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Haircut Capital Allocation as Solution of a Quadratic Optimization Problem

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Abstract: The capital allocation framework proposed by Dhaene et al. [1] presents capital allocation principles as solutions to particular optimization problems and provides a general solution of the quadratic allocation problem via a geometric proof. However, the widely used haircut allocation principle is not reconcilable with that optimization setting. In this paper we provide an alternative proof of the quadratic allocation problem based on the Lagrange multipliers method to reach the general solution. We show that the haircut allocation principle can be accommodated to the optimization setting with the quadratic optimization criterion if one of the original conditions is relaxed. Two examples are provided to illustrate the accommodation of this allocation principle.

Keywords: capital allocation problem; risk management; optimization; haircut principle; risk sharing

1. Introduction

Risk in financial and actuarial applications is sometimes defined as a random variable (r.v.) associated with costs or losses. Capital allocation problems arise when a total amount associated with the aggregate risk has to be distributed across the multiple units of risk that make it up. Examples of capital allocation problems can be found, for instance, in asset allocation strategies for portfolio selection, the allocation of the total solvency capital requirement across business lines or when distributing total claims administration costs among policies in the portfolio, among others.

A capital allocation principle is a set of guidelines that indicates how the total amount must be allocated. There is an extensive amount of capital allocation principles proposed in the literature. Some capital allocation principles have been motivated based on game theory in which capital allocation problems are interpreted as coalition games. In that context, the Aumann–Shapley value is one of the most popular capital allocation rules [2–5]. An alternative approach to derive capital allocation principles emerges from the economy theory. Capital allocation problems are interpreted as optimization problems in which a loss function of particular interest for risk managers is minimized [6–9]. Under this second approach, Dhaene et al. [1] provided a unified theoretical framework in which a capital allocation principle is the outcome of a particular optimization problem. This framework was later generalized by Zaks and Tsanakas [10] considering a hierarchical corporate structure at two organizational levels. More recently, Cai and Wang [11] considered different loss functions for capital shortfall risk and capital surplus risk when allocating capital among business lines in the hierarchical corporate structure.

In the optimization setting proposed by Dhaene et al. [1], the optimization problem has a unique solution when the quadratic optimization criterion is followed. The solution of the quadratic allocation criterion is derived via a geometric proof. In this article we provide an alternative proof of the solution to the quadratic allocation problem based on the Lagrangian method. To our knowledge, this proof has not been previously provided in the literature. Dhaene et al. [1] and Zaks and Tsanakas [10] followed geometric approaches to obtain solutions to their quadratic optimization problems. On the other hand, Cai and Wang [11] used the Lagrangian method, but their optimization problem was based on the absolute allocation criterion. That is, their loss function was based on absolute deviations, allowing for different weighting functions to apply to positive and negative deviations.

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A second contribution here is that we accommodate the haircut allocation principle into the capital allocation setting provided by Dhaene et al. [1]. Most of capital allocation principles used in practice can be accommodated to the framework proposed by Dhaene et al. [1]. However, the haircut allocation principle did not seem to be reconcilable with their general framework (Table 1 in[1]). The haircut allocation principle has been widely used in the industry due to its simplicity [12]. Under the haircut allocation principle, the portion of the aggregate capital allocated to a risk unit is computed as the proportion that the Value-at-Risk (VaR) associated to this risk unit represents in relation to the sum of VaR's for all risk units. In this paper, we prove that the haircut allocation principle can be accommodated into the quadratic optimization criterion by relaxing one of the original conditions of Dhaene et al. [1] . The general optimization framework of Dhaene et al. [1] depends on a set of non-negative auxiliary random variables with expected value equal to one which are used as weight factors to the (scaled) deviations between losses and allocated risk capitals. Previously, Belles-Sampera et al. [13] suggested a mechanism to accommodate the haircut allocation principle into the quadratic optimization framework by allowing auxiliary random variables to take negative values. However, as appointed by Cai and Wang [11], when the auxiliary random variables take negative values the loss function could be concave and the optimization problem may not have minimizers. In addition, the proof of Proposition 1 of Belles-Sampera et al. [13] was based on Theorem 1 of Dhaene et al. [1], which can be only applied with non-negative auxiliary random variables with expected value equal to one. Inspired by Belles-Sampera et al. [13], we here define a particular form of the auxiliary random variables from which the haircut allocation principle is derived. We show that the solution exits and it is unique. So, we demonstrate that the haircut allocation principle can be understood as the solution of a quadratic optimization problem. Finally, two particular examples are provided where the haircut allocation principle is obtained. The paper is structured as follows. The general optimal capital allocation framework is defined in the next section and the proof of the solution of the quadratic allocation criterion via the Lagrangian method is showed. Section 3 provides the steps to accommodate the haircut allocation principle in this framework, as the solution to a quadratic optimization problem. Two examples are provided in Section 4. Section 5 concludes.

2. Risk capital allocation as a quadratic optimization problem

Assume that a capital $K > 0$ has to be allocated across n business units denoted by $j = 1, ..., n$. The random variable X_j with finite expectation refers to the loss associated with the j−business. According to [1] (see Remark 2), most capital allocation problems can be described as the optimization problem given by

$$\min_{K_1,K_2,...,K_n} \sum_{j=1}^{n} v_j \mathbb{E}\left[\zeta_j D\left(\frac{X_j - K_j}{v_j}\right)\right] \quad s.t. \quad \sum_{j=1}^{n} K_j = K, \tag{1}$$

with the following characterizing elements:

(a) a function $D : \mathbb{R} \to \mathbb{R}^+$;
(b) a set of positive values v_j, $j = 1, ..., n$; and
(c) a set of random variables ζ_j such that $\mathbb{E}\left[\zeta_j\right] > 0$, $j = 1, ..., n$.

If $D(x) = x^2$ is selected then the optimization criterion in (1) is called the *quadratic optimization criterion*.

Proposition 1. *The solution of the minimization problem proposed in the general framework defined in* (1) *under the quadratic optimization criterion is*

$$K_i = \frac{\mathbb{E}\left[\zeta_i X_i\right]}{\mathbb{E}\left[\zeta_i\right]} + \frac{\frac{v_i}{\mathbb{E}\left[\zeta_i\right]}}{\sum_{j=1}^{n} \frac{v_j}{\mathbb{E}\left[\zeta_j\right]}} \left(K - \sum_{j=1}^{n} \frac{\mathbb{E}\left[\zeta_j X_j\right]}{\mathbb{E}\left[\zeta_j\right]}\right), \quad \textit{for} \quad i = 1, ..., n. \tag{2}$$

Proof of Proposition 1. Let rewrite expression (1) when $D(x) = x^2$ in the following way:

$$\min_{K_1,K_2,\ldots,K_n} \sum_{j=1}^{n} \mathbb{E}\left[\frac{\zeta_j}{\mathbb{E}\left[\zeta_j\right]} \frac{\left(X_j - K_j\right)^2}{\frac{v_j}{\mathbb{E}\left[\zeta_j\right]}} \right] \quad s.t. \quad \sum_{j=1}^{n} K_j = K. \tag{3}$$

Now, let $\eta_j = \frac{\zeta_j}{\mathbb{E}\left[\zeta_j\right]}$ and $w_j = \frac{v_j}{\mathbb{E}\left[\zeta_j\right]}$ for all $j = 1, \ldots, n$, so $\mathbb{E}\left[\eta_j\right] = 1$ for all j. The expression (3) can be rewritten as

$$\min_{K_1,K_2,\ldots,K_n} \sum_{j=1}^{n} \mathbb{E}\left[\eta_j \frac{\left(X_j - K_j\right)^2}{w_j} \right] \quad s.t. \quad \sum_{j=1}^{n} K_j = K. \tag{4}$$

Note that η_j is a random variable, while w_j is a constant. A similar procedure inspired in the proof of Theorem 1 by Dhaene et al. [1] is followed. Let consider that,

$$\begin{aligned}
\mathbb{E}\left[\eta_j \left(X_j - K_j\right)^2\right] &= \mathbb{E}\left[\eta_j X_j^2 - 2\eta_j X_j K_j + \eta_j K_j^2\right] \\
&= \mathbb{E}\left[\eta_j X_j^2\right] - 2\mathbb{E}\left[\eta_j X_j\right] K_j + K_j^2 \text{ (because } \mathbb{E}\left[\eta_j\right] = 1 \text{)} \\
&= \mathbb{E}\left[\eta_j X_j^2\right] - 2\mathbb{E}\left[\eta_j X_j\right] K_j + K_j^2 + \mathbb{E}\left[\eta_j X_j\right]^2 - \mathbb{E}\left[\eta_j X_j\right]^2 \\
&= \left(\mathbb{E}\left[\eta_j X_j\right]^2 - 2\mathbb{E}\left[\eta_j X_j\right] K_j + K_j^2\right) + \mathbb{E}\left[\eta_j X_j\right]^2 - \mathbb{E}\left[\eta_j X_j^2\right] \\
&= \left(\mathbb{E}\left[\eta_j X_j\right] - K_j\right)^2 + \mathbb{E}\left[\eta_j X_j\right]^2 - \mathbb{E}\left[\eta_j X_j^2\right].
\end{aligned}$$

The last two elements do not depend on K_j. So, the minimization problem in (4) is equivalent to

$$\min_{K_1,K_2,\ldots,K_n} \sum_{j=1}^{n} \frac{\left(\mathbb{E}\left[\eta_j X_j\right] - K_j\right)^2}{w_j} \quad s.t. \quad \sum_{j=1}^{n} K_j = K. \tag{5}$$

Following a similar strategy to Zaks et al. [8], the notation $x_j = \frac{K_j - \mathbb{E}\left[\eta_j X_j\right]}{\sqrt{w_j}}$ is introduced[1]. Note that

$$\sum_{j=1}^{n} \sqrt{w_j} x_j = \sum_{j=1}^{n} K_j - \sum_{j=1}^{n} \mathbb{E}\left[\eta_j X_j\right],$$

so, the optimization problem (5) is equivalent to

$$\min_{x_1,x_2,\ldots,x_n} \sum_{j=1}^{n} x_j^2 \quad s.t. \quad \sum_{j=1}^{n} \sqrt{w_j} x_j = K - \sum_{j=1}^{n} \mathbb{E}\left[\eta_j X_j\right]. \tag{6}$$

The selected method to solve problem (6) is the Lagrange multipliers' method. Consider the Lagrangian function

$$\mathcal{L}(\lambda, x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} x_j^2 + \lambda \left(\sum_{j=1}^{n} \sqrt{w_j} x_j - K + \sum_{j=1}^{n} \mathbb{E}\left[\eta_j X_j\right] \right)$$

[1] Note that $\sqrt{w_j}$ is properly defined since $w_j = \frac{v_j}{\mathbb{E}\left[\zeta_j\right]} > 0$.

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The partial derivatives of function $\mathcal{L}$ with respect to x_i and λ are

$$\frac{\partial \mathcal{L}}{\partial x_i} = 2x_i + \lambda \sqrt{w_i}, \quad i = 1, \ldots, n, \tag{7}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \sum_{j=1}^{n} \sqrt{w_j} x_j - K + \sum_{j=1}^{n} \mathbb{E}\left[\eta_j X_j\right]. \tag{8}$$

From equaling the first partial derivative (7) to zero, we obtain

$$x_{0j} = -\frac{\lambda_0}{2} \sqrt{w_j}. \tag{9}$$

Then, $x_{0j} = -\frac{\lambda_0}{2} \sqrt{w_j}$ and from (8) equal to zero, we obtain

$$-\frac{\lambda_0}{2} = \frac{\left[K - \sum_{j=1}^{n} \mathbb{E}\left[\eta_j X_j\right]\right]}{\sum_{j=1}^{n} w_j},$$

and substituting in (9),

$$x_{0i} = \frac{\sqrt{w_i}}{\sum_{j=1}^{n} w_j} \left[K - \sum_{j=1}^{n} \mathbb{E}\left[\eta_j X_j\right]\right]. \tag{10}$$

The objective function and constraints in (6) are convex functions, so the solution is unique. Changing notation from (6) to (5), then (10) can be expressed as

$$\frac{K_i - \mathbb{E}\left[\eta_i X_i\right]}{\sqrt{w_i}} = \frac{\sqrt{w_i}}{\sum_{j=1}^{n} w_j} \left[K - \sum_{j=1}^{n} \mathbb{E}\left[\eta_j X_j\right]\right].$$

The solution to problems (4) and (5) is

$$K_i = \mathbb{E}\left[\eta_i X_i\right] + \frac{w_i}{\sum_{j=1}^{n} w_j} \left[K - \sum_{j=1}^{n} \mathbb{E}\left[\eta_j X_j\right]\right].$$

Finally, the solution of problem (3) is

$$K_i = \frac{\mathbb{E}\left[\zeta_i X_i\right]}{\mathbb{E}\left[\zeta_i\right]} + \frac{\frac{v_i}{\mathbb{E}\left[\zeta_i\right]}}{\sum_{j=1}^{n} \frac{v_j}{\mathbb{E}\left[\zeta_j\right]}} \left[K - \sum_{j=1}^{n} \frac{\mathbb{E}\left[\zeta_j X_j\right]}{\mathbb{E}\left[\zeta_j\right]}\right].$$

□

A proof that solution $\mathbf{x_0} = (x_{01}, x_{02}, \ldots, x_{0n})$ in (10) is a minimum is provided. The bordered Hessian matrix of $\mathcal{L}$, $H_{\mathcal{L}}(\lambda, \mathbf{x})$ is:

$$H_{\mathcal{L}}(\lambda, \mathbf{x}) = \begin{pmatrix} 2 & 0 & \ldots & 0 & \sqrt{w_1} \\ 0 & 2 & \ldots & 0 & \sqrt{w_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 2 & \sqrt{w_n} \\ \sqrt{w_1} & \sqrt{w_2} & \ldots & \sqrt{w_n} & 0 \end{pmatrix}$$

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The characteristics of $H_{\mathcal{L}}(\lambda, \mathbf{x})$ matrix do not depend neither on λ nor on $\mathbf{x}$. The point $\mathbf{x_0} = (x_{01}, x_{02}, \ldots, x_{0n})$ with $x_{0i} = \frac{\sqrt{w_i}}{\sum_{j=1}^{n} w_j}\left[K - \sum_{j=1}^{n}\mathbb{E}\left[\eta_j X_j\right]\right]$ for all i, $i = 1, \ldots, n$, is a minimum if all minors Δ_k

$$\Delta_k = \begin{vmatrix} 2 & 0 & \ldots & 0 & \sqrt{w_1} \\ 0 & 2 & \ldots & 0 & \sqrt{w_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 2 & \sqrt{w_k} \\ \sqrt{w_1} & \sqrt{w_2} & \ldots & \sqrt{w_k} & 0 \end{vmatrix} \tag{11}$$

of $H_{\mathcal{L}}(\lambda_0, \mathbf{x_0})$ have sign equal to -1 for $k = 2, \ldots, n$. As it is shown in the Appendix A, Δ_k are equal to:

$$\Delta_k = -\left(2^{k-1}\sum_{j=1}^{k} w_j\right), \quad \forall k = 2, \ldots, n.$$

So, it is satisfied that $sign(\Delta_k) = -1$ for all $k \geq 2$, because $\sum_{j=1}^{k} w_j > 0$ for all $k \geq 2$ due to $w_j > 0$ for all j. Therefore, $\mathbf{x_0}$ is a minimum in (6).

An alternative proof is given by Dhaene et al. [1]. Authors indicate that problem (6) can be understood as finding the closest point to the origin that belongs to the hyperplane

$$\left\{(x_1, x_2, \ldots, x_n) \mid \sum_{j=1}^{n}\sqrt{w_j}x_j = K - \sum_{j=1}^{n}\mathbb{E}\left[\eta_j X_j\right]\right\}.$$

So, the solution $\mathbf{x_0}$ in (10) is unique and a minimum.

Remark 1. *The proof of the Proposition 1 requires that* $\frac{v_j}{\mathbb{E}[\zeta_j]} > 0$, $j = 1, \ldots, n$. *This is satisfied with conditions (b) and (c). However, a more general framework may be defined with the conditions (b) and (c) expressed as follows:*

(b) a set of weights v_j, $j = 1, ..., n$; *and*
(c) a set of random variables ζ_j, $j = 1, ..., n$, *with* $\frac{v_j}{\mathbb{E}[\zeta_i]} > 0$.

and the proof of the proposition still holds. However, the interpretation of a negative weight v_j *and a negative expected value of* ζ_j *in the context of risk management is not as straightforward as with positive values.*

Remark 2. *The original allocation problem proposed by Dhaene et al. [1] considered (b) and (c) in (1) as follows,*

(b) a set of non-negative weights v_j, $j = 1, ..., n$, *such that* $\sum_{i=1}^{n} v_i = 1$; *and*
(c) a set of non-negative random variables ζ_j, $j = 1, ..., n$, *with* $\mathbb{E}\left[\zeta_j\right] = 1$.

Under these constraints, solution (2) can be simplified as,

$$K_i = \mathbb{E}\left[\zeta_i X_i\right] + v_i\left(K - \sum_{j=1}^{n}\mathbb{E}\left[\zeta_j X_j\right]\right), \quad \textit{for all} \quad i = 1, ..., n.$$

3. Haircut allocation principle

In this section it is showed that the haircut allocation principle can be accommodated into the capital allocation setting (1).

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The haircut allocation principle is defined as follows. If a capital $K > 0$ has to be allocated across n business units, the haircut allocation principle states that the capital K_i assigned to each business unit is

$$K_i = K \frac{F_{X_i}^{-1}(\alpha)}{\sum_{j=1}^{n} F_{X_j}^{-1}(\alpha)} \quad \forall i = 1, \ldots, n, \tag{12}$$

where X_i is the random loss linked to the ith-business unit, $F_{X_i}^{-1}$ is the inverse of the cumulative distribution function of X_i and $\alpha \in (0,1)$ is a given confidence level.

To accommodate the haircut allocation principle into (1), we first introduce the following lemma.

Lemma 1. *Consider a constant $c \in \mathbb{R}$ and two random variables X and Y such that $\mathbb{E}[X] < \infty$, $\mathbb{E}[Y] < \infty$, $\mathbb{E}[XY] < \infty$ and $\mathbb{E}[XY] \neq \mathbb{E}[X]\mathbb{E}[Y]$. Let us define ζ as,*

$$\zeta = \frac{(Y - \mathbb{E}[Y])\, c + \mathbb{E}[XY] - \mathbb{E}[X]\, Y}{\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]} \tag{13}$$

which satisfies

a) $\mathbb{E}[\zeta] = 1$, *and*
b) $\mathbb{E}[\zeta X] = c$.

Proof of Lemma 1. Taking expectations in (13),

$$\begin{aligned}\mathbb{E}[\zeta] &= \frac{(\mathbb{E}[Y] - \mathbb{E}[Y])\, c + \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]}{\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]} \\ &= \frac{\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]}{\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]} \\ &= 1 \quad .\end{aligned}$$

Now, the numerator of ζ is multiplied by X,

$$(XY - X\mathbb{E}[Y])\, c + \mathbb{E}[XY]\, X - XY\mathbb{E}[X] .$$

The expectation of the previous expression is,

$$(\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y])\, c + \mathbb{E}[XY]\mathbb{E}[X] - \mathbb{E}[XY]\mathbb{E}[X] = (\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y])\, c.$$

Therefore, it holds that:

$$\begin{aligned}\mathbb{E}[\zeta X] &= \frac{(\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y])\, c}{\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]} \\ &= c \quad .\end{aligned}$$

□

Remark 3. *The r.v. ζ defined in (13) may take non-positive values. Note that values of ζ lie on the straight line:*

$$z = ay + b$$

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where

$$a = \frac{c - \mathbb{E}[X]}{\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]} \quad \text{and} \quad b = \frac{\mathbb{E}[XY] - c\mathbb{E}[Y]}{\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]} \quad .$$

If $c \neq \mathbb{E}[X]$, the intersection of the line with the horizontal axis, $z = 0$, is given by

$$y = \frac{-b}{a} = \frac{c\mathbb{E}[Y] - \mathbb{E}[XY]}{c - \mathbb{E}[X]} \quad .$$

Therefore, there are four scenarios in which ζ takes non-positive values depending on the value taken by Y. The four scenarios are summarized in Table 1.

Table 1. Scenarios that r.v. ζ defined in (13) takes non-positive values.

	$\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] > 0$	$\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] < 0$
$c - \mathbb{E}[X] > 0$	$Y(\omega) \leq \frac{c\mathbb{E}[Y] - \mathbb{E}[XY]}{c - \mathbb{E}[X]}$	$Y(\omega) \geq \frac{c\mathbb{E}[Y] - \mathbb{E}[XY]}{c - \mathbb{E}[X]}$
$c - \mathbb{E}[X] < 0$	$Y(\omega) \geq \frac{c\mathbb{E}[Y] - \mathbb{E}[XY]}{c - \mathbb{E}[X]}$	$Y(\omega) \leq \frac{c\mathbb{E}[Y] - \mathbb{E}[XY]}{c - \mathbb{E}[X]}$

Let us consider the following proposition.

Proposition 2. *The three characterizing elements required to represent the haircut allocation principle (12) in the general framework defined by (3) are:*

(a) $D(x) = x^2$,
(b) $v_i = \dfrac{\mathbb{E}[\zeta_i X_i]}{\sum_{j=1}^{n} \mathbb{E}\left[\zeta_j X_j\right]}$, *$i = 1, \ldots, n$; and*
(c) $\zeta_i = \dfrac{(Y_i - \mathbb{E}[Y_i]) F_{X_i}^{-1}(\alpha) + \mathbb{E}[X_i Y_i] - \mathbb{E}[X_i] Y_i}{\mathbb{E}[X_i Y_i] - \mathbb{E}[X_i]\mathbb{E}[Y_i]}$, *where Y_i is a random variable such that $\mathbb{E}[X_i Y_i] \neq \mathbb{E}[X_i]\mathbb{E}[Y_i]$, for all $i = 1, \ldots, n$.*

Proof of Proposition 2. By Lemma 1, $\mathbb{E}[\zeta_i] = 1$ and $\mathbb{E}[\zeta_i X_i] = F_{X_i}^{-1}(\alpha)$ for all i, so the general solution (2) is equal to,

$$\begin{aligned} K_i &= \mathbb{E}[\zeta_i X_i] + \frac{\mathbb{E}[\zeta_i X_i]}{\sum_{j=1}^{n} \mathbb{E}\left[\zeta_j X_j\right]}\left[K - \sum_{j=1}^{n} \mathbb{E}\left[\zeta_j X_j\right]\right] \\ &= K \frac{\mathbb{E}[\zeta_i X_i]}{\sum_{j=1}^{n} \mathbb{E}\left[\zeta_j X_j\right]} \\ &= K \frac{F_{X_i}^{-1}(\alpha)}{\sum_{j=1}^{n} F_{X_j}^{-1}(\alpha)} \quad , \end{aligned} \tag{14}$$

which is the haircut allocation principle (12). □

Remark 4. *Note that the condition* $\sum_{j=1}^{n} \mathbb{E}\left[\zeta_j X_j\right] \neq 0$ *is implicitly assumed in Proposition 2 to obtain well-defined weights* v_i*. In fact, the equivalent condition* $\sum_{j=1}^{n} F_{X_j}^{-1}(\alpha) \neq 0$ *is implicitly required to apply the haircut allocation principle (12).*

4. Examples of ζ in the haircut allocation

In this section two examples of random variable Y that could be used in the definition of ζ in Proposition 2 to obtain the haircut allocation principle are provided.

Example 1. *Suppose that* $\inf\{X_i\} < F_{X_i}^{-1}(\alpha) < \sup\{X_i\}$ *and* $F_{X_i}^{-1}(\alpha) > \mathbb{E}[X_i]$ *for all* $i = 1, \ldots, n$*. If* $Y_i = \mathbb{1}\left[X_i \mid X_i \leq F_{X_i}^{-1}(\alpha)\right]$*, then* ζ_i *is defined as follows to represent the haircut allocation principle (12) in the general framework defined by (1):*

$$\zeta_i = \frac{\left(\mathbb{1}\left[X_i \leq F_{X_i}^{-1}(\alpha)\right] - \alpha\right) F_{X_i}^{-1}(\alpha) + \alpha \mathbb{E}\left[X_i \mid X_i \leq F_{X_i}^{-1}(\alpha)\right] - \mathbb{1}\left[X_i \leq F_{X_i}^{-1}(\alpha)\right] \mathbb{E}\left[X_i\right]}{\alpha \mathbb{E}\left[X_i \mid X_i \leq F_{X_i}^{-1}(\alpha)\right] - \alpha \mathbb{E}\left[X_i\right]}$$

Proof of Example 1. Intuitively, we can state that r.v. Y_i satisfies the necessary conditions stated in Lemma (1) because $\left\{\omega \mid X_i(\omega) \leq F_{X_i}^{-1}(\alpha)\right\} \subsetneq \Omega$, due to the assumption that $\inf\{X_i\} < F_{X_i}^{-1}(\alpha) < \sup\{X_i\}$, and also because $F_{X_i}^{-1}(\alpha) > \mathbb{E}\left[X_i\right]$ (a detailed proof is provided in the Appendix B). So,

$$\begin{aligned} \mathbb{E}\left[X_i Y_i\right] &= \mathbb{E}\left[X_i \mathbb{1}\left[X_i \leq F_{X_i}^{-1}(\alpha)\right]\right] \\ &= \alpha \mathbb{E}\left[X_i \mid X_i \leq F_{X_i}^{-1}(\alpha)\right] \\ \text{(see the Appendix B)} &\neq \alpha \mathbb{E}\left[X_i\right] \\ &= \mathbb{E}\left[Y_i\right] \mathbb{E}\left[X_i\right] \quad . \end{aligned}$$

By Lemma 1, it holds that:

$$\begin{aligned} \mathbb{E}\left[\zeta_i\right] &= 1 \text{ , and} \\ \mathbb{E}\left[\zeta_i X_i\right] &= F_{X_i}^{-1}(\alpha) \quad . \end{aligned}$$

Taking into account these results and assumptions of Proposition 2, the solution of the problem (1) given by expression (2) is the haircut allocation principle (12). □

Some remarks can be made in relation to Example 1.

Remark 5. *Conditions* $\inf\{X_i\} < F_{X_i}^{-1}(\alpha) < \sup\{X_i\}$ *for all* $i = 1, \ldots, n$ *are often read as 'random variable* X_i *has a bounded risk at* α *confidence level'. In addition, considering that positive values of* X_i *represent losses, if* $F_{X_i}^{-1}(\alpha) \leq 0$ *then there is no risk of loss for the ith random variable at* α *confidence level.*

Remark 6. *Example 1 accommodates the haircut allocation principle in (almost) the original framework of Dhaene et al. [1]. This 'almost' is for those cases in which* $F_{X_i}^{-1}(\alpha) < \mathbb{E}\left[X_i\right]$ *and also because* ζ_i *in Example 1 is not restricted to be a positive random variable.*

An additional example of a random variable Y_i satisfying conditions of Proposition 2 is provided.

Example 2. *Suppose* X_i *is a non-negative r.v. with finite variance* ($0 < \mathbb{V}\left[X_i\right] = \mathbb{E}[X^2] - \mathbb{E}\left[X\right]^2 < +\infty$) *for* $i = 1, \ldots, n$*. The r.v* Y_i *is defined as* $Y_i = X_i$*, then expression of* ζ_i *is:*

$$\zeta_i = \frac{\left(X_i - \mathbb{E}\left[X_i\right]\right) F_{X_i}^{-1}(\alpha) + \mathbb{E}[X_i^2] - X_i \mathbb{E}[X_i]}{\mathbb{V}\left[X_i\right]} \quad .$$

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Note that $\mathbb{E}[X_i Y_i] \neq \mathbb{E}[X_i]\mathbb{E}[Y_i]$ *because the variance is greater than zero.*

Remark 7. *According to Dhaene et al. [1], a proportional capital allocation principle is a 'business unit driven proportional allocation principle' when* ζ_i *depends on* X_i*, and an 'aggregate portfolio driven proportional allocation principle' when* ζ_i *depends on* $S = \sum_{i=1}^{n} X_i$*, for* $i = 1, \ldots, n$*. Under this classification the haircut allocation principle defined in Proposition 2 is a 'business unit driven proportional allocation principle.' Let suposse* Y_i *is equal to* S *for all* i *in the definition of* ζ_i*. Since the r.v.* S *can be mathematically dependent of* X_i *in some cases, the haircut allocation principle would be classified as 'business unit driven proportional allocation principle' and also 'aggregate portfolio driven proportional allocation principle,' which seems counter-intuitive. Here, we propose to define a 'aggregate portfolio driven' as follows:* ζ_i *and* $\mathbb{E}[\zeta_i X_i]$ *must depend on* S*. Now, the haircut allocation principle is a 'business unit driven proportional allocation principle' but not a 'aggregate portfolio driven proportional allocation principle.'*

5. Conclusions

In this paper we generalize the capital allocation framework proposed by Dhaene et al. [1]. We prove that the haircut capital allocation principle can now be accommodated in that general optimization framework. Under this general capital allocation setting, we provide an alternative and interpretable form to obtain the optimal solution to the quadratic optimization problem that complements the existing geometrical proof. All required steps to obtain the optimal solution to that capital allocation framework are described in order to be easy to follow by a broad (not necessarily expert) audience. We argue that the majority of relevant scenarios from a risk management perspective can be represented in our capital allocation framework.

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Appendix A

Let Δ_k, $k \geq 2$, be the determinant in expression (11). We want to show that

$$\Delta_k = -2^{k-1} \sum_{j=1}^{k} w_j, \quad \forall k = 2, \ldots, n.$$

We propose a proof by induction. Starting from $k = 2$,

$$\Delta_2 = \begin{vmatrix} 2 & 0 & \sqrt{w_1} \\ 0 & 2 & \sqrt{w_2} \\ \sqrt{w_1} & \sqrt{w_2} & 0 \end{vmatrix} = (-1)^{3+2}\sqrt{w_2}\begin{vmatrix} 2 & \sqrt{w_1} \\ 0 & \sqrt{w_2} \end{vmatrix} + (-1)^{2+2} 2 \begin{vmatrix} 2 & \sqrt{w_1} \\ \sqrt{w_1} & 0 \end{vmatrix}$$

$$= -2w_2 - 2w_1 = -2(w_1 + w_2) = -2^{2-1} \sum_{j=1}^{2} w_j.$$

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Now, let us assume that $\Delta_{k-1} = -2^{k-2}\sum_{j=1}^{k-1} w_j$. In this case, note that

$$\Delta_k = \begin{vmatrix} 2 & 0 & \dots & 0 & \sqrt{w_1} \\ 0 & 2 & \dots & 0 & \sqrt{w_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 2 & \sqrt{w_k} \\ \sqrt{w_1} & \sqrt{w_2} & \dots & \sqrt{w_k} & 0 \end{vmatrix}$$

$$= (-1)^{k+1+k}\sqrt{w_k}\begin{vmatrix} 2 & 0 & \dots & 0 & \sqrt{w_1} \\ 0 & 2 & \dots & 0 & \sqrt{w_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \sqrt{w_k} \end{vmatrix} + (-1)^{k+k}2\Delta_{k-1}$$

$$= (-1)^{k+1+k}\sqrt{w_k}(-1)^{k+k}\sqrt{w_k}2^{k-1} + (-1)^{k+k}2(-2^{k-2}\sum_{j=1}^{k-1} w_j)$$

$$= -2^{k-1}w_k - 2^{k-1}\sum_{j=1}^{k-1} w_j = -2^{k-1}\sum_{j=1}^{k} w_j. \square$$

Appendix B

This appendix proves that Y_i defined in Example 1 satisfies that $\mathbb{E}[X_iY_i] \neq \mathbb{E}[X_i]\mathbb{E}[Y_i]$, $i = 1, \dots, n$. Note it holds that $\mathbb{E}[X_iY_i] = \mathbb{E}\left[X_i\mathbb{1}\left[X_i \leq F_{X_i}^{-1}(\alpha)\right]\right] = \alpha\mathbb{E}\left[X_i \mid X_i \leq F_{X_i}^{-1}(\alpha)\right]$ and $\mathbb{E}[Y_i]\mathbb{E}[X_i] = \alpha\mathbb{E}[X_i]$. Therefore, $Y_i = \mathbb{1}\left[X_i \leq F_{X_i}^{-1}(\alpha)\right]$ satisfies $\mathbb{E}[X_iY_i] \neq \mathbb{E}[X_i]\mathbb{E}[Y_i]$ is equivalent to prove that:

$$\alpha\mathbb{E}[X_i] - \alpha\mathbb{E}\left[X_i \mid X_i \leq F_{X_i}^{-1}(\alpha)\right] \neq 0 \quad .$$

Let us rewrite the previous inequality as,

$$\mathbb{E}[X_i] - (1-\alpha)\mathbb{E}[X_i] - \alpha\mathbb{E}\left[X_i \mid X_i \leq F_{X_i}^{-1}(\alpha)\right] \neq 0 \quad . \tag{A1}$$

Now, two cases are considered: $F_{X_i}^{-1}(\alpha) > 0$ and $F_{X_i}^{-1}(\alpha) \leq 0$.

Case $F_{X_i}^{-1}(\alpha) > 0$:

Following Belles-Sampera et al. [14] and Denuit et al. [15], $\mathbb{E}\left[X_i\mathbb{1}\left[X_i \leq F_{X_i}^{-1}(\alpha)\right]\right]$ is equal to,

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$$
\begin{aligned}
\mathbb{E}\left[X_i \mathbb{1}\left[X_i \le F_{X_i}^{-1}(\alpha)\right]\right] &= \int_{-\infty}^{F_{X_i}^{-1}(\alpha)} x dF_{X_i} \\
&= \int_{-\infty}^{0} x dF_{X_i} + \int_{0}^{F_{X_i}^{-1}(\alpha)} x dF_{X_i} \\
&= \int_{-\infty}^{0} \left(\int_{0}^{x} dt\right) dF_{X_i} + \int_{0}^{F_{X_i}^{-1}(\alpha)} \left(\int_{0}^{x} dt\right) dF_{X_i} \\
\text{(Fubini's Theorem)} &= \int_{t=-\infty}^{t=0} \left(\int_{x=t}^{x=-\infty} dF_{X_i}\right) dt \\
&+ \int_{t=0}^{t=F_{X_i}^{-1}(\alpha)} \left(\int_{x=t}^{x=F_{X_i}^{-1}(\alpha)} dF_{X_i}\right) dt \\
&= \int_{t=-\infty}^{t=0} \left(\int_{x=-\infty}^{x=t} dS_{X_i}\right) dt + \int_{t=0}^{t=F_{X_i}^{-1}(\alpha)} \left(F_{X_i}\left(F_{X_i}^{-1}(\alpha)\right) - F_{X_i}(t)\right) dt \\
&= \int_{-\infty}^{0} \left(S_{X_i}(t) - 1\right) dt + \int_{0}^{F_{X_i}^{-1}(\alpha)} \left(1 - F_{X_i}(t) - (1-\alpha)\right) dt \\
&= \int_{-\infty}^{0} \left(S_{X_i}(t) - 1\right) dt + \int_{0}^{F_{X_i}^{-1}(\alpha)} S_{X_i}(t) dt - (1-\alpha) F_{X_i}^{-1}(\alpha) \quad .
\end{aligned}
$$

Therefore, expression (A1) can be represented as:

$$
\begin{aligned}
&\mathbb{E}\left[X_i\right] - (1-\alpha)\mathbb{E}\left[X_i\right] - \alpha \mathbb{E}\left[X_i \mid X_i \le F_{X_i}^{-1}(\alpha)\right] \\
=&\mathbb{E}\left[X_i\right] - (1-\alpha)\mathbb{E}\left[X_i\right] - \int_{-\infty}^{0} \left(S_{X_i}(t) - 1\right) dt - \int_{0}^{F_{X_i}^{-1}(\alpha)} S_{X_i}(t) dt + (1-\alpha) F_{X_i}^{-1}(\alpha) \\
&\left(\text{alternative expression of } \mathbb{E}[X_i]\right) \\
=& \int_{-\infty}^{0} \left(S_{X_i}(t) - 1\right) dt + \int_{0}^{+\infty} S_{X_i}(t) dt - (1-\alpha)\mathbb{E}\left[X_i\right] \\
&- \int_{-\infty}^{0} \left(S_{X_i}(t) - 1\right) dt - \int_{0}^{F_{X_i}^{-1}(\alpha)} S_{X_i}(t) dt + (1-\alpha) F_{X_i}^{-1}(\alpha) \\
=& \int_{F_{X_i}^{-1}(\alpha)}^{+\infty} S_{X_i}(t) dt + (1-\alpha)\left(F_{X_i}^{-1}(\alpha) - \mathbb{E}\left[X_i\right]\right) \\
>&0 \quad ,
\end{aligned}
$$

which is positive. The last inequality holds because $S_{X_i} \ge 0, F_{X_i}^{-1}(\alpha) < \sup\{X_i\}, \alpha \ne 1$ and $F_{X_i}^{-1}(\alpha) > \mathbb{E}\left[X_i\right]$ for all $i = 1, \dots, n$. □

Case $F_{X_i}^{-1}(\alpha) \le 0$ **:** $\mathbb{E}\left[X_i \mathbb{1}\left[X_i \le F_{X_i}^{-1}(\alpha)\right]\right]$ is equal to:

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$$
\begin{aligned}
\mathbb{E}\left[X_i \mathbb{1}\left[X_i \leq F_{X_i}^{-1}(\alpha)\right]\right] &= \int_{-\infty}^{F_{X_i}^{-1}(\alpha)} x dF_{X_i} \\
&= \int_{-\infty}^{0} x dF_{X_i} - \int_{F_{X_i}^{-1}(\alpha)}^{0} x dF_{X_i} \\
&= \int_{-\infty}^{0} x dF_{X_i} - \int_{F_{X_i}^{-1}(\alpha)}^{0} \left(\int_{0}^{x} dt\right) dF_{X_i} \\
\text{(Fubini's Theorem)} &= \int_{-\infty}^{0} x dF_{X_i} - \int_{t=F_{X_i}^{-1}(\alpha)}^{t=0} \left(\int_{x=t}^{x=F_{X_i}^{-1}(\alpha)} dF_{X_i}\right) dt \\
&= \int_{-\infty}^{0} x dF_{X_i} - \int_{t=F_{X_i}^{-1}(\alpha)}^{t=0} \left(\int_{x=F_{X_i}^{-1}(\alpha)}^{x=t} dS_{X_i}\right) dt \\
&= \int_{-\infty}^{0} x dF_{X_i} - \int_{F_{X_i}^{-1}(\alpha)}^{0} \left(S_{X_i}(t) - (1-\alpha)\right) dt \\
&= \int_{-\infty}^{0} x dF_{X_i} - \int_{F_{X_i}^{-1}(\alpha)}^{0} \left(S_{X_i}(t) - 1\right) dt + \alpha F_{X_i}^{-1}(\alpha) \\
&= \int_{-\infty}^{0} \left(S_{X_i}(t) - 1\right) dt - \int_{F_{X_i}^{-1}(\alpha)}^{0} \left(S_{X_i}(t) - 1\right) dt + \alpha F_{X_i}^{-1}(\alpha) \\
&= \int_{-\infty}^{F_{X_i}^{-1}(\alpha)} \left(S_{X_i}(t) - 1\right) dt + \alpha F_{X_i}^{-1}(\alpha)
\end{aligned}
$$

Expression (A1) can be now represented as:

$$
\begin{aligned}
&\mathbb{E}\left[X_i\right] - (1-\alpha)\mathbb{E}\left[X_i\right] - \alpha\mathbb{E}\left[X_i \mid X_i \leq F_{X_i}^{-1}(\alpha)\right] \\
=&\mathbb{E}\left[X_i\right] - (1-\alpha)\mathbb{E}\left[X_i\right] - \int_{-\infty}^{F_{X_i}^{-1}(\alpha)} \left(S_{X_i}(t) - 1\right) dt - \alpha F_{X_i}^{-1}(\alpha) \\
&\left(\text{alternative expression of } \mathbb{E}[X_i]\right) \\
=&\int_{-\infty}^{0} \left(S_{X_i}(t) - 1\right) dt + \int_{0}^{+\infty} S_{X_i}(t) dt - (1-\alpha)\mathbb{E}\left[X_i\right] \\
&- \int_{-\infty}^{F_{X_i}^{-1}(\alpha)} \left(S_{X_i}(t) - 1\right) dt - \alpha F_{X_i}^{-1}(\alpha) + F_{X_i}^{-1}(\alpha) - F_{X_i}^{-1}(\alpha) \\
=&(1-\alpha)\left(F_{X_i}^{-1}(\alpha) - \mathbb{E}\left[X_i\right]\right) - F_{X_i}^{-1}(\alpha) + \int_{F_{X_i}^{-1}(\alpha)}^{0} \left(S_{X_i}(t) - 1\right) dt + \int_{0}^{+\infty} S_{X_i}(t) dt \\
&\left(\text{because } S_{X_i}(t) - 1 \geq -1 \text{ and } F_{X_i}^{-1}(\alpha) > \inf\{X_i\} \geq -\infty\right) \\
\geq&(1-\alpha)\left(F_{X_i}^{-1}(\alpha) - \mathbb{E}\left[X_i\right]\right) - F_{X_i}^{-1}(\alpha) - \left[0 - F_{X_i}^{-1}(\alpha)\right] + \int_{0}^{+\infty} S_{X_i}(t) dt \\
=&(1-\alpha)\left(F_{X_i}^{-1}(\alpha) - \mathbb{E}\left[X_i\right]\right) + \int_{0}^{+\infty} S_{X_i}(t) dt \\
>&0 \quad .
\end{aligned}
$$

The last inequality holds because $S_{X_i} \geq 0$, $\alpha \neq 1$, and $F_{X_i}^{-1}(\alpha) > \mathbb{E}\left[X_i\right]$ for all $i = 1, \ldots, n$. □

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