

Article

Not peer-reviewed version

Timelike Thin Shells: Local Lorentzian Geometry from Timelike Boundaries

[Axel G. Schubert](#)*

Posted Date: 7 May 2026

doi: 10.20944/preprints202605.0433.v1

Keywords: general relativity; timelike thin shells; Brown--York stress tensor; quasilocal energy; junction conditions; boundary stress tensor; cut balance law



Preprints.org is a free multidisciplinary platform providing preprint service that is dedicated to making early versions of research outputs permanently available and citable. Preprints posted at Preprints.org appear in Web of Science, Crossref, Google Scholar, Scilit, Europe PMC, OpenAlex.

Copyright: This open access article is published under a [Creative Commons CC BY 4.0 license](#), which permit the free download, distribution, and reuse, provided that the author and preprint are cited in any reuse.

Disclaimer/Publisher's Note: The statements, opinions, and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions, or products referred to in the content.

Article

Timelike Thin Shells: Local Lorentzian Geometry from Timelike Boundaries

Axel G. Schubert

Independent Researcher, 40589 Düsseldorf, Germany; schubert.ag@posteo.de

Abstract

Timelike boundaries provide a natural setting for organizing geometric, quasilocal, and coarse-grained information in general relativity. This work develops a cut-level reference framework for finite-radius timelike interfaces in Lorentzian spacetime. Starting from a timelike boundary, a tangent observer field, and observer-adapted spatial cuts, the construction assigns selected boundary quantities, coarse-grained reference structures, channel-specific comparison values, resolved deviations, local event closure, and cut-level response terms to the same geometric surface. The framework is local in its physical reading. The coarse-grained reference structure is not treated as a single resolved boundary record, but as the macroscopic comparison structure relative to which local deviations are defined. A local boundary event is represented by a boundary-relative deviation that becomes resolvable at the candidate event. The causal condition fixes the Lorentzian admissibility domain; it does not by itself define a resolved trajectory or microscopic propagation history between spacetime points. In the classical realization developed here, the selected variables are supplied by the Brown–York cut-level dictionary. Observer-adapted projections of the boundary stress tensor define surface energy density, momentum density, spatial cut stress, and isotropic pressure. A coarse-grained boundary reference package specifies which variables are resolved, on which cut they are evaluated, and which reference structure serves as their comparison level. The corresponding deviation map and channel-dependent resolution norms identify the locally resolved boundary content. The same cut-level variables also enter a classical balance structure in which cut-energy variation separates into normal exchange and tangential mechanical response. In isotropic spherical symmetry, this response reduces to the pressure–area form, linking cut-level stress to the area-response channel of a timelike shell. Timelike thin-shell dynamics and macroscopic shell-balance laws then appear as concrete realizations of the general reference-cut structure. The resulting formulation provides a classical boundary-reference language for finite-radius timelike systems, relating local Lorentzian geometry, quasilocal stress, coarse-grained reference structure, resolved deviations, causal admissibility, and area response within one common cut-level framework.

Keywords: general relativity; timelike thin shells; Brown–York stress tensor; quasilocal energy; junction conditions; boundary stress tensor; cut balance law

1. Introduction: Area-Based Boundary Thermodynamics and Reference Cuts

Boundary structures play a central role in gravitational thermodynamics. For black-hole horizons, the area of a null boundary carries thermodynamic information through the Bekenstein–Hawking entropy relation [1,2]. Local thermodynamic readings of gravity further suggest that heat flow, causal boundaries, and area response provide a natural language for organizing gravitational dynamics [3]. These results motivate the following question: which part of this boundary logic can be formulated on a finite-radius timelike interface, where local proper time, observer-adapted cuts, surface stress, and quasilocal response quantities are available?

Timelike thin-shell interfaces provide a natural setting for this question. A timelike shell is a Lorentzian hypersurface separating two spacetime regions, with its surface stress controlled by junction

geometry through the Israel formalism [4,5]. In spherical symmetry, such a shell admits a proper-time foliation, round spatial cuts, an areal radius, and a quasilocal energy balance. Previous work used this setting to formulate a finite-radius timelike boundary model with regular interior geometry, static-patch restrictions, bounded curvature scalars, an entropy-like area functional, redshifted observables, and frequency bounds for near-boundary modes [6].

A complementary boundary-balance formulation developed the finite-response role of the shell. There, an interior coarse-grained reference sector is represented by a reference density $\rho_{\text{ref}}(\chi)$ and reference energy E_{ref} . Reference-energy descent is registered at the shell through the effective areal input

$$\Phi_{\text{eff}} = -\frac{1}{A_{\Sigma}} \dot{E}_{\text{ref}}, \quad (1)$$

and the shell partitions this input according to the macroscopic balance

$$\dot{E}_{\Sigma} = A_{\Sigma}(\Phi_{\text{eff}} - \Phi_{\text{out}}) - P\dot{A}_{\Sigma}. \quad (2)$$

This balance organizes shell storage, exterior release, and pressure–area work on the same finite-radius boundary [7].

The present manuscript extracts the cut-level reference structure underlying these finite-radius boundary descriptions. It isolates the common reference-cut organization on which timelike boundary dynamics and macroscopic shell-balance formulations are based. The starting point is a timelike boundary Σ in Lorentzian spacetime, an observer field u^a tangent to Σ , and observer-adapted spacelike cuts $C_{\tau_{\Sigma}} \subset \Sigma$. On these cuts, boundary quantities can be selected, assigned a coarse-grained reference structure, read in channel-specific comparison objects, and compared through resolved deviations. The organizing sequence is

$$\Sigma \longrightarrow C_{\tau_{\Sigma}} \longrightarrow Q_{\Sigma} \longrightarrow \mathcal{R}_{\Sigma} \longrightarrow q_{\Sigma}^{\text{ref}} \longrightarrow \mathcal{D}_{\Sigma}[q] \longrightarrow \text{local event closure}. \quad (3)$$

Here Q_{Σ} denotes the selected set of boundary quantities, \mathcal{R}_{Σ} is the coarse-grained reference structure assigned to the cut, q_{Σ}^{ref} is the channel-specific comparison object induced by that reference structure, and $\mathcal{D}_{\Sigma}[q]$ is the corresponding boundary-relative deviation.

This chain should be read as a cut-level assignment structure. The observer-adapted cut is not a second spacetime and does not replace the Lorentzian event geometry. It supplies the local surface on which selected boundary quantities can be compared with a coarse-grained reference structure. Causality fixes which prior data are admissible for a candidate local boundary event; the event-level content is supplied by the local resolution of a boundary-relative deviation. Thus Lorentzian admissibility and resolved boundary content are kept conceptually distinct.

The construction is local in its physical reading. The coarse-grained reference structure is assigned on the observer-adapted cut and supplies the macroscopic comparison structure relative to which local deviations are defined. It is not treated as a single resolved boundary record. Rather, it represents the comparison level of the cut-level description. A local event is therefore not identified with a resolved intermediate history between two spacetime points. It is represented by a deviation that becomes locally resolvable at the candidate boundary event.

This gives the timelike boundary a macroscopic reference role analogous in spirit to horizon thermodynamics, where boundary descriptions use a restricted set of accessible state variables without resolving all interior degrees of freedom. The analogy is structural rather than identificatory. The boundary considered here is timelike, not null: it carries local proper time, observer-adapted cuts, surface stress, and finite response channels. These structures allow deviations from the coarse-grained reference level to be assigned locally on the same boundary surface on which the quasilocal response variables are defined.

The value of the chain in Eq. (3) is therefore that the reference structure, the channel-specific comparison value, the resolved boundary content, and the local response condition are assigned to the

same observer-adapted cut. Causality fixes the admissible Lorentzian domain; the event-level content is supplied by the local resolution of a boundary-relative deviation from the coarse-grained reference structure.

The classical realization of the selected boundary quantities is supplied by the Brown–York quasilocal stress tensor on a timelike boundary [8,9]. Once an observer field and observer-adapted cuts are fixed, projections of the boundary stress tensor define a cut energy density, momentum density, spatial cut stress, and isotropic pressure. These quantities provide a concrete classical choice of \mathcal{Q}_Σ , while the pressure–area reduction supplies the local mechanical link between cut stress and boundary area response.

The central object introduced below is the coarse-grained boundary reference package

$$GC_\Sigma := (\mathcal{Q}_\Sigma, C_{\tau_\Sigma}, \mathcal{R}_\Sigma). \quad (4)$$

It combines the selected boundary quantities, the observer-adapted cut on which they are evaluated, and the coarse-grained reference structure that supplies their comparison level. For any selected boundary quantity $q_\Sigma \in \mathcal{Q}_\Sigma$, this assignment provides

$$q_\Sigma^{\text{ref}}[C_{\tau_\Sigma}] := \Pi_q[\mathcal{R}_\Sigma](C_{\tau_\Sigma}) = \langle q_\Sigma \rangle_{GC}. \quad (5)$$

The deviation map is then

$$\mathcal{D}_\Sigma[q] := q_\Sigma - q_\Sigma^{\text{ref}}. \quad (6)$$

Thus the reference structure fixes the channel-specific comparison value, while the deviation is the locally resolvable departure from that value.

The local Lorentzian structure supplies the causal part of the construction. In an orthonormal frame adapted to u^a , the boundary geometry admits a local tangent-frame representation in which u^a defines the proper-time direction and the cut C_{τ_Σ} appears as the spacelike section orthogonal to the boundary observer field. A candidate local event B is represented by a resolved boundary-relative deviation at B . The causal relation does not by itself define a resolved trajectory or microscopic propagation history from A to B . It specifies the Lorentzian admissibility domain of the candidate event. In its minimal classical form, local event closure is expressed by

$$A \in J^-(B), \quad \|\mathcal{D}_\Sigma[q](B)\|_q \geq \epsilon_q, \quad (7)$$

where ϵ_q is the resolution scale associated with the selected quantity q_Σ , and $\|\cdot\|_q$ denotes the corresponding channel norm. The first condition supplies Lorentzian causal admissibility. The second condition supplies local cut-level resolution at B . The comparison itself is local: the selected boundary quantity at B is compared with its channel-specific reference value on the same observer-adapted cut.

The main contribution of the paper is therefore a classical cut-level reference framework for timelike boundaries. It relates local Lorentzian geometry, Brown–York cut quantities, coarse-grained reference structures, resolved deviations, local event closure, and pressure–area response within one common boundary language. Timelike thin-shell dynamics and the macroscopic shell balance appear as concrete realizations of this general reference-cut structure.

The paper is organized as follows. Section 2 fixes the Lorentzian boundary geometry, observer field, and observer-adapted cuts. Section 3 introduces the Brown–York cut-level quantities used as the classical realization of \mathcal{Q}_Σ . Section 4 defines the coarse-grained boundary reference package. Section 5 introduces the deviation map and local resolution condition. Section 6 formulates local event closure. Section 7 records the cut-level balance structure. Section 8 connects the general reference-cut construction to timelike thin-shell dynamics and the macroscopic boundary-balance framework. The discussion relates the resulting timelike boundary structure to area-based gravitational thermodynamics and finite-response TTS systems.

2. Timelike Boundaries in Lorentzian Spacetime

This section fixes the geometric stage used throughout the manuscript. We start from a timelike boundary Σ embedded in a Lorentzian spacetime and introduce the intrinsic boundary metric, an observer field tangent to the boundary, and observer-adapted spatial cuts. These objects provide the geometric support for the cut-level variables introduced in Section 3 and for the reference package defined in Section 4. The notation follows standard hypersurface geometry in general relativity [5,10].

Let (M, g_{ab}) be a four-dimensional Lorentzian spacetime with signature $(-+++)$. A timelike boundary or interface $\Sigma \subset M$ is specified by a spacelike unit normal field n^a ,

$$n^a n_a = 1. \quad (8)$$

The induced metric on Σ is

$$h_{ab} = g_{ab} - n_a n_b. \quad (9)$$

It defines the intrinsic Lorentzian geometry of the timelike boundary.

A future-directed unit timelike observer field u^a , tangent to Σ , satisfies

$$u^a u_a = -1, \quad u^a n_a = 0. \quad (10)$$

The field u^a fixes the local temporal direction on the boundary. Together, n^a and u^a determine the normal direction away from the interface and the intrinsic observer direction along the interface.

The construction is local unless a global foliation is explicitly assumed. Thus the observer-adapted cuts used below are understood either as local spatial sections orthogonal to u^a or, in applications where a global cut structure is required, as arising from an observer field whose orthogonal spatial distribution within Σ is integrable. This distinction is useful because the cut-level reference construction only requires local boundary sections, while special symmetric realizations, such as spherical timelike shells, provide a natural global foliation.

The purpose of the following subsections is to turn these geometric data into a cut-level description. First, the observer-adapted cuts are defined. Second, the local Lorentzian representation is recorded as the tangent-frame language used for the diagrams and for later causal statements. Finally, a schematic setup figure is reserved for the geometric data introduced here.

2.1. Observer-Adapted Cuts

Given the observer field u^a , we consider observer-adapted cuts

$$C_{\tau\Sigma} \subset \Sigma. \quad (11)$$

These cuts are spacelike two-surfaces orthogonal to u^a within Σ , whenever such a foliation is available. Locally, they are the spatial sections of the boundary tangent space selected by the observer field. They provide the geometric support on which cut-level boundary data are assigned.

The induced metric on $C_{\tau\Sigma}$ is

$$\sigma_{ab} = h_{ab} + u_a u_b. \quad (12)$$

Thus σ_{ab} projects tensors onto the spatial cut. Unlike the Lorentzian boundary metric h_{ab} , the cut metric σ_{ab} is positive on the spatial cut. It will therefore also provide the natural metric used later to define positive norms for cut-vector and cut-tensor deviations.

The corresponding area functional is

$$A[C_{\tau\Sigma}] = \int_{C_{\tau\Sigma}} \sqrt{\sigma} \, d^2x, \quad (13)$$

where $\sigma := \det(\sigma_{AB})$ denotes the determinant of the induced metric in local coordinates on the cut.

The cut C_{τ_Σ} is the local support for the boundary quantities used below. It fixes the section of Σ on which reference structures, channel-specific comparison values, deviations, and cut-level response quantities are evaluated.

2.2. Local Lorentzian Representation

At each point of Σ , an orthonormal frame adapted to u^a and n^a gives the local tangent-frame form of the spacetime metric,

$$ds^2 = -c^2 d\tau^2 + d\eta^2 + \sigma_{AB} dy^A dy^B, \quad (14)$$

where η denotes the local normal direction away from the boundary and y^A are coordinates on the spatial cut. Restricting to the boundary, $d\eta = 0$, the intrinsic line element becomes

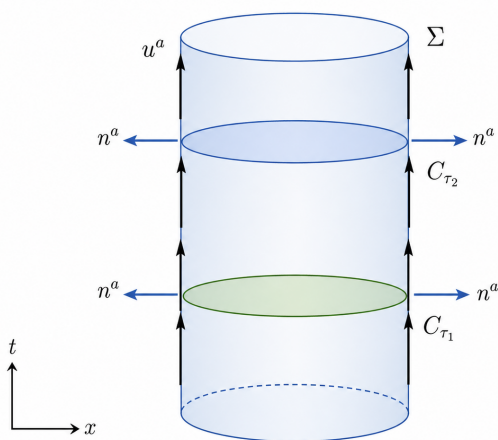
$$ds_\Sigma^2 = -c^2 d\tau^2 + \sigma_{AB} dy^A dy^B. \quad (15)$$

The observer field u^a defines the local proper-time direction, while C_{τ_Σ} is represented by the spatial section orthogonal to that direction within Σ .

In this local Lorentzian form, timelike, null, and spacelike directions have their standard causal classification. The construction below uses this causal structure to specify admissible local event closure, while the spatial cut metric σ_{ab} supplies the positive geometry of the cut on which reference values, deviations, and response quantities are evaluated.

2.3. Geometric Setup Diagram

Figure 1 summarizes the geometric data introduced in this section. It shows the timelike boundary Σ , the spacelike normal n^a , the observer field u^a , and two observer-adapted cuts C_{τ_1} and C_{τ_2} .



Geometric setup of a timelike boundary with observer-adapted cuts.

Figure 1. Geometric setup of a timelike boundary Σ with spacelike normal n^a , observer field u^a , and observer-adapted cuts C_{τ_1} and C_{τ_2} . The figure gives a local tangent-frame visualization of the boundary and its cut structure.

The diagram records the geometric stage used throughout the manuscript. The causal event-closure diagram is introduced later, after the reference package and deviation map have been defined in Sections 4 and 5.

The next section uses the cut geometry fixed here to introduce the classical boundary quantities. In particular, the Brown–York boundary stress tensor and its cut projections provide a natural classical choice of the selected boundary quantities Q_Σ .

3. Classical Cut-Level Boundary Quantities

The previous section fixed the geometric support of the construction: a timelike boundary Σ , an observer field u^a , and observer-adapted cuts C_{τ_Σ} . The present section turns this cut geometry into a

classical set of boundary quantities. The guiding object is the Brown–York boundary stress tensor, whose projections on u^a and C_{τ_Σ} define the energy, momentum, and stress variables used later as a classical realization of the selected boundary quantities \mathcal{Q}_Σ .

On each side of a timelike interface, the Brown–York boundary stress tensor is written as

$$\tau_{ab}^\pm = \frac{1}{8\pi G} (K_{ab}^\pm - K^\pm h_{ab}), \quad (16)$$

where K_{ab}^\pm are the one-sided extrinsic curvatures of Σ and $K^\pm = h^{ab}K_{ab}^\pm$. This is the Brown–York quasilocal stress tensor on a timelike boundary, written with the sign convention fixed by the chosen orientation of the spacelike normal n^a [8,9]. Reference or subtraction terms may be included when a specific quasilocal normalization is required. In that case, the subtraction prescription is understood as part of the chosen definition of τ_{ab}^\pm before the cut projections are formed.

For the cut-level bookkeeping used below, we separate the two-sided boundary data into an averaged sector and a jump sector. The averaged Brown–York tensor is defined by

$$\tau_{ab} := \langle \tau_{ab} \rangle = \frac{1}{2} (\tau_{ab}^+ + \tau_{ab}^-). \quad (17)$$

This averaged tensor is the object projected on the observer-adapted cuts in the present reference construction. It provides the quasilocal cut variables used to define reference values and deviations.

This use of the averaged sector should be distinguished from the surface stress tensor of an ideal thin shell. The latter is controlled by the jump of the extrinsic curvature through the Israel junction conditions. Thus the averaged tensor in Eq. (17) is not introduced as a replacement for the Israel surface layer. Rather, it supplies the cut-level quasilocal sector relative to which the reference package and deviation map are defined. The complementary jump sector records the discontinuity data associated with the shell layer itself.

The structure of this section is therefore simple. First, the cut projections of τ_{ab} define energy density, momentum density, and cut stress. Second, their isotropic reduction gives the pressure variable. Third, the pressure–area form of the mechanical response prepares the connection to the area variable used in Section 8.

3.1. Cut Projections

The observer field u^a and the cut projector σ_{ab} define the cut-level decomposition of the averaged boundary stress tensor. The energy surface density measured by the boundary observers is

$$\varepsilon := \tau_{ab} u^a u^b. \quad (18)$$

The momentum surface density intrinsic to the cut is

$$j_a := -\sigma_a^c \tau_{cd} u^d, \quad (19)$$

and the spatial cut stress is

$$\pi_{ab} := \sigma_a^c \sigma_b^d \tau_{cd}. \quad (20)$$

Together, these projections give the observer-adapted cut-level dictionary

$$\tau_{ab} \longrightarrow (\varepsilon, j_a, \pi_{ab}). \quad (21)$$

The isotropic cut pressure is defined by the trace of the spatial stress on the two-dimensional cut,

$$P := \frac{1}{2} \sigma^{ab} \pi_{ab}. \quad (22)$$

The corresponding cut energy is

$$E[C_{\tau_{\Sigma}}] = \int_{C_{\tau_{\Sigma}}} \sqrt{\sigma} \varepsilon \, d^2x. \quad (23)$$

The decomposition in Eqs. (18)–(20) is also the decomposition used later for local resolution. The energy density is a scalar observer channel, j_a is a cut-vector channel, and π_{ab} is a spatial cut-tensor channel. This separation avoids treating the full Lorentzian tensor τ_{ab} as a single positive-norm object.

These quantities provide a natural classical choice of selected boundary quantities,

$$\mathcal{Q}_{\Sigma, \text{BY}} = \{h_{ab}, u^a, \sigma_{ab}, \tau_{ab}, \varepsilon, j_a, \pi_{ab}, P\}. \quad (24)$$

This set will be used in Section 4 as the Brown–York realization of the selected boundary quantities entering the coarse-grained reference package.

3.2. Pressure–Area Response

The cut-level stress variables also determine the mechanical response of the boundary under a deformation of the cut. For an isotropic cut stress, vanishing cut momentum, and normal cut evolution along the observer field, the response term reduces to the pressure–area form

$$\delta_{\zeta^a} W[C] = -P \delta_{\zeta^a} A[C]. \quad (25)$$

Here ζ^a denotes the cut-evolution generator, and $A[C]$ is the area functional defined in Eq. (13). The cut-level derivation of Eq. (25) is summarized in Appendix B.

Equation (25) is the local mechanical bridge between the Brown–York cut stress and the area variable of the boundary. It is not a general equation of state for the shell. It is the isotropic pressure–area reduction of the cut-level response term under the assumptions stated above. In spherical timelike thin-shell realizations, where the cuts are round two-spheres, this pressure–area response becomes the mechanical channel associated with changes of $A_{\Sigma} = 4\pi R^2$. It is the cut-level precursor of the area-based variable $S_{\Sigma} = \alpha A_{\Sigma}$ used in Section 8.

The next section uses the variables collected in Eq. (24) to define the coarse-grained boundary reference package.

4. Coarse-Grained Boundary Reference Package

The previous section identified a classical set of cut-level boundary quantities. The present section adds the reference layer used throughout the rest of the manuscript. The aim is to distinguish the selected boundary quantities themselves from the coarse-grained reference structure and the channel-specific comparison values relative to which local deviations will be evaluated in Section 5.

A cut-level reference structure is obtained by assigning a coarse-grained comparison structure to selected boundary quantities on an observer-adapted cut. The coarse-grained boundary reference package is

$$\text{GC}_{\Sigma} := (\mathcal{Q}_{\Sigma}, C_{\tau_{\Sigma}}, \mathcal{R}_{\Sigma}). \quad (26)$$

Here \mathcal{Q}_{Σ} is the selected set of boundary quantities, $C_{\tau_{\Sigma}}$ is the observer-adapted cut, and \mathcal{R}_{Σ} denotes the coarse-grained reference structure associated with that cut. The symbol \mathcal{R}_{Σ} is used for the comparison structure itself. It is not assumed to be an additional local matter density unless a specific physical closure supplies such an interpretation.

The selection of \mathcal{Q}_{Σ} fixes the resolution and scope of the boundary description. For example, in the Brown–York realization introduced in Section 3, a natural classical choice is

$$\mathcal{Q}_{\Sigma, \text{BY}} = \{h_{ab}, u^a, \sigma_{ab}, \tau_{ab}, \varepsilon, j_a, \pi_{ab}, P\}. \quad (27)$$

Other applications may use a smaller or enlarged set of quantities, depending on which boundary data are resolved.

The role of \mathcal{R}_Σ is to supply the comparison structure of the cut-level description. For each selected quantity $q_\Sigma \in \mathcal{Q}_\Sigma$, a channel-specific reading of this reference structure provides a comparison value on the same observer-adapted cut,

$$\Pi_q[\mathcal{R}_\Sigma] : (q_\Sigma, C_{\tau_\Sigma}) \mapsto q_\Sigma^{\text{ref}}[C_{\tau_\Sigma}] := \Pi_q[\mathcal{R}_\Sigma](C_{\tau_\Sigma}) = \langle q_\Sigma \rangle_{GC}. \quad (28)$$

The assigned value q_Σ^{ref} has the same geometric character as q_Σ . Thus a scalar boundary quantity is assigned a scalar reference value, a cut vector is assigned a cut-vector reference value, and a cut tensor is assigned a tensorial reference value on the same cut.

In the Brown–York realization, this gives reference values such as

$$\varepsilon^{\text{ref}} = \langle \varepsilon \rangle_{GC}, \quad P^{\text{ref}} = \langle P \rangle_{GC}, \quad \pi_{ab}^{\text{ref}} = \langle \pi_{ab} \rangle_{GC}. \quad (29)$$

These reference values are not individual resolved boundary records. They encode the channel-specific representation of the unresolved coarse-grained reference structure. The locally resolved content is the departure from this value, not the reference structure itself.

The concrete prescription for \mathcal{R}_Σ depends on the closure chosen for the boundary system. In a classical realization it may be supplied by a smoothed cut-level field, an effective stationary profile, or a macroscopic reference solution. In a statistical realization it may be generated by an ensemble or maximum-entropy prescription for the selected boundary data. The minimal cut-level construction used here requires only that the assigned reference values are defined on the same cut as the corresponding q_Σ , have the same tensorial type, and are available as comparison values for the deviation map introduced below.

The reference package therefore separates four ingredients:

$$\text{selected quantities} + \text{cut} + \text{reference structure} + \text{channel-specific reading}. \quad (30)$$

The selected quantities specify what is being resolved, the cut specifies where the comparison is made, the reference structure specifies the coarse-grained level of comparison, and the channel-specific reading specifies how that structure is represented in the selected boundary quantity.

This organization is useful because the reference structure is not itself a local event. It is the cut-level comparison structure for the selected boundary quantity. The observable boundary content used below appears in the deviation from the corresponding channel-specific reference value, which is introduced in Section 5.

5. Deviation Map and Local Resolution

The reference package introduced in Section 4 assigns a channel-specific reference value to each selected boundary quantity. The present section defines the corresponding deviation map and the local resolution condition. The deviation map is the step from reference data to resolved boundary content: a boundary quantity becomes locally relevant through its departure from the cut-level reference value.

For any selected boundary quantity $q_\Sigma \in \mathcal{Q}_\Sigma$, with reference value

$$q_\Sigma^{\text{ref}}[C_{\tau_\Sigma}] := \Pi_q[\mathcal{R}_\Sigma](C_{\tau_\Sigma}) = \langle q_\Sigma \rangle_{GC}, \quad (31)$$

define the boundary-relative deviation in the additive representation by

$$\mathcal{D}_\Sigma[q] := q_\Sigma - q_\Sigma^{\text{ref}}. \quad (32)$$

More generally, one may regard $\mathcal{D}_\Sigma[q]$ as the output of a channel-dependent departure operation Δ_q acting on q_Σ and \mathcal{R}_Σ . The additive form in Eq. (32) is the classical realization used in the remainder of this manuscript.

The reference value q_{Σ}^{ref} is assigned on the observer-adapted cut $C_{\tau_{\Sigma}}$. The deviation $\mathcal{D}_{\Sigma}[q]$ is therefore a cut-level quantity: it records how the selected boundary quantity differs from the coarse-grained reference level on the same local boundary section.

A deviation becomes resolved only relative to the resolution scale of the selected quantity. For scalar quantities the size of the deviation is the ordinary absolute value. For vector or tensor quantities, however, the size of the deviation must be evaluated with a positive channel norm. We therefore write the local resolution condition as

$$\|\mathcal{D}_{\Sigma}[q](B)\|_q \geq \epsilon_q, \quad (33)$$

where B denotes the local boundary point or candidate event at which the deviation is evaluated, ϵ_q is the resolution scale associated with q_{Σ} , and $\|\cdot\|_q$ denotes the norm appropriate to that selected channel. The value of ϵ_q depends on the selected boundary quantity, the coarse-graining prescription, and the operational resolution of the cut-level description.

The channel norm is defined after the observer-adapted decomposition of the selected boundary quantity. For scalar channels, it reduces to the ordinary absolute value. For cut-vector and cut-tensor channels, it is induced by the positive spatial cut metric σ_{ab} . This avoids using the Lorentzian boundary metric h_{ab} itself as a positive resolution norm. The Lorentzian metric fixes the causal structure and proper-time direction, while the spatial cut metric supplies the positive geometry used to evaluate cut-level resolution.

For the Brown–York realization in Eq. (27), the scalar energy and pressure deviations are

$$\mathcal{D}_{\Sigma}[\varepsilon] = \varepsilon - \varepsilon^{\text{ref}}, \quad (34)$$

$$\mathcal{D}_{\Sigma}[P] = P - P^{\text{ref}}, \quad (35)$$

with corresponding resolution magnitudes

$$\|\mathcal{D}_{\Sigma}[\varepsilon]\|_{\varepsilon} = |\varepsilon - \varepsilon^{\text{ref}}|, \quad (36)$$

$$\|\mathcal{D}_{\Sigma}[P]\|_P = |P - P^{\text{ref}}|. \quad (37)$$

The sign of these scalar deviations records whether the local value lies above or below the coarse-grained reference level, while the absolute value gives the resolution magnitude.

For the cut-momentum channel,

$$\mathcal{D}_{\Sigma}[j]_a = j_a - j_a^{\text{ref}}, \quad (38)$$

the positive cut norm is

$$\|\mathcal{D}_{\Sigma}[j]\|_j = \left(\sigma^{ab} \mathcal{D}_{\Sigma}[j]_a \mathcal{D}_{\Sigma}[j]_b \right)^{1/2}. \quad (39)$$

For the spatial cut-stress channel,

$$\mathcal{D}_{\Sigma}[\pi]_{ab} = \pi_{ab} - \pi_{ab}^{\text{ref}}, \quad (40)$$

the corresponding cut-tensor norm is

$$\|\mathcal{D}_{\Sigma}[\pi]\|_{\pi} = \left(\sigma^{ac} \sigma^{bd} \mathcal{D}_{\Sigma}[\pi]_{ab} \mathcal{D}_{\Sigma}[\pi]_{cd} \right)^{1/2}. \quad (41)$$

Since the isotropic part of the cut stress is already represented by P , one may also separate the trace-free spatial stress

$$s_{ab} := \pi_{ab} - P\sigma_{ab}, \quad \sigma^{ab} s_{ab} = 0, \quad (42)$$

and use the corresponding norm

$$\|\mathcal{D}_\Sigma[s]\|_s = \left(\sigma^{ac} \sigma^{bd} \mathcal{D}_\Sigma[s]_{ab} \mathcal{D}_\Sigma[s]_{cd} \right)^{1/2}. \quad (43)$$

This form separates the isotropic pressure channel from the anisotropic cut-stress channel.

This structure mirrors a familiar statistical distinction. A mean or coarse-grained reference value is not identical with a single resolved record. It supplies the level relative to which deviations are organized. In the present setting, the quantities being compared are boundary-geometric or boundary-stress quantities; their deviations are defined on a timelike boundary and evaluated on observer-adapted cuts.

The deviation map supplies the local boundary content used in the event-closure construction. The next section combines the resolution condition in Eq. (33) with the causal structure fixed in Section 2.

6. Local Event Closure

The previous section introduced the deviation map and the local resolution condition. The present section adds the causal part of the construction. Local event closure combines two logically distinct ingredients. The first ingredient is Lorentzian causal admissibility: prior data must belong to the causal past of the candidate event. The second ingredient is cut-level resolution: a selected boundary quantity must depart from its coarse-grained reference value by an amount resolvable in the corresponding channel.

The causal condition alone does not supply the local boundary content. It only specifies the admissible Lorentzian domain. Conversely, a resolved deviation alone does not define a causal relation. A local boundary event is closed only when a boundary-relative deviation becomes resolvable at the candidate event within the admissible causal domain.

The construction is therefore organized from the candidate event B backward to its admissible causal past. The set $J^-(B)$ identifies which prior data are causally admissible for the local closure at B . The event itself is not represented by a resolved intermediate history between spacetime points. It is represented locally at B by a resolved deviation relative to the selected cut-level reference structure.

Let A denote a preparation, source, or prior boundary event, and let B denote a candidate local event. The Lorentzian causal structure determines whether A belongs to the admissible prior domain of B . Causal admissibility is expressed by

$$A \in J^-(B), \quad (44)$$

where $J^-(B)$ is the causal past of B .

The closure test itself is local at B . On the observer-adapted cut through B , the selected boundary quantity is compared with its channel-specific reference value,

$$\mathcal{D}_\Sigma[q](B) = q_\Sigma(B) - q_\Sigma^{\text{ref}}(B). \quad (45)$$

A local boundary event is formed when this deviation exceeds the corresponding resolution scale. The minimal local event-closure condition therefore combines causal admissibility with local cut-level resolution:

$$A \in J^-(B), \quad \|\mathcal{D}_\Sigma[q](B)\|_q \geq \epsilon_q. \quad (46)$$

The first condition supplies Lorentzian causal admissibility. The second condition is evaluated locally at B . Thus the causal relation does not by itself define a resolved trajectory or microscopic propagation history from A to B . It specifies the admissible Lorentzian domain within which local closure can occur. The event is closed at B by the local comparison between the selected boundary quantity and its coarse-grained reference value on the same observer-adapted cut.

The cut C_{τ_Σ} supplies the local section on which the relevant boundary data are assigned, while the causal past $J^-(B)$ supplies the admissible Lorentzian domain of prior data.

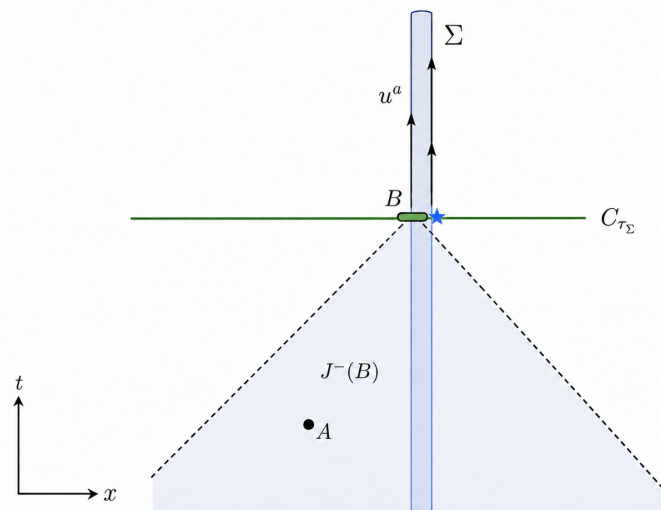
For several selected quantities $q_\Sigma^I \in \mathcal{Q}_\Sigma$, the resolution part of Eq. (46) may be written channel by channel as

$$\|\mathcal{D}_\Sigma[q^I](B)\|_I \geq \epsilon_I \quad \text{for at least one resolved channel } I. \quad (47)$$

The set of resolved channels depends on the selected boundary quantities and on the coarse-graining prescription. For example, in the Brown–York realization, local closure may involve a resolved deviation in surface energy density, pressure, cut momentum, spatial stress, or another projected cut quantity.

6.1. Diagrammatic Representation of Local Event Closure

Figure 2 illustrates the local event-closure structure introduced in this section. The diagram is organized from the candidate event B backward to its admissible causal past. The region $J^-(B)$ identifies the prior data that are causally admissible for the local closure at B , while the observer-adapted cut C_{τ_Σ} specifies the boundary section on which the relevant quantities are evaluated.



Schematic local event-closure diagram.

Figure 2. Schematic local event-closure diagram. The candidate event B is shown as a cut-level element on the observer-adapted cut C_{τ_Σ} of the timelike boundary Σ . The shaded region is the causal past $J^-(B)$, containing admissible prior data such as A . The marker at B denotes the resolved boundary-relative deviation, corresponding to the local resolution condition $\|\mathcal{D}_\Sigma[q](B)\|_q \geq \epsilon_q$. The event is closed locally at B , while $J^-(B)$ specifies the admissible prior causal domain.

The diagram combines the geometric and reference-level ingredients introduced in the previous sections. The boundary cut fixes where the selected quantities are evaluated, the reference package fixes the comparison level, and the deviation map identifies the resolved content at B . The causal past $J^-(B)$ supplies the admissible Lorentzian domain of prior data; the event itself is represented by the local closure of the resolved boundary-relative deviation at B .

The next section turns from event closure to the cut-level balance structure. There, the same boundary quantities are organized into exchange and response terms, providing the classical response framework associated with the reference-cut construction.

7. Cut Balance as Classical Response Structure

The preceding sections introduced the boundary reference package, the deviation map, and local event closure. The present section records the corresponding classical response structure on the same

cut-level geometry. The purpose is to show how the selected boundary quantities can be organized into exchange and response terms once a cut flow has been specified.

The starting point is the averaged Brown–York boundary stress tensor τ_{ab} introduced in Section 3. Together with the observer field u^a and the cut metric σ_{ab} , it defines the cut energy $E[C]$ in Eq. (23). A deformation of the cut is generated by a vector field tangent to the boundary,

$$\zeta^a = \alpha u^a, \quad (48)$$

where α is the lapse associated with the cut evolution.

For the normal cut flow used here, the cut evolution balance takes the form

$$\delta_{\zeta} E[C] = \delta_{\zeta} Q[C] + \delta_{\zeta} W[C]. \quad (49)$$

This identity is the cut-level response statement used below. It separates the variation of the cut energy into a normal exchange term and a tangential mechanical response term. The derivation follows from the Brown–York boundary conservation identity and the Stokes/Gauss step summarized in Appendix B [8,9].

The exchange term is

$$\delta_{\zeta} Q[C] = \int_C \sqrt{\sigma} \alpha \langle \Phi \rangle d^2x, \quad (50)$$

where $\langle \Phi \rangle$ denotes the two-sided averaged normal energy-flux density through the interface, evaluated with the same averaging convention as the cut-level Brown–York sector. This term records the normal exchange contribution to the cut-energy variation.

The response term is

$$\delta_{\zeta} W[C] = - \int_C \sqrt{\sigma} \alpha \tau^{ab} D_a u_b d^2x. \quad (51)$$

Here D_a denotes the covariant derivative compatible with the boundary metric h_{ab} . The response term is determined by the contraction of the cut-level boundary stress tensor with the kinematics of the observer field on Σ . It is therefore a tangential mechanical response term of the boundary system, not an independent equation of state.

The split in Eq. (49) is useful for the reference-cut construction because the same boundary data enter both the reference structure and the response structure. The projections of τ_{ab} define the cut-level quantities used to assign reference values and deviations in Sections 4 and 5. The same projected stress data also determine how the cut energy changes under a specified cut flow. Thus the reference comparison and the classical response are built from the same observer-adapted boundary quantities.

In the isotropic spherical reduction, the cut momentum vanishes and the spatial cut stress is proportional to the cut metric. Equivalently,

$$j_a = 0, \quad \pi_{ab} = P \sigma_{ab}. \quad (52)$$

Under these assumptions, Eq. (51) reduces to the pressure–area form already recorded in Eq. (25),

$$\delta_{\zeta} W[C] = -P \delta_{\zeta} A[C]. \quad (53)$$

This relation is the isotropic pressure–area reduction of the cut-level mechanical response. It is not a separate thermodynamic postulate. For round cuts of a spherical shell, $A[C]$ becomes $A_{\Sigma} = 4\pi R^2$, and the pressure–area term becomes the mechanical area–response channel of the shell.

The next section connects this general cut-level response structure to the timelike thin-shell dynamics and macroscopic boundary-balance framework. There, reference-state change is represented by an effective boundary input, while the same pressure–area channel appears as part of the finite boundary response.

8. Connection to Timelike Thin-Shell Dynamics and Boundary Balance

The preceding sections developed the cut-level reference structure in general boundary language. The present section connects this structure to the timelike thin-shell setting. The purpose is to identify how the abstract ingredients

$$C_{\tau_\Sigma}, \quad Q_\Sigma, \quad \mathcal{R}_\Sigma, \quad q_\Sigma^{\text{ref}}, \quad \mathcal{D}_\Sigma[q]$$

are realized in spherical TTS dynamics and in the macroscopic boundary-balance framework.

In the spherical TTS realization, the observer-adapted cuts are round two-spheres of areal radius $R(\tau_\Sigma)$. Their area is

$$A_\Sigma = 4\pi R^2. \quad (54)$$

This area is the geometric cut quantity that connects the spherical shell description with the cut-level area functional introduced in Eq. (13).

The corresponding macroscopic area variable is written as

$$S_\Sigma = \alpha A_\Sigma, \quad (55)$$

where α fixes the normalization of the coarse-grained area variable. At this stage S_Σ is an area-response variable. It becomes an entropy-like variable only after a thermodynamic closure has fixed the interpretation and normalization of α . Thus Eq. (55) records the macroscopic area content of the timelike boundary without assuming a microscopic state count.

The pressure–area response from Eq. (25) becomes, for spherical shell cuts,

$$\delta_{\xi} W[C] = -P \delta_{\xi} A_\Sigma. \quad (56)$$

This identifies the pressure–area term as the mechanical channel associated with changes of the shell area. For an evolution parametrized by the shell proper time, the same channel appears in rate form as $P\dot{A}_\Sigma$ in the macroscopic shell balance.

The general reference assignment \mathcal{R}_Σ supplies the coarse-grained comparison structure of the boundary description. In the TTS balance realization, the change of the relevant interior reference sector is represented by a coarse-grained reference energy E_{ref} . The effective areal input associated with reference-energy descent is

$$\Phi_{\text{eff}} = -\frac{1}{A_\Sigma} \dot{E}_{\text{ref}}. \quad (57)$$

With this sign convention, a decrease of the reference energy, $\dot{E}_{\text{ref}} < 0$, corresponds to a positive effective input into the boundary. Thus Φ_{eff} is the TTS-specific representation of a reference-sector change registered at the timelike shell.

The finite boundary response is organized by the shell balance

$$\dot{E}_\Sigma = A_\Sigma(\Phi_{\text{eff}} - \Phi_{\text{out}}) - P\dot{A}_\Sigma. \quad (58)$$

Here E_Σ is the quasilocal shell energy, Φ_{out} is the outward release flux into exterior-accessible channels, and $P\dot{A}_\Sigma$ is the pressure–area work rate. Equation (58) partitions the effective reference-sector input into shell storage, exterior release, and mechanical area response.

The relation to the cut-level construction can now be read directly. The spherical cut C_{τ_Σ} carries the area A_Σ . The Brown–York cut variables provide a classical realization of Q_Σ . The reference assignment supplies the coarse-grained comparison values q_Σ^{ref} . In the TTS balance realization, the proper-time change of the reference sector is encoded by E_{ref} and registered at the boundary through Φ_{eff} . The shell balance then supplies the finite-response law associated with these boundary quantities.

In this sense, the TTS balance is a concrete realization of the sequence

$$\text{reference-sector change} \longrightarrow \Phi_{\text{eff}} \longrightarrow (\dot{E}_{\Sigma}, \Phi_{\text{out}}, P\dot{A}_{\Sigma}). \quad (59)$$

The first arrow identifies how a change of the coarse-grained reference sector is registered as an effective areal input at the boundary. The second arrow identifies how the timelike shell redistributes this input into storage, release, and area response.

The same structure gives the response interpretation of the area variable. Using $S_{\Sigma} = \alpha A_{\Sigma}$, the work term in Eq. (58) may be written as

$$P\dot{A}_{\Sigma} = \frac{P}{\alpha} \dot{S}_{\Sigma}. \quad (60)$$

Thus the area variable enters the finite boundary response through the same mechanical channel that appears in the cut-level pressure–area relation. A thermodynamic interpretation of S_{Σ} requires an additional closure fixing α and the associated state variables.

This section completes the bridge from the general reference-cut construction to the concrete TTS setting. The resulting picture is a finite-radius timelike boundary system in which area, quasilocal quantities, reference-sector change, and local response are organized on the same geometric surface.

9. Discussion

The construction developed in the preceding sections gives a cut-level reference formulation for timelike boundaries in Lorentzian spacetime. Starting from the geometric data of a timelike interface Σ , an observer field u^a , and observer-adapted cuts $C_{\tau_{\Sigma}}$, the framework assigns selected boundary quantities, coarse-grained reference structures, channel-specific comparison values, resolved deviations, local event closure, and cut-level response terms to the same geometric surface.

The resulting structure can be summarized as

$$\Sigma \longrightarrow C_{\tau_{\Sigma}} \longrightarrow \mathcal{Q}_{\Sigma} \longrightarrow \mathcal{R}_{\Sigma} \longrightarrow q_{\Sigma}^{\text{ref}} \longrightarrow \mathcal{D}_{\Sigma}[q] \longrightarrow \text{local event closure}. \quad (61)$$

This chain is the organizing principle of the manuscript. The boundary Σ supplies the geometric support, the cut $C_{\tau_{\Sigma}}$ supplies the local section, the selected quantities \mathcal{Q}_{Σ} fix what is resolved, the reference structure supplies the coarse-grained comparison level, the channel-specific reference value specifies how this structure is represented in a selected variable, and the deviation map identifies the boundary content that can enter a local event.

9.1. Cut-Level Reference Structure

The central role of the cut is to localize the reference comparison. Boundary quantities are evaluated on $C_{\tau_{\Sigma}}$, and their reference values are assigned relative to the same cut-level structure. This gives a local organization of boundary data.

The Brown–York realization provides a classical example of this organization. The selected boundary quantities may be taken as

$$\mathcal{Q}_{\Sigma, \text{BY}} = \{h_{ab}, u^a, \sigma_{ab}, \tau_{ab}, \varepsilon, j_a, \pi_{ab}, P\}. \quad (62)$$

The reference package GC_{Σ} then assigns coarse-grained reference values to these quantities. For each selected quantity,

$$q_{\Sigma}^{\text{ref}}[C_{\tau_{\Sigma}}] := \Pi_q[\mathcal{R}_{\Sigma}](C_{\tau_{\Sigma}}) = \langle q_{\Sigma} \rangle_{\text{GC}}, \quad (63)$$

and the resolved boundary-relative deviation is

$$\mathcal{D}_{\Sigma}[q] = q_{\Sigma} - q_{\Sigma}^{\text{ref}}. \quad (64)$$

This distinction between reference structure, channel-specific reference value, and resolved deviation is the main conceptual step. The reference structure is not itself a local event. It supplies the coarse-grained comparison level of the selected boundary description. The event-level content enters through deviations from its channel-specific representation.

The use of \mathcal{R}_Σ is therefore deliberately weaker than a complete microscopic or thermodynamic closure. It specifies the reference structure needed by the cut-level framework. A later closure may identify this assignment with a stationary, equilibrium, thermal, statistical, or vacuum-like reference sector. The construction developed here only requires that the assigned reference values are defined on the same cut as the corresponding boundary quantities and have the same geometric character.

9.2. Local Resolution and Channel Norms

The local resolution condition expresses when a deviation becomes a resolved boundary record. For a selected channel q_Σ , this condition is

$$\|\mathcal{D}_\Sigma[q](B)\|_q \geq \epsilon_q. \quad (65)$$

The norm is channel dependent. For scalar channels, such as ϵ or P , it reduces to the ordinary absolute value. For cut-vector and cut-tensor channels, the norm is induced by the positive spatial cut metric σ_{ab} .

This separation is important because the boundary metric h_{ab} is Lorentzian. It determines causal classification and proper-time structure, but it does not provide a positive resolution norm for arbitrary tensorial deviations. The observer-adapted decomposition

$$\tau_{ab} \longrightarrow (\epsilon, j_a, \pi_{ab}) \quad (66)$$

therefore has both geometric and operational significance. It separates the full boundary stress tensor into scalar, cut-vector, and spatial cut-tensor channels that can be assigned reference values and resolution thresholds separately.

In this sense, local resolution is not a statement about the norm of a full Lorentzian tensor. It is a statement about the size of a projected boundary deviation in a specified observer-adapted channel.

9.3. Local Lorentzian Event Closure

The causal part of the construction is supplied by the Lorentzian structure. For a candidate boundary event B , the causal past $J^-(B)$ specifies the admissible prior domain. The minimal event-closure condition is

$$A \in J^-(B), \quad \|\mathcal{D}_\Sigma[q](B)\|_q \geq \epsilon_q. \quad (67)$$

The two conditions have different roles. The first condition supplies causal admissibility. The second condition supplies local cut-level resolution at B . The causal relation does not by itself define a resolved trajectory or microscopic propagation history from A to B . It specifies the Lorentzian domain within which local closure at B is admissible.

The comparison itself is local. At the candidate event B , the selected boundary quantity $q_\Sigma(B)$ is compared with its coarse-grained reference value $q_\Sigma^{\text{ref}}(B)$ on the same observer-adapted cut. The event-level content is therefore not the causal connection itself, but the local resolution of a boundary-relative deviation within the admissible Lorentzian domain.

This is the sense in which the present framework treats local boundary physics. The boundary observer does not resolve an entire intermediate history. The observer resolves a local deviation from the cut-level reference structure.

9.4. Area Thermodynamics: Horizon and Timelike Boundary

The event horizon is the standard null-boundary paradigm of gravitational area thermodynamics. Its entropy is proportional to the horizon area,

$$S_{\text{BH}} = \frac{k_B A_H}{4\ell_P^2}. \quad (68)$$

The timelike thin shell realizes a different but structurally related area-carrying boundary. It is a finite-radius timelike hypersurface with local proper time, observer-adapted cuts, quasilocal surface quantities, and explicit response channels.

In the TTS setting, the corresponding macroscopic area variable is

$$S_\Sigma = \alpha A_\Sigma. \quad (69)$$

At the level of the present framework, S_Σ should be read as an area-response variable. It becomes an entropy-like variable only after a thermodynamic closure has fixed the meaning and normalization of α . Thus the present construction does not assume a microscopic state count for the shell.

The common structure is the assignment of macroscopic information to a boundary area. The geometric character of the boundary is different in the two cases: the horizon is null, while the TTS is timelike. This difference gives the TTS its cut-level resolution. The timelike boundary carries local proper-time cuts, surface stress, pressure–area response, redshifted observables, and finite boundary dynamics.

The pressure–area relation

$$\delta_\zeta W[C] = -P \delta_\zeta A[C] \quad (70)$$

is the mechanical link between the cut-level stress description and the area variable. For spherical TTS cuts, this becomes the pressure–area channel associated with changes of $A_\Sigma = 4\pi R^2$. Thus the area variable enters the finite boundary response through a classical quasilocal work term.

The analogy with horizon thermodynamics is therefore structural rather than identificatory. The TTS is not a null event horizon. The common point is the macroscopic boundary logic: unresolved internal information is represented by boundary-accessible state variables and response channels, while local physical content appears through resolved deviations from a reference level.

9.5. TTS as a Finite-Response Boundary System

The TTS realization gives the reference-cut construction a concrete finite-response setting. The boundary carries quasilocal shell energy E_Σ , pressure P , area A_Σ , effective input Φ_{eff} , exterior release Φ_{out} , and an area-response channel. The macroscopic balance law

$$\dot{E}_\Sigma = A_\Sigma(\Phi_{\text{eff}} - \Phi_{\text{out}}) - P\dot{A}_\Sigma \quad (71)$$

organizes these quantities on the same timelike boundary.

The effective input

$$\Phi_{\text{eff}} = -\frac{1}{A_\Sigma} \dot{E}_{\text{ref}} \quad (72)$$

is the TTS-specific representation of reference-sector change at the boundary. With the sign convention used here, reference-energy descent, $\dot{E}_{\text{ref}} < 0$, gives positive effective input into the shell.

The shell response then partitions this input into storage, release, and pressure–area work:

$$\text{reference-sector change} \longrightarrow \Phi_{\text{eff}} \longrightarrow (\dot{E}_\Sigma, \Phi_{\text{out}}, P\dot{A}_\Sigma). \quad (73)$$

This gives the TTS an autonomous boundary role. It is a finite-radius timelike system in which reference data, quasilocal stress quantities, area response, and exterior-accessible quantities are organized at the same interface.

The reference-cut language introduced here isolates this local organization and connects it to the general cut-level structure developed above. It does not require a microscopic interpretation of the reference sector. Such an interpretation belongs to a specific closure.

9.6. Relation to Existing Boundary Frameworks

The present construction builds on, but is distinct from, several standard boundary frameworks in gravitational physics.

First, the Brown–York formalism supplies the quasilocal stress tensor and the observer-adapted projection dictionary used in the classical realization of Q_Σ . The new element introduced here is not a new definition of Brown–York energy. It is the organization of selected Brown–York cut quantities into a reference package, a deviation map, a local resolution condition, and a cut-level response structure on the same observer-adapted surface.

Second, the construction is related in spirit to the membrane paradigm and to stretched-horizon descriptions, because it assigns effective boundary quantities and response channels to a timelike surface. However, the timelike boundary considered here is not introduced as a limiting surface just outside a null horizon. It is treated as a finite-radius timelike interface with its own proper time, observer-adapted cuts, coarse-grained reference assignment, and local deviation structure.

Third, the framework differs from local causal horizon constructions. Local causal horizons use null boundary elements and local horizon thermodynamics to relate heat flow, area change, and spacetime dynamics. The present construction instead uses a timelike boundary with spacelike cuts, Brown–York-type quasilocal variables, and finite response channels. Causality enters through the admissible Lorentzian domain $J^-(B)$, while the event-level content is supplied by local resolution of a boundary-relative deviation at B .

Finally, the construction should also be distinguished from dynamical and isolated horizon frameworks. Those frameworks characterize null or spacelike horizon hypersurfaces and their evolution laws. The present framework does not assume that the boundary is a horizon. Its basic object is a finite-radius timelike interface, and its central purpose is to organize reference values, resolved deviations, and response terms on observer-adapted timelike cuts.

Thus the contribution of the present work is not a replacement for these existing boundary frameworks. It is a complementary cut-level reference language for timelike finite-radius boundaries, designed to keep the coarse-grained reference level, local deviations, causal admissibility, and mechanical response on the same boundary surface.

9.7. Outlook

The framework developed here is a classical cut-level reference structure. Its next step is the specification of concrete closures. A thermodynamic closure may fix the normalization α , introduce a boundary temperature, and determine response coefficients for storage, release, and relaxation. A statistical closure may generate the reference values q_Σ^{ref} from an ensemble, maximum-entropy prescription, smoothing operation, or effective reference solution. A finite-thickness closure may replace the idealized interface by a regulated boundary layer with additional relaxation channels.

In all such extensions, the local character of the construction should be preserved. The causal relation $A \in J^-(B)$ specifies the Lorentzian admissibility domain, while the event-level content is the resolved deviation from the coarse-grained reference level at B . Thus future closures should specify the reference assignment, the channel norms, the resolution scales, and the relaxation laws by which local deviations become resolved and return to the macroscopic boundary state.

The structural analogy with horizon thermodynamics may also be developed further. The timelike shell is not a null horizon, but it provides a finite-radius boundary system with proper time, quasilocal stress, area response, and redshifted observables. It may therefore serve as a useful test setting for local thermodynamic readings of gravity on timelike boundaries.

The same cut-level structure may also provide a useful language for later applications in which local records, operational resolution, or measurement-like boundary assignments are considered. Such

extensions would require additional statistical or operational closure rules. The present manuscript does not introduce those rules. It supplies the classical Lorentzian boundary structure on which causal admissibility, reference assignment, resolved deviation, and finite response can be defined together.

10. Conclusion

This manuscript formulated a cut-level reference structure for timelike boundaries in Lorentzian spacetime. Starting from a timelike interface Σ , an observer field u^a , and observer-adapted cuts C_{τ_Σ} , the construction assigns selected boundary quantities, coarse-grained reference structures, channel-specific reference values, resolved deviations, and local event closure to the same geometric surface.

The central object is the coarse-grained boundary reference package

$$GC_\Sigma = (\mathcal{Q}_\Sigma, C_{\tau_\Sigma}, \mathcal{R}_\Sigma). \quad (74)$$

It combines the selected boundary quantities, the local cut on which the comparison is made, and the coarse-grained reference structure used to assign reference values. For any selected quantity $q_\Sigma \in \mathcal{Q}_\Sigma$, the corresponding boundary content is represented by the deviation

$$\mathcal{D}_\Sigma[q] = q_\Sigma - \Pi_q[\mathcal{R}_\Sigma](C_{\tau_\Sigma}) = q_\Sigma - \langle q_\Sigma \rangle_{GC}. \quad (75)$$

This deviation map supplies the local quantity that enters the resolution and event-closure conditions.

In the classical realization developed here, the selected boundary quantities are naturally supplied by the Brown–York cut-level variables. The averaged boundary stress tensor τ_{ab} , together with the observer field and cut projector, defines the surface energy density ε , momentum density j_a , spatial cut stress π_{ab} , and isotropic pressure P . These quantities provide the classical boundary data to which coarse-grained reference values can be assigned.

Local event closure combines the deviation map with the Lorentzian causal structure. A candidate event B is represented by a resolved boundary-relative deviation whose prior data lie in the causal past of B :

$$A \in J^-(B), \quad \|\mathcal{D}_\Sigma[q](B)\|_q \geq \varepsilon_q. \quad (76)$$

Thus the cut-level construction links three ingredients: causal admissibility, local resolution, and reference-relative boundary content.

The same boundary quantities also carry a classical response structure. The cut balance

$$\delta_{\bar{\zeta}} E[C] = \delta_{\bar{\zeta}} Q[C] + \delta_{\bar{\zeta}} W[C] \quad (77)$$

organizes cut-energy variation into normal exchange and tangential mechanical response. In the isotropic spherical reduction, the response term takes the pressure–area form

$$\delta_{\bar{\zeta}} W[C] = -P \delta_{\bar{\zeta}} A[C]. \quad (78)$$

This relation connects the cut-level stress description to area-based boundary thermodynamics.

The timelike thin-shell setting provides a concrete realization of the framework. For spherical cuts, $A_\Sigma = 4\pi R^2$, and the macroscopic area variable

$$S_\Sigma = \alpha A_\Sigma \quad (79)$$

records the coarse-grained area response. The boundary-balance law

$$\dot{E}_\Sigma = A_\Sigma(\Phi_{\text{eff}} - \Phi_{\text{out}}) - P \dot{A}_\Sigma \quad (80)$$

then appears as a finite-response realization of the general cut-level structure: reference-state change is registered at the shell, redistributed into storage and release, and coupled to the area channel through pressure–area work.

The resulting picture is a classical boundary-reference formulation. A timelike boundary is treated as a local cut-level carrier of geometric, quasilocal, and coarse-grained reference data. Its value lies in making the reference level, the resolved deviations, the causal event structure, and the mechanical boundary response part of one common Lorentzian boundary framework.

Appendix A. Notation Summary

This appendix summarizes the symbols used repeatedly in the cut-level reference framework. The notation is grouped according to boundary geometry, Brown–York cut quantities, reference data, local resolution, and the TTS realization.

Table A1. Notation summary for the cut-level reference framework.

Symbol	Meaning
<i>Boundary geometry</i>	
Σ	timelike boundary or interface
g_{ab}	spacetime metric on M
n^a	spacelike unit normal to Σ
h_{ab}	induced Lorentzian metric on Σ
D_a	covariant derivative compatible with h_{ab}
u^a	future-directed observer field tangent to Σ
$C_{\tau\Sigma}$	observer-adapted cut of Σ
σ_{ab}	positive spatial metric induced on $C_{\tau\Sigma}$
$A[C_{\tau\Sigma}]$	area functional of the cut
ξ^a	cut-evolution generator
α	lapse associated with the cut flow
η	local coordinate in the normal direction away from Σ
y^A	local coordinates on the spatial cut $C_{\tau\Sigma}$
<i>Brown–York cut quantities</i>	
τ_{ab}^{\pm}	one-sided Brown–York boundary stress tensor
τ_{ab}	averaged Brown–York boundary stress tensor used for cut-level projections
ε	cut energy surface density measured by u^a
j_a	cut momentum surface density
π_{ab}	spatial cut stress
s_{ab}	trace-free spatial cut stress, $s_{ab} = \pi_{ab} - P\sigma_{ab}$
P	isotropic cut pressure, $P = \frac{1}{2}\sigma^{ab}\pi_{ab}$
$E[C]$	cut energy
Φ	normal energy-flux density through the interface
$\langle\Phi\rangle$	two-sided averaged normal energy-flux density
$\delta_{\xi}Q[C]$	exchange contribution to cut-energy variation
$\delta_{\xi}W[C]$	mechanical response contribution to cut-energy variation
<i>Reference structure and deviations</i>	
\mathcal{Q}_{Σ}	selected set of boundary quantities
$\mathcal{Q}_{\Sigma, \text{BY}}$	Brown–York realization of the selected boundary quantities
\mathcal{R}_{Σ}	coarse-grained boundary reference structure assigned to the cut

Continued on next page

Symbol	Meaning
$\Pi_q[\mathcal{R}_\Sigma]$	channel-specific reading of the reference structure in the selected channel q
GC_Σ	boundary reference package, $GC_\Sigma = (Q_\Sigma, C_{\tau_\Sigma}, \mathcal{R}_\Sigma)$
$\langle q_\Sigma \rangle_{GC}$	coarse-grained reference value assigned to q_Σ
q_Σ^{ref}	channel-specific reference value of q_Σ , $q_\Sigma^{\text{ref}} = \Pi_q[\mathcal{R}_\Sigma] = \langle q_\Sigma \rangle_{GC}$
$\mathcal{D}_\Sigma[q]$	boundary-relative deviation, $\mathcal{D}_\Sigma[q] = q_\Sigma - q_\Sigma^{\text{ref}}$ in the additive realization
$\ \mathcal{D}_\Sigma[q]\ _q$	channel norm of the deviation in the selected q_Σ channel
ϵ_q	local resolution scale for q_Σ
A	preparation, source, or prior boundary event
B	candidate local boundary event
$J^-(B)$	causal past of the candidate event B
<i>Spherical TTS realization</i>	
$R(\tau_\Sigma)$	areal radius of the spherical shell as a function of shell proper time
A_Σ	shell area, $A_\Sigma = 4\pi R^2$
S_Σ	macroscopic area-response variable, $S_\Sigma = \alpha A_\Sigma$
E_Σ	quasilocal shell energy
E_{ref}	coarse-grained reference energy
Φ_{eff}	effective boundary input in the TTS balance realization
Φ_{out}	exterior release flux in the TTS balance realization
T_Σ	local shell temperature in a thermodynamic closure
C_{eff}	trajectory-dependent reference-boundary response coefficient in a finite-response closure

Appendix B. Cut-Level Brown–York Balance Formulae

This appendix collects the cut-level balance identities used in Section 7. The notation follows the Brown–York boundary-stress formulation on a timelike interface and the observer-adapted cut geometry fixed in Sections 2 and 3. The formulae are written for the normal cut flow $\zeta^a = \alpha u^a$ and for the slab orientation and work convention used in the main text.

The boundary stress tensor entering the balance is the averaged Brown–York tensor τ_{ab} introduced in Eq. (17). The normal energy-flux density through the interface is defined by

$$\Phi := -T^{ab}u_a n_b. \quad (\text{A1})$$

The brackets $\langle \cdot \rangle$ denote the corresponding two-sided average across the interface.

With the cut flow

$$\tilde{\zeta}^a = \alpha u^a, \quad (\text{A2})$$

the Brown–York cut-energy balance may be written as

$$\delta_{\tilde{\zeta}} E[C] = \int_C \sqrt{\sigma} \alpha \left(\langle \Phi \rangle - \tau^{ab} D_a u_b \right) d^2 x. \quad (\text{A3})$$

Here D_a is the covariant derivative compatible with the boundary metric h_{ab} . The first term is the normal exchange contribution. The second term is the mechanical response associated with the kinematics of the observer field on the boundary.

Accordingly, the balance is written in the form

$$\delta_{\tilde{\zeta}} E[C] = \delta_{\tilde{\zeta}} Q[C] + \delta_{\tilde{\zeta}} W[C], \quad (\text{A4})$$

with

$$\delta_{\xi} Q[C] = \int_C \sqrt{\sigma} \alpha \langle \Phi \rangle d^2x, \quad (\text{A5})$$

$$\delta_{\xi} W[C] = - \int_C \sqrt{\sigma} \alpha \tau^{ab} D_a u_b d^2x. \quad (\text{A6})$$

Thus $\delta_{\xi} Q[C]$ is the normal exchange contribution to the cut-energy variation, while $\delta_{\xi} W[C]$ is the tangential mechanical response contribution.

The sign in Eq. (A6) is the work convention used throughout the manuscript. With this convention, an isotropic expansion of the cut gives the standard pressure–area form $-P \delta_{\xi} A[C]$.

To see this reduction explicitly, decompose the cut stress using the observer-adapted split of Section 3.1. For isotropic symmetry reductions with vanishing cut momentum and isotropic spatial cut stress,

$$j_a = 0, \quad \pi_{ab} = P \sigma_{ab}. \quad (\text{A7})$$

The area variation of the cut under the flow $\zeta^a = \alpha u^a$ is

$$\delta_{\xi} A[C] = \int_C \sqrt{\sigma} \alpha \sigma^{ab} D_a u_b d^2x. \quad (\text{A8})$$

Under the assumptions in Eq. (A7), the mechanical response term becomes

$$\delta_{\xi} W[C] = - \int_C \sqrt{\sigma} \alpha P \sigma^{ab} D_a u_b d^2x. \quad (\text{A9})$$

For a uniform isotropic pressure on the cut, this reduces to

$$\delta_{\xi} W[C] = -P \delta_{\xi} A[C]. \quad (\text{A10})$$

This is the pressure–area form used in Sections 3.2, 7, and 8.

References

1. Bekenstein, J.D. Black Holes and Entropy. *Physical Review D* **1973**, *7*, 2333–2346. <https://doi.org/10.1103/PhysRevD.7.2333>.
2. Hawking, S.W. Particle Creation by Black Holes. *Communications in Mathematical Physics* **1975**, *43*, 199–220. <https://doi.org/10.1007/BF02345020>.
3. Jacobson, T. Thermodynamics of Spacetime: The Einstein Equation as an Equation of State. *Physical Review Letters* **1995**, *75*, 1260–1263, [gr-qc/9504004]. <https://doi.org/10.1103/PhysRevLett.75.1260>.
4. Israel, W. Singular Hypersurfaces and Thin Shells in General Relativity. *Il Nuovo Cimento B* **1966**, *44*, 1–14. <https://doi.org/10.1007/BF02710419>.
5. Poisson, E. *A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics*; Cambridge University Press: Cambridge, 2004. <https://doi.org/10.1017/CBO9780511606601>.
6. Schubert, A.G. Timelike Thin-Shell Evolution in Gravitational Collapse: Classical Dynamics and Thermodynamic Interpretation. *Entropy* **2026**, *28*, 96. <https://doi.org/10.3390/e28010096>.
7. Schubert, A.G. Coarse-Grained Vacuum Boundaries in Gravitational Collapse: A Timelike Thin-Shell Balance Framework, 2026. Preprint, posted 28 April 2026, <https://doi.org/10.20944/preprints202604.1941.v1>.
8. Brown, J.D.; York, J.W. Quasilocal energy and conserved charges derived from the gravitational action. *Physical Review D* **1993**, *47*, 1407–1419. <https://doi.org/10.1103/PhysRevD.47.1407>.
9. Szabados, L.B. Quasi-Local Energy-Momentum and Angular Momentum in General Relativity. *Living Reviews in Relativity* **2009**, *12*. <https://doi.org/10.12942/lrr-2009-4>.
10. Wald, R.M. *General Relativity*; University of Chicago Press: Chicago, 1984.

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.