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Regularity of a Randomly Stopped Sum Determines Regularity of the Stopping Moment

Jonas Šiaulys [1,*,†], Aistė Elijio [1,†], Remigijus Leipus [2,†] and Neda Nakliuda [1,†]

1 Institute of Mathematics, Vilnius University, Naugarduko 24, LT-03225 Vilnius, Lithuania
2 Institute of Applied Mathematics, Vilnius University, Naugarduko 24, LT-03225 Vilnius, Lithuania
* Correspondence: jonas.siaulys@mif.vu.lt (J.Š.)
† All authors contributed equally to this work.

Abstract: The paper investigates the randomly stopped sums. Primary random variables are supposed to be nonnegative, independent, and identically distributed, whereas the stopping moment is supposed to be a nonnegative, integer-valued, and nondegenerate at zero random variable, independent of primary random variables. We find the conditions under which dominated variation or extended regularity of randomly stopped sum determines the stopping moment to belong to the class of dominatedly varying distributions. In the case of extended regularity, we derive the asymptotic inequalities for the ratio of tails of the distributions of randomly stopped sums and a stopping moment. The obtained results generalize analogous statements recently obtained for regularly varying distributions. Compared with the previous studies, we apply new methods to the proofs of the main statements. At the end, we provide one example that illustrates the theoretical results.

Keywords: randomly stopped sum; primary random variables; stopping moment; dominatedly varying distribution; regular variation

MSC: 60E05, 60F10, 60G40

1. Introduction

This paper is devoted to randomly stopped sums in which the primary random variables (r.v.s) are nonnegative, independent, and identically distributed (i.i.d.). The stopping moment is supposed to be nonnegative, integer-valued, and independent of the primary r.v.s. Such objects appear when the number of random variables under consideration is unknown and is described by some random integer. In particular, randomly stopped sums appear in insurance and financial mathematics, survival analysis, risk theory, computer and communication networks, etc. The area of randomly stopped sums for various subclasses of heavy-tailed r.v.s has been well developed for more than 50 years and covers mainly the case of i.i.d. r.v.s. In this paper, we continue to consider this standard model.

Specifically, suppose that $\{\xi_1, \xi_2, \dots\}$ is a sequence of r.v.s defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define the sequence of partial sums $\{S_n, n \geqslant 0\}$ by

$$S_0 := 0, \;\; S_n := \xi_1 + \cdots + \xi_n, \; n \geq 1. \tag{1}$$

The main subject of the paper is the study of *randomly stopped sums*

$$S_\nu := \xi_1 + \cdots + \xi_\nu,$$

where n in (1) is replaced by a random variable (r.v.) ν taking values in $\mathbb{N}_0 := \{0, 1, 2, \dots\}$. Throughout the paper, we assume that $\{\xi_1, \xi_2, \dots\}$ is a sequence of independent copies of r.v. ξ. In addition, we suppose that the generating r.v. ξ is nonnegative, i.e., $\mathbb{P}(\xi_1 \geqslant 0) = 1$, and the counting r.v. ν is nondegenerate at zero, i.e., $\mathbb{P}(\nu \geqslant 1) > 0$. We call such ν a *counting random variable* or a *stopping moment* and suppose throughout the paper that it is independent of the sequence $\{\xi_1, \xi_2, \dots\}$. According to our assumptions, r.v.s $\xi_1, \xi_2, \dots$ are i.i.d. with common distribution function

$$F_\xi(x) = \mathbb{P}(\xi \leqslant x).$$

In such a case the distribution function (d.f.) of the randomly stopped sum has the form

$$F_{S_\nu}(x) = \mathbb{P}(S_\nu \leqslant x) = \sum_{n=0}^{\infty} \mathbb{P}(S_n \leqslant x)\mathbb{P}(\nu = n),$$

and the tail function (t.f.) of the randomly stopped sum for positive x has the form

$$\overline{F}_{S_\nu}(x) = \sum_{n=1}^{\infty} \mathbb{P}(S_n > x)\mathbb{P}(\nu = n).$$

The usual problem considered in many papers is to give conditions that guarantee that F_{S_ν} belongs to some regularity class, provided that d.f. F_ξ or F_ν belongs to the regularity class under consideration. A detailed overview of regularity classes can be found in the books [1–6] and references therein, where all the main properties of those regularity classes are described and analyzed. In this paper, both in the main results and in the discussion of similar results, we essentially limit ourselves to four regularity classes: the class of long-tailed distributions $\mathscr{L}$, the class of distributions with dominantly varying tails $\mathscr{D}$, the class of distributions with consistently varying tails $\mathscr{C}$, and the class of regularly varying distributions $\mathscr{R}$. Here we only note that all the mentioned regularity classes of distributions are subclasses of the heavy-tailed distributions $\mathscr{H}$.

2. Brief Discussion on the Regularity Classes under Consideration

In this section, we present definitions of the regularity classes under consideration and discuss known results related to the closure of these classes under random stopped sums. As mentioned earlier, in this paper, we only study the properties of nonnegative random variable distributions. The results presented below are undoubtedly correct for nonnegative random variable distributions, whereas the results presented for distributions of r.v.s that can take negative values can be incorrect. However, all the definitions below can be applied to the distributions of r.v.s with both positive and negative values. We begin with the narrowest class of distributions $\mathscr{R}$.

• *A d.f. F with t.f. $\overline{F} = 1 - F$ is said to be regularly varying with index $\alpha \geqslant 0$, denoted $F \in \mathscr{R}_\alpha$, if for all $y > 0$,*

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha}.$$

The class of all regularly varying distributions is

$$\mathscr{R} = \bigcup_{\alpha\geqslant 0} \mathscr{R}_\alpha.$$

It is evident that d.f.s

$$F_1(x) = \frac{1}{(1+x)^3}\,\mathbb{I}_{[0,\infty)}(x), \quad F_2(x) = \frac{\log^2(1+x)}{x^4}\,\mathbb{I}_{[1,\infty)}(x),$$
$$F_3(x) = \frac{1}{\log(1+x)}\,\mathbb{I}_{[2,\infty)}(x),\; F_4(x) = \left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{-1}\sum_{1\leqslant n\leqslant x}\frac{1}{n^2}$$

are regularly varying.

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Problems related to the randomly stopped sums of regularly varying d.f.s are considered in [7–10,21], among others. For instance, Proposition 4.1 in [10] states that

$$F_\xi \in \mathscr{R}_\alpha \text{ with } \alpha > 0, \mathbb{E}\nu < \infty, \overline{F}_\nu(x) = o\big(\overline{F}_\xi(x)\big) \;\Rightarrow\; \overline{F}_{S_\nu}(x) \underset{x\to\infty}{\sim} \mathbb{E}\nu \overline{F}_\xi(x).$$

The direct generalization of the class $\mathscr{R}$ is the class $\mathscr{ERV}$ introduced in [11] and further considered in [1,12–19].

• *A d.f. F is said to be extended regularly varying with indices $0 \leqslant \alpha \leqslant \beta < \infty$, denoted $F \in \mathscr{ERV}_{\{\alpha,\beta\}}$, if for all $y > 1$,*

$$y^{-\beta} \leqslant \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \leqslant \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \leqslant y^{-\alpha}.$$

The class of all extended regularly varying distributions is

$$\mathscr{ERV} = \bigcup_{0\leqslant\alpha\leqslant\beta<\infty} \mathscr{ERV}_{\{\alpha,\beta\}}.$$

Since $\overline{F}(xy) \leqslant \overline{F}(x)$ for any d.f. F and all $x > 0$ and $y \geqslant 1$, the distribution F belongs to the class $\mathscr{ERV}$ if

$$\liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \geqslant y^{-\beta}$$

for all $y > 1$ and some $\beta < \infty$; see [20].

Another somewhat broader class of distributions is the class of distributions with consistently varying tails $\mathscr{C}$.

• *A d.f. F is said to have a consistently varying tail, denoted $F \in \mathscr{C}$, if*

$$\lim_{y\uparrow 1} \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1.$$

The definitions presented imply that $\mathscr{R} \subset \mathscr{ERV} \subset \mathscr{C}$. Therefore all d.f.s F_1, F_2, F_3, and F_4, presented above, have consistently varying tails. Due to the considerations in [22–24], d.f.

$$F_5(x) = (1-p)\left(\frac{1-p^{\lfloor \log_2 x\rfloor}}{1-p} + \left(\frac{x}{2^{\lfloor \log_2 x\rfloor}} - 1\right)p^{\lfloor \log_2 x\rfloor}\right)\mathbb{I}_{[1,\infty)}(x)$$

belongs to the class $\mathscr{C}$ for any parameter $p \in (0,1)$, where $\lfloor a\rfloor$ is the integer part of $a \in \mathbb{R}$. Namely, for $x \in [2^n, 2^{n+1})$ and $1/2 < y \leqslant 1$, we have that $\lfloor \log_2 x\rfloor = n$ and $\lfloor \log_2 xy\rfloor \in \{n-1, n\}$. If $\lfloor \log_2 x\rfloor = \lfloor \log_2 xy\rfloor = n$, then

$$\begin{aligned}\frac{\overline{F}_5(xy)}{\overline{F}_5(x)} &= 1 + \frac{(1-p)(1-y)\frac{x}{2^n}}{2-p-(1-p)\frac{x}{2^n}}\\ &\leqslant 1 + \frac{2(1-p)(1-y)}{p}.\end{aligned}$$

If $\lfloor \log_2 x\rfloor = n$ and $\lfloor \log_2 xy\rfloor = n-1$, then

$$\begin{aligned}\frac{\overline{F}_5(xy)}{\overline{F}_5(x)} &= 1 + \frac{(1-p)\Big((2-p)\big(1-\frac{xy}{2^n}\big) + p(1-y)\frac{x}{2^n}\Big)}{p\Big(2-p-(1-p)\frac{x}{2^n}\Big)}\\ &\leqslant 1 + \frac{(1-p)(1-y)(2+p)}{p^2}.\end{aligned}$$

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It is clear that the derived inequalities imply that $F_5 \in \mathscr{C}$. We also note that both inclusions in the relation $\mathscr{R} \subset \mathscr{ERV} \subset \mathscr{C}$ are proper. Due to the results of [20], the distribution with t.f.

$$\overline{F}_6(x) = \exp\Big\{ -\lfloor \log(1+x) \rfloor + \big(\log(1+x) - \lfloor \log(1+x) \rfloor\big)^2 \Big\},\ x \geqslant 0,$$

belongs to $\mathscr{ERV} \setminus \mathscr{R}$, and the distribution with t.f.

$$\overline{F}_7(x) = \exp\Big\{ -\lfloor \log(1+x) \rfloor + \big(\log(1+x) - \lfloor \log(1+x) \rfloor\big)^{1/2} \Big\},\ x \geqslant 0,$$

belongs to $\mathscr{C} \setminus \mathscr{ERV}$.

Problems related to the randomly stopped sums of d.f.s from class $\mathscr{C}$ are considered in [22,25–27], among others. For instance, [26, Theorem 3.4] states the following:

$$F_\xi \in \mathscr{C}, 0 < \mathbb{E}\nu^{q+1} < \infty \text{ for some } q > J^+_{F_\xi} \Rightarrow F_{S_\nu} \in \mathscr{C} \text{ and } \overline{F}_{S_\nu}(x) \underset{x\to\infty}{\sim} \mathbb{E}\nu \overline{F}_\xi(x),$$

where

$$J^+_{F_\xi} = -\lim_{y\to\infty} \frac{1}{\log y} \log \liminf_{x\to\infty} \frac{\overline{F}_\xi(xy)}{\overline{F}_\xi(x)}$$

is the upper Matuszewska index of d.f. F_ξ; see [28,29].

Another, even broader class is the class of distributions with dominatedly varying tails $\mathscr{D}$, described more than half a century ago in [30].

• *A d.f. F is said to have a dominatedly varying tail, denoted $F \in \mathscr{D}$, if*

$$\limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty$$

for all (equivalently, for some) $y \in (0,1)$.

It is obvious that $\mathscr{R} \subset \mathscr{ERV} \subset \mathscr{C} \subset \mathscr{D}$ and that $F \in \mathscr{D} \Leftrightarrow J^+_F < \infty$. Hence all the above-mentioned d.f.s $F_1, F_2, F_3, F_4, F_5, F_6$, and F_7 belong to the class $\mathscr{D}$. As noted in [31], the Peter and Paul distribution is an example of a distribution belonging to the class $\mathscr{D}$ but not to the class $\mathscr{C}$. We recall that r.v. ξ is said to be distributed according to the generalized Peter and Paul distribution with parameters $a > 0$ and $b > 1$ if its t.f. for $x \geqslant 1$ has the following expression:

$$\overline{F}_8(x) = \overline{F}_\xi(x) = (b^a - 1) \sum_{k\leqslant 1, b^k > x} b^{-ak} = \big(b^{-a}\big)^{\lfloor \log_b x \rfloor}.$$

For this distribution, we derive (for details, see [23]) that

$$\lim_{y\uparrow 1} \limsup_{x\to\infty} \frac{\overline{F_8}(xy)}{\overline{F}_8(x)} = b^a \neq 1,$$

which implies $F_\xi \in \mathscr{D} \setminus \mathscr{C}$.

The asymptotic properties of randomly stopped sums with summands that vary dominatedly are considered in [32–36]. For instance, [33, Theorem 5] states that

$$F_\xi \in \mathscr{D}, \mathbb{E}\xi < \infty, 0 < \mathbb{E}\nu < \infty \Rightarrow \Big\{ F_{S_\nu} \in \mathscr{D} \Longleftrightarrow \min\{F_\xi, F_\nu\} \in \mathscr{D} \Big\}.$$

Class $\mathscr{C}$ can be extended in the other direction instead of distributions with dominatedly varying tails $\mathscr{D}$ by choosing the class of long-tailed distributions $\mathscr{L}$. The class of long-tailed distributions was introduced by Chistyakov [37] in the context of branching processes and became one of the most important subclasses of heavy-tailed distributions.

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- *A distribution F is said to belong to a class of long-tailed distributions $\mathscr{L}$ if*

$$\overline{F}(x+y) \underset{x\to\infty}{\sim} \overline{F}(x)$$

for all (equivalently, for some) $y \in \mathbb{R}$.

However, all the definitions below can be applied to distributions of random variables taking on both positive and negative values.

According to the above considerations, $\mathscr{R} \subset \mathscr{ERV} \subset \mathscr{C} \subset L \cap \mathscr{D} \subset \mathscr{L}$ with all proper inclusions. To see that $\mathscr{L} \cap \mathscr{D} \setminus \mathscr{C} \neq \emptyset$, we can take d.f. F_9 from [20] with t.f.

$$\overline{F}_9(x) = \exp\Big\{ -\lfloor \log x \rfloor - \min\big\{ \log x(\log x - \lfloor \log x \rfloor), 1\big\}\Big\},\ x \geqslant 1,$$

and a simple Weibull distribution with d.f.

$$F_{10}(x) = (1 - \mathrm{e}^{-\sqrt{x}})\mathbb{I}_{[0,\infty)}(x)$$

shows that inclusion $\mathscr{L} \cap \mathscr{D} \subset \mathscr{L}$ is also proper.

The asymptotic properties of randomly stopped sums with summands with long-tailed distributions are considered in [33,36,38–43]. Asymptotic properties in the above papers are found either exclusively for summands with long tails or for summands satisfying additional conditions. For instance, [40, Theorem 8] states that

$$F_\xi \in \mathscr{L} \cap \mathscr{D}, \mathbb{E}\xi > 0, F_\nu \in \mathscr{C} \ \Rightarrow\ \overline{F}_{S_\nu}(x) \underset{x\to\infty}{\sim} \mathbb{E}\nu \overline{F}_\xi(x) + \overline{F}_\nu\Big(\frac{x}{\mathbb{E}\xi}\Big).$$

It was mentioned earlier that all discussed regularity classes consist of heavy-tailed distributions.

- *A d.f. F is said to be heavy-tailed, denoted by $F \in \mathscr{H}$, if*

$$\int_{-\infty}^{\infty} \mathrm{e}^{\delta x} \mathrm{d}F(x) = \infty$$

for all $\delta > 0$. Otherwise, F is said to be light-tailed.

Due to the inclusions $\mathscr{R} \subset \mathscr{ERV} \subset \mathscr{C} \subset L \cap \mathscr{D} \subset \mathscr{L}$, all the participating classes are heavy-tailed because $\mathscr{L} \subset \mathscr{H}$. This fact can be easily observed because of the property

$$F \in \mathscr{L} \ \Rightarrow\ \mathrm{e}^{\lambda x}\overline{F}(x) \underset{x\to\infty}{\to} \infty \text{ for all } \lambda > 0$$

(see, e.g., comments in [44]) and the criteria

$$F \in \mathscr{H} \ \Leftrightarrow\ \limsup_{x\to\infty} \mathrm{e}^{\lambda x}\overline{F}(x) = \infty \text{ for all } \lambda > 0$$

(see, e.g., [45, Lemma 1]).

The randomly stopped sum with heavy-tailed nonnegative i.i.d. summands $\{\xi_1, \xi_2, \ldots\}$ satisfies the following simple estimate:

$$\overline{F}_{S_\nu}(x) = \sum_{n=1}^{\infty} \mathbb{P}(S_n > x)\mathbb{P}(\nu = n) \geqslant \overline{F}_{\xi_1}(x)\mathbb{P}(\nu \geqslant 1).$$

Hence we have the following statement for any counting r.v. ν:

$$F_\xi \in \mathscr{H} \ \Rightarrow\ F_{S_\nu} \in \mathscr{H}.$$

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In addition, for any positive λ and i.i.d. summands $\{\xi_1, \xi_2, \dots\}$, we have

$$\mathbb{E}\, e^{\lambda S_\nu} \geqslant \sum_{n=1}^{\infty} \mathbb{E}\, e^{\lambda S_n} \mathbb{P}(\nu = n) = \sum_{n=1}^{\infty} \left(\mathbb{E}\, e^{\lambda \xi_1}\right)^n \mathbb{P}(\nu = n) = \mathbb{E}\, e^{\nu(\log \mathbb{E}\, e^{\lambda \xi_1})}.$$

This leads to the following statement for any nondegenerate at zero distribution F_ξ of $\{\xi_1, \xi_2, \dots\}$:

$$F_\nu \in \mathscr{H} \;\Rightarrow\; F_{S_\nu} \in \mathscr{H}.$$

3. Inverse-Type Statements in Class $\mathscr{R}$

In Section 2, we described the results of direct type on the regularity of the randomly stopped sums. In all the mentioned results, we specified the conditions on random summands and counting r.v. ensuring that randomly stopped sums fall into the desired regularity class. However, the paper [10] also presents results of the inverse type. Namely, Proposition 4.9 of [10] consists of the following two statements.

Theorem 1. *Let $\{\xi_1, \xi_2, \dots\}$ be a sequence of i.i.d. copies of a nonnegative r.v. ξ, and let ν be a counting r.v. independent of $\{\xi_1, \xi_2, \dots\}$. If d.f. of the randomly stopped sum $F_{S_\nu} \in \mathscr{R}_\alpha$ with $\alpha > 0$, $\mathbb{E}\xi < \infty$, $\mathbb{E}\nu < \infty$, and $\overline{F}_\xi(x) = o\left(\overline{F}_{S_\nu}(x)\right)$, then d.f. $F_\nu \in \mathscr{R}_\alpha$, and*

$$\overline{F}_{S_\nu}(x) \underset{x\to\infty}{\sim} (\mathbb{E}\xi)^\alpha\, \overline{F}_\nu(x).$$

Theorem 2. *Let the sequence of r.v.s $\{\xi_1, \xi_2, \dots\}$ and r.v ν satisfy the basic conditions of Theorem 1. If $F_{S_\nu} \in \mathscr{R}_1$, $\mathbb{E}\xi < \infty$, $\mathbb{E}\nu = \infty$, and $x\overline{F}_\xi(x) = o\left(\overline{F}_{S_\nu}(x)\right)$, then $F_\nu \in \mathscr{R}_1$, and*

$$\overline{F}_{S_\nu}(x) \underset{x\to\infty}{\sim} \mathbb{E}\xi\, \overline{F}_\nu(x).$$

In this paper, we prove analogous statements in the broader classes of distributions $\mathscr{ERV}$ and $\mathscr{D}$. We present exact formulations of the statements in Section 4.

4. Main Results

As mentioned above, in this section, we formulate the main results of the work on how the membership of the random stopped sum distribution in classes $\mathscr{ERV}$ and $\mathscr{D}$ affects the membership of the counting r.v. distribution in the class $\mathscr{D}$. If the randomly stopped sum distribution belongs to the class $\mathscr{ERV}$, then we also obtain asymptotic formulas similar to those in Theorems 1 and 2. Our first statement is an analog of Theorem 1 for the wider regularity class $\mathscr{ERV}$.

Theorem 3. *Let $\{\xi_1, \xi_2, \dots\}$ be a sequence of i.i.d. copies of a nonnegative r.v. ξ, and let ν be a counting r.v. independent of $\{\xi_1, \xi_2, \dots\}$. If d.f. of the randomly stopped sum $F_{S_\nu} \in \mathscr{ERV}_{\{\alpha,\beta\}}$ $(0 \leqslant \alpha \leqslant \beta < \infty)$, $\mathbb{E}\xi < \infty$, $\mathbb{E}\nu < \infty$, and $\overline{F}_\xi(x) = o\left(\overline{F}_{S_\nu}(x)\right)$, then d.f. $F_\nu \in \mathscr{D}$, and*

$$(\mathbb{E}\xi)^\alpha\, \overline{F}_\nu(x) \underset{x\to\infty}{\lesssim} \overline{F}_{S_\nu}(x) \underset{x\to\infty}{\lesssim} (\mathbb{E}\xi)^\beta\, \overline{F}_\nu(x) \text{ if } \mathbb{E}\xi > 1, \tag{2}$$

$$(\mathbb{E}\xi)^\beta\, \overline{F}_\nu(x) \underset{x\to\infty}{\lesssim} \overline{F}_{S_\nu}(x) \underset{x\to\infty}{\lesssim} (\mathbb{E}\xi)^\alpha\, \overline{F}_\nu(x) \text{ if } \mathbb{E}\xi < 1, \tag{3}$$

$$\overline{F}_{S_\nu}(x) \underset{x\to\infty}{\sim} \overline{F}_\nu(x) \text{ if } \mathbb{E}\xi = 1, \tag{4}$$

where the symbol $\underset{x\to\infty}{\lesssim}$ between two positive functions $a(x)$ and $b(x)$ means that

$$\limsup_{x\to\infty} \frac{a(x)}{b(x)} \leqslant 1.$$

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It is evident that Theorem 3 implies the statement of Theorem 1 because $\alpha = \beta$ in the case where d.f. F_ξ of r.v. ξ belongs to the class $\mathscr{R}_\alpha$.

Remark 1. *The conditions of Theorem 3 imply that the expectation of the primary r.v. ξ is positive, i.e., $\mathbb{E}\xi > 0$.*

Proof of Remark. Suppose on the contrary that $\mathbb{E}\xi = 0$, that is, $\mathbb{P}(\xi = 0) = 1$. In such a case, $\mathbb{P}(S_n = 0) = 1$ for all n. Hence

$$\overline{F}_{S_\nu}(x) = \sum_{n=1}^{\infty} \mathbb{P}(S_n > x)\mathbb{P}(\nu = n) = 0$$

for $x > 0$. This contradicts to the condition $F_{S_\nu} \in \mathscr{D}$. Therefore $\mathbb{P}(\xi > 0) > 0$, implying $\mathbb{E}\xi > 0$. The remark is proved. □

Remark 2. *The conditions of Theorem 3 imply that the support of the counting r.v. ν is infinite, i.e., $\overline{F}_\nu(x) > 0$ for all $x \in \mathbb{R}$.*

Proof of Remark. We will prove the statement by contradiction. Let the opposite statement be true, i.e., $\text{supp}\nu \subset \{0, 1, \ldots, K\}$ for some finite $K \geqslant 1$. In such a case, for $x > 0$, we have

$$\overline{F}_{S_\nu}(x) = \sum_{n=1}^{K} \mathbb{P}(S_n > x)\mathbb{P}(\nu = n) = \sum_{n=1}^{K} \mathbb{P}(\xi_1 + \xi_2 + \cdots + \xi_n > x)\mathbb{P}(\nu = n).$$

For each $1 \leqslant n \leqslant K$, we have

$$\begin{aligned}
\mathbb{P}(\xi_1 + \xi_2 + \cdots + \xi_n > x) &\leqslant \mathbb{P}\left(\bigcup_{k=1}^{n} \left\{\xi_k > \frac{x}{n}\right\}\right) \\
&\leqslant n\mathbb{P}\left(\xi > \frac{x}{n}\right) \\
&\leqslant n\overline{F}_\xi\left(\frac{x}{K}\right).
\end{aligned}$$

Therefore, for all $x > 0$,

$$\overline{F}_{S_\nu}(x) \leqslant \overline{F}_\xi\left(\frac{x}{K}\right) \sum_{k=1}^{K} n = \frac{K(K+1)}{2}\overline{F}_\xi\left(\frac{x}{K}\right),$$

implying that

$$\overline{F}_\xi(x) \geqslant \frac{2}{K(K+1)}\overline{F}_{S_\nu}(xK),\ x > 0,$$

and

$$\begin{aligned}
\liminf_{x\to\infty} \frac{\overline{F}_\xi(x)}{\overline{F}_{S_\nu}(x)} &\geqslant \frac{2}{K(K+1)} \liminf_{x\to\infty} \frac{\overline{F}_{S_\nu}(xK)}{\overline{F}_{S_\nu}(x)} \\
&= \frac{2}{K(K+1)} \left(\limsup_{x\to\infty} \frac{\overline{F}_{S_\nu}(\frac{x}{K})}{\overline{F}_{S_\nu}(x)}\right)^{-1} > 0
\end{aligned}$$

by the condition $F_{S_\nu} \in \mathscr{D}$. This resulting estimate contradicts the condition $\overline{F}_\xi(x) = o(\overline{F}_{S_\nu}(x))$. Hence the counting r.v. ν must have an infinite support. The statement of the remark is proved. □

Just as Theorem 3 generalizes Theorem 1, the following theorem generalizes Theorem 2 in a similar way.

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Theorem 4. *Suppose a sequence of r.v.s $\{\xi_1, \xi_2, \ldots\}$ and counting r.v. ν satisfy the basic conditions of Theorem 3. If $F_{S_\nu} \in \mathscr{ERV}_{\{1,\beta\}}$ $(1 \leqslant \beta < \infty)$ and $x\overline{F}_\xi(x) = o\Big(\overline{F}_{S_\nu}(x)\Big)$, then d.f. $F_\nu \in \mathscr{D}$, and the asymptotic inequalities (2)–(4) are satisfied with $\alpha = 1$.*

Remark 3. *The conditions $F_{S_\nu} \in \mathscr{ERV}_{1,\beta}$ and $x\overline{F}_\xi(x) = o\Big(\overline{F}_{S_\nu}(x)\Big)$ of the theorem imply that $\mathbb{E}\xi$ is finite and, moreover, that $\mathbb{E}(\xi)^r$ is also finite for some $r > 1$.*

Both theorems can be proved based on the propositions below. Note that these two propositions are interesting in themselves and that we prove them in a completely different way compared to the proofs of Theorems 1 and 2 given in [10]. In addition, we observe that Propositions 2 and 3 present statements analogous to Theorems 1 and 2 but for class $\mathscr{D}$ instead of $\mathscr{R}$. Section 5 presents detailed proofs of the theorems and propositions below.

Proposition 1. *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of i.i.d. copies of a nonnegative r.v. ξ with positive mean $\mu = \mathbb{E}\xi$, and let ν be a counting r.v. with infinite support and independent of the sequence $\{\xi_1, \xi_2, \ldots\}$. Then*

$$\overline{F}_{S_\nu}\big((1-\varepsilon)\mu x\big) \underset{x\to\infty}{\gtrsim} \overline{F}_\nu(x)$$

for all $\varepsilon \in (0,1)$.

Proposition 2. *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of i.i.d. copies of a nonnegative r.v. ξ with positive mean $\mu = \mathbb{E}\xi$ and d.f. F_ξ, and let ν be a counting r.v. with $\mathbb{E}\nu < \infty$ and independent of the sequence $\{\xi_1, \xi_2, \ldots\}$. If $F_{S_\nu} \in \mathscr{D}$ and $\overline{F}_\xi(x) = o\big(\overline{F}_{S_\nu}(x)\big)$, then $F_\nu \in \mathscr{D}$, and*

$$\overline{F}_{S_\nu}\big((1+\varepsilon)\mu x\big) \underset{x\to\infty}{\lesssim} \overline{F}_\nu(x) \tag{5}$$

for all $\varepsilon > 0$.

Proposition 3. *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of i.i.d. copies of a nonnegative r.v. ξ with positive mean $\mu = \mathbb{E}\xi$, finite moment $\mathbb{E}\,\xi^r$ of order $r > 1$, and d.f. F_ξ. Let ν be a counting r.v. independent of the sequence $\{\xi_1, \xi_2, \ldots\}$ such that $\mathbb{E}\nu = \infty$. If $F_{S_\nu} \in \mathscr{D}$ and $x\overline{F}_\xi(x) = o\big(\overline{F}_{S_\nu}(x)\big)$, then $F_\nu \in \mathscr{D}$, and the asymptotic relation (5) holds.*

5. Proofs

In this section, we present proofs of the propositions and theorems stated in Section 4.

5.1. Proof of Proposition 1

For all $x > 0$ and $\varepsilon \in (0,1)$, we have

$$\begin{aligned}
\overline{F}_{S_\nu}\big((1-\varepsilon)\mu x\big) &= \sum_{n=1}^{\infty} \mathbb{P}\big(S_n > (1-\varepsilon)\mu x\big)\mathbb{P}(\nu = n) \\
&\geqslant \sum_{n>x} \mathbb{P}\big(S_n > (1-\varepsilon)\mu x\big)\mathbb{P}(\nu = n) \\
&\geqslant \sum_{n>x} \mathbb{P}\big(S_{\lfloor x \rfloor} > (1-\varepsilon)\mu \lfloor x \rfloor\big)\mathbb{P}(\nu = n) \\
&= \mathbb{P}\left(\frac{S_{\lfloor x \rfloor} - \mu \lfloor x \rfloor}{\lfloor x \rfloor} > -\varepsilon\mu\right)\overline{F}_\nu(x).
\end{aligned}$$

Since $\overline{F}_\nu(x) > 0$ for all $x \in \mathbb{R}$, we derive that

$$\liminf_{x\to\infty} \frac{\overline{F}_{S_\nu}\big((1-\varepsilon)\mu x\big)}{\overline{F}_\nu(x)} \geqslant \liminf_{x\to\infty} \mathbb{P}\left(\frac{S_{\lfloor x \rfloor} - \mu \lfloor x \rfloor}{\lfloor x \rfloor} > -\varepsilon\mu\right) = 1$$

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due to the law of large numbers. Proposition 1 is proved.

5.2. Auxiliary Lemmas for the Proofs of Propositions 2 and 3

In this section, we state three lemmas we use in the proof of Proposition 2. The first lemma contains a construction of a special sequence of random variables with dominatedly varying distributions.

Lemma 1. *Let $\{X_1, X_2, \ldots\}$ be a sequence of nonnegative i.i.d. r.v.s with common d.f. F such that $\overline{F}(x) = o(\overline{H}(x))$ for some d.f. $H \in \mathscr{D}$. Then there exists a sequence of i.i.d. r.v.s $\{Z_1, Z_2, \ldots\}$ with common d.f. $G \in \mathscr{D}$ such that $X_k \leqslant Z_k$ for $k \in \{1, 2, \ldots\}$ and $\overline{G}(x) = o(\overline{H}(x))$.*

Proof. Here we present a detailed proof of the lemma, which is based on the ideas of Lemma 4.4 from [10] and considerations in [46].

Let

$$L(x) = 1 \text{ if } x \in \left[0, x_0 = \max\left\{1, \sup\left\{z : \frac{\overline{F}(z)}{\overline{H}(z)} > 1\right\}\right\}\right],$$
$$L(x) = 2 \text{ if } x \in \left[x_0, x_1 = \max\left\{2x_0, \sup\left\{z : \frac{\overline{F}(z)}{\overline{H}(z)} > \frac{1}{2^2}\right\}\right\}\right],$$
$$L(x) = 3 \text{ if } x \in \left[x_1, x_2 = \max\left\{3x_1, \sup\left\{z : \frac{\overline{F}(z)}{\overline{H}(z)} > \frac{1}{3^2}\right\}\right\}\right],$$
$$\ldots$$
$$L(x) = k \text{ if } x \in \left[x_{k-2}, x_{k-1} = \max\left\{kx_{k-2}, \sup\left\{z : \frac{\overline{F}(z)}{\overline{H}(z)} > \frac{1}{k^2}\right\}\right\}\right],$$
$$L(x) = k+1 \text{ if } x \in \left[x_{k-1}, x_k = \max\left\{(k+1)x_{k-1}, \sup\left\{z : \frac{\overline{F}(z)}{\overline{H}(z)} > \frac{1}{(k+1)^2}\right\}\right\}\right],$$
$$L(x) = k+2 \text{ if } x \in \left[x_k, x_{k+1} = \max\left\{(k+2)x_k, \sup\left\{z : \frac{\overline{F}(z)}{\overline{H}(z)} > \frac{1}{(k+2)^2}\right\}\right\}\right],$$
$$\ldots$$

According to the presented construction,

$$L(x) \underset{x\to\infty}{\to} \infty, \text{ and } L(x)\frac{\overline{F}(x)}{\overline{H}(x)} \underset{x\to\infty}{\to} 0,$$

because the sequence $\{x_k\}$ is unboundedly increasing, and

$$L(x)\frac{\overline{F}(x)}{\overline{H}(x)} \leqslant \frac{k+1}{k^2},\ x \in (x_{k-1}, x_k],$$

for $k \in \{1, 2, \ldots\}$.

In addition, the function L slowly varies because for each fixed $y > 1$ and large x, either both points x and yx belong to the same interval $(x_{k-1}, x_k]$, or $x \in (x_{k-1}, x_k]$ and $yx \in (x_k, x_{k+1}]$. In the first case,

$$\frac{L(yx)}{L(x)} = 1,$$

and in the second case,

$$\frac{L(yx)}{L(x)} = \frac{k+2}{k+1}.$$

In both cases,

$$\lim_{x\to\infty} \frac{L(yx)}{L(x)} = 1,$$

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implying that the function L slowly varies by the classical definition; see Section 1 in [1] for details.

Let $\{W_1, W_2, \ldots\}$ be a sequence of i.i.d. positive r.v.s independent of the sequence $\{X_1, X_2, \ldots\}$ with tail function

$$\overline{D}(x) = \begin{cases} \frac{\overline{H}(x)}{L(x)} & \text{if } x \geqslant 1, \\ 1 & \text{if } x < 1. \end{cases}$$

By the construction of functions D and L we have that

$$D \in \mathscr{D},\ \overline{D}(x) = o\big(\overline{H}(x)\big),\ \overline{F}(x) = o\big(\overline{D}(x)\big). \tag{6}$$

Now let us suppose that

$$Z_n = X_n + W_n,\ n \in \mathbb{N}.$$

For this sequence of r.v.s, we have the following properties:
(i) $X_n \leqslant Z_n$. This is obvious because $W_n \geqslant 0$ due to construction of r.v.s W_n, $n \in \mathbb{N}$.
(ii) The sequence $\{Z_1, Z_2, \ldots\}$ consists of i.i.d. r.v.s because for any real numbers $\{x_1, x_2, \ldots, x_n\}$, $n \in \mathbb{N}$, we have

$$\begin{aligned} &\mathbb{P}\big(X_1 + W_1 \leqslant x_1, \ldots, X_n + W_n \leqslant x_n\big) \\ = &\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathbb{P}\big(X_1 + y_1 \leqslant x_1, \ldots, X_n + y_n \leqslant x_n\big) \mathrm{d}\mathbb{P}\big(W_1 \leqslant y_1, \ldots, W_n \leqslant y_n\big) \\ = &\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{j=1}^{n} \mathbb{P}\big(X_j + y_j \leqslant x_j\big) \mathrm{d}F_{W_1}(y_1) \ldots \mathrm{d}F_{W_n}(y_n) \\ = &\prod_{j=1}^{n} \mathbb{P}\big(X_j + W_j \leqslant x_j\big). \end{aligned}$$

(iii) D.f. $G(x) = \mathbb{P}(Z_1 \leqslant x)$ satisfies the condition $\overline{G}(x) = o\big(\overline{H}(x)\big)$ because

$$\begin{aligned} \overline{G}(x) &= \mathbb{P}\big(X_1 + W_1 > x\big) \\ &= \int_{[0,x/2]} \overline{F}(x-y)\mathrm{d}D(y) + \int_{(x/2,\infty)} \overline{F}(x-y)\mathrm{d}D(y) \\ &\leqslant \overline{F}\Big(\frac{x}{2}\Big)\mathbb{P}\Big(0 \leqslant W_1 \leqslant \frac{x}{2}\Big) + \overline{D}\Big(\frac{x}{2}\Big) \\ &\leqslant \overline{F}\Big(\frac{x}{2}\Big) + \overline{D}\Big(\frac{x}{2}\Big), \end{aligned} \tag{7}$$

and therefore

$$\limsup_{x\to\infty} \frac{\overline{G}(x)}{\overline{H}(x)} \leqslant \limsup_{x\to\infty} \frac{\overline{F}\big(\frac{x}{2}\big)}{\overline{H}\big(\frac{x}{2}\big)} \frac{\overline{H}\big(\frac{x}{2}\big)}{\overline{H}(x)} + \limsup_{x\to\infty} \frac{\overline{D}\big(\frac{x}{2}\big)}{\overline{D}(x)} \frac{\overline{D}(x)}{\overline{H}(x)} = 0$$

by (6) and the conditions of the lemma.
(iv) Finally, d.f. G belongs to the class $\mathscr{D}$ because

$$\begin{aligned} \limsup_{x\to\infty} \frac{\overline{G}\big(\frac{x}{2}\big)}{\overline{G}(x)} &\leqslant \limsup_{x\to\infty} \frac{\overline{F}\big(\frac{x}{4}\big) + \overline{D}\big(\frac{x}{4}\big)}{\mathbb{P}(X_1 + W_1 > x)} \\ &\leqslant \limsup_{x\to\infty} \frac{\overline{F}\big(\frac{x}{4}\big) + \overline{D}\big(\frac{x}{4}\big)}{\overline{D}(x)} \\ &\leqslant \limsup_{x\to\infty} \frac{\overline{F}\big(\frac{x}{4}\big)}{\overline{D}\big(\frac{x}{4}\big)} \frac{\overline{D}\big(\frac{x}{4}\big)}{\overline{D}(x)} + \limsup_{x\to\infty} \frac{\overline{D}\big(\frac{x}{4}\big)}{\overline{D}(x)} < \infty \end{aligned}$$

by relations (6) and (7). This finishes proof of the lemma. □

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The second lemma presents a special property of i.i.d. r.v.s distributed according to the dominatedly varying distribution. This exceptional property plays an essential role in the proof of Proposition 2. The proof of the lemma is presented in [47, Theorem 1] and is generalized for a dependence structure in [32, Corollary 3.1].

Lemma 2. *Let $F \in \mathscr{D}$ be a d.f. on $\mathbb{R}$ with finite mean*

$$m = \int_{-\infty}^{\infty} y \, \mathrm{d}F(y).$$

Then for all $\gamma > \max\{0, m\}$, there exists a constant $D(\gamma) > 0$, independent of x and n, such that

$$\overline{F^{*n}}(x) \leqslant D(\gamma) n \overline{F}(x)$$

for all $x \geqslant \gamma n$ and $n \in \mathbb{N}$.

Remark 4. *For a d.f. $F : \mathbb{R} \to [0,1]$, let us define the support of F as*

$$\operatorname{supp} F = \big\{x : F(x+\delta) - F(x-\delta) > 0 \text{ for all } \delta > 0\big\}.$$

If $\operatorname{supp} F \subset [0,\infty)$, then we say that the distribution (or d.f.) F is on $\mathbb{R}^+$. If $\operatorname{supp} F \not\subset [0,\infty)$, then we say that the distribution (or d.f.) F is on $\mathbb{R}$.

Remark 5. *If d.f. F belongs to the class $\mathscr{D}$, then $F(x) < 1$ or, equivalently, $\overline{F}(x) > 0$ for all $x \in \mathbb{R}$. This follows from the definition of class $\mathscr{D}$ because the condition*

$$\limsup_{x\to\infty} \frac{\overline{F}(x/2)}{\overline{F}(x)} < \infty$$

cannot be satisfied in the case $\overline{F}(x_0) = 0$ for some $x_0 \in \mathbb{R}$.

The last lemma in this section is another version of Lemma 2.

Lemma 3. *Let F be a d.f. on $\mathbb{R}$ with finite mean m. Then for all $\gamma > \max\{0, m\}$, there exists a positive constant $D(\gamma)$, independent of x and n, such that*

$$\max_{1\leqslant n\leqslant \frac{x}{\gamma}} \frac{\overline{F^{*n}}(x)}{n\overline{F}(x)} \leqslant D(\gamma) \;\text{ for }\; x \geqslant \gamma.$$

Proof. By Lemma 2 we have

$$\begin{aligned}
x \geqslant \gamma &\Rightarrow \frac{\overline{F^{*1}}(x)}{\overline{F}(x)} \leqslant D(\gamma),\\
x \geqslant 2\gamma &\Rightarrow \frac{\overline{F^{*2}}(x)}{2\overline{F}(x)} \leqslant D(\gamma),\\
x \geqslant 3\gamma &\Rightarrow \frac{\overline{F^{*3}}(x)}{3\overline{F}(x)} \leqslant D(\gamma),\\
&\dots\\
x \geqslant N\gamma &\Rightarrow \frac{\overline{F^{*N}}(x)}{N\overline{F}(x)} \leqslant D(\gamma).
\end{aligned}$$

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For $x \geqslant \gamma$, we have $x \geqslant \lfloor \frac{x}{\gamma} \rfloor \gamma$. Hence the above estimations imply that

$$\max_{1 \leqslant n \leqslant \lfloor x/\gamma \rfloor} \frac{\overline{F^{*n}}(x)}{n\overline{F}(x)} \leqslant D(\gamma).$$

The inequality of the lemma is proved. □

We use the following lemma in the proof of Proposition 3. This lemma is proved in [32] (see Lemma 2.3) for possibly dependent i.i.d. r.v.s. Discussions on similar inequalities, which can also be used in the proof of Proposition 3, can be found in [48,49].

Lemma 4. *Let $\{X_1, X_2, \ldots\}$ be independent copies of r.v. X with distribution F, mean 0, and $\mathbb{E}(X^+)^r < \infty$ for some $r > 1$. Then for all $\gamma > 0$ and $p > 0$, there exist positive numbers a and $C = C(\gamma, p)$, independent of x and n, such that*

$$\mathbb{P}\left(\sum_{k=1}^{n} X_k > x\right) \leqslant n\overline{F}(ax) + \frac{C}{x^p}$$

for all $x \geqslant \gamma n$ and $n \in \mathbb{N}$.

The last lemma is related to the tail property of distributions from class $\mathscr{D}$. It follows from Proposition 2.2.1 of [1] and is used in the proof of Proposition 3. Some details of the proof can be found in Lemma 3.5 of [29].

Lemma 5. *Let $F \in \mathscr{D}$ be a distribution with upper Matuszewska index J_F^+. Then for all $p > J_F^+$, we have $x^{-p} = o\big(\overline{F}(x)\big)$.*

5.3. Proof of the Proposition 2

For all $x > 0$ and $\varepsilon \in (0,1)$, we have

$$\begin{aligned}\overline{F}_{S_\nu}\big((1+\varepsilon)\mu x\big) &= \left(\sum_{1 \leqslant n \leqslant x} + \sum_{n > x}\right)\mathbb{P}\big(S_n > (1+\varepsilon)\mu x\big)\mathbb{P}(\nu = n) \\ &:= \mathcal{J}_1(x) + \mathcal{J}_2(x). \end{aligned} \tag{8}$$

For the term $\mathcal{J}_2(x)$, we get

$$\begin{aligned}\frac{\mathcal{J}_2(x)}{\overline{F}_\nu(x)} &\leqslant \sup_{n > x} \mathbb{P}\big(S_n > (1+\varepsilon)\mu x\big) \\ &= \sup_{n > x} \mathbb{P}\big(S_n - \mu n > \mu x - \mu n + \varepsilon \mu x\big) \\ &\leqslant \sup_{n > x} \mathbb{P}\left(\frac{S_n}{n} - \mu > -\mu\right), \end{aligned}$$

which implies that

$$\limsup_{x \to \infty} \frac{\mathcal{J}_2(x)}{\overline{F}_\nu(x)} \leqslant 1 \tag{9}$$

due to the law of large numbers.

Therefore it remains to prove that

$$\limsup_{x \to \infty} \frac{\mathcal{J}_1(x)}{\overline{F}_\nu(x)} = 0. \tag{10}$$

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Let $\{\widehat{\xi}_1, \widehat{\xi}_2, \ldots\}$ be a sequence of nonnegative i.i.d. r.v.s with the properties

$$\xi_k \leqslant \widehat{\xi}_k,\ k \in \{1,2,\ldots\};\ \ F_{\widehat{\xi}_1} \in \mathscr{D};\ \ \overline{F}_{\widehat{\xi}_1}(x) = o\Big(\overline{F}_{S_\nu}(x)\Big).$$

Such a construction is possible due to Lemma 1.

For

$$\widehat{S}_n = \sum_{k=1}^{n} \widehat{\xi}_k,$$

we have

$$\begin{aligned}
\mathcal{J}_1(x) &\leqslant \sum_{1\leqslant n\leqslant \mu x/\widehat{\mu}} \mathbb{P}\big(\widehat{S}_n > (1+\varepsilon)\mu x\big)\mathbb{P}(\nu = n) \\
&\quad + \sum_{\mu x/\widehat{\mu}<n\leqslant x} \mathbb{P}\big(S_n > (1+\varepsilon)\mu x\big)\mathbb{P}(\nu = n) \\
&=: \mathcal{J}_{11}(x) + \mathcal{J}_{12}(x), \qquad (11)
\end{aligned}$$

where $\mu \leqslant \widehat{\mu} = \mathbb{E}\widehat{\xi}_1 < \infty$ because $\mathbb{E}S_\nu = \mu\nu < \infty$ by the conditions of the theorem, and $\overline{F}_{\widehat{\xi}_1}(x) = o\big(\overline{F}_{S_\nu}(x)\big)$ by construction of r.v.s $\{\widehat{\xi}_1, \widehat{\xi}_2, \ldots\}$. It is clear that

$$\begin{aligned}
\mathcal{J}_{11}(x) &= \sum_{1\leqslant n\leqslant \mu x/\widehat{\mu}} \mathbb{P}\big(\widehat{S}_n - n\widehat{\mu} > -n\widehat{\mu} + \mu x + \varepsilon\mu x\big)\mathbb{P}(\nu = n) \\
&\leqslant \sum_{1\leqslant n\leqslant \mu x/\widehat{\mu}} \mathbb{P}\big(\widehat{S}_n - n\widehat{\mu} > \varepsilon\mu x\big)\mathbb{P}(\nu = n).
\end{aligned}$$

By using Lemma 3 for r.v.s $\{\widehat{\xi}_1 - \widehat{\mu}, \widehat{\xi}_2 - \widehat{\mu}, \ldots\}$ we obtain that

$$\begin{aligned}
\mathcal{J}_{11}(x) &\leqslant \sum_{1\leqslant n\leqslant \varepsilon\mu x/(\varepsilon\widehat{\mu})} \mathbb{P}\big(\widehat{S}_n - n\widehat{\mu} > \varepsilon\mu x\big)\mathbb{P}(\nu = n) \\
&\leqslant C_1\overline{F}_{\widehat{\xi}_1-\widehat{\mu}}(\varepsilon\mu x) \sum_{1\leqslant n\leqslant \mu x/\widehat{\mu}} n\mathbb{P}(\nu = n) \\
&\leqslant C_1\mathbb{E}\nu\overline{F}_{\widehat{\xi}_1}(\varepsilon\mu x)
\end{aligned}$$

for $x \geqslant \varepsilon\widehat{\mu}$ with some positive quantity $C_1 = C_1(\varepsilon, \widehat{\mu})$. The last estimate implies that

$$\limsup_{x\to\infty} \frac{\mathcal{J}_{11}(x)}{\overline{F}_\nu(x)} \leqslant C_1\mathbb{E}\nu \limsup_{x\to\infty} \frac{\overline{F}_{\widehat{\xi}_1}(\varepsilon\mu x)}{\overline{F}_{\widehat{\xi}_1}(x)} \limsup_{x\to\infty} \frac{\overline{F}_{\widehat{\xi}_1}(x)}{\overline{F}_{S_\nu}(x)} \limsup_{x\to\infty} \frac{\overline{F}_{S_\nu}(x)}{\overline{F}_\nu(x)} = 0 \qquad (12)$$

because

$$\limsup_{x\to\infty} \frac{\overline{F}_{\widehat{\xi}_1}(\varepsilon\mu x)}{\overline{F}_{\widehat{\xi}_1}(x)} < \infty$$

by the condition $F_{\widehat{\xi}_1} \in \mathscr{D}$,

$$\limsup_{x\to\infty} \frac{\overline{F}_{\widehat{\xi}_1}(x)}{\overline{F}_{S_\nu}(x)} = 0$$

by the construction of r.v.s $\{\widehat{\xi}_1, \widehat{\xi}_2, \ldots\}$, and

$$\limsup_{x\to\infty} \frac{\overline{F}_{S_\nu}(x)}{\overline{F}_\nu(x)} < \infty$$

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by the following estimate:

$$\begin{aligned}\overline{F}_{S_\nu}(x) &= \sum_{n=1}^{\infty} \mathbb{P}(S_n > x)\mathbb{P}(\nu = n)\\ &\leqslant \sum_{n>x} \mathbb{P}(S_n > x)\mathbb{P}(\nu = n)\\ &\leqslant \sum_{n>x} \mathbb{P}(\nu = n) = \overline{F}_\nu(x).\end{aligned}$$

Further, let us consider the term $\mathcal{J}_{12}(x)$. It is evident that

$$\begin{aligned}\mathcal{J}_{12}(x) &= \sum_{\mu x/\widehat{\mu} < n \leqslant x} \mathbb{P}(S_n > (1+\varepsilon)\nu x)\mathbb{P}(\nu = n)\\ &= \sum_{\mu x/\widehat{\mu} < n \leqslant x} \mathbb{P}(S_n - n\mu > -n\mu + \mu x + \varepsilon\mu x)\mathbb{P}(\nu = n)\\ &\leqslant \sum_{\mu x/\widehat{\mu} < n \leqslant x} \mathbb{P}(S_n - n\mu > \varepsilon\mu x)\mathbb{P}(\nu = n)\\ &\leqslant \sup_{n > \mu x/\widehat{\mu}} \mathbb{P}\left(\frac{S_n}{n} - \mu > \varepsilon\mu\right)\overline{F}_\nu\left(\frac{\mu x}{\widehat{\mu}}\right). \end{aligned} \tag{13}$$

By choosing $\varepsilon = \frac{1}{2}$ from decompositions (8) and (11) and estimates (9), (12), and (13) it follows that

$$\begin{aligned}\overline{F}_{S_\nu}\left(\frac{3}{2}\mu x\right) &= \mathcal{J}_{11}(x) + \mathcal{J}_{12}(x) + \mathcal{J}_2(x)\\ &\leqslant \frac{1}{2}\overline{F}_\nu(x) + \frac{1}{2}\overline{F}_\nu\left(\frac{\mu x}{\widehat{\mu}}\right) + 2\overline{F}_\nu(x)\\ &\leqslant 3\overline{F}_\nu\left(\frac{\mu x}{\widehat{\mu}}\right)\end{aligned}$$

for sufficiently large x, say $x \geqslant x_1$.

Meanwhile, from the estimate of Proposition 1 with the same $\varepsilon = \frac{1}{2}$ we have that

$$\overline{F}_{S_\nu}\left(\frac{\mu x}{2}\right) \geqslant \frac{1}{2}\overline{F}_\nu(x)$$

for sufficiently large x, say $x \geqslant x_2$. The last two inequalities imply $F_\nu \in \mathscr{D}$ because

$$\frac{\overline{F}_\nu\left(\frac{x}{2}\right)}{\overline{F}_\nu(x)} \leqslant \frac{6\overline{F}_{S_\nu}\left(\frac{\mu x}{4}\right)}{\overline{F}_{S_\nu}\left(\frac{3\widehat{\mu}x}{2}\right)}$$

for $x \geqslant \max\{\mu x_1/\widehat{\mu}, 2x_2\}$, and $F_{S_\nu} \in \mathscr{D}$ by the conditions of the proposition.

Now, since $F_\nu \in \mathscr{D}$, from estimate (6) we get

$$\limsup_{x\to\infty} \frac{\mathcal{J}_{12}(x)}{\overline{F}_\nu(x)} = 0$$

according to the law of large numbers. The last equality, together with decomposition (11) and equality (12), implies the desired relation (10). Proposition 2 is proved.

5.4. **Proof of Proposition 3**

According to decomposition (8), for all $x > 0$ and $\varepsilon > 0$, we have

$$\overline{F}_{S_\nu}((1+\varepsilon)\mu x) \leqslant \mathcal{J}_1(x) + \overline{F}_\nu(x) \tag{14}$$

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with

$$
\begin{aligned}
\mathcal{J}_1(x) &= \sum_{1\leqslant n\leqslant x} \mathbb{P}(S_n > (1+\varepsilon)\mu x)\mathbb{P}(\nu = n) \\
&\leqslant \mathbb{P}(S_{\lfloor x\rfloor} > (1+\varepsilon)\mu x) \\
&\leqslant \mathbb{P}\Big(\bigcup_{n\leqslant\lfloor x\rfloor} \{\xi_n > \varepsilon x\}\Big) + \mathbb{P}\Big(\bigcap_{n\leqslant\lfloor x\rfloor} \{\xi_n \leqslant \varepsilon x\},\, S_{\lfloor x\rfloor} > (1+\varepsilon)\mu x\Big) \\
&=: \mathcal{K}_1(x) + \mathcal{K}_2(x).
\end{aligned} \tag{15}
$$

Since $F_{S_\nu} \in \mathscr{D}$, the conditions of Proposition 3 imply that

$$
\begin{aligned}
\limsup_{x\to\infty} \frac{\mathcal{K}_1(x)}{\overline{F}_{S_\nu}((1+\varepsilon)\mu x)} &\leqslant \limsup_{x\to\infty} \frac{x\overline{F}_{\xi}(\varepsilon x)}{\overline{F}_{S_\nu}((1+\varepsilon)\mu x)} \\
&= \frac{1}{\varepsilon} \limsup_{x\to\infty} \frac{\varepsilon x\,\overline{F}_{\xi}(\varepsilon x)}{\overline{F}_{S_\nu}(\varepsilon x)} \limsup_{x\to\infty} \frac{\overline{F}_{S_\nu}(\varepsilon x)}{\overline{F}_{S_\nu}((1+\varepsilon)\mu x)} = 0.
\end{aligned} \tag{16}
$$

For the second term in (14), we have

$$
\begin{aligned}
\mathcal{K}_2(x) &= \mathbb{P}\Big(\bigcap_{n\leqslant\lfloor x\rfloor} \{\xi_n \leqslant \varepsilon x\},\, \sum_{n=1}^{\lfloor x\rfloor} \Big(\xi_n \mathbb{I}_{\{\xi_n\leqslant\varepsilon x\}} + \xi_n \mathbb{I}_{\xi_n\leqslant\varepsilon x}\Big) > (1+\varepsilon)\mu x\Big) \\
&= \mathbb{P}\Big(\bigcap_{n\leqslant\lfloor x\rfloor} \{\xi_n \leqslant \varepsilon x\},\, \sum_{n=1}^{\lfloor x\rfloor} \xi_n \mathbb{I}_{\{\xi_n\leqslant\varepsilon x\}} > (1+\varepsilon)\mu x\Big) \\
&\leqslant \mathbb{P}\Big(\sum_{n=1}^{\lfloor x\rfloor} \xi_n \mathbb{I}_{\{\xi_n\leqslant\varepsilon x\}} > (1+\varepsilon)\mu x\Big) \\
&= \mathbb{P}\Big(\sum_{n=1}^{\lfloor x\rfloor} \Big(\xi_n \mathbb{I}_{\{\xi_n\leqslant\varepsilon x\}} - \mathbb{E}\,\xi_n \mathbb{I}_{\{\xi_n\leqslant\varepsilon x\}}\Big) > (1+\varepsilon)\mu x - \sum_{n=1}^{\lfloor x\rfloor} \mathbb{E}\,\xi_n \mathbb{I}_{\{\xi_n\leqslant\varepsilon x\}}\Big) \\
&\leqslant \mathbb{P}\Big(\sum_{n=1}^{\lfloor x\rfloor} \Big(\xi_n \mathbb{I}_{\{\xi_n\leqslant\varepsilon x\}} - \mathbb{E}\,\xi_n \mathbb{I}_{\{\xi_n\leqslant\varepsilon x\}}\Big) > \varepsilon\mu x\Big)
\end{aligned}
$$

because

$$
\sum_{n=1}^{\lfloor x\rfloor} \mathbb{E}\,\xi_n \mathbb{I}_{\{\xi_n\leqslant\varepsilon x\}} \leqslant \mu x
$$

for all $\varepsilon > 0$.

The random variables

$$
Y_n := \xi_n \mathbb{I}_{\xi_n\leqslant\varepsilon x} - \mathbb{E}\xi_n \mathbb{I}_{\xi_n\leqslant\varepsilon x},\; n \in \mathbb{N}
$$

satisfy the conditions of Lemma 4, that is, the sequence $\{Y_1, Y_2, \ldots\}$ consists of i.i.d. r.v.s, $\mathbb{E}Y_1 = 0$, and

$$
\begin{aligned}
\mathbb{E}(Y_1^+)^r &= \mathbb{E}\Big(\Big(\xi \mathbb{I}_{\{\xi\leqslant\varepsilon x\}} - \mathbb{E}\xi \mathbb{I}_{\{\xi\leqslant\varepsilon x\}}\Big)^+\Big)^r \\
&\leqslant \mathbb{E}\Big(\big(\xi \mathbb{I}_{\{\xi\leqslant\varepsilon x\}}\big)^+\Big)^r \\
&\leqslant \mathbb{E}\,\xi^r < \infty
\end{aligned}
$$

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for $r > 1$ by the conditions of Proposition 3. By using Lemma 4 with $\gamma = \varepsilon\mu/2$ and $p > J^{+}_{F_{S_\nu}}$ we get that

$$\mathcal{K}_2(x) \leqslant \mathbb{P}\bigg(\sum_{n=1}^{\lfloor x \rfloor} Y_n > \varepsilon\mu x\bigg) \leqslant \lfloor x \rfloor \overline{F}_{Y_1}(a\varepsilon\mu x) + \frac{C_2}{(\varepsilon\mu)^p x^p} \tag{17}$$

with constants a and C_2 depending on $\mathbb{E}(Y_1^+)^r$, ε, and p, but independent of x. By the definition of r.v. Y_1, for $x > 0$, we have

$$\begin{aligned}\overline{F}_{Y_1}(a\varepsilon\mu x) &= \mathbb{P}(Y_1 > a\varepsilon\mu x) = \mathbb{P}\big(\xi\mathbb{I}_{\{\xi\leqslant\varepsilon x\}} > a\varepsilon\mu x + \mathbb{E}\xi\mathbb{I}_{\{\xi\leqslant\varepsilon x\}}\big)\\ &\leqslant \mathbb{P}\big(\xi\mathbb{I}_{\{\xi\leqslant\varepsilon x\}} > a\varepsilon\mu x, \xi \leqslant \varepsilon x\big) + \mathbb{P}\big(\xi\mathbb{I}_{\{\xi\leqslant\varepsilon x\}} > a\varepsilon\mu x, \xi > \varepsilon x\big)\\ &\leqslant \mathbb{P}\big(\xi > a\varepsilon\mu x\big)\\ &= \overline{F}_{\xi}(a\varepsilon\mu x).\end{aligned}$$

Therefore by estimate (17), the conditions $x\overline{F}_{\xi}(x) = o\big(\overline{F}_{S_\nu}(x)\big)$ and $F_{S_\nu} \in \mathscr{D}$, and Lemma 5 we derive

$$\begin{aligned}\limsup_{x\to\infty}\frac{\mathcal{K}_2(x)}{\overline{F}_{S_\nu}((1+\varepsilon)\mu x)} &\leqslant \frac{1}{a\varepsilon\mu}\limsup_{x\to\infty}\frac{a\varepsilon\mu x\overline{F}_{\xi}(a\varepsilon\mu x)}{\overline{F}_{S_\nu}(a\varepsilon\mu x)}\limsup_{x\to\infty}\frac{\overline{F}_{S_\nu}(a\varepsilon\mu x)}{\overline{F}_{S_\nu}((1+\varepsilon)\mu x)}\\ &+ \frac{C_2}{(\varepsilon\mu)^p}\limsup_{x\to\infty}\frac{\overline{F}_{S_\nu}(x)}{\overline{F}_{S_\nu}((1+\varepsilon)\mu x)}\limsup_{x\to\infty}\frac{1}{x^p\overline{F}_{S_\nu}(x)} = 0.\end{aligned} \tag{18}$$

Now let $\delta \in (0,1)$. Decompositions (14) and (15) and limiting relations (16) and (17) imply that

$$\overline{F}_{S_\nu}((1+\varepsilon)\mu x) \leqslant \delta\overline{F}_{S_\nu}((1+\varepsilon)\mu x) + \overline{F}_\nu(x) \tag{19}$$

for sufficiently large x, say $x \geqslant x_3(\varepsilon,\delta)$. In the case $\varepsilon = \delta = 1/2$, we get

$$\frac{1}{2}\overline{F}_{S_\nu}\Big(\frac{3}{2}\mu x\Big) \leqslant \overline{F}_\nu(x)$$

for $x \geqslant x_3(1/2,1/2)$. Meanwhile, Proposition 1 implies

$$\overline{F}_{S_\nu}\Big(\frac{1}{2}\mu x\Big) \geqslant \frac{1}{2}\overline{F}_\nu(x)$$

for large x, say $x \geqslant x_4$. The last two estimates show that

$$\limsup_{x\to\infty}\frac{\overline{F}_\nu\big(\frac{x}{2}\big)}{\overline{F}_\nu(x)} \leqslant \limsup_{x\to\infty}\frac{4\overline{F}_{S_\nu}\big(\frac{\mu x}{2}\big)}{\overline{F}_{S_\nu}\big(\frac{3\mu x}{2}\big)} < \infty$$

implying that $F_\nu \in \mathscr{D}$ because $F_{S_\nu} \in \mathscr{D}$ by the conditions of the proposition.

To finish the proof, we observe that by inequality (19)

$$\limsup_{x\to\infty}\frac{\overline{F}_{S_\nu}((1+\varepsilon)\mu x)}{\overline{F}_\nu(x)} \leqslant \frac{1}{1-\delta}$$

for all $\varepsilon \in (0,1)$ and $\delta \in (0,1)$. Hence

$$\limsup_{x\to\infty}\frac{\overline{F}_{S_\nu}((1+\varepsilon)\mu x)}{\overline{F}_\nu(x)} \leqslant 1$$

due to the arbitrary choice of parameter $\delta \in (0,1)$. This finishes the proof of the proposition.

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5.5. Proof of Theorems 3 and 4

Let d.f. $F_{S\nu} \in \mathscr{ERV}_{\{\alpha,\beta\}}$, and suppose that $\mu = \mathbb{E}\xi > 1$. In such a case, $(1-\varepsilon)\mu > 1$ if $\varepsilon \in (0,1)$ is sufficiently small. Since $\mathscr{ERV} \subset \mathscr{D}$, we have that all conditions of Propositions 1 and 2 are satisfied. According to Proposition 1,

$$\limsup_{x\to\infty} \frac{\overline{F}_\nu(x)}{\overline{F}_{S_\nu}((1-\varepsilon)\mu x)} \leqslant 1,$$

and the condition $F_{S_\nu} \in \mathscr{ERV}_{\{\alpha,\beta\}}$ implies that

$$\limsup_{x\to\infty} \frac{\overline{F}_{S_\nu}((1-\varepsilon)\mu x)}{\overline{F}_{S_\nu}(x)} \leqslant ((1-\varepsilon)\mu)^{-\alpha}$$

for $\varepsilon \in (0, 1-1/\mu)$. Therefore, for such ε,

$$\limsup_{x\to\infty} \frac{\mu^\alpha \overline{F}_\nu(x)}{\overline{F}_{S_\nu}(x)} \leqslant \mu^\alpha \limsup_{x\to\infty} \frac{\overline{F}_\nu(x)}{\overline{F}_{S_\nu}((1-\varepsilon)\mu x)} \limsup_{x\to\infty} \frac{\overline{F}_{S_\nu}((1-\varepsilon)\mu x)}{\overline{F}_{S_\nu}(x)} \leqslant (1-\varepsilon)^{-\alpha},$$

implying that

$$\mu^\alpha \overline{F}_\nu(x) \underset{x\to\infty}{\lesssim} \overline{F}_{S_\nu}(x). \tag{20}$$

Due to Proposition 2,

$$\limsup_{x\to\infty} \frac{\overline{F}_{S_\nu}((1+\varepsilon)\mu x)}{\overline{F}_\nu(x)} \leqslant 1$$

for $\varepsilon > 0$, and the condition $F_{S_\nu} \in \mathscr{ERV}_{\{\alpha,\beta\}}$ implies that

$$\limsup_{x\to\infty} \frac{\overline{F}_{S_\nu}(x)}{\overline{F}_{S_\nu}((1+\varepsilon)\mu x)} = \left(\liminf_{x\to\infty} \frac{\overline{F}_{S_\nu}((1+\varepsilon)\mu x)}{\overline{F}_{S_\nu}(x)}\right)^{-1} \leqslant ((1+\varepsilon)\mu)^\beta.$$

Therefore

$$\limsup_{x\to\infty} \frac{\overline{F}_{S_\nu}(x)}{\mu^\beta \overline{F}_\nu(x)} \leqslant \frac{1}{\mu^\beta} \limsup_{x\to\infty} \frac{\overline{F}_{S_\nu}(x)}{\overline{F}_{S_\nu}((1+\varepsilon)\mu x)} \limsup_{x\to\infty} \frac{\overline{F}_{S_\nu}((1+\varepsilon)\mu x)}{\overline{F}_\nu(x)} \leqslant (1+\varepsilon)^\beta,$$

implying that

$$\overline{F}_{S_\nu}(x) \underset{x\to\infty}{\lesssim} \mu^\beta \overline{F}_\nu(x). \tag{21}$$

Relations (20) and (21) imply the first asymptotic inequality (2) of Theorem 3.

Now let again d.f. $F_{S\nu} \in \mathscr{ERV}_{\{\alpha,\beta\}}$, but let $\mu = \mathbb{E}\xi < 1$. In such a case, $1/(1+\varepsilon)\mu > 1$ if $\varepsilon \in (0,1)$ is sufficiently small. From the inclusion $\mathscr{ERV} \subset \mathscr{D}$ we have that the conditions of Propositions 1 and 2 are satisfied. According to Proposition 2,

$$\limsup_{x\to\infty} \frac{\overline{F}_{S_\nu}((1+\varepsilon)\mu x)}{\overline{F}_\nu(x)} \leqslant 1,$$

and the condition $F \in \mathscr{ERV}_{\alpha,\beta\}}$ implies that

$$\limsup_{x\to\infty} \frac{\overline{F}_{S_\nu}(x)}{\overline{F}_{S_\nu}((1+\varepsilon)\mu x)} = \limsup_{z\to\infty} \frac{\overline{F}_{S_\nu}\left(\frac{z}{(1+\varepsilon)\mu}\right)}{\overline{F}_{S_\nu}(z)} \leqslant ((1+\varepsilon)\mu)^{-\alpha}$$

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if $\varepsilon \in \left(0, \frac{1}{\mu} - 1\right)$. Consequently,

$$\limsup_{x\to\infty} \frac{\overline{F}_{S_\nu}(x)}{\mu^\alpha \overline{F}_\nu(x)} \leqslant \frac{1}{\mu^\alpha} \limsup_{x\to\infty} \frac{\overline{F}_{S_\nu}((1+\varepsilon)\mu x)}{\overline{F}_\nu(x)} \limsup_{x\to\infty} \frac{\overline{F}_{S_\nu}(x)}{\overline{F}_{S_\nu}((1+\varepsilon)\mu x)} \leqslant (1+\varepsilon)^{-\alpha},$$

and

$$\overline{F}_{S_\nu}(x) \underset{x\to\infty}{\lesssim} \mu^\alpha \overline{F}_\nu(x) \tag{22}$$

due to the arbitrariness of $\varepsilon \in \left(0, \frac{1}{\mu} - 1\right)$.

If $\mu < 1$, then $1/(1-\varepsilon)\mu > 1$ for all $\varepsilon \in (0,1)$. Hence the condition $F \in \mathscr{ERV}_{\{\alpha,\beta\}}$ implies that

$$\limsup_{x\to\infty} \frac{\overline{F}_{S_\nu}((1-\varepsilon)\mu x)}{\overline{F}_{S_\nu}(x)} = \limsup_{z\to\infty} \frac{\overline{F}_{S_\nu}(z)}{\overline{F}_{S_\nu}\left(\frac{z}{(1-\varepsilon)\mu}\right)} = \left(\limsup_{z\to\infty} \frac{\overline{F}_{S_\nu}\left(\frac{z}{(1-\varepsilon)\mu}\right)}{\overline{F}_{S_\nu}(z)}\right)^{-1} \leqslant ((1-\varepsilon)\mu)^{-\beta}.$$

By Proposition 1 we get that

$$\limsup_{x\to\infty} \frac{\mu^\beta \overline{F}_\nu(x)}{\overline{F}_{S_\nu}(x)} \leqslant \mu^\beta \limsup_{x\to\infty} \frac{\overline{F}_\nu(x)}{\overline{F}_{S_\nu}((1-\varepsilon)x)} \limsup_{x\to\infty} \frac{\overline{F}_{S_\nu}((1-\varepsilon)x)}{\overline{F}_{S_\nu}(x)} \leqslant (1-\varepsilon)^{-\beta},$$

which implies that

$$\mu^\beta \overline{F}_\nu(x) \underset{x\to\infty}{\lesssim} \overline{F}_{S_\nu}(x). \tag{23}$$

The asymptotic inequalities (22) and (23) imply the second relation (3) of the theorem.

Let us consider the last case $\mu = \mathbb{E}\xi = 1$. Since $F_{S_\nu} \in \mathscr{ERV} \subset \mathscr{D}$ and $\mu = 1$, Propositions 1 and 2 imply that

$$\overline{F}_{S_\nu}((1+\varepsilon)x) \underset{x\to\infty}{\lesssim} \overline{F}_\nu(x) \underset{x\to\infty}{\lesssim} \overline{F}_{S_\nu}((1-\varepsilon)x)$$

for all $\varepsilon \in (0,1)$. By the definition of the class $\mathscr{ERV}_{\{\alpha,\beta\}}$ we have

$$\overline{F}_{S_\nu}((1+\varepsilon)x) \underset{x\to\infty}{\gtrsim} (1+\varepsilon)^{-\beta} \overline{F}_{S_\nu}(x)$$

and

$$\overline{F}_{S_\nu}((1-\varepsilon)x) \underset{x\to\infty}{\lesssim} (1-\varepsilon)^{-\beta} \overline{F}_{S_\nu}(x).$$

Consequently, for all $\varepsilon \in (0,1)$,

$$(1+\varepsilon)^{-\beta} \overline{F}_{S_\nu}(x) \underset{x\to\infty}{\lesssim} \overline{F}_\nu(x) \underset{x\to\infty}{\lesssim} (1-\varepsilon)^{-\beta} \overline{F}_{S_\nu}(x),$$

which implies relation (4). Theorem 3 is proved.

Note that the proof of Theorem 4 is completely analogous to that of Theorem 3. We only need to refer to Proposition 3 instead of Proposition 2 because the conditions of Theorem 4 are consistent with those of Proposition 3.

6. Illustrating Example

Example 1. *Let us consider the following model. The sequence of r.v.s $\{\xi_1, \xi_2, \ldots\}$ consists of the independent copies of r.v. ξ distributed according to the exponential law, i.e.*

$$\overline{F}_\xi(x) = \mathbb{P}(\xi > x) = \mathrm{e}^{-x},\ x \geqslant 0.$$

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The counting r.v. ν independent of the sequence $\{\xi_1, \xi_2, \ldots\}$ is distributed according to the zeta law, i.e.

$$\mathbb{P}(\nu = n) = \frac{1}{\zeta(2)} \frac{1}{n^2},\ n \in \{1, 2, \ldots\},$$

where

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

In the case under consideration, we have the following:

$$\begin{aligned}
&\mathbb{E}\xi = 1,\\
&\mathbb{E}\xi^r < \infty \text{ for any } r > 1,\\
&\mathbb{E}\nu = \infty,\\
\limsup_{x\to\infty} \frac{x\overline{F}_\xi(x)}{\overline{F}_{S_\nu}(x)} = \limsup_{x\to\infty} &\frac{x\mathrm{e}^{-x}}{\frac{1}{\zeta(2)}\sum_{n=1}^{\infty}\frac{1}{n^2}\overline{F_\xi^{*n}}(x)} \leqslant \limsup_{x\to\infty} \frac{9\zeta(2)x\mathrm{e}^{-x}}{(1 + x + \frac{x^2}{2!})\mathrm{e}^{-x}} = 0.
\end{aligned}$$

In addition, for positive x

$$\begin{aligned}
\overline{F}_{S_\nu}(x) &= \frac{1}{\zeta(2)} \sum_{n=1}^{\infty} \frac{1}{n^2} \overline{F_\xi^{*n}}(x)\\
&= \frac{1}{\zeta(2)} \sum_{n=1}^{\infty} \frac{1}{n^2} \mathrm{e}^{-x} \sum_{k=0}^{n-1} \frac{x^k}{k!}\\
&= \frac{\mathrm{e}^{-x}}{\zeta(2)} \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{n=k+1}^{\infty} \frac{1}{n^2}.
\end{aligned}$$

Since

$$\frac{1}{k+1} \leqslant \sum_{n=k+1}^{\infty} \frac{1}{n^2} \leqslant \frac{1}{k}$$

for any $k \geqslant 1$, we derive that

$$\begin{aligned}
\overline{F}_{S_\nu}(x) &= \frac{\mathrm{e}^{-x}}{\zeta(2)} \left(\sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{k=1}^{\infty} \frac{x^k}{k!} \sum_{n=k+1}^{\infty} \frac{1}{n^2} \right)\\
&\geqslant \frac{\mathrm{e}^{-x}}{\zeta(2)} \left(1 + \sum_{k=1}^{\infty} \frac{x^k}{(k+1)!} \right)\\
&= \frac{\mathrm{e}^{-x}}{\zeta(2)} \sum_{k=0}^{\infty} \frac{x^k}{(k+1)!}\\
&= \frac{\mathrm{e}^{-x}}{\zeta(2)} \frac{\mathrm{e}^{x} - 1}{x} \underset{x\to\infty}{\gtrsim} \frac{1}{\zeta(2)x},
\end{aligned}$$

and

$$\begin{aligned}
\overline{F}_{S_\nu}(x) &\leqslant \mathrm{e}^{-x} + \frac{\mathrm{e}^{-x}}{\zeta(2)} \sum_{k=1}^{\infty} \frac{x^k}{k!k}\\
&= \mathrm{e}^{-x} + \frac{\mathrm{e}^{-x}}{\zeta(2)} \left(\sum_{k=1}^{\infty} \frac{x^k}{(k+1)!} + \sum_{k=1}^{\infty} \frac{x^k}{k(k+1)!} \right)\\
&\leqslant \mathrm{e}^{-x} + \frac{\mathrm{e}^{-x}}{\zeta(2)} \left(\frac{\mathrm{e}^{x} - 1}{x} + \sum_{k=1}^{\infty} \frac{x^k}{k(k+1)!} \right).
\end{aligned}$$

The last estimate implies that

$$\overline{F}_{S_\nu}(x) \underset{x\to\infty}{\lesssim} \frac{1}{\zeta(2)x}$$

because

$$\begin{aligned}\frac{\mathrm{e}^{-x}\sum_{k=1}^{\infty}\frac{x^k}{k(k+1)!}}{\frac{1}{x}} &= x\mathrm{e}^{-x}\left(\sum_{k=1}^{K}\frac{x^k}{k(k+1)!}+\sum_{k=K+1}^{\infty}\frac{x^k}{k(k+1)!}\right)\\ &\leqslant x^{K+1}\mathrm{e}^{-x}\sum_{k=1}^{K}\frac{1}{k(k+1)!}+x\mathrm{e}^{-x}\frac{\mathrm{e}^x-1}{x}\frac{1}{K}\end{aligned}$$

for all $x > 0$ and $K \geqslant 2$.

We can see from the obtained relations that

$$\overline{F}_{S_\nu}(x) \underset{x\to\infty}{\sim} \frac{1}{\zeta(2)x} \underset{x\to\infty}{\sim} \mathbb{E}\xi\,\overline{F}_\nu(x)$$

which is consistent with assertions of theorems 2 and 4.

7. Concluding Remarks

In this paper, we generalize Proposition 4.9 from [10], where cases are found where a randomly stopped sum belonging to the class of regular distributions induces the regularity of the counting random variable together with an asymptotic formula of a special form. We have shown the transfer of regularity from a randomly stopped sum to a counting random variable for a broader class of distributions $\mathscr{D}$. For this class, we have also obtained asymptotic formulas of some special form relating the tail of a randomly stopped sum to the tail of a counting random variable. In our formulas we incorporate an additional free parameter. For the class of distributions $\mathscr{ERV}$, we derive more precise formulas relating the tail of a randomly stopped sum to the tail of a counting random variable and the mean of the primary random variable generator. This class of distributions is intermediate between the regular distributions considered in [10] and the class $\mathscr{D}$ of dominatedly varying distributions. Similar "inverse" problems for other transformations of distributions are considered in [41,50–58].

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