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[Dumitru Adam](#) *

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Article

On the Method for Proving the RH Using the Alcantara-Bode Equivalence

Dumitru Adam

Independent Researcher; dumitru_adam@yahoo.ca

Abstract

This is the final version of the preprint series dedicated to provide a robust functional-numerical analysis method for investigation of the injectivity of linear bounded operators on separable Hilbert spaces. The aim has been to solve the Alcantara-Bode equivalent formulation of the Riemann Hypothesis. This equivalent states that RH holds if and only if the null space of a specific integral operator on $L^2(0,1)$ related to the Riemann Zeta function, contains only the null element or equivalently, the operator is injective. Or, such problems could be solved outside the area of the number theory. Our solution, a functional-numerical method is built on a generic result introduced (Theorem 1): a linear bounded operator on a separable Hilbert space strict positive definite on a dense set is injective. We updated and extended the results from [1] by separating the analysis of the operator restrictions from their operator approximations on finite-dimension subspaces. With the numerical analysis criteria we reduced the injectivity problem to the existence of a strict positive bound modulo the square of the mesh of the positivity parameters defined as the diagonal entries in the sparse matrices associated to the integral operator. Applying this method we were able to prove the Alcantara-Bode equivalent having as effect the proof of RH that it holds.

Keywords: integral operators; approximation methods; riemann hypothesis

JEL Classification: 31A10; 45P05; 47G10; 65R99; 11M26

1. Introduction

The first result obtained (Theorem 1) shows that a linear bounded operator T strictly positive on a dense set S in a separable Hilbert space H , is injective. Equivalently, its null space does not contain non null elements: $N_T = \{0\}$. In order to use it, we built a framework like in the multilevel (multigrid) techniques in which the problem of injectivity is moved on finite dimension subspaces allowing us to define numerical methods for dealing with.

The positivity of a linear bounded operator T on S , $\langle Tv, v \rangle > 0 \ \forall v \in S$ not null, ensures that the null space of T contains from S only the element 0, i.e., $N_T \cap S = \{0\}$. Thus, a zero of T could be only in the difference set $E := H \setminus S$ considering that a linear combination of $u \in S$ and $v \in E$ not null, is inside the difference set. Let observe that (T^*T) is non negative on the entire space making the method useful for any linear bounded operator provided that the operator is strict positive on the finite dimension subspaces of the family whose union is dense.

The idea is to consider the dense set in H be the union of a family F of finite dimension including subspaces $S_n, S_n \subset S_{n+1}, n \geq 1$, once the positivity on each subspace from the family F will induce the positivity on the dense set. For obtaining the necessary criteria for injectivity, we will exploit the relationship between the orthogonal projections of the eligible elements; i.e., elements outside the dense set onto the family subspaces and the positivity parameters of the operator or its operator approximations on these subspaces.

Now, a linear bounded operator T positive on a finite dimension subspace is in fact strictly positive on it, i.e., there exists $\alpha_n(T) > 0$ such that $\langle Tv, v \rangle \geq \alpha_n(T) \|v\|^2 \ \forall v \in S_n$. Suppose T be

positive in each subspace $S_n \in F$. If there exists $\alpha > 0$ such that $\alpha_n \geq \alpha$ for any $n \geq 1$ then T is strict positive on the dense set S and, by Theorem 1 introduced below, $N_T = \{0\}$. In this case is no need for further investigations.

If the sequence of the positivity parameters of T is not bounded, $\alpha_n(T) \xrightarrow{n \rightarrow \infty} 0$, we consider two directions to continue:

- involving the adjoint operator restrictions on the subspaces of the family, improving in the new context with Lemma 2 below the criteria introduced in [1] or,
- considering a sequence of positive operator approximations on subspaces.

An inferior bound of the positivity parameters of operator approximations, ensures strict positivity of the operator on the dense set. Lemma 1 addresses this case. Both cases are analysed in the next paragraph.

The third paragraph is dedicated to analyse the dense set most appropriate for obtaining operator or operator approximations having sparse matrix representations on the finite dimension subspaces whose union is dense in $L^2(0,1)$.

In the last paragraph we showed that on these subspaces the operator used by Alcantara-Bode in [2] has the sequence of positivity parameters bounded inferior and so, verifying the criteria introduced in the third paragraph. For minimising the volume of computations we choose in $L^2(0,1)$ the finite dimension subspaces spanned by indicator interval functions of the domain partitions, indicators having on each subspace disjoint support. And so, obtaining sparse matrix representations for the integral operator. The theory behind the numerical method follows.

2. Two Theorems on Injectivity

Let H be a separable Hilbert space and denote with $\mathcal{L}(H)$ the class of the linear bounded operators on H . If $T \in \mathcal{L}(H)$ is positive on a dense set $S \subset H$, i.e., $\langle Tv, v \rangle > 0 \forall v \in S$ not null, then T has no zeros in the dense set. Otherwise, if there exists $w \in S$ such that $Tw = 0$ then $\langle Tw, w \rangle = 0$ contradicts its positivity.

Follows: its 'eligible' zeros are all in the difference set $E := H \setminus S$, i.e., $N_T \subset E$. In our analysis we will take in consideration only the collection of eligible zeros that are on the unit sphere, without restricting the generality once for an element that is not null $w \in H$, both w and $w/\|w\|$ are or are not together in N_T .

Theorem 1. *If $T \in \mathcal{L}(H)$ is strictly positive on a dense set of a separable Hilbert space then T is injective, equivalently $N_T = \{0\}$.*

Proof.

The set $S \subset H$ is dense if its closure coincides with H . Then, if $w \in E := H \setminus S$, for every $\varepsilon > 0$ there exists $u_{\varepsilon,w} \in S$ such that $\|w - u_{\varepsilon,w}\| < \varepsilon$. Now, the (1) results as follows. If $\|w\| \geq \|u_{\varepsilon,w}\|$:

$$0 \leq \|w\| - \|u_{\varepsilon,w}\| = \|w - u_{\varepsilon,w} + u_{\varepsilon,w}\| - \|u_{\varepsilon,w}\| \leq \|w - u_{\varepsilon,w}\| + \|u_{\varepsilon,w}\| - \|u_{\varepsilon,w}\| < \varepsilon.$$

If $\|u_{\varepsilon,w}\| \geq \|w\|$ instead, then:

$$0 \leq \|u_{\varepsilon,w}\| - \|w\| = \|u_{\varepsilon,w} - w + w\| - \|w\| \leq \|u_{\varepsilon,w} - w\| < \varepsilon.$$

Therefore, given $w \in E$, for every $\varepsilon > 0$ there exists $u_{\varepsilon,w} \in S$ such that

$$|\|w\| - \|u_{\varepsilon,w}\|| < \varepsilon \quad (1)$$

Let w be an eligible element from the unit sphere, $\|w\| = 1$ and take $\varepsilon_n = 1/n$.

Then there exists at least one element $u_{\varepsilon_n,w} \in S$ such that $\|u_{\varepsilon_n,w} - w\| < \varepsilon_n$ holds. From (1), $|1 - \|u_{\varepsilon_n,w}\|| < 1/n$ showing that, for any choices of a sequence approximating w , $u_{\varepsilon_n,w} \in S, n \geq 1$, it verifies $\|u_{\varepsilon_n,w}\| \rightarrow 1$.

If $T \in \mathcal{L}(H)$ is strict positive on S , then there exists $\alpha > 0$ such that $\forall u \in S, \langle Tu, u \rangle \geq \alpha \|u\|^2$. Suppose that there exists $w \in E \cap N_T, \|w\| = 1$ and consider a sequence of approximations of w ,

$u_{\varepsilon_n, w} \in S, n \geq 1$ that, as we showed, has its normed sequence converging in norm to 1. From the positivity of T on dense set S , follows:

$$\alpha \|u_{\varepsilon_n, w}\|^2 \leq \langle Tu_{\varepsilon_n, w}, u_{\varepsilon_n, w} \rangle = \langle T(u_{\varepsilon_n, w} - w), u_{\varepsilon_n, w} \rangle < \varepsilon_n \|T\| \|u_{\varepsilon_n, w}\| \quad (2)$$

With $c = \|T\|/\alpha$, we obtain $\|u_{\varepsilon_n, w}\| \leq c/\varepsilon_n$. Then, $\|u_{\varepsilon_n, w}\| \rightarrow 0$ with $n \rightarrow \infty$, contradicting its convergence $\|u_{\varepsilon_n, w}\| \rightarrow 1$ with $n \rightarrow \infty$.

This occurs for any choice of the sequence of approximations of w , verifying

$$\|w - u_{\varepsilon_n, w}\| < \varepsilon_n, n \geq 1, \text{ when } Tw = 0.$$

Thus $w \notin N_T$, valid for any $w \in E, \|w\| = 1$, proving the theorem because no zeros of T there are in S either. $\square$

Suppose that the dense set S is the result of an union of finite dimension subspaces of a family F : $S = \bigcup_{n \geq 1} S_n, \bar{S} = H$. It is not mandatory but will ease our proofs considering that the subspaces are including: $S_n \subset S_{n+1}, n \geq 1$.

Observation 1. Let $\beta_n(u) := \|u - u_n\|$ be the normed residuum of element $u \in E$ after its orthogonal projection onto S_n . Then, $\beta_n(u) \rightarrow 0$ with $n \rightarrow \infty$.

Proof.

Given $\epsilon > 0$, from the density of the set S in H there exists $u_\epsilon \in S$ verifying $\|u - u_\epsilon\| < \epsilon$, as per the observations made in the proof of the Theorem 1. Let S_{n_ϵ} be the coarsest subspace, i.e., with the smallest dimension, from the family of subspaces containing u_ϵ . Because the best approximation of u in S_{n_ϵ} is its orthogonal projection, we obtain

$\beta_{n_\epsilon}(u) := \|u - P_{n_\epsilon}u\| \leq \|u - u_\epsilon\| < \epsilon$, valid for every $\epsilon > 0$, proving our assertion. Rewriting this, $\beta_n(u) := \|u - P_nu\| = \|(I - P_n)u\| \leq \|I - P_n\| \|u\| \rightarrow 0$ for $n \rightarrow \infty$ for any $u \in H$ with P_n the orthogonal projection onto S_n . $\square$

Theorem 2. Suppose that $T \in \mathcal{L}(H)$ has a sequence of operator approximations on the dense set $S, \{S_n, n \geq 1\}$, having the following properties:

- i) $\epsilon_n := \|T - T_n\| \rightarrow 0$ with $n \rightarrow \infty$,
- ii) $\langle T_nv, v \rangle \geq \alpha_n \|v\|^2, \forall v \in S_n, S_n \in F$.

If T is positive on S and there exists $\alpha > 0$ such that

- iii) $\alpha_n \geq \alpha > 0, n \geq 1$

then $N_T = \{0\}$.

Proof.

Being positive on S , the operator does not have zeros in the dense set.

For $u \in E := H \setminus S, \|u\| = 1$ denoting the not null orthogonal projection over S_n by $u_n := P_nu, n \geq n_0 := n_0(u)$, then on any subspace $S_n, 1 = \|u\|^2 = \|u_n\|^2 + \beta_n^2(u)$. If there exists $u \in N_T \cap E, \|u\| = 1$, for it denoting $\beta_n := \beta_n(u)$ we have from ii):

$$0 < \alpha_n \|u_n\|^2 \leq \langle T_nu_n, u_n \rangle \leq \|T_nu_n\| \|u_n\|.$$

Estimating $\|T_nu_n\|$,

$$\begin{aligned} \|T_nu_n\| &= \|T_nu_n - Tu_n + Tu_n - Tu\| \leq (\|T - T_n\| \|u_n\| + \|T\| \|u - u_n\|) \\ &= (\epsilon_n \|u_n\| + \|T\| \beta_n), \end{aligned}$$

we observe that $\|T_nu_n\| \rightarrow 0$ because $\|u_n\| \rightarrow 1, \epsilon_n \rightarrow 0$ (from i)) and, $\beta_n \rightarrow 0$ (from Observation 1).

Now, from iii)

$$\alpha \|u_n\|^2 \leq (\epsilon_n + \|T\| \beta_n / \|u_n\|) \|u_n\|^2.$$

From Observation 1 we have $\beta_n / \|u_n\| = \beta_n / \sqrt{1 - \beta_n^2} \rightarrow 0$. So,

$$\alpha \leq (\epsilon_n + \|T\| \beta_n / \|u_n\|) \rightarrow 0.$$

The inequality is violated from a range $n_1 \geq n_0$, involving $u \notin N_T$, valid for any supposed zero of T in E . Because T has no zeros in the dense set, $N_T = \{0\}$. $\square$

Let $T := T_\rho$ be a Hilbert-Schmidt integral operator. A technique for obtaining approximations for $T := T_\rho$ to verify i) was used in [5], [6]. When $T_n, n \geq 1, T_n := P_n^r(T)$ are approximations of T_ρ on the subspaces of family F obtained through a class of finite rank operators - that are orthogonal projection integral operators $\{P_n^r, n \geq 1\}$, then from $\|I - P_n^r\| \rightarrow 0$, we obtain the property i). In the next paragraph we show that $\{P_n^r, n \geq 1\}$ is a collection of finite rank projection operators on a family of finite dimension subspaces (see [5]) whose union is dense in $H := L^2(0, 1)$. Moreover, if the operator approximations $\{T_n, n \geq 1\}$ verifies ii), we can show that the operator T is strictly positive on the dense set S provided that their positivity parameters sequence is bounded.

Lemma 1. (Criteria for operator approximations). *If the finite rank approximations of a positive Hilbert-Schmidt integral operator T_ρ verify the conditions ii) and iii) from Theorem 2, then T_ρ is strictly positive on the dense set.*

Proof.

From the convergence to zero of the sequence $\epsilon_n, n \geq 1$ there exists ϵ_0 a parameter ϵ_0 such that $\epsilon_0 := \max_n \{\epsilon_n; \epsilon_n < \alpha\}$, corresponding to a subspace $S_{n_0}, n_0 < \infty$. This parameter is independent of any $v \in S$ and, because of the inclusion property, for any $n < n_0$ we have $S_n \subset S_{n_0}$. We could consider S_{n_0} to be S_1 discarding a finite number of subspaces or, we could consider v to be inside of S_{n_0} . Then:

$$\alpha_n \geq \alpha > \epsilon_0 \geq \epsilon_n \text{ for } n \geq 1, \text{ resulting } (\alpha_n - \epsilon_n) > (\alpha - \epsilon_0) > 0 \forall n \geq 1.$$

For an arbitrary $v \in S$ there exists a coarser subspace (i.e., with a smaller dimension) $S_n, n \geq n_1 := n_1(v)$, for which $v \in S_n$. For it, with $T := T_\rho$ we have:

$$\langle Tv, v \rangle = \langle T_n v, v \rangle - \langle (T_n - T)v, v \rangle > 0. \text{ Since } T_n \text{ is positive on } S_n,$$

$$\langle Tv, v \rangle \geq \alpha_n \|v\|^2 - \langle (T_n - T)v, v \rangle.$$

Because T and T_n are positive on S_n , the inner product in the right side of the inequality is real valued and, $|\langle (T_n - T)v, v \rangle| \leq \epsilon_n \|v\|^2$.

So, if $\langle (T_n - T)v, v \rangle \geq 0$, then $\langle (T_n - T)v, v \rangle \leq \epsilon_n \|v\|^2$. From $\epsilon_n < \alpha_n$, follows:

$$\langle Tv, v \rangle \geq (\alpha_n - \epsilon_n) \|v\|^2 \geq (\alpha - \epsilon_0) \|v\|^2.$$

Now, if $\langle (T_n - T)v, v \rangle < 0$, then $\langle Tv, v \rangle \geq \alpha_n \|v\|^2 \geq \alpha \|v\|^2 \geq (\alpha - \epsilon_0) \|v\|^2$.

Thus, taking $\alpha(T) = (\alpha - \epsilon_0)$, for any $v \in S$ we obtain

$$\langle Tv, v \rangle \geq \alpha(T) \|v\|^2, \text{ i.e., } T_\rho \text{ is strict positive on the dense set } S. \quad \square$$

Corollary. *If $Q \in \mathcal{L}(H)$ is an Hermitian Hilbert-Schmidt operator verifying on a dense set S the properties ii) and iii) from Theorem 2, then Q is injective.*

Proof.

Being Hermitian, the operator verifies $\langle Qv, v \rangle \geq 0$, for every $v \in H$. Being Hilbert-Schmidt it could be approximated on a dense family of finite dimension subspaces, its sequence of operator approximations verifying i). Then,

$\langle Qv, v \rangle = \langle Q_n v, v \rangle - \langle (Q_n - Q)v, v \rangle \geq 0$ for any $v \in S$. Following the steps from the proof of Lemma 1 we obtain that:

$\langle Qv, v \rangle \geq (\alpha - \epsilon_0) \|v\|^2$ meaning that Q is strictly positive on the dense set. Thus, due to Theorem 1/Lemma 1, we obtain $N_Q = \{0\}$. $\square$

Now, reformulating the injectivity criteria introduced in [1], we have the following lemma, useful when a sequence of operator approximations could not be obtained.

Lemma 2. (Criteria for operator restrictions.) *Let $T \in \mathcal{L}(H)$ positive on the subspaces $S_n, n \geq 1$ whose union S is a dense set S , verifying: $\langle Tv, v \rangle \geq \alpha_n \|v\|^2$ for every $v \in S_n$, where $\alpha_n \rightarrow 0$ with $n \rightarrow \infty$. Consider now the parameters:*

$$\mu_n := \alpha_n(T) / \omega_n \text{ where } \omega_n \text{ verifies } \|T^* v\| \leq \omega_n \|v\|, \forall v \in S_n, n \geq 1.$$

If $C > 0$ exists such that $\mu_n \geq C$ for every $n \geq 1$, then $N_T = \{0\}$.

Proof.

Suppose that there exists $u \in (H \setminus S) \cap N_T$, $\|u\| = 1$ and let u_n its orthogonal projection on S_n , $n \geq 1$. Then, denoting with $\beta_n := \beta_n(u) = \|u - u_n\|$, we obtain from the (strict) positivity of T on each of the subspaces S_n , $n \geq 1$ (as in (2)),

$$\alpha_n(T)\|u_n\|^2 \leq \langle Tu_n, u_n \rangle = \langle T(u_n - u), u_n \rangle = \langle (u_n - u), T^*u_n \rangle \leq \beta_n \omega_n \|u_n\|$$

Rewriting,

$C \leq \mu_n \leq \beta_n / \sqrt{1 - \beta_n^2} \rightarrow 0$ that is a contradiction from a range $n_0(u)$. Thus, $u \notin N_T \forall u \in H \setminus S$. Follows: $N_T = \{0\}$. $\square$

3. Approximations on Subspaces

Let $H := L^2(0, 1)$. The semi-open intervals of equal lengths $h = 2^{-m}$, $m \in \mathbb{N}$, $nh = 1$, $\Delta_{h,k} = ((k-1)/2^m, k/2^m]$, $k = 1, n-1$ together with the open $\Delta_{h,n}$ define for $m \geq 1$ a partition of $(0, 1)$, $k=1, n$, $n = 2^m$, $nh = 1$. Consider the interval indicator functions that have as support these intervals ($k=1, n$), $nh=1$:

$$I_{h,k}(t) = 1 \text{ for } t \in \Delta_{h,k} \text{ and } 0 \text{ otherwise} \quad (3)$$

The family F of finite dimensional subspaces $\{S_h, nh = 1, n \geq 2\}$ that are the linear spans of interval indicator functions of the h -partitions defined by (3) with disjoint supports, $S_h = \text{span}\{I_{h,k}; k = 1, n, nh = 1\}$, built on a multi-level structure, are including $S_h \subset S_{h/2}$ by halving the mesh h . In fact, this property is obtained from (3) observing that any $I_{h,i} \in S_h$, $i = 1, n, nh = 1$ can be rewritten as

$$I_{h,i} = I_{h/2,2i-1} + I_{h/2,2i} \in S_{h/2}.$$

Observation 2: Any pair from $\{I_{h,k}, k = 1, n, nh = 1\}$ has disjoint supports. Moreover,

- for any $f \in L^2(0, 1)$, $I_{h,k}(t)f = 0$ and $fI_{h,k}(t) = 0$ if $t \notin \Delta_{h,k}$, $k=1, n$;
- for any $\rho \in L^2(0, 1)^2$, $I_{h,j}(y)\rho = 0$ if $y \notin \Delta_{h,j}$ and, $\rho I_{h,k}(x) = 0$ if $x \notin \Delta_{h,k}$;
- supposing that $\rho \in L^2(0, 1)^2$ verifies the requests to apply the Fubini theorem, then:
 $\langle T_\rho I_{h,k}, I_{h,j} \rangle = \int_0^1 [\int_0^1 \rho(y, x) I_{h,k}(x) dx] I_{h,j}(y) dy \stackrel{\text{Fubini}}{=} \int_{(0,1)^2} \rho(y, x) \delta_{k,j}(y, x) dx dy = 0$ because $\delta_{k,j}(y, x) := I_{h,k}(x) I_{h,j}(y) = 0$. due to the disjoint supports for $k \neq j$ of the indicators. The property attracts the 1-diagonal sparsity of matrix representations for operator approximations on finite dimension subspaces.

Citing [5], (pg 986), integral operator P_h^r , $n \geq 1$ with the kernel function:

$$r_h(y, x) = h^{-1} \sum_{k=1, n} I_{h,k}(y) I_{h,k}(x) \quad (4)$$

is a finite rank integral operator orthogonal projection having the spectrum $\{0, 1\}$ with eigenvalue 1 of multiplicity n ($nh=1$) corresponding to the orthogonal eigenfunctions $I_{h,k}$, $k = 1, n$. We will show it, by proving that $\forall u \in H$, $P_h^r u \in S_h$ and, as a consequence, obviously $(P_h^r)^2 = P_h^r$ for $n \geq 2$, $nh = 1$. For any $u \in H$,

$$\begin{aligned} (P_h^r u)(y) &= \int_{x \in (0,1)} (h^{-1} \sum_{k=1, n} I_{h,k}(y) I_{h,k}(x)) u(x) dx \\ &= h^{-1} \sum_{k=1, n} c_k I_{h,k}(y), \quad \text{where } c_k := \int_{\Delta_{h,k}} u(x) dx, \end{aligned}$$

that has the form of the standard orthogonal projection from $L^2(0, 1)$ onto S_h , i.e., for any $f \in L^2(0, 1)$, the linear operator P_h^r defined as

$$P_h^r f = \sum_{k=1, n} I_{h,k} \frac{1}{|\Delta_{h,k}|} \int_{\Delta_{h,k}} f dx = h^{-1} \sum_{k=1, n} c_k I_{h,k}, \text{ where } c_k = \int_{\Delta_{h,k}} f dx.$$

We will note the difference between the projection of an element $w \in L^2(0, 1)$ on a subspace, denoting it with: $P_h w = \sum_{j=1, n} \langle w, I_{h,j} \rangle I_{h,j}$ that we will use it in connection with operator restrictions to S_h .

Now, for $I_{h,j} \in S_h$,

$$P_h^r(I_{h,j}) = h^{-1} \sum_{k=1, n} c_k I_{h,k}, \text{ with } c_k = \int_{\Delta_{h,k}} I_{h,j}(x) dx = 0 \text{ for } k \neq j, \text{ and valued to } c_k = h \text{ for } k=j.$$

Follows:

$$P_h^r(I_{h,j}) = I_{h,j} \text{ and therefore, } P_h^r(v_h) = v_h \text{ for every } v_h \in S_h \text{ involving } (P_h^r)^2 u = P_h^r u \text{ for any } u \in H.$$

Now, due to the including properties of the finite dimension subspaces whose union is dense,

$$\|I - P_h^r\| \rightarrow 0 \text{ for } n \rightarrow \infty, nh = 1.$$

Then, from $(T_\rho - P_h^r(T_\rho)) = (I - P_h^r)T_\rho$ the property i) in Theorem 2 holds for any integral operator $T_\rho \in \mathcal{L}(H)$ on the family of finite dimension subspaces spanned by indicator interval functions associated with partitions defined by (3).

In fact formally, with the notations $T = T_\rho$ and $T_n = P_h^r(T_\rho)$,

$$\begin{aligned} \epsilon_n &:= \|T - T_n\| = \sup_{u \in H, \|u\|=1} \|(T - T_n)(u)\| = \sup_{u \in H, \|u\|=1} \|(I - P_h^r)(Tu)\| \\ &\leq \|(I - P_h^r)\| \|T\| \rightarrow 0, \quad \text{proving that } T_n, n \geq 1 \text{ is a collection of operator approximations verifying the property i) in Theorem 2/Lemma 1.} \end{aligned}$$

Suppose that T_ρ is positive on the dense set. Otherwise, we should replace it with its associate Hermitian and, use the Corollary of the Lemma 1.

The integral operator approximation of T_ρ on S_h denoting it by T_{ρ_h} , is a finite rank operator approximation, with a kernel function ([5]):

$$\rho_h(y, x) = h^{-1} \sum_{k=1, n} I_{h,k}(y) \rho(y, x) I_{h,k}(x) := h^{-1} \sum_{k=1, n} \rho_h^k(y, x) \quad (5)$$

where the pieces $\rho_h^k, k = 1, n$ of the kernel function ρ_h in the sum have disjoint supports in $L^2(0, 1)^2$, namely $\Delta_{h,k} \times \Delta_{h,k}, k = 1, n, nh = 1$. Thus, follows:

Remark 2. The matrix representation of T_{ρ_h} is a sparse 1-diagonal matrix.

Evaluating $T_{\rho_h} v$ for $v = I_{h,i}$, we obtain

$$\begin{aligned} (T_{\rho_h} I_{h,i})(y) &= h^{-1} \left(\sum_{k=1, n} \int_0^1 \rho_h^k(y, x) I_{h,i}(x) dx \right) (y) \\ &= h^{-1} \omega_{h,i}(y) I_{h,i}(y) \text{ with } \omega_{h,i}(y) = \left(\int_{x \in \Delta_{h,i}} \rho(y, x) I_{h,i}(x) dx \right). \end{aligned}$$

$\langle T_{\rho_h} I_{h,i}, I_{h,j} \rangle = 0$ for $i \neq j$ (see also Observation 2) and the matrix representation of the finite rank operator $P_h^r(T_\rho) := T_{\rho_h}$, is: $M_h^r(T_\rho) = h^{-1} \text{diag}[d_{kk}^h]_{k=1, n}$. It is a sparse diagonal matrix because $d_{ij}^h = 0$ for $i \neq j$ and, having the diagonal entries

$$d_{kk}^h = \int_{\Delta_{h,k}} \int_{\Delta_{h,k}} I_{h,k}(y) \rho(y, x) I_{h,k}(x) dx dy := \int_{\Delta_{h,k}} \int_{\Delta_{h,k}} \rho(y, x) dx dy, \quad k = 1, n. \quad (6)$$

Now, if $d_{kk}^h > 0, k = 1, n$,

$\langle T_{\rho_h} v_h, v_h \rangle \geq \alpha_h(T_{\rho_h}) \|v_h\|^2$ where $\alpha_h(T_{\rho_h})$ is the positivity parameter of the finite rank operator approximation T_{ρ_h} given by

$$\alpha_h(T_{\rho_h}) = h^{-2} \min_{(k=1, n)} d_{kk}^h \quad (7)$$

In order to apply Lemma 1 or Lemma 2, we need the positivity of T_ρ on S . It is obtained from the positivity of its restrictions on the subspaces of the family, as follows. Otherwise, we have to apply the Corollary of Lemma 1.

Observation 3. The restrictions of the integral operator T_ρ to the subspaces of F , where its kernel is positive valued on $(0, 1)^2$ excepting a set of measure Lebesgue zero, are positive definite.

Proof.

We will use Observation 2 applying Fubini theorem in order to obtain matrix representations sparse 1-diagonal. Like in the cases of operator approximations based on the domain partitions, we expect to obtain sparse matrix representations. The number of the diagonal non empty is given by the number of the function elements sharing their support, in our case any pair of indicator interval functions has disjoint supports so, the matrix representations are 1-diagonal:

$$\langle T_\rho I_{h,k}(x), I_{h,j}(y) \rangle = 0 \text{ for } k \neq j. \text{ If the kernel } \rho \text{ is positive valued, then:}$$

$$\langle T_\rho v_h, v_h \rangle = \sum_k c_k \bar{c}_k \int_{\Delta_{h,k}} \int_{\Delta_{h,k}} \rho(y, x) dx dy > 0 \text{ for any } v_h = \sum_j c_j I_{h,j} \in S_h, \text{ i.e., } T_\rho|_{S_h} \text{ is positive on } S_h, nh = 1 \text{ for any } S_h \in F \text{ involving the positivity of the integral operator on } S. \quad \square$$

Moreover, $\langle T_\rho v_h, v_h \rangle = \sum_{k=1, n} c_k \bar{c}_k d_{kk}^h = h \langle T_{\rho_h} v_h, v_h \rangle$ involving

$$\alpha_h(T_\rho) = h\alpha_h(T_{\rho_h}) \xrightarrow{n \rightarrow \infty} 0, nh = 1$$

4. Proof of the Alcantara-Bode Equivalent

Alcantara-Bode ([2], pg. 151) in his theorem of the equivalent formulation obtained from the Beurling equivalent formulation ([4]) of RH, states:

$$\text{The Riemann Hypothesis holds if and only if } N_{T_\rho} = \{0\}$$

where T_ρ is a Hilbert-Schmidt integral operator ([2]) whose kernel $\rho(y, x) = \{y/x\}$ is the fractional part function of the ratio (y/x) . The kernel function $\rho \in L^2(0, 1)^2$ is continue almost everywhere and its discontinuities in $(0, 1)^2$ consist in a set of numerable one dimensional lines of the form $y = kx, k \in \mathbb{N}$, so with Lebesgue measure zero. The entries in the diagonal matrix representation $M_h(T_{\rho_h})$ of the finite rank integral operator T_{ρ_h} are given by: $d_{kk}^h = \int_{\Delta_{h,k}} \int_{\Delta_{h,k}} \{y/x\} dx dy$, as valued in [1]:

$$d_{11}^h = h^2(3 - 2\gamma)/4; \quad d_{kk}^h = \frac{h^2}{2}(-1 + \frac{2k-1}{k-1} \ln(\frac{k}{k-1})^{k-1}), \text{ for } k \geq 2 \quad (8)$$

where γ is the Euler-Mascheroni constant ($\simeq 0.5772156\dots$).

The formulae in (9) were computed using the suggestion found in [4] for the fractional part: for $0 < a < b < 2a$, $\{b/a\} = (b/a) - 1$.

Subsequently, splitting the interval of integration in variable x , in $((k-1)h, y]$ and $(y, kh]$, we obtain:

$\int_{\Delta_{h,k}} \int_{\Delta_{h,k}} \{y/x\} dx dy = \int_{\Delta_{h,k}} [\int_{\Delta_{h,k}} (y/x) dx - \int_{(k-1)h}^y dx] dy$. The sequence $f(k) := h^{-2}d_{kk}^h = (-1 + \frac{2k-1}{k-1} \ln(\frac{k}{k-1})^{k-1})/2$ monotonically decrease for $k \geq 2$ and converges to 0.5 for $k \rightarrow \infty$. When $k \geq 2$, we have: $d_{kk}^h > h^2/2 > d_{11}^h$. Then:

$$\alpha_h(T_{\rho_h}) = h^{-2}d_{11}^h = (3 - 2\gamma)/4 > 0, \text{ for any } h, nh = 1. \quad (9)$$

showing that the positivity parameters of the sequence of operator approximations $\{T_{\rho_h}, nh = 1, n \geq 2\}$ verifies ii) and iii) properties in Lemma 1 (Theorem 2).

Theorem 3. *The Alcantara-Bode equivalent holds, involving that RH is true.*

Proof.

From Observation 3 we obtain the positivity of the operator on the dense set and with (9), we obtain the bound of the positivity parameters of the operator approximations on the subspaces of the family F . Then, by Lemma 1 follows the strict positivity of the operator on the dense set and subsequently, from Theorem 1, $N_{T_\rho} = \{0\}$. Then half of Alcantara-Bode equivalent formulation of the Riemann Hypothesis holds involving the other half should hold. Therefore the Riemann Hypothesis is true. $\square$

Theorem 3.a Injectivity Criteria: *The Alcantara-Bode equivalent holds.*

Proof.

The injectivity criteria introduced in [1] could be used in our context applied to the restrictions of T_ρ .

It is easier to observe (see also [11]) that the matrix representations of the operator restrictions differ from the representations of the finite rank approximations by the scaling factor h : $M_h(T_\rho) = hM_h^r(T_\rho)$. Thus, we have

$\alpha_h(T_\rho) = h\alpha_h(T_{\rho_h}) = h(3 - 2\gamma)/4 \rightarrow 0$ for $n \rightarrow \infty, nh = 1$. So, for applying Lemma 2 we need to involve the adjoint operator whose kernel function is

$\rho^*(y, x) = \overline{\rho(x, y)} = \rho(x, y)$ in order to obtain the parameters $\omega_h, nh = 1$ that is already done in [11]. However, here are the steps.

For $v_h = \sum_{k=1, n} c_k I_h^k \in S_h$,

$T_\rho^* v_h = \sum_{k=1,n} c_k \int_{\Delta_{h,k}} \rho(x,y) I_h^k(y) dy = \sum_{k=1,n} c_k \int_{\Delta_{h,k}} \rho_{h,k}(x,y) dy$,
 where $\rho_{h,k} = I_{h,k}(x) \rho(x,y) I_{h,k}(y)$. Follows:

$$\begin{aligned} \|T_\rho^* v_h\|^2 &= \langle \sum_{k=1,n} c_k \int_{\Delta_{h,k}} \rho_{h,k}(x,y) dy, \sum_{j=1,n} c_j \int_{\Delta_{h,j}} \rho_{h,j}(x,y) dy \rangle \\ &= \sum_{k=1,n} c_k \bar{c}_k \left(\int_{\Delta_{h,k}} \left[\int_{\Delta_{h,k}} \rho(x,y) I_{h,k}(y) dy \right]^2 I_{h,k}(x) dx \right). \end{aligned}$$

Because $\rho(x,y)$ is valued in $[0,1]$, $\rho(x,y) < 1$ for every $x, y \in (0,1)$, obtaining:

$$\|T_\rho^* v_h\|^2 \leq \sum_{k=1,n} c_k \bar{c}_k h^3 = h^2 \|v_h\|^2 \text{ and, } \|T_\rho^* v_h\| \leq h \|v_h\| \text{ for every } v_h \in S_h.$$

With $\omega_h(T_\rho^*) = h$, the injectivity parameter of T on S_h given by $\mu_h = \alpha_h(T_\rho) / \omega_h(T_\rho^*)$ is evaluated as

$$\mu_h = (3 - 2\gamma)/4 > 0, \text{ for any } h, nh = 1. \quad (10)$$

a constant on every subspace. Then, applying Lemma 2 we obtain $N_{T_\rho} = \{0\}$. $\square$

Comments.

· A connection between Zeta function ζ and the integral operator T_ρ can be observed in [4] by reformulating the left term in the expression as $(T_\rho x^{s-1})(\theta)$:

$$\int_0^1 \rho(\theta/x) x^{s-1} dx = \theta/(s-1) - \theta^s \zeta(s)/s, \quad \sigma > 0, s = \sigma + it.$$

· Considering the indicator of semi-open intervals functions of a partition of the domain, the subspaces are including $(S_h \subset S_{h/2})$ ensuring the monotony of the positivity parameters. If we replace the indicator open-interval functions for generating the subspace S_h^0 as well as the indicator closed-interval functions generating the subspace S_h^c , $nh = 1, n \geq 1$ then both sets S^0 and S^c are still dense like S losing instead the including subspaces property. Information on the density of the set S^0 could be found in textbooks of functional analysis. On the density of S , we showed in V4 of [11] that *if one of the sets S , S^c and S^0 is dense, then others are dense*. A sketch of proof follows. Let S^0 be dense. If f is orthogonal on any $I_{h,k} \in S$ then:

$|\langle f, I_h^k \rangle| = |\langle f, I_h^k - I_{h,k} \rangle| \leq \|f\| \|I_h^k - I_{h,k}\| = 0, k=1,n, \forall n, nh = 1$, showing that f is orthogonal to any I_h^k and so f should be 0 because S^0 is dense. So, S is dense.

· The dense sets S and S^c have been used in [5] and [6] to obtain optimal evaluations of the decay rate of convergence to zero of the eigenvalues of Hermitian integral operators having a kernel function such as Mercer kernels ([9]).

· The associated Hermitian having the kernel $\rho^*(y,x) := \int_0^1 \rho(y,t) \rho(t,x) dt$ verifies automatically the requests of positivity on the dense set and the existence of the finite rank approximations. For proving its strict positivity on S we should compute the positivity parameters on finite dimension subspaces S_h and determine their inferior bound. We could choose the line used in [1] or to approximate its kernel on subspaces like: for $y, x \in \Delta_{h,k}, t \in \Delta_{h,j}, j=2,k-1$ and $j=k+1, n$, for $\varphi(y,x,t)$, as

$$\varphi(y,x,t) = (y/t - \lfloor y/t \rfloor)(t/x - \lfloor t/x \rfloor) = (y/t - \lfloor y/t \rfloor)(t/x) \text{ for } j < k \text{ and}$$

$(y/t)(t/x - \lfloor t/x \rfloor)$ for $j > k+1$ where $\lfloor \cdot \rfloor$ is the floor function. We could continue by integrating in variable t first and then in x and y , to obtain on a level h for a k diagonal entry from $d_{kk}^* = \min_j d_{kk}^{*j}$, $d_{kk}^{*j} = \int_{\Delta_{h,k}} \int_{\Delta_{h,k}} \left(\int_{\Delta_{h,j}} \varphi(y,x,t) dt \right) dx dy$ as $d_{kk}^* = n \times d_{kk}^{*j}$ that is of h^2 order. Thus, the positivity parameter on S_h of the Hermitian is

$$\alpha_h = h^{-2} \min_{k=1,n} d_{kk}^{*j} \text{ where } d_{kk}^{*j} \text{ for } j=1, \text{ and } j=k \text{ should also be computed.}$$

The references [13-16] are related to other RH equivalents, [8] to exotic integrals and [12] to multi-level discretisations on separable Hilbert spaces.

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backed by results obtained in a classical fashion of functional analysis. However, while waiting for a result to RH in the number theory field, this solution is addressed to those in need to have a response whether RH is true and, it is in accordance with the principle of Clay Inst. of Math. expressed as (citing [7]): "A proof that it is true for every interesting solution would shed light on many of the mysteries surrounding the distribution of prime numbers."

Conflicts of Interest: No Competing Interests. Nothing to declare.

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