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Mean-field theory of the Meissner effect in bulk revisited.

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Abstract

We show that the implementation of the $\frac{1}{c^2}$ transverse current-current interaction between electrons into the standard self-consistent electron BCS model in bulk under thermal equilibrium ensures in the stable superconductive phase the full compensation of a constant external magnetic field by the internal magnetic field created by the electrons i.e. one has an ideal diamagnet. However, no proof of the phenomenological London equation emerges within the bulk approach.

keywords: BCS superconductivity, mean field, current-current interaction, internal/external fields, stability, Meissner effect

1 Introduction

Since its discovery by Kamerling-Onnes [1] at the beginning of the 20th century superconductivity became a particularly important field of physics with wide technical applications. The identification of superconductors as perfect diamagnets i.e. the Meissner effect [2] was discovered in the thirties and was soon followed by the beautiful phenomenological electromagnetic theory of London [3, 4]. The fundamental theoretical breakthrough however is due to Bardeen,Cooper and Schrieffer [5]. They have shown that the origin of the superconductive phase transition lies in the correlation between electrons of opposite momenta and spin resulted from phonon exchange. Wile the most important and striking feature is the absence of resistance below a critical temperature, there is no deep understanding of it. Within the present unsatisfactory status of non-equilibrium statistical mechanics irreversibility (dissipation) is introduced "by hand", although there are significant recent progresses in understanding the treatment of open systems [6], [7]. Therefore, the understanding at least of the equilibrium properties is a central point. The most important in this respect is the theory of the Meissner effect to which we devote our discussion.

The standard modeling of the BCS idea within self-consistent electron theories [8, 9, 10] offers no convincing results for the Meissner effect. The failure is due to the incorrect implementation of electromagnetism and the insistence to relate the results to the London equation in the context of the bulk, where the magnetic field should be identically null. Starting from the non-relativistic quantum electrodynamics a current-current magnetic interaction may be derived [11, 12]. We show in this paper that its inclusion improves the standard bulk mean field theory explaining within this frame the perfect diamagnetism. In this sense we contradict our recent skepticism [12].

2 The standard theory of the Meissner effect in bulk.

The revelation of BCS that the source of superconductivity lies in the correlation of electrons of opposite momenta and spin due to interaction with phonons was decisive for the theory. However, to pursue this idea within the many-body theory including phonons seems difficult. Therefore, one tried to construct pure electron many-body theories with a built-in "potential" giving rise to such correlations and implicitly to a superconducting phase transition.

Such a model Hamiltonian is due to Rickayzen [8, 9] that we describe here. The specific version of Bogolyubov-de Gennes [10] is included in this frame.

Since one is concentrated on equilibrium properties in a μ-system one considers $\mathcal{H} \equiv H - \mu N$ instead of the Hamiltonian H. Rickayzen introduces besides the kinetic energy in the presence of classical time-independent given vector $\vec{\mathcal{A}}(\vec{x})$ and scalar $\phi(\vec{x})$ potentials, an electron-electron interaction by a "correlating" potential $W(\vec{x})$, that produces no bound states, but gives rise to correlations of BCS type:

$$
\begin{aligned}
\mathcal{H} &\equiv H - \mu N \\
&= \sum_{\sigma=\pm\frac{1}{2}} \int d\vec{x} \psi_\sigma^+(\vec{x}) \left\{ \frac{1}{2m} \left(-\imath\hbar\nabla - \frac{e}{c}\vec{\mathcal{A}}(\vec{x}) \right)^2 + e\phi(\vec{x}) - \mu \right\} \psi_\sigma(\vec{x}) \\
&+ \frac{1}{2} \int d\vec{x} \int d\vec{x}\,' W(\vec{x} - \vec{x}\,') \{ \psi_{\frac{1}{2}}^+(\vec{x}) \psi_{-\frac{1}{2}}^+(\vec{x}\,') \psi_{\frac{1}{2}}(\vec{x}\,') \psi_{-\frac{1}{2}}(\vec{x}) + h.c. \} \ .
\end{aligned} \tag{1}
$$

This Hamiltonian is invariant against time independent gauge transformations of the vector potential $\vec{\mathcal{A}}(\vec{x})$. We shall keep the discussion in a rather general frame without choosing a definite potential $W(\vec{x})$ and ignore the Coulomb interactions (including a positive background) since they play no role in the next steps to follow. Further, one resorts to a self-consistent Hartree-Fock approximation including anomalous averages like $\langle \psi_{\frac{1}{2}}^+(\vec{x}) \psi_{-\frac{1}{2}}^+(\vec{x}\,') \rangle$.

Then, the self-consistent Hamiltonian is

$$
\begin{aligned}
\mathcal{H}_{s.c.} &= \sum_\sigma \int d\vec{x} \psi_\sigma^+(\vec{x}) \left\{ \frac{1}{2m} \left(-\imath\hbar\nabla - \frac{e}{c}\vec{\mathcal{A}}(\vec{x}) \right)^2 + \phi(\vec{x}) - \mu \right\} \psi_\sigma(\vec{x}) \\
&+ \frac{1}{2} \int d\vec{x} \int d\vec{x}\,' W(\vec{x} - \vec{x}\,') \Big\{ \langle \psi_{\frac{1}{2}}^+(\vec{x}) \psi_{-\frac{1}{2}}^+(\vec{x}\,') \rangle \psi_{-\frac{1}{2}}(\vec{x}\,') \psi_{\frac{1}{2}}(\vec{x}) \\
&+ \langle \psi_{-\frac{1}{2}}(\vec{x}\,') \psi_{\frac{1}{2}}(\vec{x}) \rangle \psi_{\frac{1}{2}}^+(\vec{x}) \psi_{-\frac{1}{2}}^+(\vec{x}\,') - \langle \psi_{\frac{1}{2}}^+(\vec{x}) \psi_{-\frac{1}{2}}^+(\vec{x}\,') \rangle \langle \psi_{-\frac{1}{2}}(\vec{x}\,') \psi_{\frac{1}{2}}(\vec{x}) \rangle \Big\} \ .
\end{aligned} \tag{2}
$$

One can show within this frame, that in the absence of the fields $\vec{\mathcal{A}}(\vec{x})$, $\phi(\vec{x})$ a phase transition may occur under a critical temperature, provided the potential $W(\vec{x})$ ensures a non-vanishing solution for the symmetry breaking gap parameter ($\Delta(\vec{k}) \neq 0$) of the largely described "gap equation" we do not give here. This condition is equivalent to the vanishing of the first derivative of the free energy with respect to $\Delta(\vec{k})$.

The next step is to use equilibrium linear response theory to get the relationship between the average $\langle \vec{j}(\vec{x}) \rangle$ of the current density operator

$$
\vec{j}(\vec{x}) \equiv \frac{e}{2m} \psi^+(\vec{x}) \left(-\imath\hbar\nabla + \frac{e}{c}\vec{\mathcal{A}}(\vec{x}) \right) \psi(\vec{x}) + h.c. \tag{3}
$$

and the weak vector potential $\vec{\mathcal{A}}(\vec{x})$ in the Coulomb gauge $\nabla\vec{\mathcal{A}}(\vec{x}) = 0$ and $\phi(\vec{x}) = 0$.

A peculiarity of the linear response within self-consistent theories is that the deviation of the averages from their equilibrium value constitute so called induced perturbations. This means, that the true perturbation (to first order in $\vec{\mathcal{A}}$) is

$$
\mathcal{H}'_{s.c.} = -\frac{1}{c} \int d\vec{x} \vec{j}(\vec{x}) \vec{\mathcal{A}}(\vec{x}) + \int d\vec{x} \int d\vec{x}\,' W(\vec{x} - \vec{x}\,') \left[\eta(\vec{x}, \vec{x}\,') \psi_{-\frac{1}{2}}(\vec{x}\,') \psi_{\frac{1}{2}}(\vec{x}) + h.c. \right] \tag{4}
$$

with

$$
\eta(\vec{x}, \vec{x}\,') \equiv \langle \psi_{\frac{1}{2}}^+(\vec{x}) \psi_{-\frac{1}{2}}^+(\vec{x}\,') \rangle - \langle \psi_{\frac{1}{2}}^+(\vec{x}) \psi_{-\frac{1}{2}}^+(\vec{x}\,') \rangle_0 \tag{5}
$$

being the deviations of the anomalous averages from their values in the absence of the field $\vec{\mathcal{A}}$.

The resulting linear relationship between the Fourier Transforms of the two transverse vectors looks as

$$
\langle \tilde{j}_\mu(\vec{k}) \rangle = \kappa(k) \tilde{\mathcal{A}}_\mu(\vec{k}); \qquad (\mu = 1, 2, 3) \tag{6}
$$

with the scalar coefficient $\kappa(k)$ in an infinite, homogeneous, isotropic system being a function of $k = \sqrt{\vec{k}^2}$. Its explicit expression however was calculated explicitly [8] only after neglecting the induced perturbations.

One is interested in this relationship only for small wave vectors (slowly varying behavior in the coordinate space!). If in the absence of the perturbation one had an anomalous superconducting phase

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with a non-vanishing gap $\Delta(\vec{k})$ at $\vec{k}=0$ one gets (without induced induced perturbations [8]) that $\kappa(0)$ is finite and strictly negative

$$\kappa(0) = -\frac{1}{c\Lambda} < 0 \ . \tag{7}$$

The same result is described in the frame of the anomalous Green functions, under similar approximations by Schrieffer [13].

It may be shown however [14], that under the condition of stability of the superconducting phase (which amounts to a non-negative second derivative of the free energy with respect to the gap paramater $\Delta(\vec{k})$ at $k=0$) Λ is indeed positive and the contribution produced by consideration of $\eta(\vec{x},\vec{x}\,')$ does not change Rickayzen's result. For $\vec{k} \neq 0$ no sound results are available, since it requires the full calculation considering the induced perturbation terms.

So far the calculations are OK. The problem resides in the interpretation of these results. According to the standard interpretation (see [15]) Eqs.6,7 have to be compared to the second London equation that reads as

$$\nabla \times \vec{i} = -\frac{1}{c\Lambda}\vec{B} \qquad or \qquad \vec{i} = -\frac{1}{c\Lambda}\vec{A} \ ,$$

with $\vec{i}(\vec{x})$ being the macroscopic super-current density and $\vec{A}(\vec{x})$ the total macroscopic field in the superconductor. Thus the Meissner effect would be explained within this model Hamiltonian. Even more, the parameter Λ might be interpreted as the London penetration length.

Unfortunately, such a reasoning is **misleading**. The total magnetic field $\vec{B}$ as well as the supercurrent are identically null in the bulk if Meissner effect occurs! The reference to the London equation is totally misplaced here. On the other hand, the classical field $\vec{\mathcal{A}}$ in this model has undefined sources and no magnetic field produced by the electrons is present at all in this theory. Therefore it cannot be identified with the total macroscopic magnetic vector potential $\vec{A}$. Besides, this Hamiltonian is invariant with respect to time independent gauge transformations and therefore $\vec{\mathcal{A}}$ has to be identified with the external field. In the frame of the non-relativistic QED the Hamiltonian is defined in a fixed gauge, namely in the Coulomb gauge for the quantized ("radiation") electromagnetic field.

The primary task of the microscopical theory in the bulk is just to show, that in thermal equilibrium a homogeneous constant external magnetic field is compensated completely by the internal one produced by the electrons. Obviously, some ingredients are still missing. One is at the range of validity of the ordinary quantum mechanics, that takes no magnetic interactions between the electrons into account. Steps towards the non-relativistic quantum electrodynamics of charged particles are compulsory. This criticism of the electromagnetic aspects is pertinent also to the Ginzburg-Landau non-linear theory of superconductivity [16].

3 The current-current interaction and the theory of superconductivity

Already 100 years ago Darwin [17] has argued in the frame of the classical electrodynamics of point-like electrons, that up to order $1/c^2$ one might separate the motion of the particles from that of the electromagnetic field. From this separation emerges a magnetic electronic current-current interaction. Since the classical electrodynamics of point-like charged particles is a vicious theory having neither Lagrangian, nor Hamiltonian formulation, his derivation lacked any rigor, nevertheless it contains a grain of truth and was considered later also by Landau and Lifshitz [18]. They have shown that Darwin implicitly has chosen a certain very strange non-linear choice of gauge. Such a gauge however imposes constraints on the velocities of the particles and therefore no ordinary canonical formalism is allowed. See in this context Dirac's theory of canonical formalism with constraints [19, 20].

Actually one needs a new analysis of the problem in the frame of the non-relativistic QED (see [21, 22] for a modern presentation). A natural choice of the gauge is here the Coulomb one. Indeed, in this gauge one has to do only with the two physical degrees of freedom of the photons and one is free from artificial constraints. One must restrict the attention to the subspace of states without photons in order to have a pure electron theory. The next necessary step is to neglect retardation effects and this restricts the validity of the results to order $1/c^2$. A good object to perform such a discussion is either the S-matrix, or the theory of Green functions with the help of the Feynman diagrams. In an early paper Holstein, Norton and Pincus [23] have shown, that without retardation the photon propagator reduces to the Coulomb potential and the most important Feynman diagram is a transverse current-current interaction between the electrons. Recently one has shown [11, 12] that the two ingredients : restriction to the electronic subspace and ignoring the retardation in the photon propagator leads to an electronic Hamiltonian, that besides the usual Coulomb interaction between the electrons contains also a magnetic transverse current-current interaction

$$-\frac{1}{2}\int d\vec{x}\int d\vec{x}\,'\frac{N\left[\vec{j}_\perp(\vec{x})\vec{j}_\perp(\vec{x}\,')\right]}{c^2|\vec{x}-\vec{x}\,'|} \;, \tag{8}$$

where $\vec{j}_\perp(\vec{x})$ denotes the transverse part of the current density operator

$$\vec{j}(\vec{x}) = \frac{e}{2m}\left(\psi^+(\vec{x})\left(\frac{\hbar}{\imath}\nabla - \frac{e}{c}\vec{A}_{ext}(\vec{x})\right)\psi(\vec{x}) + h.c.\right) \;. \tag{9}$$

Here we introduced the notation $N[\cdots]$ for the normal product of operators. This is nothing else but the well-known Biot-Savart law. One may argue that due to the smallness of the velocities in the condensed matter such an $1/c^2$ term may be neglected. This is obviously false. Our everyday experience teaches us, that a macroscopic number of slow electrons may create enormous magnetic fields. We shall show, that exactly this term is necessary to complete the theory of the preceding Section leading to the exact cancellation of a total constant magnetic field in the bulk.

Supplementing the Hamiltonian of Eq. 1 with the above term and taking into account the presence of time independent external fields $\vec{A}_{ext}$ and V_{ext}, the new BCS-Hamiltonian looks now as

$$\begin{aligned}
\mathcal{H}^{new} \;=\; & -\int d\vec{x}\psi^+(\vec{x})\left[\left(\frac{\hbar}{\imath}\nabla - \frac{e}{c}\vec{A}_{ext}(\vec{x})\right)^2 + V_{ext}(\vec{x}) - \mu\right]\psi(\vec{x}) \\
& + \frac{1}{2}\int d\vec{x}\int d\vec{x}\,'\frac{N\left[\rho(\vec{x})\rho(\vec{x}\,')\right]}{|\vec{x}-\vec{x}\,'|} - \frac{1}{2}\int d\vec{x}\int d\vec{x}\,'\frac{N\left[\vec{j}_\perp(\vec{x})\vec{j}_\perp(\vec{x}\,')\right]}{c^2|\vec{x}-\vec{x}\,'|} \\
& + \frac{1}{2}\int d\vec{x}\int d\vec{x}\,'W(\vec{x}-\vec{x}\,')\{\psi^+_{\frac{1}{2}}(\vec{x})\psi^+_{-\frac{1}{2}}(\vec{x}\,')\psi_{\frac{1}{2}}(\vec{x}\,')\psi_{-\frac{1}{2}}(\vec{x}) + h.c.\} \;.
\end{aligned} \tag{10}$$

For sake of completeness and symmetry we included here also the Coulomb interaction between the electrons, however as before we ignore it in what follows.

Then, within the mean-field approximation of the current-current term in Eq.10 the extension of Eq.2 is

$$\begin{aligned}
\mathcal{H}^{new}_{s.c.} \;=\; & \int d\vec{x}\psi^+(\vec{x})\left[\left(\frac{\hbar}{\imath}\nabla - \frac{e}{c}\vec{A}_{ext}(\vec{x})\right)^2 + eV_{ext}(\vec{x}) - \mu\right]\psi(\vec{x}) \\
& + \frac{1}{2}\int d\vec{x}\int d\vec{x}'W(\vec{x}-\vec{x}\,')\left\{\langle\psi^+_{\frac{1}{2}}(\vec{x})\psi^+_{-\frac{1}{2}}(\vec{x}\,')\rangle\psi_{-\frac{1}{2}}(\vec{x}\,')\psi_{\frac{1}{2}}(\vec{x})\right. \\
& + \left.\langle\psi_{-\frac{1}{2}}(\vec{x}\,')\psi_{\frac{1}{2}}(\vec{x})\rangle\psi^+_{\frac{1}{2}}(\vec{x})\psi^+_{-\frac{1}{2}}(\vec{x}\,') - \langle\psi^+_{\frac{1}{2}}(\vec{x})\psi^+_{-\frac{1}{2}}(\vec{x}\,')\rangle\langle\psi_{-\frac{1}{2}}(\vec{x}\,')\psi_{\frac{1}{2}}(\vec{x})\rangle\right\} \;. \\
& - \int d\vec{x}\int d\vec{x}\,'\frac{\vec{j}_\perp(\vec{x})\langle\vec{j}_\perp(\vec{x}\,')\rangle}{c^2|\vec{x}-\vec{x}\,'|}
\end{aligned} \tag{11}$$

In this expression one may already identify the internal transverse vector field

$$\vec{A}_{int}(\vec{x}) = \int d\vec{x}\,'\frac{\langle\vec{j}_\perp(\vec{x}\,')\rangle}{|\vec{x}-\vec{x}\,'|} \;, \tag{12}$$

or in Fourier transform

$$\vec{A}_{int}(\vec{k}) = \frac{4\pi}{k^2}\langle\vec{j}(\vec{k})\rangle \;. \tag{13}$$

As before we consider no external scalar potential ($V_{ext} = 0$) and the external vector potential $\vec{A}_{ext}$ we choose again to be transverse ($\nabla\vec{A}_{ext} = 0$). Its longitudinal part would have been anyway irrelevant for the magnetic field $\vec{B}_{ext} = \nabla\times\vec{A}_{ext}$. Thus all the vectors are implicitly transverse an no need to mention it in the notations.

The terms of first order in $\vec{A}^{ext}$ differ from those of the previous approach only by the replacement

$$\vec{j}\vec{\mathcal{A}} \quad \Rightarrow \quad \vec{j}(\vec{A}_{ext} + \vec{A}_{int})$$

in the perturbation Eq. 4. (However, it is important to stress, that this holds only in the first order terms!).

Therefore, due to the new induced term now we get instead of Eq. 6 with the **same previously defined** $\kappa(k)$ the relationship

$$\langle \tilde{j}^{\mu}(\vec{k}) \rangle = \kappa(k) \left(\tilde{A}^{\mu}_{ext}(\vec{k}) + \tilde{A}^{\mu}_{int}(\vec{k}) \right); \qquad (\mu = 1,2,3) \ . \tag{14}$$

Multiplying this equation with $\frac{4\pi}{k^2}$ one recovers on the left-hand side again the internal field $\tilde{A}^{\mu}_{int}(\vec{k})$, therefore

$$\tilde{A}^{\mu}_{int}(\vec{k}) = \frac{\frac{4\pi}{k^2}\kappa(k)}{1 - \frac{4\pi}{k^2}\kappa(k)} \tilde{A}^{\mu}_{ext}(\vec{k}); \qquad (\mu = 1,2,3) \ . \tag{15}$$

For the internal and external magnetic fields holds similarly

$$\tilde{B}^{\mu}_{int}(\vec{k}) = \frac{\frac{4\pi}{k^2}\kappa(k)}{1 - \frac{4\pi}{k^2}\kappa(k)} \tilde{B}^{\mu}_{ext}(\vec{k}); \qquad (\mu = 1,2,3) \ . \tag{16}$$

This last equation could have been obtained also by Zubarev's reasoning [24] based just on the macroscopic Maxwell equations, without any reference to a Hamiltonian. He obtains the linear response to the field $\vec{B}$ from the known linear response to the field $\vec{H}$.

Since it was already proven [8, 14] that under the condition of a stable superconductive phase $\kappa(0)$ is finite, it follows

$$\tilde{B}^{\mu}_{int}(0) = -\tilde{B}^{\mu}_{ext}(0) \qquad (\mu = 1,2,3) \ . \tag{17}$$

This proves that in the frame of this modified s.c. BCS model in the stable superconductive phase no constant magnetic field in the bulk may survive! (Actually, the vanishing of the constant magnetic field in real space still needs the easily accepted assumption that a homogeneous external field induces also a homogeneous magnetic field in the bulk.) The internal magnetic field of the electrons compensates fully the applied external field. The finite value of the coefficient $\kappa(0)$ of Eq. 7 remains the criterion for the Meissner effect [15], however, it cannot be related rigorously to the London penetration length and its sign is irrelevant for our proof.

4 Conclusions

By improving the standard bulk model of BCS superconductivity in its electromagnetic aspects, we have proven that in its mean-field approximation, if the superconductive phase transition is stable, the Meissner effect results in the sense of the perfect compensation of a constant external magnetic field by the internal magnetic field. This means, we have an ideal diamagnet achieved by considering the magnetic transverse current-current interaction deduced from the non-relativistic QED. Over the whole derivation no definite "correlating potential" $W(\vec{x})$ was considered, but just the general requirement of the minimum of the free energy with a non-vanishing gap parameter. Our result implies no supplementary assumptions nor new approximations, but just the implementation of the ordinary non-relativistic QED.

Of course, a microscopic derivation of the London equations remains highly desirable, but within this bulk approach it is an almost impossible task. One needs anyway to consider a specific potential $W(\vec{x})$ and for such a purpose one should consider either a boundary problem (in a half space) with a homogeneous magnetic external field, or a local magnetic field in the bulk. The second variant in the absence of a magnetic charge may be realized only with an external solenoid immersed in the bulk. This gives rise to complicated 3D equations. Even in the first variant one needs more ingredients in the Hamiltonian as those already considered. Linear response in coordinate space without strongly localized correlations leads nowhere. The discussed model cannot ensure such a property. Therefore, one needs one more ingredient, that destroys spatial long-range correlations typical for a phase transition, without destroying the phase transition itself.

In this context it is useful to remind the situation with the Debye electric field penetration in a semiconductor. It may be obtained in a model of free classical electrons with a s.c. potential in equilibrium (Maxwell-Boltzmann distribution) within the linear approximation. However, a quantum mechanical derivation within linear response needs short range correlations in real space, that the free electron model cannot deliver. Here also some ingredient is necessary, although it is easier to imagine one since no phase transition is involved.

To conclude, with the implementation of the current-current interaction, the standard mean field theory can explain ideal diamagnetism in bulk, but not yet the London equation.

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