Short Note

Not peer-reviewed version

Ultrametric Bourgain-Figiel-Milman, Enflo Type and Mendel-Naor Cotype Problems

K. Mahesh Krishna *

Posted Date: 9 April 2026

doi: 10.20944/preprints202604.0674.v1

Keywords: bourgain-figiel-milman theorem; enflo type; mendel-naor cotype; ultrametric space

doi:10.20944/preprints202604.0674.v1

Short Note

Ultrametric Bourgain-Figiel-Milman, Enflo Type and Mendel-Naor Cotype Problems

K. Mahesh Krishna

School of Mathematics and Natural Sciences, Chanakya University Global Campus, NH-648, Haraluru Village, Devanahalli Taluk, Bengaluru North District, Karnataka State, 562 110, India; kmaheshak@gmail.com

Abstract

We ask for ultrametric version of following three: (1) Bourgain-Figiel-Milman Theorem, (2) Enflo Type, (3) Mendel-Naor Cotype.

Keywords: bourgain-figiel-milman theorem; enflo type; mendel-naor cotype; ultrametric space

MSC: 47B10; 47L20; 26E30; 46S10; 47S10; 12J25

---

1. Ultrametric Bourgain-Figiel-Milman Problem

Let $\mathcal{X}$ and $\mathcal{Y}$ be finite dimensional normed linear spaces such that $\dim(\mathcal{X}) = \dim(\mathcal{Y})$. The **Banach-Mazur distance** between $\mathcal{X}$ and $\mathcal{Y}$ is defined as

$$d_{BM}(\mathcal{X},\mathcal{Y}) := \inf\{\|T\|\|T^{-1}\| : T : \mathcal{X} \to \mathcal{Y} \text{ is invertible linear operator}\}.$$

For $n \in \mathbb{N}$, let $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ be the standard Euclidean Hilbert space. Following wonderful result says that every finite dimensional Banach space has a subspace which is close (in the Banach-Mazur distance) to the Euclidean space.

Theorem 1. [1,6] ***(Dvoretzky-Milman Theorem)*** *There is a universal constant $C > 0$ satisfying the following property: If $\mathcal{X}$ is any n-dimensional real Banach space and $0 < \varepsilon < \frac{1}{3}$, then for every natural number*

$$k \leq C\frac{\varepsilon^2}{|\log \varepsilon|}\log n,$$

there exists a k-dimensional Banach subspace $\mathcal{Y}$ of $\mathcal{X}$ such that

$$d_{BM}(\mathcal{Y}, (\mathbb{R}^k, \langle\cdot,\cdot\rangle)) < 1 + \varepsilon.$$

We refer [8–13,22,23,26–28,30–32,34] for more information on Dvoretzky-Milman theorem. With Theorem 1, we ask about the metric space version of that. Let $\mathcal{M}$ and $\mathcal{N}$ be finite metric spaces with $o(\mathcal{M}) = o(\mathcal{N})$. Given a map $f : \mathcal{M} \to \mathcal{N}$, define

$$\|f\|_{\text{Lip}} := \sup\left\{\frac{d(f(x), f(y))}{d(x,y)} : x, y \in \mathcal{M}, x \neq y\right\}.$$

The **Bourgain-Figiel-Milman distance** between $\mathcal{M}$ and $\mathcal{N}$ is defined as

$$d_{BFM}(\mathcal{M},\mathcal{N}) := \inf\{\|f\|_{\text{Lip}}\|f^{-1}\|_{\text{Lip}} : f : \mathcal{M} \to \mathcal{N} \text{ is injective}\}.$$

In 1986, Bourgain, Figiel and Milman proved the following nonlinear Dvoretzky theorem [3].

 doi:10.20944/preprints202604.0674.v1

Theorem 2. *[3] (**Bourgain-Figiel-Milman Theorem**) For every $\varepsilon > 0$, there is a universal constant $C(\varepsilon) > 0$ satisfying the following: Every finite metric space $\mathcal{M}$ contains a subset $\mathcal{N}$ satisfying following conditions.*

(i) *There is an injective map $f : \mathcal{N} \to \ell^2(\mathbb{N}, \mathbb{R})$ such that*

$$d_{BFM}(\mathcal{N}, f(\mathcal{N})) \leq 1 + \varepsilon.$$

(ii)

$$o(\mathcal{N}) \geq C(\varepsilon) \log(o(\mathcal{M})).$$

Further, following holds: If $\varepsilon < 1$, then we can take

$$C(\varepsilon) = \frac{c_1 \varepsilon}{\log(\frac{c_2}{\varepsilon})} \quad \textit{for some } c_1 > 0, c_2 > 0.$$

We will formulate problem based on Theorem 2 to ultrametric spaces.

Definition 1. *[33] Let $\mathcal{M}$ be a set. A map $d : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ is said to be an **ultrametric** if it satisfies following conditions.*

(i) $d(z, z) = 0$ *for all* $z \in \mathcal{M}$. *If* $x, y \in \mathcal{M}$ *are such that* $d(x, y) = 0$, *then* $x = y$.

(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in \mathcal{M}$.

(iii) $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ *for all* $x, y, z \in \mathcal{M}$.

*In this case, $(\mathcal{M}, d)$ is called as an **ultrametric space**.*

Let $\mathbb{K}$ be a non-Archimedean valued field. We define

$$c_0(\mathbb{N}, \mathbb{K}) := \left\{ \{a_n\}_n : a_n \in \mathbb{K}, \forall n \in \mathbb{N}, \lim_{n \to \infty} a_n = 0 \right\}$$

equipped with the ultra-norm

$$\|\{a_n\}_n\| := \max_{n \in \mathbb{N}} |a_n|, \quad \forall \{a_n\}_n \in c_0(\mathbb{N}, \mathbb{K})$$

and the p-adic inner product

$$\langle \{a_n\}_n, \{b_n\}_n \rangle := \sum_{n=1}^{\infty} a_n b_n, \quad \forall \{a_n\}_n, \{b_n\}_n \in c_0(\mathbb{N}, \mathbb{K}).$$

Then $c_0(\mathbb{N}, \mathbb{K})$ is a non-Archimedean Banach space which is also a p-adic Hilbert space [17,29]. We now ask for ultrametric version of Bourgain-Figiel-Milman theorem.

Problem 1. *(**Ultrametric Bourgain-Figiel-Milman Problem**) $\mathbb{K}$ be a non-Archimedean valued field. For every $\varepsilon > 0$, whether there is a universal constant $C(\varepsilon, \mathbb{K}) > 0$ satisfying the following: Every finite ultrametric space $\mathcal{M}$ contains a subset $\mathcal{N}$ satisfying following conditions.*

(i) *There is an injective map $f : \mathcal{N} \to c_0(\mathbb{N}, \mathbb{K})$ such that*

$$d_{BFM}(\mathcal{N}, f(\mathcal{N})) \leq 1 + \varepsilon.$$

(ii)

$$o(\mathcal{N}) \geq C(\varepsilon, \mathcal{A}) \log(o(\mathcal{M})).$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 9 April 2026 doi:10.20944/preprints202604.0674.v1

2. Ultrametric Enflo Type and Mendel-Naor Cotype Problems

Let $\mathcal{H}$ be a Hilbert space, $n \in \mathbb{N}$. Recall that for any n points $h_1, \dots, h_n \in \mathcal{H}$, we have

$$\frac{1}{2^n} \sum_{\varepsilon_1,\dots,\varepsilon_n \in \{-1,1\}} \left\| \sum_{j=1}^{n} \varepsilon_j h_j \right\|^2 = \sum_{j=1}^{n} \|h_j\|^2. \tag{1}$$

Equality (1) motivated the definition of Type and Cotype for Banach spaces.

Definition 2. *[1] Let $1 \leq p \leq 2$. A Banach space $\mathcal{X}$ is said to be of **(Rademacher) Type** p if there exists $T_p(\mathcal{X}) > 0$ such that*

$$\left(\frac{1}{2^n} \sum_{\varepsilon_1,\dots,\varepsilon_n \in \{-1,1\}} \left\| \sum_{j=1}^{n} \varepsilon_j x_j \right\|^p \right)^{\frac{1}{p}} \leq T_p(\mathcal{X}) \left(\sum_{j=1}^{n} \|x_j\|^p \right)^{\frac{1}{p}}, \quad \forall x_1, \dots, x_n \in \mathcal{X}, \ \forall n \in \mathbb{N}.$$

Definition 3. *[1] Let $2 \leq q < \infty$. A Banach space $\mathcal{X}$ is said to be of **(Rademacher) Cotype** q if there exists $C_q(\mathcal{X}) > 0$ such that*

$$\left(\sum_{j=1}^{n} \|x_j\|^q \right)^{\frac{1}{q}} \leq C_q(\mathcal{X}) \left(\frac{1}{2^n} \sum_{\varepsilon_1,\dots,\varepsilon_n \in \{-1,1\}} \left\| \sum_{j=1}^{n} \varepsilon_j x_j \right\|^q \right)^{\frac{1}{q}}, \quad \forall x_1, \dots, x_n \in \mathcal{X}, \ \forall n \in \mathbb{N}.$$

There is a vast literature on Type-Cotype of Banach spaces, see [1,5,14–16,19–21,24,35].

In 1970, Enflo introduced the notion of Type for metric spaces [7,25].

Definition 4. *[7,25] Let $p > 0$. A metric space $(\mathcal{M}, d)$ is said to be of Enflo Type p if there exists $E_p(\mathcal{M}) > 0$ satisfying following conditions: For every $n \in \mathbb{N}$ and for every $f : \{-1,1\}^n \to \mathcal{M}$ it holds*

$$\left(\frac{1}{2^n} \sum_{\varepsilon_1,\dots,\varepsilon_n \in \{-1,1\}} d(f(\varepsilon_1,\dots,\varepsilon_n), d(-\varepsilon_1,\dots,-\varepsilon_n))^p \right)^{\frac{1}{p}} \leq$$

$$E_p(\mathcal{M}) \left(\sum_{j=1}^{n} \frac{1}{2^n} \sum_{\varepsilon_1,\dots,\varepsilon_n \in \{-1,1\}} d(f(\varepsilon_1,\dots,\varepsilon_{j-1},\varepsilon_j,\varepsilon_{j+1},\dots,\varepsilon_n), d(\varepsilon_1,\dots,\varepsilon_{j-1},-\varepsilon_j,\varepsilon_{j+1},\dots,\varepsilon_n))^p \right)^{\frac{1}{p}}.$$

In 2008, Mendel and Naor introduced the notion of Cotype for metric spaces [25].

Definition 5. *[25] Let $q > 0$. A metric space $(\mathcal{M}, d)$ is said to be of Mendel-Naor Cotype q if there exists $\Gamma_q(\mathcal{M}) > 0$ satisfying following conditions: For every $n \in \mathbb{N}$, there exists an even integer $m \in \mathbb{N}$ such that for every $f : \mathbb{Z}_m^n \to \mathcal{M}$ it holds*

$$\begin{aligned} &\sum_{j=1}^{n} \frac{1}{m^n} \sum_{x \in \mathbb{Z}_m^n} d\left(f\left(x + \frac{m}{2}e_j, f(x)\right)\right)^q \\ &\leq \Gamma_q(\mathcal{M})^q m^q \frac{1}{m^n} \sum_{x \in \mathbb{Z}_m^n} \frac{1}{3^n} \sum_{\varepsilon \in \{-1,0,1\}^n} d(f(x+\varepsilon), f(x))^q, \quad \varepsilon = (\varepsilon_1, \dots, \varepsilon_n), \end{aligned}$$

where $\{e_j\}_{j=1}^n$ is the standard basis for $\mathbb{R}^n$.

We now ask for ultrametric versions of Enflo Type and Mendel-Naor Cotype.

Problem 2. ***(Ultrametric Enflo Type Problem)*** *Whether there is a way to define ultrametric Enflo Type?*

Problem 3. ***(Ultrametric Mendel-Naor Cotype Problem)*** *Whether there is a way to define ultrametric Mendel-Naor Cotype?*

Note that there are also notions of Type for metric spaces by Bourgain, Milman and Wolfson [4] and by Ball [2]. These lead to following problems.

Problem 4. ***(Ultrametric Bourgain-Milman-Wolfson Type Problem)*** *Whether there is a way to define ultrametric Bourgain-Milman-Wolfson Type?*

Problem 5. ***(Ultrametric Markov/Ball Type Problem)*** *Whether there is a way to define ultrametric Markov/Ball Type?*

Remark 1. *In 2026, we formulated non-Archimedean John, non-Archimedean Dvoretzky-Milman, non-Archimedean Type-Cotype and non-Archimedean Kwapien problems which are still open [18].*

References

1. Fernando Albiac and Nigel J. Kalton. *Topics in Banach space theory*, volume 233 of *Graduate Texts in Mathematics*. Springer, [Cham], 2016.
2. K. Ball. Markov chains, Riesz transforms and Lipschitz maps. *Geom. Funct. Anal.*, 2(2):137–172, 1992.
3. J. Bourgain, T. Figiel, and V. Milman. On Hilbertian subsets of finite metric spaces. *Isr. J. Math.*, 55:147–152, 1986.
4. J. Bourgain, V. Milman, and H. Wolfson. On type of metric spaces. *Trans. Am. Math. Soc.*, 294:295–317, 1986.
5. Joe Diestel, Hans Jarchow, and Andrew Tonge. *Absolutely summing operators*, volume 43 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995.
6. Aryeh Dvoretzky. Some results on convex bodies and Banach spaces. In *Proc. Internat. Sympos. Linear Spaces (Jerusalem, 1960)*, pages 123–160. Jerusalem Academic Press, Jerusalem; Pergamon, Oxford, 1961.
7. Per Enflo. On the nonexistence of uniform homeomorphisms between L^p-spaces. *Ark. Mat.*, 8:103–105, 1970.
8. T. Figiel. Some remarks on Dvoretzky's theorem on almost spherical sections of convex bodies. *Colloq. Math.*, 24:241–252, 1971/72.
9. T. Figiel. A short proof of Dvoretzky's theorem. In *Séminaire Maurey-Schwartz 1974–1975: Espaces L^p, applications radonifiantes et géométrie des espaces de Banach, Exp. No. XXIII*, pages 6 pp. (erratum, p. 3). 1975.
10. T. Figiel. A short proof of Dvoretzky's theorem on almost spherical sections of convex bodies. *Compositio Math.*, 33(3):297–301, 1976.
11. T. Figiel, J. Lindenstrauss, and V. D. Milman. The dimension of almost spherical sections of convex bodies. *Acta Math.*, 139(1-2):53–94, 1977.
12. Y. Gordon, O. Guedon, and M. Meyer. An isomorphic Dvoretzky's theorem for convex bodies. *Studia Math.*, 127(2):191–200, 1998.
13. Yehoram Gordon. Gaussian processes and almost spherical sections of convex bodies. *Ann. Probab.*, 16(1):180–188, 1988.
14. Paata Ivanisvili, Ramon van Handel, and Alexander Volberg. Rademacher type and Enflo type coincide. *Ann. Math. (2)*, 192(2):665–678, 2020.
15. W. B. Johnson and J. Lindenstrauss, editors. *Handbook of the geometry of Banach spaces. Vol. I*. North-Holland Publishing Co., Amsterdam, 2001.
16. W. B. Johnson and J. Lindenstrauss, editors. *Handbook of the geometry of Banach spaces. Vol. 2*. North-Holland, Amsterdam, 2003.
17. G. K. Kalisch. On p-adic Hilbert spaces. *Ann. Math. (2)*, 48:180–192, 1947.
18. K. Mahesh Krishna. Non-Archimedean John, non-Archimedean Dvoretzky-Milman, non-Archimedean type-cotype and non-Archimedean Kwapien problems. *Preprints: https://doi.org/10.20944/preprints202603.1137.v1*, 2026.
19. Rafal Latala and Krzysztof Oleszkiewicz. On the best constant in the Khinchin-Kahane inequality. *Studia Math.*, 109(1):101–104, 1994.
20. Daniel Li and Herve Queffelec. *Introduction to Banach spaces: analysis and probability. Vol. 1*, volume 166 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2018.

doi:10.20944/preprints202604.0674.v1

21. Daniel Li and Herve Queffelec. *Introduction to Banach spaces: analysis and probability. Vol. 2*, volume 167 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2018.
22. J. Matousek. *Lectures on discrete geometry*, volume 212 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2002.
23. Ivan Matsak and Anatolij Plichko. Dvoretzky's theorem by Gaussian method. In *Functional analysis and its applications*, volume 197 of *North-Holland Math. Stud.*, pages 171–184. Elsevier Sci. B. V., Amsterdam, 2004.
24. Bernard Maurey. Type, cotype and K-convexity. In *Handbook of the geometry of Banach spaces. Volume 2*, pages 1299–1332. Amsterdam: North-Holland, 2003.
25. Manor Mendel and Assaf Naor. Metric cotype. *Ann. Math. (2)*, 168(1):247–298, 2008.
26. V. Milman. Dvoretzky's theorem—thirty years later. *Geom. Funct. Anal.*, 2(4):455–479, 1992.
27. V. D. Milman. A new proof of A. Dvoretzky's theorem on cross-sections of convex bodies. *Funkcional. Anal. i Priložen.*, 5(4):28–37, 1971.
28. Vitali D. Milman and Gideon Schechtman. *Asymptotic theory of finite-dimensional normed spaces*, volume 1200 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986.
29. C. Perez-Garcia and W. H. Schikhof. *Locally convex spaces over non-Archimedean valued fields*, volume 119 of *Camb. Stud. Adv. Math.* Cambridge: Cambridge University Press, 2010.
30. Gilles Pisier. Probabilistic methods in the geometry of Banach spaces. In *Probability and analysis (Varenna, 1985)*, volume 1206 of *Lecture Notes in Math.*, pages 167–241. Springer, Berlin, 1986.
31. Gilles Pisier. *The volume of convex bodies and Banach space geometry*, volume 94 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1989.
32. Gideon Schechtman. A remark concerning the dependence on ϵ in Dvoretzky's theorem. In *Geometric aspects of functional analysis (1987–88)*, volume 1376 of *Lecture Notes in Math.*, pages 274–277. Springer, Berlin, 1989.
33. W. H. Schikhof. *Ultrametric calculus. An introduction to p-adic analysis*, volume 4 of *Camb. Stud. Adv. Math.* Cambridge: Cambridge University Press, 2006.
34. Andrzej Szankowski. On Dvoretzky's theorem on almost spherical sections of convex bodies. *Israel J. Math.*, 17:325–338, 1974.
35. S. J. Szarek. On the best constants in the Khinchin inequality. *Studia Math.*, 58(2):197–208, 1976.