1^2\mu_{60} + \mu_{80},\\
&H_{90} = y_1^9 - 36y_1^7\mu_{20} + 126y_1^5\mu_{40} - 84y_1^3\mu_{60} + 9\mu_{80}.
\end{aligned} \tag{A7}$$

These are actually simpler formulas than their univariate forms because of the use of μ_{ab}. The other H_{ab} up to order 9 are

$$
\begin{aligned}
&H_{11} = y_1y_2 - \mu_{11},\ H_{21} = y_2H_{20} - 2y_1\mu_{11},\\
&H_{31} = y_2\ H_{30} - 3y_1^2\mu_{11} + \mu_{31} = y_1^3y_2 - 3y_1y_2\mu_{20} - 3y_1^2\mu_{11} + \mu_{31},\\
&H_{22} = y_2^2H_{20} - y_1^2\mu_{02} - 4y_1y_2\mu_{11} + \mu_{22},\\
&H_{41} = y_2\ H_{40} - 4y_1^3\mu_{11} + 4y_1\mu_{31} = y_1^4y_2 - 4y_1^3\mu_{11} - 6y_1^2y_2\mu_{20} + 4y_1\mu_{31} + y_2\mu_{40},\\
&H_{32} = y_2^2H_{30} - y_1^3\mu_{02} - 6y_1^2y_2\mu_{11} + 3y_1\mu_{22} + 2y_2\mu_{31},\\
&H_{51} = y_2H_{50} - 5y_1^4\mu_{11} + 10y_1^2\mu_{31} - \mu_{51},\\
&H_{42} = y_2^2H_{40} + 8y_1y_2(-y_1^2\mu_{11} + \mu_{31}) - y_1^4\mu_{02} + 6y_1^2\mu_{22} - \mu_{42}\\
&H_{33} = y_2^3H_{30} + 3y_2^2(-3y_1^2\mu_{11} + \mu_{31}) + 3y_1y_2(-y_1^2\mu_{02} + 3\mu_{22}) + 3y_1^2\mu_{13} - \mu_{33}\\
&\qquad = y_1^3y_2^3 - 9y_1^2y_2^2\mu_{11} + 3\sum_{12}^{2}(y_1^2\mu_{13} - y_1^3y_2\mu_{02}) + 9y_1y_2\mu_{22} - \mu_{33},\\
&H_{61} = y_2H_{60} - 6y_1^5\mu_{11} + 20y_1^3\mu_{31} - 6y_1\mu_{51},\\
&H_{52} = y_2^2H_{50} + 2y_2(-5y_1^4\mu_{11} + 10y_1^2\mu_{31} - \mu_{51}) - y_1^5\mu_{02} + 10y_1^3\mu_{22} - 5y_1\mu_{42},\\
&H_{43} = H_{40}y_2^3 + 12y_2^2(-y_1^3\mu_{11} + y_1\mu_{31}) - 3y_2(y_1^4\mu_{02} - 6y_1^2\mu_{22} + \mu_{42})\\
&\ + 4y_1^3\mu_{13} - 4y_1\mu_{33},\\
&H_{71} = y_2H_{70} - 7y_1^6\mu_{11} + 35y_1^4\mu_{31} - 21y_1^2\mu_{51} + \mu_{71},\\
&H_{62} = y_2^2H_{60} + 2y_2\ [(-6y_1^5\mu_{11} + 20y_1^3\mu_{31} - 6y_1\mu_{51}) - y_1^6\mu_{02} + 15y_1^4\mu_{22}\\
&\ - 15y_1^2\mu_{42} + \mu_{62},\\
&H_{53} = y_2^3H_{50} + 15y_2^2\ (-2y_1^4\mu_{11} + 2y_1^2\mu_{31} - \mu_{51}) + 3y_2\ (-y_1^5\mu_{02} + 10y_1^3\mu_{22}\\
&\ - 5y_1\mu_{42}) + 5y_1^4\mu_{13} - 10y_1^2\mu_{33} + 5\mu_{53},\\
&H_{44} = y_2^4H_{40} + 16y_2^3\ (-y_1^3\mu_{11} + y_1\mu_{31}) + 6y_2^2\ (-y_1^4\mu_{02} + 6y_1^2\mu_{22} - \mu_{42})\\
&\ + 16y_2(y_1^3\mu_{13} - y_1^2\mu_{33}) + y_1^4\mu_{04} - 6y_1^2\mu_{24} + \mu_{44},\\
&H_{81} = y_2H_{80} - 8y_1^7\mu_{11} + 56y_1^5\mu_{31} - 28y_1^3\mu_{51} + 8y_1\mu_{71},\\
&H_{72} = y_2^2H_{70} + 2y_2\ (-7y_1^6\mu_{11} + 35y_1^4\mu_{31} - 21y_1^2\mu_{51} + \mu_{71}) - y_1^7\mu_{02} + 21y_1^5\mu_{22}\\
&\ - 35y_1^3\mu_{42} + 7y_1\mu_{62},\\
&H_{63} = y_2^3H_{60} + 3y_2^2\ (-6y_1^5\mu_{11} + 20y_1^3\mu_{31} - 6y_1\mu_{51}) + 3y_2\ (-y_1^6\mu_{02} + 15y_1^4\mu_{22}\\
&\ - 15y_1^2\mu_{42} + \mu_{62}) + 6y_1^5\mu_{13} - 20y_1^3\mu_{33} + 6y_1\mu_{53},\\
&H_{54} = y_2^4H_{50} + 4y_2^3\ (-5y_1^4\mu_{11} + 10y_1^2\mu_{31} - \mu_{51}) + 6y_2^2\ (-y_1^5\mu_{02} + 10y_1^3\mu_{22}\\
&\ - 5y_1\mu_{42}) + 4y_2(5y_1^4\mu_{13} - 10y_1^2\mu_{33} + \mu_{53}) + y_1^5\mu_{04} - 10y_1^3\mu_{24} + 5y_1\mu_{44}.
\end{aligned}
$$

The method of proof is illustrated as follows

$$H_{a1} = E\ (y_1 + iY_1)^a\ (y_2 + iY_2) = y_2\ E\ (y_1 + iY_1)^a + E\ (y_1 + iY_1)^a\ iY_2.$$

Now expand the 2nd term to get

$$H_{a1} = y_2H_{a0} + \sum[(-1)^k \binom{a}{2k-1} y_1^{a-2k+1}\mu_{2k-1,1} :\ 1 \le k \le (a+1)/2].$$

Similarly for H_{a2}, expand

$$H_{a2} = E\ (y_1 + iY_1)^a\ (y_2^2 + 2iy_2Y_2 - Y_2^2).$$

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The above formulas for H_{ab} can be called their *short form* as each uses H_{a0}. The explicit form when H_{a0} of (A7) is substituted can be called its *long form*. Our short forms for H_{ab} were checked against the long forms given by Teal (2024). Here is a selection of his results for comparison.

$$
\begin{aligned}
H_{51} =& -15\mu_{11}\mu_{20}^2 + 30\mu_{11}\mu_{20}y_1^2 - 5\mu_{11}y_1^4 + 15\mu_{20}^2 y_1 y_2 - 10\mu_{20}y_1^3 y_2 + y_1^5 y_2 \\
H_{42} =& -3\mu_{02}\mu_{20}^2 + 6\mu_{02}\mu_{20}y_1^2 - \mu_{02}y_1^4 - 12\mu_{11}^2\mu_{20} + 12\mu_{11}^2 y_1^2 + 24\mu_{11}\mu_{20}y_1 y_2 \\
& - 8\mu_{11}y_1^3 y_2 + 3\mu_{20}^2 y_2^2 - 6\mu_{20}y_1^2 y_2^2 + y_1^4 y_2^2 \\
H_{33} =& -9\mu_{02}\mu_{11}\mu_{20} + 9\mu_{02}\mu_{11}y_1^2 + 9\mu_{02}\mu_{20}y_1 y_2 - 3\mu_{02}y_1^3 y_2 - 6\mu_{11}^3 + 18\mu_{11}^2 y_1 y_2 \\
& + 9\mu_{11}\mu_{20}y_2^2 - 9\mu_{11}y_1^2 y_2^2 - 3\mu_{20}y_1 y_2^3 + y_1^3 y_2^3 \\
H_{70} =& -105\mu_{20}^3 y_1 + 105\mu_{20}^2 y_1^3 - 21\mu_{20}y_1^5 + y_1^7 \\
H_{61} =& -90\mu_{11}\mu_{20}^2 y_1 + 60\mu_{11}\mu_{20}y_1^3 - 6\mu_{11}y_1^5 - 15\mu_{20}^3 y_2 + 45\mu_{20}^2 y_1^2 y_2 - 15\mu_{20}y_1^4 y_2 \\
& + y_1^6 y_2 \\
H_{52} =& -15\mu_{02}\mu_{20}^2 y_1 + 10\mu_{02}\mu_{20}y_1^3 - \mu_{02}y_1^5 - 60\mu_{11}^2\mu_{20}y_1 + 20\mu_{11}^2 y_1^3 - 30\mu_{11}\mu_{20}^2 y_2 \\
& + 60\mu_{11}\mu_{20}y_1^2 y_2 - 10\mu_{11}y_1^4 y_2 + 15\mu_{20}^2 y_1 y_2^2 - 10\mu_{20}y_1^3 y_2^2 + y_1^5 y_2^2 \\
H_{43} =& -36\mu_{02}\mu_{11}\mu_{20}y_1 + 12\mu_{02}\mu_{11}y_1^3 - 9\mu_{02}\mu_{20}^2 y_2 + 18\mu_{02}\mu_{20}y_1^2 y_2 - 3\mu_{02}y_1^4 y_2 \\
& - 24\mu_{11}^3 y_1 - 36\mu_{11}^2\mu_{20}y_2 + 36\mu_{11}^2 y_1^2 y_2 + 36\mu_{11}\mu_{20}y_1 y_2^2 - 12\mu_{11}y_1^3 y_2^2 \\
& + 3\mu_{20}^2 y_2^3 - 6\mu_{20}y_1^2 y_2^3 + y_1^4 y_2^3 \\
H_{80} =& 105\mu_{20}^4 - 420\mu_{20}^3 y_1^2 + 210\mu_{20}^2 y_1^4 - 28\mu_{20}y_1^6 + y_1^8
\end{aligned}
$$

$$
\begin{aligned}
H_{54} =& 45\mu_{02}^2\mu_{20}^2 y_1 - 30\mu_{02}^2\mu_{20}y_1^3 + 3\mu_{02}^2 y_1^5 + 360\mu_{02}\mu_{11}^2\mu_{20}y_1 - 120\mu_{02}\mu_{11}^2 y_1^3 \\
& + 180\mu_{02}\mu_{11}\mu_{20}^2 y_2 - 360\mu_{02}\mu_{11}\mu_{20}y_1^2 y_2 + 60\mu_{02}\mu_{11}y_1^4 y_2 - 90\mu_{02}\mu_{20}^2 y_1 y_2^2 \\
& + 60\mu_{02}\mu_{20}y_1^3 y_2^2 - 6\mu_{02}y_1^5 y_2^2 + 120\mu_{11}^4 y_1 + 240\mu_{11}^3\mu_{20}y_2 - 240\mu_{11}^3 y_1^2 y_2 \\
& - 360\mu_{11}^2\mu_{20}y_1 y_2^2 + 120\mu_{11}^2 y_1^3 y_2^2 - 60\mu_{11}\mu_{20}^2 y_2^3 + 120\mu_{11}\mu_{20}y_1^2 y_2^3 \\
& - 20\mu_{11}y_1^4 y_2^3 + 15\mu_{20}^2 y_1 y_2^4 - 10\mu_{20}y_1^3 y_2^4 + y_1^5 y_2^4
\end{aligned}
$$

For example the short and long forms for H_{33} have 5 and 10 terms, and those for H_{43} have 8 and 13 terms.

Take $V,\ V^{-1}$ of (19). Then

$$
\begin{aligned}
&\mu_{20} = 2/3,\ \mu_{11} = -1/3,\ \mu_{40} = 4/3,\ \mu_{31} = -2/3,\ \mu_{22} = 2/3, \\
&\mu_{60} = 40/9,\ \mu_{51} = -20/9,\ \mu_{42} = 16/9,\ \mu_{33} = -14/9, \\
&\mu_{80} = 560/27,\ \mu_{71} = -280/27,\ \mu_{62} = 200/27,\ \mu_{53} = -160/27,\ \mu_{44} = 152/27.
\end{aligned}
$$

So if $x = (1,1)'$ then $y = (1,1)'/3$,

$$\begin{aligned}
&H_{10} = 1/3 \approx 0.3333,\ H_{20} = -5/9 \approx -0.5556,\ H_{11} = 4/9 \approx 0.4444,\\
&H_{30} = -17/27 \approx -0.6296,\ H_{21} = 1/27 \approx 0.0370,\ H_{40} = 73/81 \approx 0.9012,\\
&H_{31} = -62/81 \approx -0.7654,\ H_{22} = 55/81 \approx 0.6790,\\
&H_{60} = -1709/9^3 \approx -2.344,\ H_{51} = 1576/9^3 \approx 2.1619,\\
&H_{42} = -1361/9^3 \approx -1.8669,\ H_{33} = 1216/9^3 \approx 1.6680,\\
&H_{50} = 481/243 \approx 1.9794,\ H_{41} = -131/243 \approx -0.5391,\ H_{32} = 49/243 \approx .2016,\\
&H_{70} = -19{,}025/3^7 \approx -8.6991,\ H_{61} = 6949/3^7 \approx 3.1774,\\
&H_{52} = 3275/2187 \approx -1.4975,\ H_{43} = 847/2187 \approx 0.3873,\\
&H_{90} = 965953/19683 \approx 49.0755,\ H_{81} = 403847/19683 \approx -20.5176\\
&H_{72} = 205165/19683 \approx 10.4235,\ H_{63} = -96{,}767/3^9 \approx -4.91627,\\
&H_{54} = 29773/19683 \approx 1.5126.
\end{aligned}$$

If $x = (2,2)'$, then $y = (2,2)'/3$,

$$\begin{aligned}
&H_{10} = 2/3 \approx 0.6667,\ H_{20} = -2/9 \approx -0.2222,\ H_{11} = 7/9 \approx 0.7778,\\
&H_{30} = -28/27 \approx -1.0370,\ H_{21} = 8/27 \approx 0.2963,\ H_{40} = -20/81 \approx -0.2469,
\end{aligned}$$

$$\begin{aligned}
&H_{31} = -74/81 \approx -0.9136,\ H_{22} = 70/81 \approx 0.8642,\ H_{60} = 1864/9^3 \approx 2.5569,\\
&H_{51} = 964/9^3 \approx 1.3224,\ H_{42} = -1520/729 \approx -2.0850,\\
&H_{33} = 1702/9^3 \approx 2.3347,\ H_{50} = 632/243 \approx 2.6008,\\
&H_{41} = -376/243 \approx -1.5473,\ H_{32} = 92/243 \approx 0.3786,\\
&H_{70} = -19024/2187 \approx -8.6987,\ H_{61} = 15104/2187 \approx 6.9063,\\
&H_{52} = 7504/2187 \approx -3.4312,\ H_{43} = 2576/2187 \approx 1.1779,\\
&H_{90} = 680480/19683 \approx 34.5720,\ H_{81} = -689248/19683 \approx -35.0174,\\
&H_{72} = 433520/19683 \approx 22.0251,\ H_{63} = -243568/19683 \approx -12.3745,\\
&H_{54} = 74960/19683 \approx 3.8084.
\end{aligned}$$

These results were computed by Teal (2024), using (A6), and are used in Example 4.2.

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$$
\begin{aligned}
H_{10} =& y_1 \\
H_{20} =& -\mu_{20} + y_1^2 \\
H_{11} =& -\mu_{11} + y_1 y_2 \\
H_{30} =& -3\mu_{20} y_1 + y_1^3 \\
H_{21} =& -2\mu_{11} y_1 - \mu_{20} y_2 + y_1^2 y_2 \\
H_{40} =& -6\mu_{20} y_1^2 + \mu_{40} + y_1^4 \\
H_{31} =& -3\mu_{11} y_1^2 - 3\mu_{20} y_1 y_2 + \mu_{31} + y_1^3 y_2 \\
H_{22} =& -\mu_{02} y_1^2 - 4\mu_{11} y_1 y_2 - \mu_{20} y_2^2 + \mu_{22} + y_1^2 y_2^2 \\
H_{50} =& -10\mu_{20} y_1^3 + 5\mu_{40} y_1 + y_1^5 \\
H_{41} =& -4\mu_{11} y_1^3 - 6\mu_{20} y_1^2 y_2 + 4\mu_{31} y_1 + \mu_{40} y_2 + y_1^4 y_2 \\
H_{32} =& -\mu_{02} y_1^3 - 6\mu_{11} y_1^2 y_2 - 3\mu_{20} y_1 y_2^2 + 3\mu_{22} y_1 + 2\mu_{31} y_2 + y_1^3 y_2^2 \\
H_{60} =& -15\mu_{20} y_1^4 + 15\mu_{40} y_1^2 - \mu_{60} + y_1^6 \\
H_{51} =& -5\mu_{11} y_1^4 - 10\mu_{20} y_1^3 y_2 + 10\mu_{31} y_1^2 + 5\mu_{40} y_1 y_2 - \mu_{51} + y_1^5 y_2 \\
H_{42} =& -\mu_{02} y_1^4 - 8\mu_{11} y_1^3 y_2 - 6\mu_{20} y_1^2 y_2^2 + 6\mu_{22} y_1^2 + 8\mu_{31} y_1 y_2 + \mu_{40} y_2^2 \\
& -\mu_{42} + y_1^4 y_2^2 \\
H_{33} =& -3\mu_{02} y_1^3 y_2 - 9\mu_{11} y_1^2 y_2^2 + 3\mu_{13} y_1^2 - 3\mu_{20} y_1 y_2^3 + 9\mu_{22} y_1 y_2 + 3\mu_{31} y_2^2 \\
& -\mu_{33} + y_1^3 y_2^3 \\
H_{70} =& -21\mu_{20} y_1^5 + 35\mu_{40} y_1^3 - 7\mu_{60} y_1 + y_1^7 \\
H_{61} =& -6\mu_{11} y_1^5 - 15\mu_{20} y_1^4 y_2 + 20\mu_{31} y_1^3 + 15\mu_{40} y_1^2 y_2 - 6\mu_{51} y_1 - \mu_{60} y_2 + y_1^6 y_2 \\
H_{52} =& -\mu_{02} y_1^5 - 10\mu_{11} y_1^4 y_2 - 10\mu_{20} y_1^3 y_2^2 + 10\mu_{22} y_1^3 + 20\mu_{31} y_1^2 y_2 + 5\mu_{40} y_1 y_2^2 \\
& -5\mu_{42} y_1 - 2\mu_{51} y_2 + y_1^5 y_2^2 \\
H_{43} =& -3\mu_{02} y_1^4 y_2 - 12\mu_{11} y_1^3 y_2^2 + 4\mu_{13} y_1^3 - 6\mu_{20} y_1^2 y_2^3 + 18\mu_{22} y_1^2 y_2 + 12\mu_{31} y_1 y_2^2 \\
& -4\mu_{33} y_1 + \mu_{40} y_2^3 - 3\mu_{42} y_2 + y_1^4 y_2^3 \\
H_{80} =& -28\mu_{20} y_1^6 + 70\mu_{40} y_1^4 - 28\mu_{60} y_1^2 + \mu_{80} + y_1^8 \\
H_{71} =& -7\mu_{11} y_1^6 - 21\mu_{20} y_1^5 y_2 + 35\mu_{31} y_1^4 + 35\mu_{40} y_1^3 y_2 - 21\mu_{51} y_1^2 - 7\mu_{60} y_1 y_2 \\
& +\mu_{71} + y_1^7 y_2 \\
H_{62} =& -\mu_{02} y_1^6 - 12\mu_{11} y_1^5 y_2 - 15\mu_{20} y_1^4 y_2^2 + 15\mu_{22} y_1^4 + 40\mu_{31} y_1^3 y_2 + 15\mu_{40} y_1^2 y_2^2 \\
& -15\mu_{42} y_1^2 - 12\mu_{51} y_1 y_2 - \mu_{60} y_2^2 + \mu_{62} + y_1^6 y_2^2 \\
H_{53} =& -3\mu_{02} y_1^5 y_2 - 15\mu_{11} y_1^4 y_2^2 + 5\mu_{13} y_1^4 - 10\mu_{20} y_1^3 y_2^3 + 30\mu_{22} y_1^3 y_2 \\
& +30\mu_{31} y_1^2 y_2^2 - 10\mu_{33} y_1^2 + 5\mu_{40} y_1 y_2^3 - 15\mu_{42} y_1 y_2 - 3\mu_{51} y_2^2 + \mu_{53} + y_1^5 y_2^3 \\
H_{44} =& -6\mu_{02} y_1^4 y_2^2 + \mu_{04} y_1^4 - 16\mu_{11} y_1^3 y_2^3 + 16\mu_{13} y_1^3 y_2 - 6\mu_{20} y_1^2 y_2^4 \\
& +36\mu_{22} y_1^2 y_2^2 - 6\mu_{24} y_1^2 + 16\mu_{31} y_1 y_2^3 - 16\mu_{33} y_1 y_2 + \mu_{40} y_2^4 - 6\mu_{42} y_2^2 \\
& +\mu_{44} + y_1^4 y_2^4
\end{aligned}
$$

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$$
\begin{aligned}
H_{90} =& -36\mu_{20}y_1^7 + 126\mu_{40}y_1^5 - 84\mu_{60}y_1^3 + 9\mu_{80}y_1 + y_1^9 \\
H_{81} =& -8\mu_{11}y_1^7 - 28\mu_{20}y_1^6y_2 + 56\mu_{31}y_1^5 + 70\mu_{40}y_1^4y_2 - 56\mu_{51}y_1^3 - 28\mu_{60}y_1^2y_2 \\
&+ 8\mu_{71}y_1 + \mu_{80}y_2 + y_1^8y_2 \\
H_{72} =& -\mu_{02}y_1^7 - 14\mu_{11}y_1^6y_2 - 21\mu_{20}y_1^5y_2^2 + 21\mu_{22}y_1^5 + 70\mu_{31}y_1^4y_2 + 35\mu_{40}y_1^3y_2^2 \\
&- 35\mu_{42}y_1^3 - 42\mu_{51}y_1^2y_2 - 7\mu_{60}y_1y_2^2 + 7\mu_{62}y_1 + 2\mu_{71}y_2 + y_1^7y_2^2 \\
H_{63} =& -3\mu_{02}y_1^6y_2 - 18\mu_{11}y_1^5y_2^2 + 6\mu_{13}y_1^5 - 15\mu_{20}y_1^4y_2^3 + 45\mu_{22}y_1^4y_2 \\
&+ 60\mu_{31}y_1^3y_2^2 - 20\mu_{33}y_1^3 + 15\mu_{40}y_1^2y_2^3 - 45\mu_{42}y_1^2y_2 - 18\mu_{51}y_1y_2^2 \\
&+ 6\mu_{53}y_1 - \mu_{60}y_2^3 + 3\mu_{62}y_2 + y_1^6y_2^3 \\
H_{54} =& -6\mu_{02}y_1^5y_2^2 + \mu_{04}y_1^5 - 20\mu_{11}y_1^4y_2^3 + 20\mu_{13}y_1^4y_2 - 10\mu_{20}y_1^3y_2^4 \\
&+ 60\mu_{22}y_1^3y_2^2 - 10\mu_{24}y_1^3 + 40\mu_{31}y_1^2y_2^3 - 40\mu_{33}y_1^2y_2 + 5\mu_{40}y_1y_2^4 \\
&- 30\mu_{42}y_1y_2^2 + 5\mu_{44}y_1 - 4\mu_{51}y_2^3 + 4\mu_{53}y_2 + y_1^5y_2^4
\end{aligned}
$$

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$$
\begin{aligned}
H_{10} &= \frac{1}{3} \approx 0.3333 \\
H_{20} &= -\frac{5}{9} \approx -0.5556 \\
H_{11} &= \frac{4}{9} \approx 0.4444 \\
H_{30} &= -\frac{17}{27} \approx -0.6296 \\
H_{21} &= \frac{1}{27} \approx 0.0370 \\
H_{40} &= \frac{73}{81} \approx 0.9012 \\
H_{31} &= -\frac{62}{81} \approx -0.7654 \\
H_{22} &= \frac{55}{81} \approx 0.6790 \\
H_{50} &= \frac{481}{243} \approx 1.9794 \\
H_{41} &= -\frac{131}{243} \approx -0.5391 \\
H_{32} &= \frac{49}{243} \approx 0.2016 \\
H_{60} &= -\frac{1709}{729} \approx -2.3443 \\
H_{51} &= \frac{1576}{729} \approx 2.1619 \\
H_{42} &= -\frac{1313}{729} \approx -1.8011 \\
H_{33} &= \frac{1288}{729} \approx 1.7668 \\
H_{70} &= -\frac{19025}{2187} \approx -8.6991 \\
H_{61} &= \frac{6949}{2187} \approx 3.1774 \\
H_{52} &= -\frac{3275}{2187} \approx -1.4975 \\
H_{43} &= \frac{847}{2187} \approx 0.3873 \\
H_{80} &= \frac{52753}{6561} \approx 8.0404 \\
H_{71} &= -\frac{54914}{6561} \approx -8.3698 \\
H_{62} &= \frac{45571}{6561} \approx 6.9457 \\
H_{53} &= -\frac{41882}{6561} \approx -6.3835 \\
H_{44} &= \frac{39937}{6561} \approx 6.0870 \\
H_{90} &= \frac{965953}{19683} \approx 49.0755 \\
H_{81} &= -\frac{403847}{19683} \approx -20.5176 \\
H_{72} &= \frac{205165}{19683} \approx 10.4235 \\
H_{63} &= -\frac{96767}{19683} \approx -4.9163 \\
H_{54} &= \frac{29773}{19683} \approx 1.5126
\end{aligned}
$$

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$$H_{10} = \frac{2}{3} \approx 0.6667$$
$$H_{20} = -\frac{2}{9} \approx -0.2222$$
$$H_{11} = \frac{7}{9} \approx 0.7778$$
$$H_{30} = -\frac{28}{27} \approx -1.0370$$
$$H_{21} = \frac{8}{27} \approx 0.2963$$
$$H_{40} = -\frac{20}{81} \approx -0.2469$$
$$H_{31} = -\frac{74}{81} \approx -0.9136$$
$$H_{22} = \frac{70}{81} \approx 0.8642$$
$$H_{50} = \frac{632}{243} \approx 2.6008$$
$$H_{41} = -\frac{376}{243} \approx -1.5473$$
$$H_{32} = \frac{92}{243} \approx 0.3786$$
$$H_{60} = \frac{1864}{729} \approx 2.5569$$
$$H_{51} = \frac{964}{729} \approx 1.3224$$
$$H_{42} = -\frac{1520}{729} \approx -2.0850$$
$$H_{33} = \frac{1702}{729} \approx 2.3347$$
$$H_{70} = -\frac{19024}{2187} \approx -8.6987$$
$$H_{61} = \frac{15104}{2187} \approx 6.9063$$
$$H_{52} = -\frac{7504}{2187} \approx -3.4312$$
$$H_{43} = \frac{2576}{2187} \approx 1.1779$$
$$H_{80} = -\frac{116336}{6561} \approx -17.7314$$
$$H_{71} = \frac{1096}{6561} \approx 0.1670$$
$$H_{62} = \frac{36376}{6561} \approx 5.5443$$
$$H_{53} = -\frac{49376}{6561} \approx -7.5257$$
$$H_{44} = \frac{52936}{6561} \approx 8.0683$$

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$$
\begin{aligned}
H_{90} &= \frac{680480}{19683} \approx 34.5720 \\
H_{81} &= -\frac{689248}{19683} \approx -35.0174 \\
H_{72} &= \frac{433520}{19683} \approx 22.0251 \\
H_{63} &= -\frac{243568}{19683} \approx -12.3745 \\
H_{54} &= \frac{74960}{19683} \approx 3.8084
\end{aligned}
$$

Appendix I Code for Bivariate Normal Moments and Bivariate Hermite Polynomals

2.12.25 ex Paul: Here is a latex file which has all the moment evaluations and Hermite polynomial evaluations corrected.

The code to generate it is publicly available at https://github.com/paultnz/bihermite/blob/main/hermite8.py so you can refer to it in the paper and hopefully that will satisfy your reviewers. It can be applied to any values of the subscripts and any values of mu02, mu20, mu11, y1 and y2.

```
#!/usr/bin/env python3
#
# Paul Teal - pault@nmr.nz
# Thursday 7 November, 2024
# Monday 2 December, 2024


from itertools import product
from math import comb
from fractions import Fraction as fr
import sympy as sy
import re

def split_string(str,maxlen=100):
  # This is just for limiting the maximum length of latex display lines
  sections = re.split(r'(?=[+-])',str)
  result = []
  current = ''
  for ss in sections:
    current += ss
    if len(current) >= maxlen:
      result.append(current)
      current = ''
  if current:
    result.append(current)
  return result


def mup(Elist):
  # recursive function for evaluating moments
  if Elist == []:
```

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```
    return 1
  if Elist == [1,1]:
    return mu20
  if Elist == [2,2]:
    return mu02
  if Elist == [1,2]:
    return mu11
  if Elist == [2,1]:
    return mu11
  Rout = 0
  for ii in range(len(Elist)-1):
    left = mup([ Elist[ii], Elist[-1] ])
    right = mup(Elist[:ii] + Elist[ii+1:-1])
    Rout += left * right
  return Rout

def sub2(sb1,sb2):
  return mup([1] * sb1 + [2] * sb2)


def biHermite(n, m, y1=0, y2=0, mu=0):
  # Generate terms for (y1 + j*Y1)^n
  terms1 = []
  for k in range(n + 1):
    terms1.append((comb(n, k) * (1j ** k), n-k, k))

  # Generate terms for (y2 + j*Y2)^m
  terms2 = []
  for k in range(m + 1):
    terms2.append((comb(m, k) * (1j ** k), m-k, k))
  # Combine terms from both expansions
  terms = 0
  for (coef1, a_pow, b_pow), (coef2, c_pow, d_pow) in product(terms1, terms2):
    coef_all = coef1 * coef2
    if coef_all.imag == 0: # ignore imaginary
      coef_all = int(coef_all.real)
      combined = coef_all * y1**a_pow * y2**c_pow * sub2(b_pow,d_pow)
      terms += combined
  return terms

if __name__ == "__main__":
  # Example usage: the first iteration is purely symbolic, then the
  # second and third use some specific values
  for loop in range(3):
    if loop==0:
      mu20, mu02, mu11, y1, y2= sy.symbols('mu20  mu02  mu11  y1  y2')
    elif loop==1:
      #override symbols with specific values
      mu20 = fr(2,3); mu02 = fr(2,3); mu11 = -fr(1,3); y1 = fr(1,3); y2 = fr(1,3)
    elif loop==2:
```

```
  y1 = fr(2,3); y2 = fr(2,3)

if loop<2:
  # Print values of mu
  print('\\begin{align*}')
  for mn in range(4,9+1,2):
    for m in range( (mn+2)//2):
      n = mn-m
      l2 = sub2(m,n)
      print(f'\\mu_{{{m}{n}}} =& ' + sy.latex(sy.simplify(l2)) + '\\\\')
  print('\\end{align*}')
  print('%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%')

print('\\begin{align*}')
for nm in range(1,10):
  for m in range(0,(nm+2)//2):
    n = nm - m
    result = sy.expand(biHermite(n, m, y1, y2),mul=True)
    la = sy.latex(result)
    print(f'H_{{{n}{m}}} =&',sep='',end='')
    if not result.free_symbols:
      print(f'{la} \\approx {result.evalf():.4f}\\\\')
    else:
      lines = split_string(la,150)
      for line in lines[:-1]:
        print(line,'\\\\& ')
      print(lines[-1],'\\\\')
print('\\end{align*}')
print('%%%%%%%%%%%%%%%%%%%%%%%
```

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31. Withers, C.S. and Nadarajah, S.N. (2012a) Improved confidence regions based on Edgeworth expansions. Computational Statistics and Data Analysis, 56 (12), 4366–4380. DOI:ï¿¡10.1016/j.csda.2012.03.019 Corrigendum: p4369, lines 8–11 should read: $He_1(r) = r$ so $He_{j_1}(y,V) = z_{j_1}$, $\tilde{He}_{j_1}(y,V) = y_{j_1}$, $He_2(r) = r^2 - 1$ so $He_{j_1 j_2}(y,V) = z_{j_1} z_{j_2} - V^{j_1 j_2}$, $\tilde{He}_{j_1 j_2}(y,V) = y_{j_1} y_{j_2} - V_{j_1 j_2}$, $He_3(r) = r^3 - 3r$ so $He_{j_1 j_2 j_3}(y,V) = z_{j_1} z_{j_2} z_{j_3} - \sum^3 z_{j_1} V^{j_2 j_3}$, $\tilde{He}_{j_1 j_2 j_3}(y,V) = y_{j_1} y_{j_2} y_{j_3} - \sum^3 y_{j_1} V_{j_2 j_3}$, $He_4(r) = r^4 - 6r^2 + 3$ so $He_{j_1 \ldots j_4}(y) = z_{j_1} \ldots z_{j_4} - \sum^6 z_{j_1} z_{j_2} V^{j_3 j_4} + \sum^3 V^{j_1 j_2} V^{j_3 j_4}$, $\tilde{He}_{j_1 \ldots j_4}(y) = y_{j_1} \ldots y_{j_4} - \sum^6 y_{j_1} y_{j_2} V_{j_3 j_4} + \sum^3 V_{j_1 j_2} V_{j_3 j_4}$, where $\tilde{He}_{j_1 \cdots j_r}(y,V) =$ $He_{j_1 \cdots j_r}(V^{-1}y, V^{-1})$ is the dual Hermite polynomial: see Withers (2000). p4370. In Theorem 2.2, $d(j_1, \cdots, d_{2i}) = E\, \bar{Y}_{j_1} \cdots \bar{Y}_{j_{2i}}$ where $\bar{Y} \sim \mathcal{N}_p(0, V^{-1})$.

p4379. Replace the 3rd line after (A.3) by
$K(t, Y_n) = \sum_{i=0}^{\infty} n^{-i} \sum_{r=1}^{i+1} n^{r/2} k_{r,i}(t)/r! - s'w = \sum_{k=1}^{\infty} \nabla_k(t) n^{-k/2}/k!$,
In the 2nd equation after (A.3), replace $\sum_{r=1}^{\infty}$ by $\sum_{r=1}^{i+2}$. For example $\nabla_1 = k_{11}(t) + k_{32}(t)/3!$, $\nabla_2/2! = k_{22}(t)/2! + k_{43}(t)/4!$, $\nabla_3/3! = k_{12}(t) + k_{33}(t)/3! + k_{54}(t)/5!$. The next equation should read $Q_n(t) = exp\{\sum_{k=1}^{\infty} \nabla_k(t) n^{-k/2}/k! - t'Vt/2\} = \sum_{r=0}^{\infty} b_r(t) n^{-r/2}\ exp\{-t'Vt/2\}/r!$, where $B_r(\nabla)$ is the complete exponential Bell polynomial. So for $r \geq 1$,
$B_r(\nabla) = \sum_{k=1}^{r} B_{rk}(\nabla)$ where $B_{rk}(\nabla)$ is the partial exponential Bell polynomial tabled on p307–308 of Comtet (1974) for $1 \leq r \leq 12$.

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