

Article

Not peer-reviewed version

Advanced Information on Multiple-Granulation Approximation Spaces

E. El-Sanowsy , [A.A. Abdallah](#) , [O.R. Sayed](#) , Y. H. Ragheb Sayed , [.Salahuddin](#) *

Posted Date: 26 June 2024

doi: 10.20944/preprints202406.1826.v1

Keywords: Multi-granulation nano generalized topology; Multi-granulation nano generalized interior; Multi-granulation nano generalized closure



Preprints.org is a free multidiscipline platform providing preprint service that is dedicated to making early versions of research outputs permanently available and citable. Preprints posted at Preprints.org appear in Web of Science, Crossref, Google Scholar, Scilit, Europe PMC.

Copyright: This is an open access article distributed under the Creative Commons Attribution License which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Article

Advanced Information on Multiple-Granulation Approximation Spaces

E. El-Sanousy ¹, A.A. Abdallah ² , O.R. Sayed ³, Y.H. Ragheb Sayed ^{1,4}  and Salahuddin ^{2,*} 

¹ Department of Mathematics, Faculty of Science, Sohag University, Sohag, 82524, Egypt; elsanowsy@yahoo.com

² Department of Mathematics, College of Science, Jazan University, P.O. Box. 114, Jazan 45142, Kingdom of Saudia Arabia; anassr@jazanu.edu.sa (elseleny_2006@yahoo.com)

³ Department of Mathematics, Faculty of Science, Assiut University, Assiut 71516, Egypt; o_sayed@aun.edu.eg (o_r_sayed@yahoo.com)

⁴ Department of Mathematics, Faculty of Science, University of Hafr Al Batin, Hafar Al Batin 39524, Saudi Arabia; yh_raghp2011@yahoo.com or yhassanien@uhb.edu.sa

* Correspondence: smohammad@jazanu.edu.sa (dr_salah12@yahoo.com)

Abstract: To define the optimistic multiple-granulation(OMG) supra interior operator $int_{\mathcal{R}}^A$ and the OMG-supra closure operator $cl_{\mathcal{R}}^A$ associated with a set A in a multiple-granulation approximation space, we used the concepts of the OMG-lower set $OL_{\sum_{i=1}^m R_i}(A)$ and the OMG-upper set $OU_{\sum_{i=1}^m R_i}(A)$. These operators create an approximate topological space known as optimistic multiple-granulation supratopological spaces. OMG-supra-open and OMG-supra-closed approximation sets are also defined, followed by OMG-supra-continuous approximation and OMG-irresolute approximation.

Keywords: multi-granulation nano generalized topology; multi-granulation nano generalized interior; multi-granulation nano generalized closure

1. Historical Background

Pawlak [9,10] proposed a term for rough sets in approximation space to address the uncertainty arising from artificial systems with lacking and incomplete real-world information. Rough sets are conceptualised in terms of the equivalent relationship of a universal finite set, which serves as the foundation for lower and upper approximations for particular subsets. Numerous researchers have investigated the connection between topological spaces and rough sets, demonstrating that the lower and upper approximation operators formed from a reflexive and transitive relation correspond to the interior and closure operators in a topology, [1,3,11,12,15,19–21]. Ibedou et al. [2] proposed semi-connectedness for nano topological and approximation spaces. Additionally, he utilized the concept of operators in approximation spaces to introduce semicontinuity and other forms of continuity more comprehensively.

Extending and generalising the rough set model is one of the most significant research avenues. Qian et al. [13,14] extended the Pawlak set model by introducing and defining multi-granulation rough sets with multiple equivalence relations in the universe set. Pawlak's rough set has only one equivalence relation. Qian's definition produces two different kinds of multi-granulation rough sets because it contains multiple equivalence relations. Initially, we consider the optimistic multi-granulation rough set. The term "optimistic" in the lower approximation refers to the requirement that at least one independent granular structure satisfy the implication relationship between the equivalency class and the indefinable set.

Second, the pessimistic multi-granulation rough set, in which the term "pessimistic" is used in the lower approximation in multiple independent granular structures, refers to the idea that every granular structure must satisfy the implication relation between the equivalence class and the undefinable set. Following that, a number of researchers looked into multi-granulation rough set models based on various types of relationships and came up with a number of interesting ideas (for example, [4–8,16,17]).

This paper focuses on the OMG-supra interior and OMG-supra closure operators in a multiple-granulation approximation space, which generates an approximation topological space known as the optimistic multiple-granulation supratopological spaces. The OMG-supra-preopen and OMG-supra-preclosed sets in multiple-granulation approximation space are defined. There are also definitions for the OMG-supra-pre interior and OMG-supra pre closure operators. The OMG-supra-continuity approximation and the OMG-supra-pre continuity approximation are defined. Some new approximation continuities are defined in a generalised form by employing special multiple-granulation approximation operators defined on multiple-granulation approximation spaces.

In this paper, let $U = \{x_1, x_2, \dots, x_n\}$ be a nonempty and finite set of objects, called a universe of discourse. $R \subseteq U \times U$ is an equivalence relation on U . $x \in U$, $[x]_R = \{y \in U : (x, y) \in R\}$ is the equivalence class that contains x . Then $U/R = \{[x]_R : x \in U\}$ creates a partition of U . Let $\mathcal{R}$ be a family of equivalence relations on U . The pair $(U, \mathcal{R})$ is described as a multiple-granulation approximation space.

Definition 1.1. [13,14] Let $(U, \mathcal{R})$ be a multiple-granulation approximation space with $R_1, R_2, R_3, \dots, R_m \in \mathcal{R}$, where m is a natural number. For any $X \subseteq U$, the OMG-lower and upper approximations of X on the family of equivalence relations $\{R_i : 1 \leq i \leq m\}$ are defined

$$OL_{\sum_{i=1}^m R_i}^m(X) = \{x \in U : [x]_{R_1} \subseteq X \text{ or } [x]_{R_2} \subseteq X \text{ or } \dots \text{ or } [x]_{R_m} \subseteq X\},$$

$$\begin{aligned} OU_{\sum_{i=1}^m R_i}^m(X) &= \sim (OL_{\sum_{i=1}^m R_i}^m(\sim X)) \\ &= \{x \in U : [x]_{R_1} \cap X \neq \emptyset \text{ and } [x]_{R_2} \cap X \neq \emptyset \dots \text{ and } [x]_{R_m} \cap X \neq \emptyset\}. \end{aligned}$$

$(OL_{\sum_{i=1}^m R_i}^m(X), OU_{\sum_{i=1}^m R_i}^m(X))$ is referred to as the optimistic multiple-granulation rough set of X . The term "optimistic" refers to $x \in OL_{\sum_{i=1}^m R_i}^m(X)$ if and only if there exists at least one equivalence relation R_i such that $[x]_{R_i} \subseteq X$ for all $x \in U$ and $1 \leq i \leq m$. The optimistic multiple-granulation boundary region of X is defined by $OBND_{\sum_{i=1}^m R_i}^m(X)$ as:

$$OBND_{\sum_{i=1}^m R_i}^m(X) = OU_{\sum_{i=1}^m R_i}^m(X) - OL_{\sum_{i=1}^m R_i}^m(X).$$

Proposition 1.3. [14] Let $(U, \mathcal{R})$ be a multiple-granulation approximation space. For any $X, Y \subseteq U$, the following properties are true:

- (1) $OL_{\sum_{i=1}^m R_i}^m(X) \subseteq X \subseteq OU_{\sum_{i=1}^m R_i}^m(X)$.
- (2) $OL_{\sum_{i=1}^m R_i}^m(\emptyset) = OU_{\sum_{i=1}^m R_i}^m(\emptyset) = \emptyset$ and $OL_{\sum_{i=1}^m R_i}^m(U) = OU_{\sum_{i=1}^m R_i}^m(U) = U$.
- (3) $OL_{\sum_{i=1}^m R_i}^m(\sim X) = \sim (OU_{\sum_{i=1}^m R_i}^m(X))$ and $OU_{\sum_{i=1}^m R_i}^m(\sim X) = \sim (OL_{\sum_{i=1}^m R_i}^m(X))$.
- (4) $OL_{\sum_{i=1}^m R_i}^m(OL_{\sum_{i=1}^m R_i}^m(X)) = OU_{\sum_{i=1}^m R_i}^m(OL_{\sum_{i=1}^m R_i}^m(X)) = OL_{\sum_{i=1}^m R_i}^m(X)$.
- (5) $OU_{\sum_{i=1}^m R_i}^m(OU_{\sum_{i=1}^m R_i}^m(X)) = OL_{\sum_{i=1}^m R_i}^m(OU_{\sum_{i=1}^m R_i}^m(X)) = OU_{\sum_{i=1}^m R_i}^m(X)$.
- (6) $X \subseteq Y \implies OL_{\sum_{i=1}^m R_i}^m(X) \subseteq OL_{\sum_{i=1}^m R_i}^m(Y)$.
- (7) $X \subseteq Y \implies OU_{\sum_{i=1}^m R_i}^m(X) \subseteq OU_{\sum_{i=1}^m R_i}^m(Y)$.

- (8) $OL_{\sum_{i=1}^m R_i}^m(X \cap Y) \subseteq OL_{\sum_{i=1}^m R_i}^m(X) \cap OL_{\sum_{i=1}^m R_i}^m(Y)$ and
 $OU_{\sum_{i=1}^m R_i}^m(X \cup Y) \supseteq OU_{\sum_{i=1}^m R_i}^m(X) \cup OU_{\sum_{i=1}^m R_i}^m(Y)$.
- (9) $OL_{\sum_{i=1}^m R_i}^m(X \cup Y) \supseteq OL_{\sum_{i=1}^m R_i}^m(X) \cup OL_{\sum_{i=1}^m R_i}^m(Y)$ and
 $OU_{\sum_{i=1}^m R_i}^m(X \cap Y) \subseteq OU_{\sum_{i=1}^m R_i}^m(X) \cap OU_{\sum_{i=1}^m R_i}^m(Y)$.
- (10) $OL_{\sum_{i=1}^m R_i}^m(X) = \cup_i L_{R_i}(X)$ and $OU_{\sum_{i=1}^m R_i}^m(X) = \cap_i U_{R_i}(X)$ for $1 \leq i \leq m$.

2. Pre-Connectedness in Multiple-Granulation Approximation Spaces

Theorem 2.1. Suppose $A \subseteq U$ is a multiple-granulation approximation space in $(U, \mathcal{R})$. For each $B \in P(U)$, we define the mapping $MGS - int_{\mathcal{R}}^A$ from $P(U)$ to $P(U)$ as:

$$MGS - int_{\mathcal{R}}^A(B) = \begin{cases} OL_{\sum_{i=1}^m R_i}^m(A) \cap OL_{\sum_{i=1}^m R_i}^m(B) & \text{for all } B \neq U, \\ U, & \text{if } B = U. \end{cases}$$

For $B, C \in P(U)$, the operator $MGS - int_{\mathcal{R}}^A$ fulfils the following conditions:

- (1) $MGS - int_{\mathcal{R}}^A(\emptyset) = \emptyset$, $MGS - int_{\mathcal{R}}^A(U) = U$.
- (2) $MGS - int_{\mathcal{R}}^A(B) \subseteq B$.
- (3) If $B \subseteq C$ then $MGS - int_{\mathcal{R}}^A(B) \subseteq MGS - int_{\mathcal{R}}^A(C)$.
- (4) $MGS - int_{\mathcal{R}}^A(B \cap C) \subseteq MGS - int_{\mathcal{R}}^A(B) \cap MGS - int_{\mathcal{R}}^A(C)$.
- (5) $MGS - int_{\mathcal{R}}^A(MGS - int_{\mathcal{R}}^A(B)) = MGS - int_{\mathcal{R}}^A(B)$.

Proof.

- (1) According to Proposition 1.3 (2),

$$MGS - int_{\mathcal{R}}^A(\emptyset) = OL_{\sum_{i=1}^m R_i}^m(A) \cap OL_{\sum_{i=1}^m R_i}^m(\emptyset) = \emptyset.$$

The second part comes directly from the definition.

- (2) $MGS - int_{\mathcal{R}}^A(B) = OL_{\sum_{i=1}^m R_i}^m(A) \cap OL_{\sum_{i=1}^m R_i}^m(B) \subseteq OL_{\sum_{i=1}^m R_i}^m(A) \cap B \subseteq B$.

- (3) If $B \subseteq C$ then $OL_{\sum_{i=1}^m R_i}^m(B) \subseteq OL_{\sum_{i=1}^m R_i}^m(C)$, implies that

$$MGS - int_{\mathcal{R}}^A(B) \subseteq MGS - int_{\mathcal{R}}^A(C).$$

- (4) For $B, C \in P(U)$ we have

$$\begin{aligned} MGS - int_{\mathcal{R}}^A(B \cap C) &= OL_{\sum_{i=1}^m R_i}^m(A) \cap OL_{\sum_{i=1}^m R_i}^m(B \cap C) \\ &\subseteq OL_{\sum_{i=1}^m R_i}^m(A) \cap \left(OL_{\sum_{i=1}^m R_i}^m(B) \cap OL_{\sum_{i=1}^m R_i}^m(C) \right) \\ &= \left(OL_{\sum_{i=1}^m R_i}^m(A) \cap OL_{\sum_{i=1}^m R_i}^m(B) \right) \cap \left(OL_{\sum_{i=1}^m R_i}^m(A) \cap OL_{\sum_{i=1}^m R_i}^m(C) \right) \\ &= MGS - int_{\mathcal{R}}^A(B) \cap MGS - int_{\mathcal{R}}^A(C). \end{aligned}$$

- (5) Obvious.

We observe that in a multiple-granulation approximation space $(U, \mathcal{R})$, a mapping $MGS - int_{\mathcal{R}}^A$ obtained in Theorem 2.1 is called the OMG-supra interior operator associated with A .

In the event that there is no misunderstanding, we write $int_{\mathcal{R}}^A(B)$ instead of $MGS - int_{\mathcal{R}}^A(B)$.

Theorem 2.2. If $int_{\mathcal{R}}^A$ is the OMG-supra interior operator associated with A in a multiple-granulation approximation space $(U, \mathcal{R})$, there exists a multiple-granulation supratopology $\mathcal{T}_{\mathcal{R}}^A \subseteq P(U)$ defined as:

$$\mathcal{T}_{\mathcal{R}}^A = \{B \in P(U) : B = int_{\mathcal{R}}^A(B)\}.$$

$(U, \mathcal{T}_{\mathcal{R}}^A)$ is called the OMG-approximation supratopological space.

Proof.

(1) Using (1) in Theorem 2.1, we have

$$\emptyset, U \in \mathcal{T}_{\mathcal{R}}^A.$$

(2) Assume $B_i \in \mathcal{T}_{\mathcal{R}}^A$, then

$$B_i = int_{\mathcal{R}}^A(B_i) \text{ for any } i \in \Delta.$$

According to Theorem 2.1(2), $int_{\mathcal{R}}^A(\cup_{i \in \Delta} B_i) \subseteq \cup_{i \in \Delta} B_i$. In contrast, $int_{\mathcal{R}}^A(B_i) \subseteq \cup_{i \in \Delta} int_{\mathcal{R}}^A(B_i)$. Theorem 2.1 (3) and (5) states that for any $i \in \Delta$,

$$\begin{aligned} int_{\mathcal{R}}^A(B_i) &= int_{\mathcal{R}}^A(int_{\mathcal{R}}^A(B_i)) \\ &\subseteq int_{\mathcal{R}}^A(\cup_{i \in \Delta} int_{\mathcal{R}}^A(B_i)). \end{aligned}$$

Thus,

$$\cup_{i \in \Delta} int_{\mathcal{R}}^A(B_i) \subseteq int_{\mathcal{R}}^A(\cup_{i \in \Delta} int_{\mathcal{R}}^A(B_i)).$$

Since $B_i = int_{\mathcal{R}}^A(B_i)$ we have

$$\cup_{i \in \Delta} B_i \subseteq int_{\mathcal{R}}^A(\cup_{i \in \Delta} B_i).$$

So $\cup_{i \in \Delta} B_i = int_{\mathcal{R}}^A(\cup_{i \in \Delta} B_i)$. This suggests that $\cup_{i \in \Delta} B_i \in \mathcal{T}_{\mathcal{R}}^A$. Hence, $\mathcal{T}_{\mathcal{R}}^A$ is a multiple-granulation supratopology.

Each element of $\mathcal{T}_{\mathcal{R}}^A$ is known as the OMG-supra open approximation set, while its complement is known as the OMG-supra closed approximation set.

Theorem 2.3. If $A \subseteq U$ is associated with a multiple-granulation approximation space $(U, \mathcal{R})$, then the mapping $MGS - cl_{\mathcal{R}}^A$ from $P(U)$ to $P(U)$ is defined by:

$$MGS - cl_{\mathcal{R}}^A(B) = \begin{cases} (OL_{\sum_{i=1}^m R_i}(A))^c \cup OU_{\sum_{i=1}^m R_i}(B) & \text{for all } B \neq \emptyset, \\ \emptyset, & \text{if } B = \emptyset. \end{cases}$$

For $B, C \in P(U)$, the operator $MGS - cl_{\mathcal{R}}^A$ has the following conditions:

- (1) $MGS - cl_{\mathcal{R}}^A(\emptyset) = \emptyset$, $MGS - cl_{\mathcal{R}}^A(U) = U$.
- (2) $B \subseteq MGS - cl_{\mathcal{R}}^A(B)$.
- (3) If $B \subseteq C$ then $MGS - cl_{\mathcal{R}}^A(B) \subseteq MGS - cl_{\mathcal{R}}^A(C)$.
- (4) $MGS - cl_{\mathcal{R}}^A(B) \cup MGS - cl_{\mathcal{R}}^A(C) \subseteq MGS - cl_{\mathcal{R}}^A(B \cup C)$.
- (5) $MGS - cl_{\mathcal{R}}^A(MGS - cl_{\mathcal{R}}^A(B)) = MGS - cl_{\mathcal{R}}^A(B)$.

Proof. Analogous to Theorem 2.1.

A mapping $MGS - cl_{\mathcal{R}}^A$ defined in Theorem 2.3 is called the OMG-supra closure operator associated with A in a multiple-granulation approximation space $(U, \mathcal{R})$.

To avoid confusion, we can write $cl_{\mathcal{R}}^A(B)$ instead of $MGS - cl_{\mathcal{R}}^A(B)$.

Lemma 2.4. Let $(U, \mathcal{R})$ be a multiple-granulation approximation space. For each $B \in P(U)$, we have

- (1) $int_{\mathcal{R}}^A(B^c) = (cl_{\mathcal{R}}^A(B))^c$.
- (2) $cl_{\mathcal{R}}^A(B^c) = (int_{\mathcal{R}}^A(B))^c$.
- (3) $cl_{\mathcal{R}}^A(OU_{\sum_{i=1}^m R_i}(B)) = cl_{\mathcal{R}}^A(B)$.
- (4) $int_{\mathcal{R}}^A(OL_{\sum_{i=1}^m R_i}(B)) = int_{\mathcal{R}}^A(B)$.
- (5) B is the OMG-supra open approximation set if and only if $B = int_{\mathcal{R}}^A(B)$.

Proof. Obvious.

$cl_{\mathcal{R}}^A(B^c) = (int_{\mathcal{R}}^A(B))^c$ implies that the OMG-supra closure operator generates the same OMG-supratopology $\mathcal{T}_{\mathcal{R}}^A$ as the following theorem.

Theorem 2.5. Let $cl_{\mathcal{R}}^A$ be the OMG-supra closure operator associated with A in a multiple-granulation approximation space $(U, \mathcal{R})$. Then there exists the OMG-supratopology $\mathcal{T}_{\mathcal{R}}^A \subseteq P(U)$ defined as:

$$\mathcal{T}_{\mathcal{R}}^A = \{B \in P(U) : B^c = cl_{\mathcal{R}}^A(B^c)\}.$$

Proof. In the same way as Theorem 2.2.

Definition 2.6. Assume $(U, \mathcal{R})$ is a multiple-granulation approximation space and $A \in P(U)$. Then,

- (1) B is the OMG-supra-preopen approximation set with A if and only if $B \subseteq int_{\mathcal{R}}^A(cl_{\mathcal{R}}^A(B))$.
- (2) B is the OMG-supra-preclosed approximation set with A if and only if $B \supseteq cl_{\mathcal{R}}^A(int_{\mathcal{R}}^A(B))$.
- (3) The OMG-supra-pre interior of B with A , denoted by $spint_{\mathcal{R}}^A(B)$, is defined as:

$$spint_{\mathcal{R}}^A(B) = \bigcup \{C : C \subseteq B, C \text{ is the OMG-supra-preopen set}\}.$$

- (4) The OMG-supra-pre closure of B with respect to A , denoted by $spcl_{\mathcal{R}}^A(B)$, is defined as:

$$spcl_{\mathcal{R}}^A(B) = \bigcap \{C : B \subseteq C, C \text{ is the OMG-supra-pre closed set}\}.$$

Theorem 2.7. Let $(U, \mathcal{R})$ be a multiple-granulation approximation space, with $A, B \in P(U)$. Then the following statements are valid:

- (1) B is the OMG-supra-preopen approximation set if and only if $B = spint_{\mathcal{R}}^A(B)$.
- (2) B is the OMG-supra-preclosed approximation set if and only if $B = spcl_{\mathcal{R}}^A(B)$.
- (3) $spint_{\mathcal{R}}^A(U) = U$ and $spcl_{\mathcal{R}}^A(\emptyset) = \emptyset$.
- (4) $int_{\mathcal{R}}^A(B) \subseteq spint_{\mathcal{R}}^A(B) \subseteq B \subseteq spcl_{\mathcal{R}}^A(B) \subseteq cl_{\mathcal{R}}^A(B)$.
- (5) $spcl_{\mathcal{R}}^A(B) \cup spcl_{\mathcal{R}}^A(C) \subseteq spcl_{\mathcal{R}}^A(B \cup C)$.
- (6) $spcl_{\mathcal{R}}^A(spcl_{\mathcal{R}}^A(B)) = spcl_{\mathcal{R}}^A(B)$.
- (7) $cl_{\mathcal{R}}^A(spcl_{\mathcal{R}}^A(B)) = spcl_{\mathcal{R}}^A(cl_{\mathcal{R}}^A(B)) = cl_{\mathcal{R}}^A(B)$.

$$(8) \text{ spint}_{\mathcal{R}}^A(B^c) = (\text{spcl}_{\mathcal{R}}^A(B))^c.$$

Proof. We demonstrate (7): Based on (4), $B \subseteq \text{spcl}_{\mathcal{R}}^A(B) \subseteq \text{cl}_{\mathcal{R}}^A(B)$; thus,

$$\text{cl}_{\mathcal{R}}^A(B) \subseteq \text{cl}_{\mathcal{R}}^A(\text{spcl}_{\mathcal{R}}^A(B)) \subseteq \text{cl}_{\mathcal{R}}^A(\text{cl}_{\mathcal{R}}^A(B)) = \text{cl}_{\mathcal{R}}^A(B).$$

Hence, we get

$$\text{cl}_{\mathcal{R}}^A(\text{spcl}_{\mathcal{R}}^A(B)) = \text{cl}_{\mathcal{R}}^A(B).$$

Furthermore, since $\text{cl}_{\mathcal{R}}^A(B)$ is the OMG-supra-preclosed approximation set, then $\text{spcl}_{\mathcal{R}}^A(\text{cl}_{\mathcal{R}}^A(B)) = \text{cl}_{\mathcal{R}}^A(B)$.

Remark. (1) Every OMG-supra-open (supra-closed) approximation set is OMG-supra-preopen (supra-preclosed) approximation set.

(2) Any union of OMG-supra-open approximation sets is OMG-supra-open approximation.

(3) Any intersection of OMG-supra-closed approximation sets constitutes an OMG-supra-closed approximation set.

The following example demonstrates that the converse of the statement is not true.

Example 2.8. Consider $P = U/R_1 = \{\{a\}, \{b, c, d\}\}$, $Q = U/R_2 = \{\{b, d\}, \{a, c\}\}$ and $L = U/R_3 = \{\{a, b, c\}, \{d\}\}$ are three partitions on the universe $U = \{a, b, c, d\}$. Then we compute the OMG-lower, OMG-upper, OMG-interior and OMG-closure with respect to $A = \{a, b, c\}$ of each subset $B \subseteq U$ in the Table 1.

Table 1. The optimistic multiple-granulation lower and upper

	$OL_{P+Q+L}(B)$	$\text{int}_{\mathcal{R}}^{\{a,b,c\}}(B)$	$OU_{P+Q+L}(B)$	$\text{cl}_{\mathcal{R}}^{\{a,b,c\}}(B)$
$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a, d\}$
$\{b\}$	$\emptyset$	$\emptyset$	$\{b\}$	$\{b, d\}$
$\{c\}$	$\emptyset$	$\emptyset$	$\{c\}$	$\{c, d\}$
$\{d\}$	$\{d\}$	$\emptyset$	$\{d\}$	$\{d\}$
$\{a, b\}$	$\{a\}$	$\{a\}$	$\{a, b, c\}$	$\{a, b, c, d\}$
$\{a, c\}$	$\{a, c\}$	$\{a, c\}$	$\{a, c\}$	$\{a, c, d\}$
$\{a, d\}$	$\{a, d\}$	$\{a\}$	$\{a, b, c, d\}$	$\{a, b, c, d\}$
$\{b, c\}$	$\emptyset$	$\emptyset$	$\{b, c\}$	$\{b, c, d\}$
$\{b, d\}$	$\{b, d\}$	$\{b\}$	$\{b, d\}$	$\{b, d\}$
$\{c, d\}$	$\{d\}$	$\emptyset$	$\{b, c, d\}$	$\{b, c, d\}$
$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c, d\}$
$\{a, b, d\}$	$\{a, b, d\}$	$\{a, b\}$	$\{a, b, c, d\}$	$\{a, b, c, d\}$
$\{a, c, d\}$	$\{a, c, d\}$	$\{a, c\}$	$\{a, b, c, d\}$	$\{a, b, c, d\}$
$\{b, c, d\}$	$\{b, c, d\}$	$\{b, c\}$	$\{b, c, d\}$	$\{b, c, d\}$

The set $B = \{a, b\}$ is the OMG-supra-preopen approximation set with $A = \{a, b, c\}$, but not the OMG-supra-open approximation set.

$$\text{int}_{\mathcal{R}}^{\{a,b,c\}}(B) = \{a\} \neq \{a, b\} = B.$$

The sets $\{a, c\}$ and $\{b, c\}$ are OMG-supra-preopen approximation sets with $A = \{a, b, c\}$, but their intersection is not OMG-supra-preopen approximation set with respect to $A = \{a, b, c\}$.

$$\{c\} = \{a, c\} \cap \{b, c\} \not\subseteq \text{int}_{\mathcal{R}}^{\{a,b,c\}}(\text{cl}_{\mathcal{R}}^{\{a,b,c\}}(\{c\})) = \emptyset.$$

Furthermore, since $\text{int}_{\mathcal{R}}^{\{a,b,c\}}(\{a, c, d\}) = \{a, c\}$, $\text{int}_{\mathcal{R}}^{\{a,b,c\}}(\{b, c, d\}) = \{b, c\}$, then

$$\begin{aligned} \emptyset = \text{int}_{\mathcal{R}}^{\{a,b,c\}}(\{c, d\}) &= \text{int}_{\mathcal{R}}^{\{a,b,c\}}(\{a, c, d\} \cap \{b, c, d\}) \\ &\not\subseteq \text{int}_{\mathcal{R}}^{\{a,b,c\}}(\{a, c, d\}) \cap \text{int}_{\mathcal{R}}^{\{a,b,c\}}(\{b, c, d\}) \\ &= \{a, c\} \cap \{b, c\} = \{c\}. \end{aligned}$$

Definition 2.9. Let $(U, \mathcal{R})$ be a multiple-granulation approximation space with $A \in P(U)$. If $B \subseteq \text{int}_{\mathcal{R}}^A(\text{cl}_{\mathcal{R}}^A(B)) \cup \text{cl}_{\mathcal{R}}^A(\text{int}_{\mathcal{R}}^A(B))$, then B is the OMG-supra- b -open approximation set with respect to A .

Proposition 2.10. Let B be the OMG-supra- b -open approximation set of $(U, \mathcal{R})$ with $\text{int}_{\mathcal{R}}^A(B) = \emptyset$. Then B and B^c are the OMG-supra-preopen approximation sets.

Proof. Given that B is the OMG-supra- b -open approximation set. Then

$$B \subseteq \text{int}_{\mathcal{R}}^A(\text{cl}_{\mathcal{R}}^A(B)) \cup \text{cl}_{\mathcal{R}}^A(\text{int}_{\mathcal{R}}^A(B)).$$

If $\text{int}_{\mathcal{R}}^A(B) = \emptyset$, then $\text{cl}_{\mathcal{R}}^A(\text{int}_{\mathcal{R}}^A(B)) = \emptyset$. Thus,

$$B \subseteq \text{int}_{\mathcal{R}}^A(\text{cl}_{\mathcal{R}}^A(B)).$$

Thus, B represents the OMG-supra-preopen approximation set. Moreover, $(\text{int}_{\mathcal{R}}^A(B))^c = U$. This indicates that $\text{cl}_{\mathcal{R}}^A(B^c) = U$. Therefore, B^c is the OMG-supra-preopen approximation set.

3. Pre-Continuity in Multiple-Granulation Approximation Spaces

Definition 3.1. Let $(U, \mathcal{R})$ $(V, \mathcal{R}^*)$ be two multiple-granulation approximation spaces, with $A \in P(U)$ and $B \in P(V)$. The mapping $f : (U, \mathcal{R}) \rightarrow (V, \mathcal{R}^*)$ is called

- (1) An OMG-supra-continuous approximation if $f^{-1}(C)$ is the OMG-supra-open approximation set in U for each OMG-supra-open approximation set in V .
- (2) An OMG-irresolute approximation (resp. the OMG-supra-pre-continuous approximation) if $f^{-1}(C)$ is an OMG-supra-preopen approximation set in U for each the OMG-supra-preopen approximation set in V (resp. $\text{int}_{\mathcal{R}^*}^B(C) = C$).
- (3) An OMG-irresolute open approximation (resp. the OMG-supra-preopen approximation) if $f(C)$ is the OMG-supra-preopen approximation set in V for each the OMG-supra-preopen approximation set in U (resp. $\text{int}_{\mathcal{R}}^A(C) = C$).
- (4) The OMG-irresolute closed approximation (resp. the OMG-supra-preclosed approximation) if $f(C)$ is the OMG-supra-preclosed approximation set in V for each the OMG-supra-preclosed approximation set in U (resp. $\text{cl}_{\mathcal{R}}^A(C) = C$).

Theorem 3.2. Let $(U, \mathcal{R})$ and $(V, \mathcal{R}^*)$ be two multiple-granulation approximation spaces, $A \in P(U)$, $B \in P(V)$ and $f : X \rightarrow Y$ be a function. Then the following statements are equivalent:

- (1) f is the OMG-supra continuous approximation.
- (2) The OMG-supra-closed approximation set $C \subseteq Y$ is $f^{-1}(C)$.
- (3) $f(\text{cl}_{\mathcal{R}}^A(K)) \subseteq \text{cl}_{\mathcal{R}^*}^B(f(K))$ for all $K \subseteq U$.
- (4) $\text{cl}_{\mathcal{R}}^A(f^{-1}(C)) \subseteq f^{-1}(\text{cl}_{\mathcal{R}^*}^B(C))$ for all $C \subseteq V$.

$$(5) f^{-1}(int_{\mathcal{R}^*}^B(C)) \subseteq int_{\mathcal{R}}^A(f^{-1}(C)) \text{ for all } C \subseteq V.$$

Proof. (1) $\Leftrightarrow$ (2) Obvious.

(2) $\Rightarrow$ (3) For every $K \subseteq U$, $cl_{\mathcal{R}^*}^B(f(K))$ is a multiple-granulation approximation closed set in V . Hence, $f^{-1}(cl_{\mathcal{R}^*}^B(f(K)))$ is a multiple-granulation approximation closed set in U . However, $K \subseteq f^{-1}(f(K)) \subseteq f^{-1}(cl_{\mathcal{R}^*}^B(f(K)))$, and then

$$\begin{aligned} cl_{\mathcal{R}}^A(K) &\subseteq cl_{\mathcal{R}}^A(f^{-1}(f(K))) \\ &\subseteq cl_{\mathcal{R}}^A(f^{-1}(cl_{\mathcal{R}^*}^B(f(K)))) \\ &= f^{-1}(cl_{\mathcal{R}^*}^B(f(K))). \end{aligned}$$

Thus, we get $f(cl_{\mathcal{R}}^A(K)) \subseteq cl_{\mathcal{R}^*}^B(f(K))$.

(3) $\Rightarrow$ (4) For any $C \subseteq V$, set $f^{-1}(C) = K$. From (3) we get

$$f(cl_{\mathcal{R}}^A(f^{-1}(C))) \subseteq cl_{\mathcal{R}^*}^B(f(f^{-1}(C))) \subseteq cl_{\mathcal{R}^*}^B(C).$$

This means that $cl_{\mathcal{R}}^A(f^{-1}(C)) \subseteq f^{-1}(cl_{\mathcal{R}^*}^B(C))$.

(4) $\Rightarrow$ (5) Straightforward.

(5) $\Rightarrow$ (1) Let C be a multiple-granulation approximation open set in V . Then, $C = int_{\mathcal{R}^*}^B(C)$. By (5) $f^{-1}(C) = f^{-1}(int_{\mathcal{R}^*}^B(C)) \subseteq int_{\mathcal{R}}^A(f^{-1}(C))$. Since $int_{\mathcal{R}}^A(f^{-1}(C)) \subseteq f^{-1}(C)$ and $int_{\mathcal{R}}^A(f^{-1}(C)) = f^{-1}(C)$. Thus, $f^{-1}(C)$ is a multiple-granulation approximation open set in U . Hence, a multiple-granulation approximation f is continuous.

The following two theorems are proved analogously to theorem 3.2.

Theorem 3.3. Let $(U, \mathcal{R})$ and $(V, \mathcal{R}^*)$ be two multiple-granulation approximation spaces with $A \in P(U)$, $B \in P(V)$ and $f : X \rightarrow Y$ a function. Then, the following statements are equivalent:

- (1) f is an OMG-irresolute approximation map.
- (2) The OMG-supra pre-closed approximation set $C \subseteq Y$ is $f^{-1}(C)$.
- (3) $f(spcl_{\mathcal{R}}^A(K)) \subseteq spcl_{\mathcal{R}^*}^B(f(K))$ for all $K \subseteq U$.
- (4) $spcl_{\mathcal{R}}^A(f^{-1}(C)) \subseteq f^{-1}(spcl_{\mathcal{R}^*}^B(C))$ for all $C \subseteq V$.
- (5) $f^{-1}(spint_{\mathcal{R}^*}^B(C)) \subseteq spint_{\mathcal{R}}^A(f^{-1}(C))$ for all $C \subseteq V$.

Theorem 3.4. Let $(U, \mathcal{R})$ and $(V, \mathcal{R}^*)$ be two multiple-granulation approximation spaces with $A \in P(U)$, $B \in P(V)$ and $f : X \rightarrow Y$ a function. Then the following statements are equivalent:

- (1) f is the OMG-supra-precontinuous approximation.
- (2) $f(spcl_{\mathcal{R}}^A(K)) \subseteq cl_{\mathcal{R}^*}^B(f(K))$ for all $K \subseteq U$.
- (3) $spcl_{\mathcal{R}}^A(f^{-1}(C)) \subseteq f^{-1}(cl_{\mathcal{R}^*}^B(C))$ for all $C \subseteq V$.
- (4) $f^{-1}(int_{\mathcal{R}^*}^B(C)) \subseteq spint_{\mathcal{R}}^A(f^{-1}(C))$ for all $C \subseteq V$.

Theorem 3.5. Let $(U, \mathcal{R})$, $(V, \mathcal{R}^*)$ be two a multiple-granulation approximation spaces with $A \in P(U)$, $B \in P(V)$ and $f : X \rightarrow Y$ a function. Then, the following statements are equivalent:

- (1) f is the OMG-irresolute open approximation map.
- (2) $f(spint_{\mathcal{R}}^A(K)) \subseteq spint_{\mathcal{R}^*}^B(f(K))$ for all $K \subseteq U$.
- (3) $spint_{\mathcal{R}}^A(f^{-1}(C)) \subseteq f^{-1}(spint_{\mathcal{R}^*}^B(C))$ for all $C \subseteq V$.
- (4) For any set $C \subseteq Y$ and any OMG-supra-preclosed approximation set $K \subseteq X$ such that $f^{-1}(C) = K$, there exists an OMG-supra-preclosed approximation set $E \subseteq Y$ with $C \subseteq E$ such that $f^{-1}(E) \subseteq K$.

Proof. (1) $\Rightarrow$ (2) Given that $spint_{\mathcal{R}}^A(K) \subseteq K$ for any $K \subseteq U$, then

$$f(spint_{\mathcal{R}}^A(K)) \subseteq f(K).$$

According to (1), $f(spint_{\mathcal{R}}^A(K))$ is the OMG-supra-pre-open approximation set in V . Therefore,

$$f(spint_{\mathcal{R}}^A(K)) \subseteq spint_{\mathcal{R}^*}^B(f(K)).$$

(2) $\Rightarrow$ (3) For any $C \subseteq V$, set $f^{-1}(C) = K$. Then,

$$f(spint_{\mathcal{R}}^A(f^{-1}(C))) \subseteq spint_{\mathcal{R}^*}^B(f(f^{-1}(C))) \subseteq spint_{\mathcal{R}^*}^B(C).$$

It implies that $spint_{\mathcal{R}}^A(f^{-1}(C)) \subseteq f^{-1}(spint_{\mathcal{R}^*}^B(C))$.

(3) $\Rightarrow$ (4) Let $K \subseteq X$ be the OMG-supra-preclosed approximation set with $f^{-1}(C) \subseteq K$, $C \subseteq Y$. Thus, $cl_{\mathcal{R}}^A(int_{\mathcal{R}}^A(K)) \subseteq K$ and $spint_{\mathcal{R}}^A(K^c) \subseteq spint_{\mathcal{R}}^A(f^{-1}(C^c))$. By (3)

$$K^c \subseteq spint_{\mathcal{R}}^A(f^{-1}(C^c)) \subseteq f^{-1}(spint_{\mathcal{R}^*}^B(C^c)).$$

This means that $f^{-1}(spcl_{\mathcal{R}^*}^B(C)) = f^{-1}((spcl_{\mathcal{R}^*}^B(C^c))^c) \subseteq C$. Thus, there exists the OMG-supra-preclosed approximation set $spcl_{\mathcal{R}^*}^B(C)$ in V with $C \subseteq spcl_{\mathcal{R}^*}^B(C)$ so that $f^{-1}(spcl_{\mathcal{R}^*}^B(C)) \subseteq K$.

(4) $\Rightarrow$ (1) Clear.

The following theorems can be proven in the same way that Theorem 3.9 was.

Theorem 3.6. Let $f : X \rightarrow Y$ be a function and $(U, \mathcal{R}), (V, \mathcal{R}^*)$ be two multiple-granulation approximation spaces with $A \in P(U), B \in P(V)$. Then, the following claims are interchangeable:

- (1) f is the OMG-supra-open approximation map.
- (2) $f(int_{\mathcal{R}}^A(K)) \subseteq int_{\mathcal{R}^*}^B(f(K))$ for all $K \subseteq U$.
- (3) $int_{\mathcal{R}}^A(f^{-1}(C)) \subseteq f^{-1}(int_{\mathcal{R}^*}^B(C))$ for all $C \subseteq V$.
- (4) For any set $C \subseteq Y$, the OMG-supra-closed approximation set $K \subseteq X$ so that $f^{-1}(C) = K$, there exists the OMG-supra-closed approximation set $E \subseteq Y$ with $C \subseteq E$ so that $f^{-1}(E) \subseteq K$.

Theorem 3.7. Let $(U, \mathcal{R}), (V, \mathcal{R}^*)$ be two multiple-granulation approximation spaces with $A \in P(U), B \in P(V)$ and $f : X \rightarrow Y$ a function. Then, the following statements are equivalent:

- (1) f is the OMG-supra-preopen approximation map.
- (2) $f(int_{\mathcal{R}}^A(K)) \subseteq spint_{\mathcal{R}^*}^B(f(K))$ for all $K \subseteq U$.
- (3) $int_{\mathcal{R}}^A(f^{-1}(C)) \subseteq f^{-1}(spint_{\mathcal{R}^*}^B(C))$ for all $C \subseteq V$.
- (4) For any set $C \subseteq Y$ and any OMG-supra-preclosed approximation set $K \subseteq X$ so that $f^{-1}(C) = K$, there exists an OMG-supra-closed approximation set $E \subseteq Y$ with $C \subseteq E$ such that $f^{-1}(E) \subseteq K$.

Theorem 3.8. Let $(U, \mathcal{R}), (V, \mathcal{R}^*)$ be two multiple-granulation approximation spaces with $A \in P(U), B \in P(V)$ and $f : X \rightarrow Y$ a function. Then, the following claims are interchangeable:

- (1) f is the OMG-irresolute closed approximation map.
- (2) $f(spcl_{\mathcal{R}}^A(K)) \supseteq spcl_{\mathcal{R}^*}^B(f(K))$ for all $K \subseteq U$.

Theorem 3.9. Let $(U, \mathcal{R}), (V, \mathcal{R}^*)$ be two multiple-granulation approximation spaces with $A \in P(U), B \in P(V)$ and $f : X \rightarrow Y$ a function. Then, the following claims are interchangeable:

- (1) f is the OMG-supra-preclosed approximation map.

- (2) $f(cl_R^A(K)) \subseteq int_{R^*}^B(cl_{R^*}^B(f(K)))$ for all $K \subseteq U$.
 (3) $spcl_{R^*}^B(f(K)) \subseteq f(cl_R^A(K))$ for all $K \subseteq U$.

Remark 3.10.

- (1) Every OMG-supra continuous approximation (respectively, the OMG-supra open approximation map or the OMG-supra closed approximation map) is the OMG-supra precontinuous approximation (respectively, the OMG-supra-preopen approximation or the OMG-supra-preclosed approximation) map.
 (2) Every OMG-irresolute approximation map has an OMG-supra-precontinuous approximation.

Example 3.11. Given $U = V$, consider $P = U/R_1 = \{\{a\}, \{b, c, d\}\}$, $Q = U/R_2 = \{\{b, d\}, \{a, c\}\}$, and $L = U/R_3 = \{\{a, b, c\}, \{d\}\}$ are three partitions on the universe $U = \{a, b, c, d\}$. We then compute the OMG-lower, OMG-upper, OMG-interior, and OMG-closure with respect to $A = \{a, b, c\}$ of each subset $B \subseteq U$ in Table 1.

A mapping $f : (U, \mathcal{R}) \rightarrow (V, \mathcal{R}^*)$ is defined as $f(a) = f(d) = b$, $f(b) = a$, and $f(c) = c$. If $\{a, c\}$ is the OMG-supra-open approximation set and $f^{-1}(\{a, c\}) = \{b, c\}$ is the OMG-supra-preopen approximation set, then f is the OMG-supra-precontinuous approximation, but not the OMG-supra continuous approximation.

Author Contributions: Methodology, O.R.Sayed.; Validation, E.El-Ssanousy; Formal analysis, Salahuddin; Investigation, O.R.Sayed; Resources, E.El-Ssanousy; Writing—original draft, A.A.Abd-Allah; Writing – review and editing, Salahuddin; Visualization, Y.H. Ragheb Sayed. All authors have read and agreed to the published version of the manuscript.

References

1. M. Chuchro, On rough sets in topological Boolean algebra. In: Ziarko, W. (ed.): Rough Sets, Fuzzy Sets and Knowledge Discovery, Springer-Verlage, New York, pp. 157-160, 1994.
2. I. Ibedou, S. E. Abbas and S. Jafari, More on Approximation spaces in the sense of Pawlak, March 2020, ResearchGate.
3. J. Kortelainen, On the relationship between modified sets, topological spaces and rough sets, Fuzzy Sets and Systems 61 (1994), 91-95.
4. J. Y. Liang, F. Wang, C. Y. Dang and Y. H. Qian, An efficient rough feature selection algorithm with a multi-granulation view, International Journal of Approximate Reasoning, 50 (2012), 912-926.
5. G. P. Lin, Y. H. Qian and J. J. Li, NMGRS: neighborhood-based multi-granulation rough sets, International Journal of Approximate Reasoning, 53 (7) (2012), 1080-1093.
6. C. H. Liu and M. Z. Wang, Covering fuzzy rough set based on multi-granulations, International Conference on Uncertainty Reasoning and Knowledge Engineering, pp. 146-149, 2011.
7. C. H. Liu, D. Q. Miao and J. Qian, On multi-granulation covering rough sets, International Journal of Approximate Reasoning, 55(6) (2014), 1404-1418.
8. J. M. Ma, W. X. Zhang, Y. Leung and X. X. Song, Granular computing and dual Galois connection, Information Sciences, 177 (2007), 5365-5377.
9. Z. Pawlak, Rough sets, International Journal of Computer and Information Sciences, 11 (1982), 341-356.
10. Z. Pawlak, Rough Sets, Theoretical Aspects of Reasoning about Data, Kluwer Academic Publishers, Dordrecht, Boston, 1991.
11. L. Polkowski, Mathematical morphology of rough sets, Bulletin of the Polish Academy Science Mathematics, 41 (3) (1993) 241-273.
12. L. Polkowski, Metric spaces of topological rough sets from countable knowledge bases, in: R. Slowinski, J. Stefanowski (Eds.), Foundations of Computing and Decision Sciences, 18 (3-4) (1993), special issue, 241-273.
13. Y. H. Qian and J. Y. Liang, Rough Set Method Based on Multi-granulations. The 5th IEEE International Conference on Cognitive Informatics, Beijing, China, (2006), 297-304.
14. Y. H. Qian, J. Y. Liang, Y. Y. Yao and C. Y. Dang, MGRS: A multi-granulation rough set, Information Sciences, 180 (6) (2010), 949-970.

15. K. Qin, J. Yang and Z. Pei, Generalized rough sets based on reflexive and transitive, *Information Science*, 178 (2008), 4138-4141.
16. Y. H. She and X. L. He, On the structure of the multi-granulation rough set model, *Knowledge-Based systems*, 36 (2012), 81-92.
17. A. A. Abdallah, O. R. Sayed, E. El-Sanousy, Y. H. Ragheb Sayed, M. N. Abushugair, Salahuddin, Multi-Granulation double fuzzy rough sets, *Symmetry*, MDPI, 15 (2023), 129.
18. M. L. Thivagar and C. Richard, On Nano forms of weakly open sets, *International Journal of Mathematics and Statistics Invention*, 1/1 (2013), 31-37.
19. Y. Y. Yao, Two views of the theory of rough sets in finite universes, *International Journal of Approximate Reasoning*, 15 (1996), 291-317.
20. Y. Y. Yao and T Y Lin, Generalization of rough sets using modal logics, *Intelligent Automation and Soft Computing*, 2 (1996), 103-120.
21. P. Zhi, D. Pei and L. Zheng, Topology vs generalized rough sets, *International Journal of Approximate Reasoning*, 52(2) (2011), 231-239.

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.