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Geometry and Constants in Finite Ring Continuum

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Abstract
We study the Euclidean prime-shell stage of the Finite Ring Continuum programme through a framed finite field and establish three main components. First, after fixing an extended frame, the shell supports a finite orbital complex combinatorially equivalent to the two-sphere; the labeled complex is frame-presented, while its cellular isomorphism class is shell-invariant. Second, its core Euclidean symmetry axes appear with increasing framing depth: additively framed negation, affinely framed reciprocal inversion, and Euclidean conjugation in the extended frame. Third, standard density and Lipschitz estimates show that sufficiently large prime-shell grids reproduce the observer-side outputs of any fixed bounded Euclidean experiment to any prescribed tolerance. The Fourier formalism is recorded strictly as a discrete Fourier transform over the shell ring, while contact with conventional continuum Fourier language is treated only as an observer-side large-prime approximation.

Keywords: symmetry-complete prime shells; framed involutions; finite geometry; combinatorial two-sphere; operational approximation; discrete Fourier transform over a ring; finitist harmonic analysis; Finite Ring Continuum

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1. Introduction

Symmetry-complete prime shells furnish a finite setting in which geometric orientation, reciprocal duality, and harmonic phase structure are all available without leaving the arithmetic of a finite field. In the foundational algebra paper published in *Axioms* [22], these shells are the first stage at which a Euclidean geometry with nontrivial involutive symmetry axes can be developed inside a strictly finitist formalism. This article studies that Euclidean prime-shell stage.

The paper works with the *Euclidean prime-shell layer*: Symmetry-Complete fields of radius t, written as framed finite fields $\mathbb{F}_{\mathsf{p}}(\mathsf{t}; 0, 1, e_{\mathsf{t}})$ when the full reference-frame data are explicit, their orbital two-dimensional geometry, the framed realization of the shell's involutive symmetry axes, the shell/reference-frame distinction for the constants i, π, and e, and the finite Fourier duality naturally carried by these shells. The orbital complex is studied in a chosen frame, so its longitude labels are frame-relative; what belongs to the shell itself is the resulting cellular isomorphism class and the shell invariants carried by that class.

The classical sphere S^2 is used only as an *observer-side large-*p *comparator*. The internal object is the finite orbital complex $\mathcal{S}_{\mathsf{p}}$, and the comparison theorem is stated at the operational level of finitely many marked points together with finite families of Lipschitz observables. Mathematically, that theorem is a standard density-and-continuity argument applied to the particular grid family supplied by prime shells.

The objective of the FRC framework is to show, step by step, that a strictly finite formalism has the complexity and explanatory power needed to account for the full range of scientific observation, while remaining internally coherent and grounded in first principles. Each paper isolates one necessary layer of that programme. The role of the present paper is to establish the Euclidean prime-shell layer: the first stage at which orientation, reciprocal duality, structural constants, sphere-like geometry, and harmonic phase structure appear together in one finite shell.

The paper pursues four objectives.

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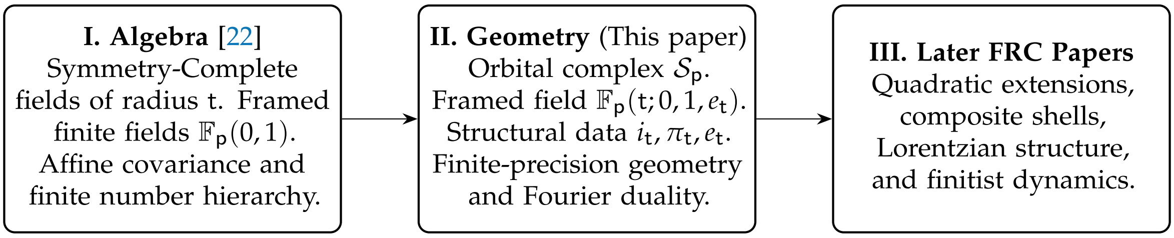

Figure 1. Programme placement of the present revision. Paper II isolates the Euclidean prime-shell layer and leaves extension, causality, and dynamics to later work.

(i) Construct a strictly finitist Euclidean orbital complex $\mathcal{S}_p$ from a Symmetry-Complete field of radius t.
(ii) Identify the core involutive symmetry axes of the Euclidean shell together with their minimal framing depth: additively framed negation $R(x) = -x$, affinely framed reciprocal inversion $I(x) = x^{-1}$ on $\mathbb{F}_p^\times$, and extended-frame Euclidean conjugation C_g.
(iii) Prove an operational finite-precision theorem showing that prime shells model each fixed bounded experiment, consisting of finitely many marked points and a finite family of Lipschitz Euclidean observables, to arbitrary prescribed accuracy.
(iv) Formulate the shell's harmonic duality strictly as a discrete Fourier transform over the finite ring $\mathbb{F}_p$.

The paper is situated in a clear scholarly setting. On the foundational side, it is closer to feasibility and bounded-arithmetic viewpoints, where the distinction between formal existence and operational accessibility is explicit [23,24], and to finitist or non-Cantorian size programmes that seek mathematically coherent alternatives to unrestricted actual infinity [25]. On the geometric side, the orbital complex and the observer-side comparison theorem belong to the broader tradition of finite or combinatorial models approximating continuum geometric outputs, including Regge's coordinate-free discretization of geometry, Dodziuk's combinatorial Hodge theory, and modern discrete differential geometry [26–28]. On the harmonic side, the Fourier formalism used here is not ad hoc: discrete Fourier transforms over finite fields, finite groups, and related finite rings are classical tools in number-theoretic and coding-theoretic analysis [29–32]. These comparisons do not identify FRC with any one prior framework, but they do locate the present claims inside recognizable mathematical terrain.

Relative to earlier work in the FRC series, the present paper isolates the first stage at which the Euclidean shell can be treated as a self-contained object. The published algebra paper [22] establishes the symmetry-complete prime shell, the quarter-turn condition $p \equiv 1 \pmod 4$, the framed-complex layer, and the basic Klein-type orbit structure. The present article builds on that algebraic base and contributes three specifically geometric and harmonic steps: a frame-presented orbital complex combinatorially equivalent to S^2, an explicit bounded-experiment approximation theorem for the resulting prime-shell grids, and a strictly finite Fourier formalism written entirely inside the shell/reference-frame language. In that sense, the novelty here is not a new abstract approximation lemma or a new Fourier identity, but the assembly of those ingredients into one Euclidean prime-shell package.

The paper treats Euclidean symmetries realized within a chosen frame hierarchy. Quadratic extensions, Lorentzian structure, shell transfer, and Dirac-type dynamics are outside the present scope. In particular, the genuinely temporal involution is not identified here with Euclidean conjugation.

Why $p = 4t + 1$.

The congruence $p \equiv 1 \pmod 4$ ensures that -1 is a square in $\mathbb{F}_p$, so the shell carries a quarter-turn subgroup of order 4. This is the minimal algebraic condition required for the Euclidean phase structure studied here. By Dirichlet's theorem there are infinitely many such primes, so the refinement process used in the finite-precision theorem is genuinely unbounded [33, Chapter 13].

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Generality and constraints.

The results are uniform across the whole family of *symmetry-complete prime shells*, i.e. prime fields $\mathbb{F}_p$ with $p = 4t + 1$, once an extended frame $\mathbb{F}_p(t; 0, 1, e_t)$ has been fixed. No claim is made here for arbitrary finite fields, for primes $p \equiv 3 \pmod 4$, or for composite moduli. Those regimes either lack the quarter-turn structure needed for the Euclidean shell developed here or require separate treatment. In particular, when the quarter-turn symmetry fails, the Euclidean shell construction studied in this paper is not asserted to survive unchanged. Thus the paper is genuinely general within its stated class, but it is not presented as a theorem about all finite rings or all finite fields indiscriminately.

What is proved here.

Sections 2–4 define the shell and its structural data, with the shell built from the additive meridian count and the multiplicative phase cycle of the framed finite field. The same sections isolate the core involutive symmetry axes already present at the Euclidean prime-shell stage and organize them by minimal framing depth. Section 5 proves the operational approximation theorem for prime-shell grids. Section 6 records the standard discrete Fourier transform facts needed on longitude cycles of length $p - 1$, specialized to the shell setting and separated from observer-side continuum language. Section 7 discusses the significance and limits of the resulting symmetry architecture, and Section 8 summarizes the main conclusions.

2. Framework: Symmetry-Complete Prime Shells and Frames

We work throughout with odd primes of the form

$$p = 4t + 1,$$

so that t is the radius / chronon parameter and $\mathbb{F}_p$ is a finite field with $\mathrm{card}(\mathbb{F}_p) = p$.

Definition 1 (Symmetry-Complete field of radius t)**.** *A* Symmetry-Complete field of radius t *is a pair* $(t, \mathbb{F}_p)$ *with* $p = 4t + 1$ *prime. Equivalently,* $\mathbb{F}_p^\times$ *is cyclic of order* $4t$*, has a unique subgroup of order* 4*, and, in any framed realization* $\mathbb{F}_p(0, 1)$*, the remaining nonzero elements split into* $t - 1$ *orbital classes of the form*

$$\{x, -x, x^{-1}, -x^{-1}\}.$$

These are the generic orbits of the commuting involution pair generated by the framed formulas for negation and reciprocal inversion, as established already in the published algebra paper [22]. When emphasizing the geometric role of the same object, we also refer to it as a symmetry-complete prime shell*.*

Definition 2 (Extended affine reference frame)**.** *An* extended affine reference frame *on a symmetry-complete prime shell is the framed finite field*

$$\mathbb{F}_p(t; 0, 1, e_t),$$

or equivalently the quadruple $(t; 0, 1, e_t)$*, consisting of the radius / chronon, the distinguished origin* $0 \in \mathbb{F}_p$*, the distinguished unit* $1 \in \mathbb{F}_p$*, and the observer-canonical exponential unit* $e_t \in \mathbb{F}_p^\times$*. Externally,* e_t *is represented by a chosen primitive generator of* $\mathbb{F}_p^\times$*. This extends the framed finite field* $\mathbb{F}_p(0, 1)$ *of the algebra paper by adjoining a phase chart.*

Remark 1. *The shell* $(t, \mathbb{F}_p)$ *and the reference frame* $\mathbb{F}_p(t; 0, 1, e_t)$ *are different kinds of data. The shell determines the finite field, its cyclic unit group, and the structural possibilities available at that grade. The framed field fixes the affine reference pair* $(0, 1)$*, and the extended frame further adds the observer-canonical exponential unit* e_t*, which fixes a phase chart on* $\mathbb{F}_p^\times$*. In the present relational reading, explicit symmetry formulas appear only after the corresponding frame data have been fixed.*

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Remark 2 (Framing depth of explicit symmetries). *The Euclidean prime-shell symmetries used in this paper come with increasing framing depth. Additive negation $x \mapsto -x$ is explicit once the origin 0 is fixed. Reciprocal inversion $x \mapsto x^{-1}$ is explicit once the affine pair $(0,1)$ is fixed. Euclidean conjugation becomes explicit only in the extended frame $\mathbb{F}_{\mathsf{p}}(\mathsf{t};0,1,e_{\mathsf{t}})$, where the observer-canonical exponential unit e_{t} determines a phase chart while the shell's oriented quarter-turn is written i_{t}.*

Remark 3 (Scope of reference-frame variation). *The algebra paper proves full affine covariance under changes of the framed pair $(0,1)$. Here the affine frame is kept fixed, and only the added exponential chart together with meridian orientation is varied. This keeps the notation simple while isolating the geometric structures studied in the present article.*

Definition 3 (Half-period and phase cycle). *For a symmetry-complete prime shell set*

$$\pi_{\mathsf{t}} := \frac{\mathsf{p}-1}{2} = 2\mathsf{t}.$$

The meridian count interval *is*

$$I_{\mathsf{p}} := \{0,1,\ldots,\pi_{\mathsf{t}}\},$$

and the phase cycle *is the cyclic group*

$$\Phi_{\mathsf{p}} := \mathbb{Z}/(\mathsf{p}-1)\mathbb{Z}.$$

Operationally, I_{p} counts discrete meridian steps from north pole to south pole, while Φ_{p} records longitude phases via the observer-canonical exponential unit e_{t}. A point $m \in \Phi_{\mathsf{p}}$ corresponds to the field element $e_{\mathsf{t}}^{m} \in \mathbb{F}_{\mathsf{p}}^{\times}$, so the prime shell contributes the two basic motions used in the geometry below: meridian advance by one count and phase advance by multiplication with e_{t}. In the language of the algebra paper, these are refinements of the additive and multiplicative symmetries of the framed finite field.

Proposition 1 (Primitive generators form one orbit). *If $\mathsf{g} \in \mathbb{F}_{\mathsf{p}}^{\times}$ is primitive, then every primitive generator of $\mathbb{F}_{\mathsf{p}}^{\times}$ has the form*

$$\mathsf{g}^{u}, \qquad u \in (\mathbb{Z}/(\mathsf{p}-1)\mathbb{Z})^{\times}.$$

Hence the set of primitive generators is a torsor for the automorphism group of the cyclic group $\mathbb{F}_{\mathsf{p}}^{\times}$.

Proof. Since $\mathbb{F}_{\mathsf{p}}^{\times}$ is cyclic of order $\mathsf{p}-1$, its generators are exactly the powers g^{u} with $\gcd(u,\mathsf{p}-1) = 1$ [34, Chapter 5]. □

Proposition 2 (Phase-frame change). *Let $\mathsf{g}' = \mathsf{g}^{u}$ with $u \in (\mathbb{Z}/(\mathsf{p}-1)\mathbb{Z})^{\times}$. Then the map*

$$\rho_u \colon \Phi_{\mathsf{p}} \to \Phi_{\mathsf{p}}, \qquad m \mapsto um$$

is a cyclic reindexing of longitude phases. Reversing the meridian orientation acts by

$$\sigma \colon I_{\mathsf{p}} \to I_{\mathsf{p}}, \qquad j \mapsto \pi_{\mathsf{t}} - j.$$

Both reindexings preserve the shell incidence structure defined below.

Proof. The first statement is immediate because u is invertible modulo $\mathsf{p}-1$. The second is the reflection of the interval I_{p}. Both maps are bijections, hence they preserve adjacency once the shell is defined by step relations in $I_{\mathsf{p}} \times \Phi_{\mathsf{p}}$. □

3. Orbital Complex and Euclidean Geometry

The central finite geometry object of this paper is the orbital complex $\mathcal{S}_\mathsf{p}$. It refines, in a stricter finitist form, the orbital complex S_p previewed in the algebra paper. It is built from the meridian counts I_p and the phase cycle Φ_p, and should be understood as the Euclidean prime-shell analogue of a longitude-latitude cellulation of the sphere. Its explicit labeled presentation depends on the chosen extended frame; what the shell determines directly is the corresponding cellular isomorphism class and the invariants carried by that class.

The arithmetic source of the construction is the finite *orbit cylinder*

$$C_\mathsf{p}^{\mathrm{orb}} := I_\mathsf{p} \times \Phi_\mathsf{p},$$

equipped with the two shell motions

$$T(j,m) = (j+1,m), \qquad P(j,m) = (j,m+1),$$

whenever the right-hand side is defined. Here T records one meridian step and P records one multiplicative phase step relative to the observer-canonical exponential unit e_t. The orbital complex itself is obtained by collapsing the two boundary phase cycles $j = 0$ and $j = \pi_\mathsf{t}$ to poles.

This construction is intentionally mixed: the longitude direction is literal phase-orbit data from $\mathbb{F}_\mathsf{p}^\times$, while the meridian direction is the operational count axis supplied by π_t. The paper claims a finitist Euclidean orbital complex built from prime-shell arithmetic and count structure, not that both directions arise from one and the same internal group action.

Definition 4 (Orbital complex)**.** *Fix an extended affine reference frame $\mathbb{F}_\mathsf{p}(\mathsf{t}; 0, 1, e_\mathsf{t})$ on a symmetry-complete prime shell. The* orbital complex $\mathcal{S}_\mathsf{p}$ *is the finite CW complex with:*

(i) *vertices*
$$V(\mathcal{S}_\mathsf{p}) = \{\mathsf{N}, \mathsf{S}\} \cup \{v_{j,m} \mid 1 \le j \le \pi_\mathsf{t} - 1,\ m \in \Phi_\mathsf{p}\},$$

(ii) *meridian edges joining $v_{j,m}$ to $v_{j+1,m}$ for $1 \le j \le \pi_\mathsf{t} - 2$, together with polar edges joining N to $v_{1,m}$ and $v_{\pi_\mathsf{t}-1,m}$ to S,*

(iii) *latitude edges joining $v_{j,m}$ to $v_{j,m+1}$ for $1 \le j \le \pi_\mathsf{t} - 1$,*

(iv) *north triangular faces $[\mathsf{N}, v_{1,m}, v_{1,m+1}]$,*

(v) *south triangular faces $[\mathsf{S}, v_{\pi_\mathsf{t}-1,m+1}, v_{\pi_\mathsf{t}-1,m}]$,*

(vi) *and quadrilateral faces*
$$[v_{j,m}, v_{j,m+1}, v_{j+1,m+1}, v_{j+1,m}]$$
for $1 \le j \le \pi_\mathsf{t} - 2$.

All longitude indices are taken modulo $\mathsf{p} - 1$.

The meridian indexed by $m \in \Phi_\mathsf{p}$ is the path

$$M_m := \{\mathsf{N}, \mathsf{S}\} \cup \{v_{j,m} : 1 \le j \le \pi_\mathsf{t} - 1\},$$

and the latitude indexed by j is the cycle

$$L_j := \{v_{j,m} : m \in \Phi_\mathsf{p}\}.$$

Figure 2 shows a concrete shell visualization in the case $\mathsf{p} = 13$. The gray meridians and orange latitudes display the longitude-latitude cellulation induced by the orbit cylinder, while N, S mark the observer origin and antipode poles, respectively.

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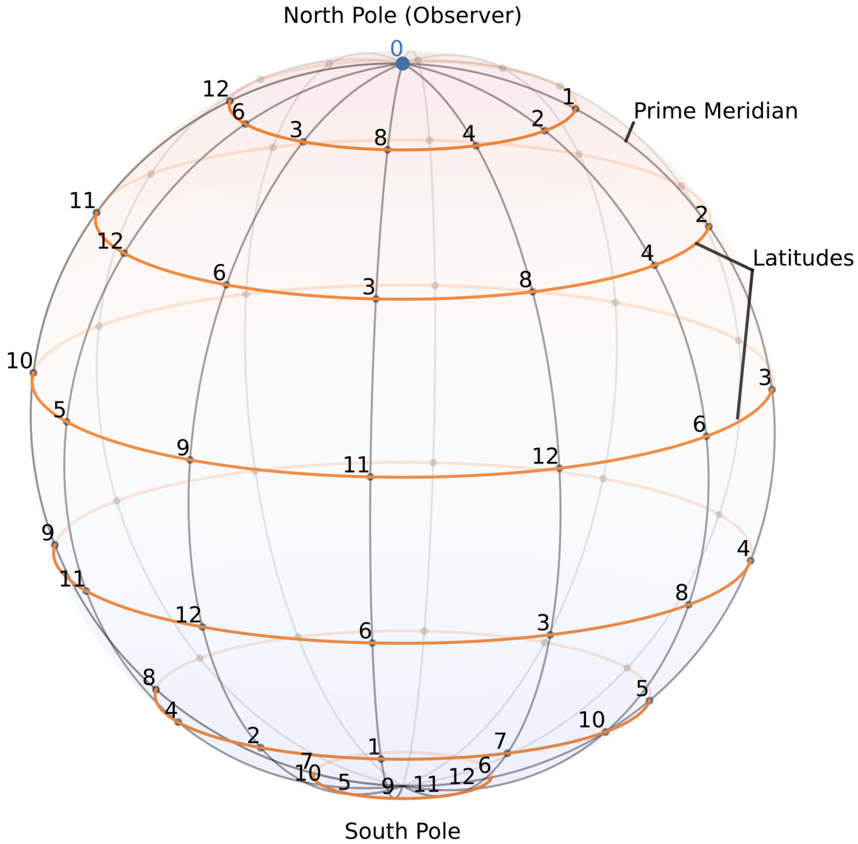

Figure 2. Observer-framed visualization of the orbital complex for the symmetry-complete prime shell $\mathbb{F}_{13}(3;0,1,2)$. Gray curves are meridians, orange curves are latitudes, while N, S mark the observer origin and antipode poles, respectively.

Theorem 1 (Combinatorial 2-sphere)**.** *For every symmetry-complete prime shell and every chosen extended affine reference frame, the resulting orbital complex $\mathcal{S}_p$ is a finite regular CW complex combinatorially isomorphic to S^2.*

Proof. Consider the orbit cylinder

$$C_p := [0, \pi_t] \times (\mathbb{R}/(p-1)\mathbb{Z}),$$

subdivided by the unit lattice in the j- and m-directions. This is the topological realization of the discrete orbit cylinder C_p^{orb}. Collapse the boundary circles

$$\{0\} \times (\mathbb{R}/(p-1)\mathbb{Z}) \quad \text{and} \quad \{\pi_t\} \times (\mathbb{R}/(p-1)\mathbb{Z})$$

to single points N and S, respectively. The quotient is homeomorphic to the suspension of a circle, hence to S^2 [35, Example 0.14]. The vertices, edges, and faces listed in Definition 4 are exactly the cells induced by this quotient cellulation, so $\mathcal{S}_p$ is a finite regular CW complex with the required combinatorial type. $\square$

Remark 4 (Cell counts)**.** *The orbital complex has*

$$\operatorname{card} V(\mathcal{S}_p) = (\pi_t - 1)(p-1) + 2, \qquad \operatorname{card} F(\mathcal{S}_p) = \pi_t(p-1).$$

The edge count is

$$\operatorname{card} E(\mathcal{S}_p) = (2\pi_t - 1)(p-1),$$

so $\chi(\mathcal{S}_\mathsf{p}) = \operatorname{card} V - \operatorname{card} E + \operatorname{card} F = 2$*, as expected for a sphere.*

Proposition 3 (Reference-frame covariance of the orbital complex)**.** *Let* $e'_\mathsf{t} = e^u_\mathsf{t}$ *with* $u \in (\mathbb{Z}/(\mathsf{p}-1)\mathbb{Z})^\times$*. Then*

$$\varphi_u(v_{j,m}) := v'_{j,um}, \qquad \varphi_u(\mathsf{N}) = \mathsf{N}', \qquad \varphi_u(\mathsf{S}) = \mathsf{S}'$$

extends to a cellular isomorphism $\mathcal{S}_\mathsf{p} \to \mathcal{S}'_\mathsf{p}$*. Likewise, meridian reversal* $j \mapsto \pi_\mathsf{t} - j$ *gives a cellular involution exchanging the poles.*

Proof. The map $m \mapsto um$ is a permutation of Φ_p, so it carries latitude and meridian adjacency in $\mathcal{S}_\mathsf{p}$ to the corresponding adjacency in $\mathcal{S}'_\mathsf{p}$. The same holds for the reflection $j \mapsto \pi_\mathsf{t} - j$, which preserves latitude edges and reverses meridians. Hence both maps are cellular automorphisms of the orbital complex geometry. □

Remark 5 (Frame-presented complex versus shell-invariant type)**.** *The orbital complex is not shell-intrinsic as a labeled CW complex. Different choices of the generator* g *produce differently labeled longitude systems, and the identification between them is supplied by the explicit reindexing map* $m \mapsto um$*. Thus the shell determines* $\mathcal{S}_\mathsf{p}$ *only up to a non-unique cellular isomorphism. The shell-level content lies in the resulting isomorphism class together with the numerical and combinatorial invariants carried by that class.*

4. Symmetry Axes and Structural Data

The structural roles of the classical constants are the following:

(i) the shell determines a half-period π_t,
(ii) the shell determines a quarter-turn subgroup $\mathcal{Q}_\mathsf{p}$ and its oriented element i_t,
(iii) the shell admits explicit involutive symmetries only after the corresponding frame data have been fixed,
(iv) the reference frame carries an observer-canonical exponential unit e_t.

Proposition 4 (Shell half-period)**.** *Let* $\pi_\mathsf{t} = (\mathsf{p}-1)/2$*. For every primitive generator* $\mathsf{g} \in \mathbb{F}_\mathsf{p}^\times$*,*

$$\mathsf{g}^{\pi_\mathsf{t}} = -1.$$

Consequently, π_t *is shell-determined data: it depends only on* p*, not on the choice of* g*.*

Proof. $\mathbb{F}_\mathsf{p}^\times$ is cyclic of even order $\mathsf{p}-1$. A cyclic group has a unique element of order 2, namely -1, and the $(\mathsf{p}-1)/2$-th power of any generator is that element. □

Definition 5 (Quarter-turn subgroup)**.** *The* quarter-turn subgroup *of the shell is*

$$\mathcal{Q}_\mathsf{p} := \{x \in \mathbb{F}_\mathsf{p}^\times : x^4 = 1\}.$$

Proposition 5 (Quarter-turn structure)**.** $\mathcal{Q}_\mathsf{p}$ *is the unique subgroup of* $\mathbb{F}_\mathsf{p}^\times$ *of order* 4*. If* g *is a chosen primitive generator, then*

$$\mathsf{g}^{(\mathsf{p}-1)/4} = \mathsf{g}^\mathsf{t}$$

has order 4*, belongs to* $\mathcal{Q}_\mathsf{p}$*, and satisfies*

$$\left(\mathsf{g}^{(\mathsf{p}-1)/4}\right)^2 = -1.$$

Consequently, $\mathcal{Q}_\mathsf{p}$ *contains exactly two square roots of* -1*. We write the distinguished oriented one as* i_t*, so that*

$$i_\mathsf{t}^2 = -1 \qquad \textit{and} \qquad \mathcal{Q}_\mathsf{p} = \{\pm 1, \pm i_\mathsf{t}\}.$$

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Proof. Because $\mathbb{F}_p^\times$ is cyclic of order $4t$, it has a unique subgroup of order 4. The element $g^{(p-1)/4}$ has order 4, hence generates that subgroup. Squaring gives

$$\left(g^{(p-1)/4}\right)^2 = g^{(p-1)/2} = -1$$

by Proposition 4. □

Remark 6 (Structural set in a chosen framed field)**.** *In the notation of the algebra paper, once the framed finite field $\mathbb{F}_p(t;0,1,e_t)$ is fixed, the corresponding structural set is*

$$\{1, i_t, -1, -i_t\} = \{i_t^k : 0 \le k < 4\}.$$

Thus the shell-determined quarter-turn subgroup $\mathcal{Q}_p$ appears in the chosen reference frame as the structural set of fourth roots of unity.

Proposition 6 (No shell-intrinsic exponential unit)**.** *There is no shell-intrinsic distinguished primitive generator e_t. More precisely, if g is primitive, then the primitive generators are exactly*

$$\{g^u : u \in (\mathbb{Z}/(p-1)\mathbb{Z})^\times\},$$

so the choice of an exponential unit belongs to the extended reference frame, not to the shell alone.

Proof. This is Proposition 1. The shell determines the cyclic group $\mathbb{F}_p^\times$, but a generator of that cyclic group is only determined after choosing a phase chart, i.e. a reference frame. □

Remark 7 (Observer-canonical exponential unit)**.** *The external non-uniqueness of primitive generators does not mean that different observers experience different effective exponential bases. In any fixed frame $\mathbb{F}_p(t;0,1,e_t)$, the local exponential unit is read internally as e_t. Under frame transport, the external representative may change, but the transported observer again reads the local affine data as $(0,1)$ and the local exponential unit as e_t. In that sense e_t is not shell-absolute, but it is observer-canonical.*

4.1. Framed involutive symmetries of the Euclidean shell

At the Euclidean prime-shell stage, the natural symmetry axes are involutions. In the present relational reading, all of their explicit formulas are framed. What the shell determines is the availability of these symmetry classes and their orbit structure; the formulas themselves appear only after the necessary frame data have been fixed.

Proposition 7 (Additively framed negation)**.** *The map*

$$R\colon \mathbb{F}_p \to \mathbb{F}_p, \qquad R(x) := 0 - x = -x,$$

is an involution relative to the chosen additive origin 0. Restricted to the additive cycle underlying reversible enumeration, it is the framed realization of shell reversal.

Proof. Clearly $R^2(x) = x$ for every $x \in \mathbb{F}_p$. On the additive structure of $\mathbb{F}_p \cong \mathbb{Z}/p\mathbb{Z}$, the map $x \mapsto -x$ reverses the direction of the cyclic order. □

Remark 8 (Origin dependence)**.** *Additive reversal is only canonical up to translation. Changing the chosen origin conjugates the formula for R by the corresponding translation, so the symmetry class is structural while its explicit coordinate expression is framed.*

Proposition 8 (Affinely framed inversion and four-orbit symmetry). *The map*

$$I\colon \mathbb{F}_{\mathsf{p}}^{\times} \to \mathbb{F}_{\mathsf{p}}^{\times}, \qquad I(x) := 1/x = x^{-1},$$

is an involution of the unit cycle relative to the chosen affine pair $(0,1)$*. Moreover, R and I commute on $\mathbb{F}_{\mathsf{p}}^{\times}$, so they generate a four-element involution action*

$$G_{\mathsf{K}} := \langle R, I\rangle = \{\mathrm{id}, R, I, R\circ I\}.$$

For every $x \in \mathbb{F}_{\mathsf{p}}^{\times} \setminus \mathcal{Q}_{\mathsf{p}}$, the G_{K}-orbit of x is

$$\{x, -x, x^{-1}, -x^{-1}\}.$$

Proof. Since $(x^{-1})^{-1} = x$, the map I is an involution on $\mathbb{F}_{\mathsf{p}}^{\times}$. For $x \in \mathbb{F}_{\mathsf{p}}^{\times}$,

$$I(R(x)) = (-x)^{-1} = -x^{-1} = R(I(x)),$$

so R and I commute. The orbit description is then exactly the orbit of x under the four commuting maps id, R, I, and $R\circ I$. If $x \notin \mathcal{Q}_{\mathsf{p}}$, these four points are distinct; otherwise degeneracies occur only on the quarter-turn subgroup. □

Remark 9 (Framed orbital classes). *Proposition 8 recovers the orbital classes already identified in the published algebra paper [22]. In the present geometry paper, these same involutions supply the framed symmetry backbone of the Euclidean prime shell.*

Remark 10 (Transport under reference-frame change). *If $\mathsf{g}' = \mathsf{g}^{u}$ with $u \in (\mathbb{Z}/(\mathsf{p}-1)\mathbb{Z})^{\times}$, then*

$$\mathsf{g}'^{(\mathsf{p}-1)/4} = \left(\mathsf{g}^{(\mathsf{p}-1)/4}\right)^{u}.$$

Since u is odd, $\mathsf{g}'^{(\mathsf{p}-1)/4}$ is again one of the two square roots of -1 in $\mathcal{Q}_{\mathsf{p}}$. The distinguished shell value i_{t} is not defined by generator choice; it remains fixed once the shell orientation is fixed. Thus changing the generator only reparametrizes the phase chart in which the quarter-turn data are written.

Remark 11 (Framed-complex layer). *Once an extended frame $\mathbb{F}_{\mathsf{p}}(\mathsf{t}; 0, 1, e_{\mathsf{t}})$ is fixed, the shell's oriented quarter-turn i_{t} gives a framed-complex presentation in the sense of the algebra paper [22]. In particular, expressions of the form*

$$a + bi_{\mathsf{t}}, \qquad a, b \in \mathbb{F}_{\mathsf{p}},$$

may be used internally as finitist framed-complex quantities. Thus complex-style elements are available inside the theory after a reference frame has been chosen, even though the phase chart in which they are written still depends on the chosen exponential unit e_{t}.

Proposition 9 (Extended-frame Euclidean conjugation). *In the same framed-complex presentation, one has the frame-relative involution on coordinate expressions*

$$C_{e_{\mathsf{t}}}(a + bi_{\mathsf{t}}) := a - bi_{\mathsf{t}}.$$

This map is an involution of the framed-complex presentation determined by e_{t}. It is entirely internal to the Euclidean shell and should be read as a static orientation-reversing duality of the chosen frame presentation.

Proof. Applying C_{g} twice gives

$$C_{e_{\mathsf{t}}}(C_{e_{\mathsf{t}}}(a + bi_{\mathsf{t}})) = C_{e_{\mathsf{t}}}(a - bi_{\mathsf{t}}) = a + bi_{\mathsf{t}},$$

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so $C_{e_t}^2 = \mathrm{id}$. The definition depends on the chosen exponential unit e_t through the framed-complex coordinate system, hence the involution is frame-relative rather than shell-intrinsic. □

Remark 12 (Hierarchy boundary)**.** *The map C_{e_t} is a Euclidean orientation-reversing involution of the chosen framed-complex presentation. No hierarchy dynamics is assumed here, and the temporal involution studied elsewhere is distinct from this Euclidean conjugation [22].*

Table 2 summarizes the status of the constants used in the present formulation. The shell determines the half-period, the quarter-turn subgroup, and the oriented quarter-turn i_t. In a chosen framed field $\mathbb{F}_p(t; 0, 1, e_t)$, the same quarter-turn data appear as the structural set of fourth roots of unity, while the exponential unit is read internally as e_t: observer-canonical within the frame, but not shell-absolute under external comparison of frames.

Together, Table 1 and Table 2 give the symmetry-centered reading of the Euclidean prime shell used in the remainder of the paper: the shell determines structural symmetry possibilities, while the chosen frame realizes them in explicit coordinate form.

Table 1. Core involutive symmetry axes of the Euclidean prime shell.

symmetry axis	action	status
additive negation	$R(x) = 0 - x$	requires origin 0
reciprocal inversion	$I(x) = 1/x$ on $\mathbb{F}_p^\times$	requires affine pair $(0, 1)$
Euclidean conjugation	$C_{e_t}(a + bi_t) = a - bi_t$	requires extended frame $(0, 1, e_t)$

Table 2. Structural roles of the classical constants in the prime-shell formulation.

classical role	prime-shell object	status
π as half-period	$\pi_t = 2t = (p-1)/2$	shell-determined
i as oriented quarter-turn	$i_t \mid i_t^2 = -1$	shell-determined
e as exponential base	$e_t \mid e_t^{\pi_t} = -1$	observer-canonical, frame-carried

5. Operational Euclidean Geometry to Arbitrary Finite Precision

The orbital complex $\mathcal{S}_p$ of the framed finite field $\mathbb{F}_p(t; 0, 1, e_t)$ is the internal finite geometry object of FRC. The classical sphere enters only as an observer-side comparator used to formulate large-p finite-precision theorems. The approximation theorem in this section is intentionally restricted to finite families of Lipschitz observables on finitely many marked points. In that sense, the section is closer to the established literature on discrete approximation of geometric output — such as Regge calculus, combinatorial Hodge theory, and discrete differential geometry — than to any claim of internal smooth reconstruction [26–28]. Mathematically, the proof is a standard grid-density argument combined with Lipschitz control, specialized here to the particular shell grids arising from symmetry-complete primes.

Definition 6 (Observer-side realization map)**.** *Fix the framed finite field $\mathbb{F}_p(t; 0, 1, e_t)$. For $0 \le j \le \pi_t$ and $m \in \Phi_p$, set*

$$\theta_j := \frac{\pi j}{\pi_t}, \qquad \phi_m := \frac{2\pi m}{p-1}.$$

Define

$$\Xi_{p,e_t}(\mathsf{N}) = (0, 0, 1), \qquad \Xi_{p,e_t}(\mathsf{S}) = (0, 0, -1),$$

and for $1 \le j \le \pi_t - 1$,

$$\Xi_{p,e_t}(v_{j,m}) := \big(\sin\theta_j \cos\phi_m, \sin\theta_j \sin\phi_m, \cos\theta_j\big) \in S^2 \subset \mathbb{R}^3.$$

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The formula for $\Xi_{\mathsf{p},e_{\mathsf{t}}}$ uses only the chart coordinates (j,m); its dependence on e_{t} enters through the meaning of the longitude label m, namely the identification of shell phase with the field element e_{t}^m.

The map $\Xi_{\mathsf{p},e_{\mathsf{t}}}$ does not endow the shell with a continuum ontology. It only compares the finite shell with the classical unit sphere at the level of observer-side outputs.

Proposition 10 (Grid-density estimate)**.** *Let d_{S^2} be the geodesic distance on the unit sphere. Every point $x \in S^2$ lies within distance*

$$\delta_{\mathsf{p}} := \frac{2\pi}{\mathsf{p}-1}$$

of some point in the finite set $\Xi_{\mathsf{p},\mathsf{g}}(V(\mathcal{S}_{\mathsf{p}}))$.

Proof. Write x in spherical coordinates (θ,ϕ) with $0 \le \theta \le \pi$ and $0 \le \phi < 2\pi$. Choose $j \in \{0,\dots,\pi_{\mathsf{t}}\}$ and $m \in \Phi_{\mathsf{p}}$ so that

$$|\theta-\theta_j| \le \frac{\pi}{\mathsf{p}-1}, \qquad |\phi-\phi_m| \le \frac{\pi}{\mathsf{p}-1}.$$

The sphere metric in these coordinates satisfies

$$d_{S^2}\big((\theta,\phi),(\theta',\phi')\big) \le |\theta-\theta'| + |\phi-\phi'|,$$

because the line element is $ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ and $\sin\theta \le 1$. Hence

$$d_{S^2}\big(x, \Xi_{\mathsf{p},e_{\mathsf{t}}}(v_{j,m})\big) \le \frac{2\pi}{\mathsf{p}-1} = \delta_{\mathsf{p}}.$$

If $j=0$, choose the north pole N; if $j=\pi_{\mathsf{t}}$, choose the south pole S. The same estimate holds because those are exactly the images of the boundary meridian circles. $\square$

Definition 7 (H-bounded observable family)**.** *Fix $H \in \mathbb{N}$. An H-*bounded observable family *on r marked points, with $r \le H$, is a finite set $\mathcal{O}$ of functions*

$$O\colon (S^2)^r \to \mathbb{R}$$

such that each $O \in \mathcal{O}$ is H-Lipschitz with respect to the product metric

$$d^{(r)}\big((x_1,\dots,x_r),(y_1,\dots,y_r)\big) := \sum_{k=1}^{r} d_{S^2}(x_k,y_k).$$

This definition captures bounded finite experiments: a finite number of marked points, a finite list of measurements, and a uniform upper bound on experimental sensitivity.

Theorem 2 (Operational finite-precision theorem for prime-shell grids)**.** *Let $H \in \mathbb{N}$, let $r \le H$, let $\mathcal{O}$ be an H-bounded observable family on r marked points, and let $\epsilon > 0$. For every configuration*

$$X = (x_1,\dots,x_r) \in (S^2)^r$$

there exists a symmetry-complete prime shell $(\mathsf{t},\mathbb{F}_{\mathsf{p}})$, a framed finite field $\mathbb{F}_{\mathsf{p}}(\mathsf{t};0,1,e_{\mathsf{t}})$, and vertices $v_1,\dots,v_r \in V(\mathcal{S}_{\mathsf{p}})$ such that

$$\big|O(X) - O\big(\Xi_{\mathsf{p},e_{\mathsf{t}}}(v_1),\dots,\Xi_{\mathsf{p},e_{\mathsf{t}}}(v_r)\big)\big| < \epsilon$$

for every $O \in \mathcal{O}$.

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Proof. By Dirichlet's theorem, there are infinitely many primes $\mathsf{p} \equiv 1 \pmod 4$ [33, Chapter 13]. Choose such a prime large enough that
$$H^2 \delta_{\mathsf{p}} < \epsilon.$$
By Proposition 10, for each k there exists $v_k \in V(\mathcal{S}_{\mathsf{p}})$ with
$$d_{S^2}\left(x_k, \Xi_{\mathsf{p},e_{\mathsf{t}}}(v_k)\right) \le \delta_{\mathsf{p}}.$$
Hence
$$d^{(r)}\Big(X, \left(\Xi_{\mathsf{p},e_{\mathsf{t}}}(v_1), \dots, \Xi_{\mathsf{p},e_{\mathsf{t}}}(v_r)\right)\Big) \le r\delta_{\mathsf{p}} \le H\delta_{\mathsf{p}}.$$
Each observable $O \in \mathcal{O}$ is H-Lipschitz, so
$$\left|O(X) - O\left(\Xi_{\mathsf{p},e_{\mathsf{t}}}(v_1), \dots, \Xi_{\mathsf{p},e_{\mathsf{t}}}(v_r)\right)\right| \le H \cdot H\delta_{\mathsf{p}} < \epsilon.$$
□

Remark 13 (Approximation-theoretic status)**.** *Theorem 2 is not presented as a new approximation method. Its mathematical content is the standard implication*

$$\textit{uniform grid density} + \textit{Lipschitz observables} \Longrightarrow \textit{uniform finite-precision approximation},$$

applied to the specific grid family $\Xi_{\mathsf{p},e_{\mathsf{t}}}(V(\mathcal{S}_{\mathsf{p}}))$*. The shell-specific contribution is that symmetry-complete prime shells supply such grids with* $\delta_{\mathsf{p}} \to 0$ *along primes* $\mathsf{p} \equiv 1 \pmod 4$*, and hence furnish a finitist operational reading of Euclidean approximation within the FRC setting.*

Remark 14 (Meaning of arbitrary precision)**.** *Theorem 2 is the precise finitist interpretation of "Euclidean geometry to arbitrary precision." The quantifiers range only over finite configurations, finite observable families, and positive real tolerances, and the convergence claim is strictly an observer-side large-*p *statement. The internal ontology remains finite at every stage.*

6. Finite Fourier Duality in a Chosen Reference Frame

In this paper Fourier formalism is strictly finitist in the sense of a discrete Fourier transform over a finite ring. The algebraic statements collected here are standard DFT facts specialized to shell phase cycles and recorded explicitly because later constructions use them in framed form. We work in the framed finite field $\mathbb{F}_{\mathsf{p}}(\mathsf{t}; 0, 1, e_{\mathsf{t}})$. The relevant ring is $\mathbb{F}_{\mathsf{p}}$, the relevant transform length is

$$n := \mathsf{p} - 1,$$

and the relevant root of unity is the observer-canonical exponential unit $e_{\mathsf{t}} \in \mathbb{F}_{\mathsf{p}}^{\times}$. Thus the shell transform lives entirely on the finite phase cycle carried by each latitude. This places the section inside the standard finite-field and finite-group Fourier tradition rather than outside it [29–32].

Proposition 11 (Standard root-of-unity facts)**.** *Let* $n = \mathsf{p} - 1$*. The observer-canonical exponential unit* $e_{\mathsf{t}} \in \mathbb{F}_{\mathsf{p}}^{\times}$ *is a principal n-th root of unity in* $\mathbb{F}_{\mathsf{p}}$*, i.e.*

$$e_{\mathsf{t}}^{n} = 1 \qquad \textit{and} \qquad \sum_{j=0}^{n-1} e_{\mathsf{t}}^{jk} = 0 \quad (1 \le k < n).$$

Moreover n is invertible in $\mathbb{F}_{\mathsf{p}}$*, with*
$$n^{-1} = -1.$$

Proof. Because e_{t} has order $\mathsf{p}-1=n$, we have $e_{\mathsf{t}}^n=1$ and $e_{\mathsf{t}}^k\neq 1$ for $1\leq k<n$. Since $\mathbb{F}_{\mathsf{p}}$ is an integral domain, the geometric-series identity gives

$$(e_{\mathsf{t}}^k-1)\sum_{j=0}^{n-1}e_{\mathsf{t}}^{jk}=e_{\mathsf{t}}^{nk}-1=0,$$

and therefore the sum is zero for $1\leq k<n$. Finally, $n=\mathsf{p}-1\equiv -1 \pmod{\mathsf{p}}$, so $n^{-1}=-1$ in $\mathbb{F}_{\mathsf{p}}$. □

Definition 8 (Ring Fourier transform on the phase cycle). *Let $V_{e_{\mathsf{t}}}:=\mathbb{F}_{\mathsf{p}}^n$. For $v=(v_0,\dots,v_{n-1})\in V_{e_{\mathsf{t}}}$, define its* ring Fourier transform

$$\mathcal{F}_{e_{\mathsf{t}}}(v)=(f_0,\dots,f_{n-1})\in V_{e_{\mathsf{t}}}$$

by

$$f_k:=\sum_{j=0}^{n-1}v_je_{\mathsf{t}}^{jk},\qquad 0\leq k<n.$$

Proposition 12 (Standard inverse formula). *For every $v\in V_{e_{\mathsf{t}}}$,*

$$v_j=-\sum_{k=0}^{n-1}\big(\mathcal{F}_{e_{\mathsf{t}}}(v)\big)_k e_{\mathsf{t}}^{-jk},\qquad 0\leq j<n.$$

Equivalently,

$$\mathcal{F}_{e_{\mathsf{t}}}^{-1}(f)_j=n^{-1}\sum_{k=0}^{n-1}f_ke_{\mathsf{t}}^{-jk}.$$

Proof. Let $f=\mathcal{F}_{e_{\mathsf{t}}}(v)$. Then

$$n^{-1}\sum_{k=0}^{n-1}f_ke_{\mathsf{t}}^{-jk}=n^{-1}\sum_{k=0}^{n-1}\sum_{j'=0}^{n-1}v_{j'}e_{\mathsf{t}}^{j'k}e_{\mathsf{t}}^{-jk}=n^{-1}\sum_{j'=0}^{n-1}v_{j'}\sum_{k=0}^{n-1}e_{\mathsf{t}}^{(j'-j)k}.$$

By Proposition 11, the inner sum is 0 when $j'\neq j$ and n when $j'=j$. Hence the right-hand side equals v_j. □

Remark 15 (Matrix formulation). *If $W_{e_{\mathsf{t}}}$ denotes the $n\times n$ matrix with entries*

$$(W_{e_{\mathsf{t}}})_{k,j}:=e_{\mathsf{t}}^{jk},$$

then $\mathcal{F}_{e_{\mathsf{t}}}(v)=W_{e_{\mathsf{t}}}v$ and

$$W_{e_{\mathsf{t}}}^{-1}=n^{-1}\big(e_{\mathsf{t}}^{-jk}\big)_{j,k}=-\big(e_{\mathsf{t}}^{-jk}\big)_{j,k}.$$

The same transform also has a direct polynomial interpretation on the finite phase cycle.

Proposition 13 (Standard polynomial reading). *For $v=(v_0,\dots,v_{n-1})\in V_{e_{\mathsf{t}}}$, define the polynomial*

$$P_v(X):=\sum_{j=0}^{n-1}v_jX^j\in\mathbb{F}_{\mathsf{p}}[X].$$

Then

$$\big(\mathcal{F}_{e_{\mathsf{t}}}(v)\big)_k=P_v(e_{\mathsf{t}}^k),\qquad 0\leq k<n.$$

Thus the ring Fourier transform exchanges coefficient data with values on the finite phase cycle $\{1,e_{\mathsf{t}},e_{\mathsf{t}}^2,\dots,e_{\mathsf{t}}^{n-1}\}\subset\mathbb{F}_{\mathsf{p}}^\times$.

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Proof. This is immediate from the definition:

$$P_v(e_{\mathsf{t}}^k) = \sum_{j=0}^{n-1} v_j (e_{\mathsf{t}}^k)^j = \sum_{j=0}^{n-1} v_j e_{\mathsf{t}}^{jk} = \big(\mathcal{F}_{e_{\mathsf{t}}}(v)\big)_k.$$

□

Every latitude L_j of the shell is a copy of the phase cycle $\Phi_{\mathsf{p}} \cong \mathbb{Z}/n\mathbb{Z}$, so Definition 8 applies verbatim to any shell quantity recorded around a latitude.

Remark 16 (Framed-complex coefficient layer). *The algebra paper constructs finitist framed-complex numbers over symmetry-complete prime shells [22]. Through the shell-determined quarter-turn element i_{t}, the same coefficient vectors and spectra may therefore be viewed inside that framed-complex layer when complex-style notation is convenient. The transform formulas themselves, however, already live over the base ring $\mathbb{F}_{\mathsf{p}}$ and do not require any external coefficient field.*

The remaining structural point is covariance under change of the chosen phase generator.

Proposition 14 (Standard covariance under generator change). *Let $e'_{\mathsf{t}} = e_{\mathsf{t}}^u$ with $u \in (\mathbb{Z}/n\mathbb{Z})^\times$. For a shell phase quantity $q\colon \mathbb{F}_{\mathsf{p}}^\times \to \mathbb{F}_{\mathsf{p}}$, let*

$$v_j^{(e_{\mathsf{t}})} := q(e_{\mathsf{t}}^j), \qquad v_j^{(e'_{\mathsf{t}})} := q(e_{\mathsf{t}}'^{\,j}).$$

Then

$$v_j^{(e'_{\mathsf{t}})} = v_{uj}^{(e_{\mathsf{t}})}$$

and the corresponding spectra agree:

$$\mathcal{F}_{e'_{\mathsf{t}}}\big(v^{(e'_{\mathsf{t}})}\big) = \mathcal{F}_{e_{\mathsf{t}}}\big(v^{(e_{\mathsf{t}})}\big).$$

Proof. The coefficient relation is immediate from $e_{\mathsf{t}}'^{\,j} = e_{\mathsf{t}}^{uj}$. Then

$$\Big(\mathcal{F}_{e'_{\mathsf{t}}}\big(v^{(e'_{\mathsf{t}})}\big)\Big)_k = \sum_{j=0}^{n-1} v_j^{(e'_{\mathsf{t}})} e_{\mathsf{t}}'^{\,jk} = \sum_{j=0}^{n-1} v_{uj}^{(e_{\mathsf{t}})} e_{\mathsf{t}}^{ujk}.$$

Since u is invertible modulo n, the substitution $m = uj$ permutes the index set $\{0, \ldots, n-1\}$. Hence

$$\Big(\mathcal{F}_{e'_{\mathsf{t}}}\big(v^{(e'_{\mathsf{t}})}\big)\Big)_k = \sum_{m=0}^{n-1} v_m^{(e_{\mathsf{t}})} e_{\mathsf{t}}^{mk} = \Big(\mathcal{F}_{e_{\mathsf{t}}}\big(v^{(e_{\mathsf{t}})}\big)\Big)_k.$$

□

Remark 17 (Status of the Fourier identities). *This section records the finite Fourier facts used in the paper. They are entirely ring-internal: the signal space, the transform kernel, the inverse transform, and the covariance statement all live inside the finite ring $\mathbb{F}_{\mathsf{p}}$. Their mathematical content is standard within finite Fourier analysis; the shell-specific contribution is the shell interpretation of the phase cycle and the explicit boundary drawn between internal algebra and observer-side continuum comparison.*

Remark 18 (Observer-side continuum contact). *Fix the framed finite field $\mathbb{F}_{\mathsf{p}}(\mathsf{t}; 0, 1, e_{\mathsf{t}})$ and set $n = \mathsf{p} - 1$. Externally, the phase index $m \in \Phi_{\mathsf{p}}$ may be read by an observer as the angle*

$$\phi_m := \frac{2\pi m}{n}, \qquad 0 \le m < n,$$

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so the phase cycle becomes a uniform n-point sampling grid on S^1*. In that observer-side reading, the ring DFT of Definition 8 is structurally parallel to the classical n-point complex DFT on sampled trigonometric data, although the internal coefficient ring and normalization remain those of* $\mathbb{F}_{\mathsf{p}}$*.*

Consequently, for any fixed trigonometric polynomial

$$F(\phi) = \sum_{|k| \le N} c_k e^{ik\phi},$$

external exact recovery by the classical sample DFT holds whenever $n > 2N$*. More generally, for periodic analytic* $F\colon S^1 \to \mathbb{C}$*, the observer-side trapezoidal or sample coefficient approximations on the grid* ϕ_m *converge rapidly to the classical Fourier coefficients as* $\mathsf{p} \to \infty$ *along primes* $\mathsf{p} \equiv 1 \pmod 4$ *[36–38]. These are observer-side sampling facts about the large-*p *phase grid; they are not internal theorems of the shell itself.*

7. Discussion

This article occupies a precise place both within the FRC programme and within the broader literature on symmetry in finite mathematical structures.

What this paper secures.

It gives the programme a coherent Euclidean prime-shell layer:

(i) a finite orbital 2-sphere $\mathcal{S}_{\mathsf{p}}$,
(ii) the shell/reference-frame separation $(\mathcal{Q}_{\mathsf{p}}, i_{\mathsf{t}}, \pi_{\mathsf{t}})$ versus e_{t},
(iii) a symmetry backbone realized through additively framed negation, affinely framed reciprocal inversion, and extended-frame Euclidean conjugation,
(iv) a strict finitist meaning of “geometry to arbitrary precision,”
(v) and finite Fourier duality on the shell.

Why the symmetry framing matters.

The present article is not only a discrete geometry paper with a symmetry wrapper. Its main constructions are organized by framed involutions and their orbit structure. The orbital complex uses the same additive and multiplicative cycle data that make the involution package visible. The structural-data section then isolates which parts of the shell are shell-determined and which appear only after fixing progressively richer frame data. The Fourier section continues the same pattern: the duality is ring-internal, while the observer-canonical exponential unit e_{t} supplies the phase chart in which that duality is written. This is the sense in which symmetry is the paper’s organizing mechanism rather than a purely expository theme.

Structural limitation of the orbital complex.

The orbital complex is a hybrid construction. Its longitude direction comes from multiplicative phase-orbit data in $\mathbb{F}_{\mathsf{p}}^{\times}$, while its meridian direction comes from the operational count axis supplied by π_{t}. The paper states this openly because it is a genuine structural limitation: $\mathcal{S}_{\mathsf{p}}$ is a coherent finitist geometry object, but it is not forced by a single internal group action in both directions.

What the paper does not claim.

The strongest theorem of the paper is operational rather than ontological. The large-p statement concerns fixed bounded experiments with finitely many marked points and finitely many Lipschitz observables. It does not amount to unrestricted Euclidean reconstruction, nor does it claim that continuum mathematics is internally recovered as such. This narrower statement is also the one most consistent with the literature on discrete approximation and sampling, and it is the one best supported by the present proofs.

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What is deliberately deferred.

Quadratic extensions, Lorentzian structure, composite shells, shell transfer, Dirac-type dynamics, and bracket-based modular Lie structure are outside the present scope. In particular, the genuinely temporal involution belongs to a different layer of structure and is not identified here with Euclidean conjugation C_{e_t}.

Programme consequence.

The prime shell is the Euclidean checkpoint of FRC. It supplies a finite geometric and harmonic vocabulary while leaving extension-stage structures to separate study. Within that scope, the present results isolate the geometric and harmonic core, while modular Lie structure over $\mathbb{F}_p$ can be introduced independently when a concrete operational need for brackets appears [39–41].

Potential application.

The present article is theoretical, but the structures it isolates are operationally relevant. The orbital complex provides a finitist model for finite-resolution spherical sampling; the phase-cycle transform provides a native shell Fourier tool for periodic shell data; and the framed symmetry package gives a controlled language for reversible finite geometry and phase processing. Within the FRC programme, this Euclidean shell layer also serves as the prerequisite for the quadratic-extension causality analysis developed in [42]. These are not yet empirical applications in the narrow sense, but they do indicate where the formalism can carry practical computational or modeling load: bounded geometric approximation, shell-based signal analysis, finite observer-side reconstruction tasks, and staged extension to algebraic spacetime models.

Future directions.

The next natural developments are now clearer. On the mathematical side, the Euclidean prime-shell layer should be extended to observer-local shell structures, composite or transferred shells, and the quadratic-extension stage where Lorentzian and temporal structures can be studied on their own terms. On the applied side, the shell Fourier formalism suggests finitist spectral constructions for bounded data, while the operational approximation theorem provides the template for further "finite first, continuum later" comparison results. These are precisely the directions in which the present Euclidean checkpoint is meant to support the broader FRC programme.

8. Conclusions

A symmetry-complete prime shell, equipped with the framed finite field $\mathbb{F}_p(t; 0, 1, e_t)$, supports a frame-presented orbital complex combinatorially equivalent to the two-sphere, a small but nontrivial family of framed involutive symmetries, shell-determined half-period and quarter-turn data, and a ring-internal Fourier duality. The labeled complex depends on the chosen frame, while its cellular type and associated shell invariants are preserved under explicit frame transport.

Equally important, the approximation claim is stated in a disciplined way. The large-p theorem is an observer-side statement about bounded experiments and prescribed tolerances, not an internal smooth-manifold claim, and its proof uses standard grid-density and Lipschitz estimates on the prime-shell grid family. Likewise, the Fourier section records standard finite DFT facts in a shell-adapted form rather than claiming new Fourier identities. In that form, the result gives the FRC programme a credible finitist Euclidean checkpoint and a technically coherent account of prime-shell geometry and harmonic duality.

That checkpoint also has a clear interpretive consequence. The paper does not prove that the classical continuum is impossible or inconsistent. It does show, however, that in the Euclidean setting treated here the continuum is not operationally forced: a strictly finite shell already carries the bounded geometric and harmonic load under study. Within this domain, the continuum is therefore best read as an observer-side idealization rather than as an automatically privileged ontology. The broader claim

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that finitism can displace continuum ontology altogether remains a programme-level thesis, but the present results shift the burden of proof in that direction.

Notations

t	Fundamental radius / chronon parameter with prime-shell law $\mathsf{p} = 4\mathsf{t} + 1$.
$\mathbb{F}_\mathsf{p}$	Symmetry-Complete field of radius t; unless stated otherwise, it is understood in its framed form $\mathbb{F}_\mathsf{p}(0,1)$ following [22]. When the full refined frame data are explicit, we write $\mathbb{F}_\mathsf{p}(\mathsf{t}; 0, 1, e_\mathsf{t})$.
$\mathcal{S}_\mathsf{p}$	Orbital complex of the prime shell, i.e. the Euclidean refinement of the orbital complex S_p previewed in the algebra paper.
$\mathcal{Q}_\mathsf{p}$	Quarter-turn subgroup of $\mathbb{F}_\mathsf{p}^\times$, namely the unique subgroup of order 4; in a chosen frame this is the structural set $\{1, i_\mathsf{t}, -1, -i_\mathsf{t}\}$.
π_t	Shell half-period, defined by $\pi_\mathsf{t} = (\mathsf{p}-1)/2 = 2\mathsf{t}$.
i_t	Distinguished oriented quarter-turn of the shell, with $i_\mathsf{t}^2 = -1$; equivalently, $\mathcal{Q}_\mathsf{p} = \{\pm 1, \pm i_\mathsf{t}\}$. In an orientation-compatible phase chart one has $\mathsf{g}^{(\mathsf{p}-1)/4} = i_\mathsf{t}$.
e_t	Observer-canonical exponential unit carried by the frame. Externally it is represented by a primitive generator of $\mathbb{F}_\mathsf{p}^\times$, but within the chosen frame it is read simply as e_t, just as the affine frame is read as $(0,1)$.
$R(x) = -x$	Additively framed negation, read on the additive shell as spatial reversal once the origin 0 is fixed.
$I(x) = x^{-1}$	Affinely framed reciprocal inversion on the unit cycle $\mathbb{F}_\mathsf{p}^\times$, available once the affine pair $(0,1)$ is fixed.
$C_{e_\mathsf{t}}(a + b i_\mathsf{t}) = a - b i_\mathsf{t}$	Euclidean conjugation in the extended frame attached to the observer-canonical exponential unit e_t; this is distinct from the temporal involution considered outside the present scope.
$\Xi_{\mathsf{p}, e_\mathsf{t}}$	Observer-side realization map from the finite orbital complex $\mathcal{S}_\mathsf{p}$ to the classical unit sphere S^2, used only for large-p finite-precision comparison statements. Its dependence on e_t is through the chosen phase chart.

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