Article

Not peer-reviewed version

A Simple Rigorous Proof of Riemann's Hypothesis

Jau Tang *

Posted Date: 19 February 2025

doi: 10.20944/preprints202502.1458.v1

Keywords: Riemann's hypothesis; Riemann's zeta function; reflection symmetry; critical line and strip; prime numbers

Article

A Simple Rigorous Proof of Riemann's Hypothesis

Jau Tang

Institute of Technological Sciences, Wuhan University, Wuhan 43007, China; jautang66@gmail.com

Abstract: We present a simple rigorous proof of Riemann's hypothesis. This hypothesis has remained unsolved since Riemann's original formulation in 1859, although numerous zeros have been found along the critical line with the assistance of computer calculations. Our analytic proof is based on the analysis of the reflection symmetry between $|\Gamma(s/2))\,\zeta(s)/\pi^{s/2}|^2$ and $|\Gamma((1-s)/2))\,\zeta(1-s)/\pi^{(1-s)/2}|^2$, although the zeta and Gamma functions are asymmetric. We show their global minimum along the x-direction throughout the critical strip, their zeros, and the non-trivial zeros of the zeta function must occur at s=1/2+iy. If the zeros were not along the critical line, we show contradictions to the properties of the symmetric functional pair would arise. Thus, we prove rigorously the validity of Riemann's conjecture.

Keywords: Riemann's hypothesis; Riemann's zeta function; reflection symmetry; critical line and strip; prime numbers

---

1. Introduction

Riemann's hypothesis, first formulated in 1859 by German mathematician B. Riemann, is one of the most profound and long-standing unsolved problems in mathematics [1–6]. He postulated the non-trivial zeros of the Riemann zeta function ζ(s) must lie along the critical line in the complex plane with $s=\frac{1}{2}+iy$. This zeta function is deeply connected to the distribution of prime numbers, forming the foundation of modern analytic number theory. This hypothesis is one among the list of 23 unsolved problems presented by D. Hilbert in 1900 at the Internal Congress of Mathematicians [4,5]. Despite numerous partial results obtained by notable mathematicians, such as Hardy [5], Selberg [6], and many others, and an astronomical number of zeros computationally identified with a zero having an imaginary part as large as 8.1×10^{34} [7], the Reimann hypothesis remains unsolved [8]. Its proof would have far-reaching implications across number theory, random matrix theory, quantum chaos, and cryptography. We present in this report simple rigorous proof of Riemann's hypothesis. Our approach is based on the analysis of the reflection symmetry between $|\Gamma(s/2)\,\zeta(s)/\pi^{s/2}|^2$ and $\Gamma((1-s)/2)\,\zeta(1-s)/\pi^{(1-s)/2}|^2$ to establish the validity of Riemann's conjecture.

2. The Proof of Riemann's Hypothesis

Riemann formulated the hypothesis in the seminal paper, entitled "On the Number of Primes Less Than a Given Magnitude". In this work, he studied the properties of the Riemann zeta function, ζ(s), as [1–3]

$$\zeta(s)=\frac{1}{\Gamma(s)}\int_0^\infty dx\,\frac{x^{s-1}}{e^x-1}=\frac{1}{\Gamma(s)}\sum_{n=1}^\infty\int_0^\infty dx\,x^{s-1}e^{-nx}=\sum_{n=1}^\infty\frac{1}{n^s}\ . \quad (1A)$$

where s is defined on the complex plane.

$$\zeta(s)=\prod_{p:\ prime}^\infty\frac{1}{1-p^{-s}}, \quad (1B)$$

which interestingly relates the zeta function to a product of terms involving prime numbers. According to Riemann's hypothesis [4–6], the zeros of the zeta function occur only along the critical line with $s=\frac{1}{2}+iy$. Because it is well-known that the zeros of the Riemann zeta function would

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 February 2025 doi:10.20944/preprints202502.1458.v1

occur along the critical strip with x between 0 and 1 [5], to prove Riemann's hypothesis one only needs to analyze the location of the minimum for $|\xi(x+iy)|^2$ in the critical strip, which happens to be at the zeros if the zeta function, must lie along the critical line.

2.1. Riemann's Reflection-Symmetric Functional Pair

To prove Riemann's hypothesis, we utilize the following Riemann's functional pair [1] that possess reflection symmetry to the critical line, given as

$$A = \frac{\Gamma(s/2)\zeta(s)}{\pi^{s/2}},\ B = \frac{\Gamma((1-s)/2)\zeta(1-s)}{\pi^{(1-s)/2}}, A = B. \tag{2A}$$

$$\bar{A} = \frac{\Gamma(\bar{s}/2)\zeta(\bar{s})}{\pi^{\bar{s}/2}}, \bar{B} = \frac{\Gamma((1-\bar{s})/2)\zeta(1-\bar{s})}{\pi^{(1-\bar{s})/2}}, \quad \bar{A} = \bar{B}. \tag{2B}$$

To achieve our goals, we propose a novel approach that utilizes the reflection symmetry of $|\Gamma(s/2))\,\zeta(s)/\pi^{s/2}\,|^2 = |\Gamma((1-s)/2)\,\zeta(1-s)/\pi^{(1-s)/2}\,|^2$, involving the Riemann zeta function and the Gamma functions.

Despite the individual zeta and Gamma functions being asymmetric, the composite function pair has a reflection symmetry. We defined the following product pair $A\bar{A}$ which equals t o $B\bar{B}$, and

$$|A|^2 = \frac{\Gamma(s/2)\Gamma(\bar{s}/2)\zeta(s)\zeta(\bar{s})}{\pi^{Re(s)}} = |B|^2 = \frac{\Gamma((1-s)/2)\Gamma((1-\bar{s})/2)\zeta(1-s)\zeta(1-\bar{s})}{\pi^{1-Re(s)}} \tag{3}$$

Defining $s = x+iy, \bar{s} = x-iy$, we obtain

$$F(x,y) = |A|^2 = \frac{\Gamma\left(\frac{x+iy}{2}\right)\Gamma\left(\frac{x-iy}{2}\right)\zeta(x+iy)\zeta(x-iy)}{\pi^x} \tag{4A}$$

$$G(x,y) = |B|^2 = \frac{\Gamma\left(\frac{1-(x+iy)}{2}\right)\Gamma\left(\frac{1-(x-iy)}{2}\right)\zeta(1-(x+iy))\zeta(1-(x-iy))}{\pi^{1-x}}. \tag{4B}$$

and $F(x,y) = G(x,y) = |A|^2 = |B|^2 \geq 0$.

Because $F(x,y)$ and $G(x,y)$ are symmetric to the critical line which is parallel to the y-axis with x = ½, from Eq. (4), one can show

$$G(1-x,y) = \frac{\Gamma\left(\frac{1-(1-x+iy)}{2}\right)\Gamma\left(\frac{1-(1-x-iy)}{2}\right)\zeta(1-(1-x+iy))\zeta(1-(1-x-iy))}{\pi^x} = \frac{\Gamma\left(\frac{x-iy}{2}\right)\Gamma\left(\frac{x+iy}{2}\right)\zeta(x-iy)\zeta(x+iy)}{\pi^x} = F(x,y) \tag{5}$$

Similarly, one can show $F(1-x,y) = G(x,y)$. Therefore, one obtains

$$F(x,y) = F(1-x,y) = G(x,y) = G(1-x,y), \tag{6}$$

which exhibits the reflection symmetry of $F(x,y)$ and $G(x,y)$ to the critical line.

2.2. Proving Lemma: $\boldsymbol{F(x,y)}$'s Global Minima and Non-Trivial Zeros Must Be at s=1/2+iy

Now, we shall prove a lemma for the minima and nontrivial zeros of $F(x,y)$ must lie along the critical line with x=1/2. Owing to the symmetry of $F(x,y) = F(1-x,y)$ according to Eq. (6), their partial derivative x is anti-symmetric to the critical line, i.e.,

$$\frac{\partial F(x,y)}{\partial x} = -\frac{\partial F(1-x,y)}{\partial x}, \frac{\partial G(x,y)}{\partial x} = -\frac{\partial G(1-x,y)}{\partial x} \tag{7A}$$

Thus, their slopes, as the derivatives along the critical line, at x= ½ must vanish, i.e.,

$$\frac{\partial}{\partial x} F\left(\frac{1}{2}, y\right) = \frac{\partial}{\partial x} G\left(\frac{1}{2}, y\right) = 0. \tag{7B}$$

We shall show that the second-order partial derivatives along x are positive definite along the critical line so that the minimum points of $F(x,y)$ must occur only along the critical line so that the zeros of $F(x,y)$ can only occur along this line. Here, we shall derive the first and second-order partial derivatives of $F(x,y)$ along x. We first define

$$F(x,y) = M(x,y)N(x,y)$$
$$M(x,y) = \Gamma\left(\frac{x+iy}{2}\right)\Gamma\left(\frac{x-iy}{2}\right)/\pi^{x} > 0 \quad (8)$$
$$N(x,y) = \zeta(x+iy)\zeta(x-iy) \geq 0.$$

And the first-order partial derivatives are given by

$$\frac{\partial F(x,y)}{\partial x} = N(x,y)\frac{\partial M(x,y)}{\partial x} + M(x,y)\frac{\partial N(x,y)}{\partial x}$$
$$\frac{\partial^2 F(x,y)}{\partial x^2} = N(x,y)\frac{\partial^2 M(x,y)}{\partial x^2} + M(x,y)\frac{\partial^2 N(x,y)}{\partial x^2} + 2\frac{\partial M(x,y)}{\partial x}\frac{\partial N(x,y)}{\partial x} \quad (9)$$

Because $d\Gamma(z)/dz = \Gamma(z)\psi(z)$, where $\psi(z)$ is the digamma function [9], one has

$$\frac{\partial M}{\partial x} = \frac{M}{2}\left(\psi\left(\frac{x+iy}{2}\right) + \psi\left(\frac{x-iy}{2}\right) - 2\ln\pi\right), \quad (10A)$$

and the 2nd-order derivative as given by

$$\frac{\partial^2 M}{\partial x^2} = \frac{M}{4}\left[\left(\psi\left(\frac{x+iy}{2}\right) + \psi\left(\frac{x-iy}{2}\right) - 2\ln\pi\right)^2 + \psi'\left(\frac{x+iy}{2}\right) + \psi'\left(\frac{x-iy}{2}\right)\right]. \quad (10B)$$

Because the tri-gamma function $\psi'(z)$ is positive in the critical strip [9], one concludes $\partial^2 M/\partial x^2 > 0$. Now, let us show the 2nd-order derivative of $N(x,y)$ is positive definite. One has

$$N(x,y) = \zeta(x+iy)\zeta(x-iy) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} e^{-x(\ln n + \ln m)) - iy(\ln n - \ln m)}$$
$$\frac{\partial^2}{\partial \quad x^2}N(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\ln n + \ln m)^2 e^{-x(\ln n + \ln m)) - iy(\ln n - \ln m)}. \quad (11A)$$

Along the critical line, one has

$$\frac{\partial^2}{\partial x^2}N(1/2\,,y) = \sum_{n=1}^{\infty}\frac{4(\ln n)^2}{n} + \sum_{n>m=1}^{\infty}\frac{2(\ln(nm))^2}{\sqrt{nm}}\cos\big(y(\ln(n/m))\big).$$

Because the second damped oscillatory off-diagonal term cannot exceed its magnitude, which is smaller than the first diagonal term, therefore, $\partial^2 N(1/2\,,y)/\partial x^2$is positive definite. According to Eq. (8) $M(x,y) = |\Gamma(x+iy)/2|^2 > 0$, $N(x,y) = |\zeta(x+iy)|^2 \geq 0$, and $\partial F(1/2,y)/\partial x = 0$ from Eq. (7B), the minimum of $F(x,y) = M(x,y)N(x,y)$ must occur along the critical line, which happens to be where the zeros are located, so the minima and zeros of $N(1/2,y) = \zeta(1/2+iy))\zeta(1/2-iy)$ and $\zeta(1/2+iy)$ can occur at x=1/2, which also implies

$$\frac{\partial N(1/2,y)}{\partial x} = 0$$
$$\frac{\partial^2 F(1/2,y)}{\partial x} = N(1/2,y\frac{\partial^2 M(1/2,y)}{\partial x^2} + M(1/2,y\frac{\partial^2 N(1/2,y)}{\partial x^2} > 0. \quad (12)$$

2.3. Proving x=1/2 as the Global Minima of $\boldsymbol{F(x,y)}$ *and Zeros of Riemann's Zeta Function Across the Entire Critical Strip*

From the above analysis, we have shown that the zeros and the local minima of $F(x,y)$ are located at x=1/2. Here, we shall further prove that the minim isn't just a local minimum along the critical line but a global minimum within the entire critical strip. According to Eq. (11A), one has

$$\frac{\partial^2}{\partial x^2}N(x,y) = \sum_{n=1}^{\infty} 4(\ln n)^2 e^{-2x\ln n}$$
$$+ \sum_{n>m} 2(\ln n + \ln m)^2 e^{-x(\ln n + \ln m)}\cos\big(y(\ln n - \ln m)\big). \quad (13)$$

The first diagonal term is strictly positive definite in the critical strip. The second oscillatory term has an exponential damping factor $e^{-x(\ln n + \ln m)}$ with an overall magnitude smaller than the first diagonal term. Therefore, the sum of both diagonal and off-diagonal terms cannot be negative. Since we have both $\partial^2 N(x,y)/\partial x^2 > 0$ and $\partial^2 M(x,y)/\partial x^2 > 0$, thus $F(x,y)$ is a convex function throughout the critical strip and its minima can lie only along the critical line. The proof of the above lemma leads naturally to Riemann's hypothesis, i.e., the zeros of the Riemann zeta function must lie along the critical line with x=1/2.

We have shown above that the minima and the zeros of $F(x,y)$, as well as the nontrivial zeros of the zeta function, must lie along the critical line. If the minima or zero do not lie along the critical line, we can show in the following that contradiction to the reflection symmetry shall arise. If one assumes $F(x_0,y) = M(x_0,y)N(x_0,y) = 0$ at $x_0 \neq 1/2$, because $M(x,y)$ is positive in the critical strip, one must have $N(x_0,y) = 0$ and $F(x_0,y)$ must be at a minimum with a vanishing slope $\partial F(x_0,y)/\partial x$. This cannot be true because of the reflection symmetry, one must have $F(1-x_0,y) = 0$, $N(1-x_0,y) = 0$, and $\partial F(1-x_0,y)/\partial x = 0$ as well, i.e.,

$$\frac{\partial F(x_0,y)}{\partial x} = \frac{\partial M(x_0,y)}{\partial x} N(x_0,y) + \frac{\partial N(x_0,y)}{\partial x} M(x_0,y) = 0$$
$$\frac{\partial F(1-x_0,y)}{\partial x} = \frac{\partial M(1-x_0,y)}{\partial x} N(1-x_0,y) + \frac{\partial N(1-x_0,y)}{\partial x} M(1-x_0,y) = 0. \quad (14A)$$

However, because $M(x_0,y)$ is positive and its derivative is non-zero in the critical strip, for the above equality to hold, one must have

$$\frac{\partial F(x_0,y)}{\partial x} = \frac{\partial N(x_0,y)}{\partial x} M(x_0,y) = 0$$
$$\frac{\partial M(1-x_0,y)}{\partial x} = \frac{\partial N(1-x_0,y)}{\partial x} M(1-x_0,y) = 0. \quad \text{s} \quad (14B)$$

Because $M(x_0,y)$ and $M(1-x_0,y)$ are non-zero, one must have

$$\frac{\partial N(x_0,y)}{\partial x} = \frac{\partial N(1-x_0,y)}{\partial x} = 0. \quad (14C)$$

However, the above criteria with the vanishing slope of $N(x_0,y)$ and $N(1-x_0,y)$ at $x_0 \neq 1/2$ is contrary to the fact that both derivatives are asymmetric, and $\partial N(x_0,y)/\partial x$ should differ from $\partial N(1-x_0,y)/\partial x$ unless $x_0 = 1/2$. This leads to contradiction to the assumption of at $x_0 \neq 1/2$. Consequently, we conclude $F(x_0,y)$ and $\xi(x_0+iy)$ cannot be zero unless $x_0 = 1/2$.

3. Conclusions

In conclusion, we utilize Riemann's reflection-symmetric functional pair $|\Gamma(s/2))\zeta(s)/\pi^{s/2}|^2$ and $|\Gamma((1-s)/2))\zeta(1-s)/\pi^{(1-s)/2}|^2$, although the individual Zeta and Gamma functions are asymmetric. We first analyze their symmetric properties to prove the lemma for the global minimum of $F(x,y)$ along the x-direction occurs at x=1/2. We then show the zeros, minima, and non-trivial zeros of $\zeta(s)$ must lie along the critical line and nowhere else throughout the entire critical strip. We further show if the zeros and the minima of $F(x,y)$ and the non-trivial zeros of $\zeta(x+iy)$ were not along the critical line, then self-contradiction to the symmetric properties of the functional pair would arise. Therefore, we have rigorously proven the validity of Riemann's hypothesis [10].

Funding: The author is a retired professor. He received no external funding.

References

1. Riemann zeta function - Wikipedia.
2. Titchmarsh, E. C., The Theory of the Riemann Zeta-Function (Oxford University Press, **1986**).
3. Ivic, A., The Riemann Zeta-Function: Theory and Applications (Dover Publications, **2003**).
4. Hilbert, D., Mathematical problems =. Bulletins of the American Mathematical Society, **1902**, 8(10), 437-479.
5. Hardy, G. H., and Littlewood, J. E. The zeros of Riemann's zeta function on the critical line, **1921**, Math. Zeitschrift.
6. Selberg, A., Collected Papers. Volumes 1 & II (Springer-Verlag, **1989)**.

 doi:10.20944/preprints202502.1458.v1

7. 7. Bober, J. W., and Hiary, G. A. New Computations of the Riemann Zeta Function on the Critical Line. Experimental Mathematics, **2016**, 27(2), 125-137.
8. Sabbagh, K., Riemann Zeta: The Greatest Unsolved Problem in Mathematics (Farrar, Straus and Giroux, **2004**).
9. NIST Digital Library of Mathematical Functions (DLMF), Sec. 5.4. https://dlmf.nist.gov.
10. Tang, J., A rigorous proof of Riemann's hypothesis, Zenodo Preprint (**2025)**, DOI: 10.5281/zenodo.14868031.