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Article

Homoclinic Orbits and Chaos in the Un-Perturbed System for Quintic Duffing Equation

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Abstract: In this paper we have investigated the driven quintic Duffing equation. Using some analysis technique, we are able to predict the number of limit cycles around the equilibrium and to develop a theoretical approach to chaos transition in damped driven systems.

Keywords: Homoclinic orbits; chaos theory; duffing equation; Hamiltonian

AMS (MOS) Subject Classifications: 82C80, 91B62, 92C55, 11xx

1. Introduction

The Duffing equation is an example of a dynamical system that exhibits chaotic behaviour [1]. The equation is given by :

$$\ddot{x} + \delta \dot{x} + \rho x + \mu x^3 = \lambda \cos(\omega t), \quad (1)$$

where the (unknown) function $x = x(t)$ is the displacement at time t . The damping factor δ controls the size of the damping, the ρ controls the size of the stiffness and the μ controls the amount of nonlinearity in the restoring force. If $\mu = 0$, the Duffing equation describes a damped and driven simple harmonic oscillator. The quantity λ controls the amplitude of the periodic driving force. If $\lambda = 0$, we have a system without driving force. The quantity ω controls the frequency of the periodic driving force.[6] In this paper the special case as the modified formula of (1) which is called cubic-quintic Duffing equation [2] is considered :

$$\ddot{x} - ax + bx^3 + cx^5 = \varepsilon(\gamma \cos \omega t - \delta \dot{x}). \quad (2)$$

2. Homoclinic Orbits in the Unperturbed System

For the unperturbed system with fractional order displacement, when $\varepsilon = 0$, the differential Equation (2) simplifies to

$$\ddot{x} - ax + bx^3 + cx^5 = 0. \quad (3)$$

Let

$$\Delta := b^2 + 4ac. \quad (4)$$

Equilibrium points for $\Delta > 0$ are :

$$(x, \dot{x}) = \left(\pm \sqrt{\frac{-b + \sqrt{b^2 + 4ac}}{2c}}, 0 \right) : \text{centers} \quad (5)$$

Define

$$x_e^+ = \sqrt{\frac{-b + \sqrt{b^2 + 4ac}}{2c}} \text{ and } x_e^- = -\sqrt{\frac{-b + \sqrt{b^2 + 4ac}}{2c}} \quad (6)$$

The energy function for (3) is

$$\frac{1}{2} \dot{x}(t)^2 - \frac{1}{2} ax(t)^2 + \frac{1}{4} bx(t)^4 + \frac{1}{6} cx(t)^6 = K \quad (7)$$

where K is the energy constant dependent on the initial amplitude $x(0) = x_0$ and initial velocity $x'(0) = \dot{x}_0$:

$$K = \frac{1}{2}\dot{x}_0^2 - \frac{1}{2}ax_0^2 + \frac{1}{4}bx_0^4 + \frac{1}{6}cx_0^6. \quad (8)$$

Dependently on K , the level sets are different. For all of them it is common that they form closed periodic orbits which surround the fixed points $(x, \dot{x}) = (x_e^+, 0)$ or $(x, \dot{x}) = (x_e^-, 0)$ or all the three fixed points $(x_e^\pm, 0)$ and $(0, 0)$. The boundary between these two groups of orbits corresponds to $K = 0$, when

$$\dot{x}_0 = \pm x_0 \sqrt{\frac{1}{6}(6a - 3bx_0^2 - 2cx_0^4)}. \quad (9)$$

The level set

$$\frac{\dot{x}^2}{2} - \frac{1}{2}ax^2 + \frac{1}{4}bx^4 + \frac{1}{6}cx^6 = 0 \quad (10)$$

is composed of two homoclinic orbits

$$\Gamma_+^0(t) \equiv (x_+^0(t), \dot{x}_+^0(t)), \quad (11)$$

$$\Gamma_-^0(t) \equiv (x_-^0(t), \dot{x}_-^0(t)), \quad (12)$$

which connect the fixed hyperbolic saddle point $(0, 0)$ to itself and contain the stable and unstable manifolds. The functions $x_\pm^0(t)$ may be evaluated using formulas of the two following homoclinic orbits:

$$x(t) = x_0 \frac{\sqrt{1+\lambda} \tanh(\sqrt{k}t)}{\sqrt{1+\lambda \tanh^2(\sqrt{k}t)}} = \frac{A \tanh(\sqrt{k}t)}{\sqrt{1+\lambda \tanh^2(\sqrt{k}t)}}, \quad A = x_0 \sqrt{1+\lambda},$$

for the choices

$$k = \frac{1}{2}(-bx_0^2 - 2cx_0^4), \quad \lambda = -\frac{2cx_0^2}{3(b+2cx_0^2)}, \quad x_0 = \sqrt{\frac{-b \pm \sqrt{b^2+4ac}}{2c}},$$

$$b^2 + 4ac > 0. \quad (13)$$

and

$$x(t) = x_0 \frac{\sqrt{1+\lambda} \operatorname{sech}(\sqrt{k}t)}{\sqrt{1+\lambda \operatorname{sech}^2(\sqrt{k}t)}} = \frac{A \operatorname{sech}(\sqrt{k}t)}{\sqrt{1+\lambda \operatorname{sech}^2(\sqrt{k}t)}}, \quad A = x_0 \sqrt{1+\lambda},$$

for the choices

$$k = a, \quad \lambda = \frac{bx_0^2 - 2a}{4a - bx_0^2}, \quad x_0 = \frac{1}{2} \sqrt{\frac{-3b \pm \sqrt{48ac + 9b^2}}{c}},$$

$$48ac + 9b^2 > 0. \quad (14)$$

see Figure 1



Figure 1. Homoclinic orbit.

The homoclinic orbit separates the phase plane into two areas. ([3],[4]) Inside the separatrix curve the orbits are around one of the centers, and outside the separatrix curve the orbits surround both the centers and the saddle point. Physically it means that for certain initial conditions the oscillations are around one steady-state position, and for others around all the steady-state solutions (two stable and an unstable). [5]

For $\epsilon \neq 0$ and $\omega(2k+1)\pi/2, k \in \mathbb{Z}$ in 2 the system become perturbed, let us treat some chaotic behavior of 2 for some values of $a, b, c, \gamma, \delta, \omega$ and ϵ . it is known that the Duffing oscillator

describes the motion of a classical particle in a double well potential such that the potential is given by $V(x) = \frac{x^4}{4} - \frac{x^2}{2} + \frac{x^6}{6}$, $V(x)$ is plotted in Figure 2

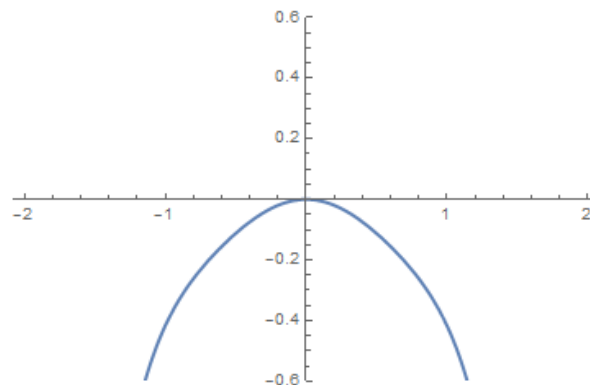


Figure 2. Potential plot.

We choose the units of length so that the minima are at $x = \pm 1$, and the units of energy so that the depth of each well is at $-1/6$. We may consider the force $F(x) = -dV/dx = x - x^3 + cx^5$. As usual we solve the second order differential equation $F = ma$ by expressing it as two first order differential equations:

$$\frac{dx}{dt} = v, \quad \frac{dv}{dt} = x - x^3 + cx^5$$

,where we set the mass equal to unity. Because of energy conservation ([3,7,8]), one can clearly never get chaos from the motion of a single degree of freedom. We therefore add both a driving force and damping, in order to remove energy conservation. The equations of motion become:

$$\frac{dx}{dt} = v, \quad \frac{dv}{dt} = x - x^3 + cx^5 - \gamma v + A \cos(\omega t),$$

where γ is the friction coefficient, and A is the strength of the driving force which oscillates at a frequency ω . We will see that a transition to chaos ([9–11]), now occurs as the strength of the driving force is increased (The values of c must be decreasing). We will fix coefficients as: $\gamma = 0.05$, $\omega = 0.05$, $A = 0.005$, $c = 0.0000015$ (which will be in the non-chaotic regime). We now start the particle off at rest at the origin and integrate the equations of motion. We will go up to $t = 200$. We obtain the following figure (see Figures 3 and 4).

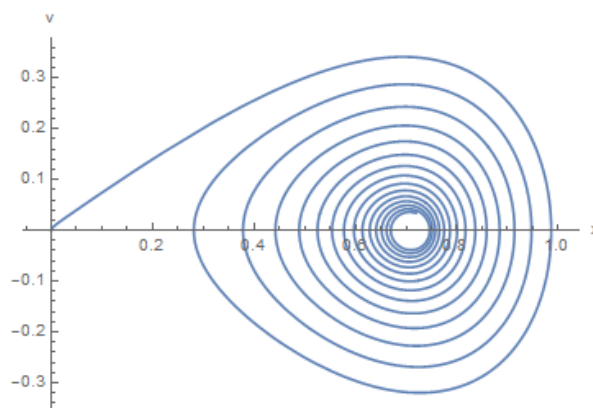


Figure 3. Transition to chaos whenever $\gamma = 0.05$, $\omega = 0.05$, $A = 0.005$, $c = 0.0000015$.

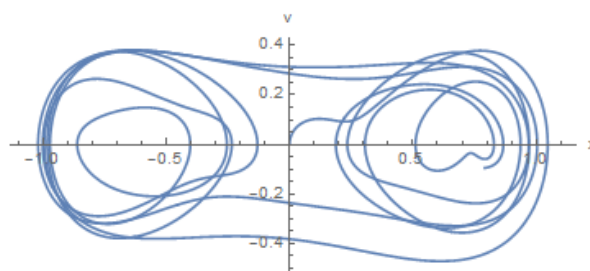


Figure 4. Transition to chaos whenever $\gamma = 0.05 = \omega = 1.2, A = 0.005, c = 0.00000005$.

3. Conclusions

Control of chaos remains an area of intensive research. Reliable forecasting of the dynamics of nonlinear systems with chaotic behaviour is a challenging task. It can be solved in several ways. For example, by localizing a chaotic attractor, while obtaining a rough forecasting, or by introducing control of unstable periodic orbits embedded in a chaotic attractor, thereby making the behavior of the system predictable for given values of its parameters .

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