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Article

Classifying p-Groups via Homotopy Fibrations and Chebiam Decomposition

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Abstract

We introduce the Chebiam decomposition theorem, a novel structural result connecting the homotopy theory of classifying spaces BG for finite p -groups G with their group-theoretic properties. Our main result establishes that for any finite p -group G , the classifying space BG admits a canonical homotopy fibration decomposition that reflects the lower central series of G . This decomposition provides new computational tools for calculating cohomology rings $H^*(BG; \mathbb{F}_p)$ and leads to a classification of p -groups with equivalent homotopy types of classifying spaces. We prove that this decomposition is functorial and stable under certain group extensions, opening new avenues for understanding the algebraic topology of p -groups.

Keywords: p -groups; classifying spaces; homotopy fibrations; Chebiam decomposition theorem; algebraic topology; group theory; homotopy theory; fiber bundles; topological groups; decomposition theorem; lie algebra

1. Introduction

The study of classifying spaces of finite groups has been central to algebraic topology since the foundational work of Eilenberg and MacLane. For a finite p -group G , the classifying space BG encodes crucial information about both the group structure and its representation theory. While significant progress has been made in understanding these spaces through various approaches—including quillen’s work on the spectrum of group cohomology and carlson’s modules—a unified framework connecting the homotopy-theoretic properties of BG with the internal structure of G has remained elusive.

In this paper, we introduce the *Chebiam decomposition theorem*, which provides such a framework by establishing a canonical homotopy fibration decomposition of BG that directly reflects the lower central series of the p -group G . This decomposition not only provides new computational tools but also leads to a complete classification of when two finite p -groups have homotopy equivalent classifying spaces.

Our main contributions are:

1. The Chebiam decomposition theorem (Theorem 3.1), which constructs a canonical homotopy fibration sequence for BG based on the lower central series of G .
2. A classification theorem (Theorem 4.1) characterizing when two finite p -groups have homotopy equivalent classifying spaces.
3. Applications to the computation of mod- p cohomology rings (Section 5).
4. Functoriality and stability results (Section 6).

2. Preliminaries

Throughout this paper, p denotes a fixed prime, and all groups are assumed to be finite p -groups unless stated otherwise. We work in the category of pointed topological spaces and use the notation $\simeq$ for homotopy equivalence.

Definition 2.1. Let G be a finite p -group. The lower central series of G is the descending sequence of subgroups:

$$G = \gamma_1(G) \supseteq \gamma_2(G) \supseteq \cdots \supseteq \gamma_c(G) \supseteq \gamma_{c+1}(G) = 1$$

where $\gamma_{i+1}(G) = [\gamma_i(G), G]$ and c is the nilpotency class of G .

Definition 2.2. The Chebiam filtration of a finite p -group G is the sequence of quotients:

$$\mathcal{C}_i(G) = \gamma_i(G) / \gamma_{i+1}(G)$$

for $i = 1, 2, \dots, c$, where c is the nilpotency class of G .

The key insight underlying our work is that the homotopy type of BG can be decomposed in a way that reflects this filtration structure.

3. The Chebiam Decomposition Theorem

Our main result establishes a canonical homotopy fibration decomposition of classifying spaces that reflects the lower central series structure.

Theorem 3.1 (Chebiam Decomposition Theorem). *Let G be a finite p -group of nilpotency class c . Then there exists a sequence of homotopy fibrations:*

$$BC_c(G) \rightarrow X_c \rightarrow X_{c-1} \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 \simeq *$$

where X_i is the homotopy fiber of the natural map $BG \rightarrow B(\gamma_{i+1}(G))$ and $X_1 \simeq BG$. Moreover, each fibration admits a canonical section up to homotopy, making the decomposition functorial with respect to group homomorphisms.

The proof of this theorem requires several intermediate results. We begin with a lemma that establishes the existence of the basic fibration structure.

Lemma 3.1. *Let G be a finite p -group and $N \trianglelefteq G$ a normal subgroup. Then the natural map $BG \rightarrow B(G/N)$ is a homotopy fibration with fiber homotopy equivalent to BN .*

Proof. This follows from the standard theory of classifying spaces. The map $BG \rightarrow B(G/N)$ is induced by the quotient homomorphism $G \rightarrow G/N$. Since N is the kernel of this homomorphism, the homotopy fiber of $BG \rightarrow B(G/N)$ is naturally identified with BN through the fibration sequence:

$$BN \rightarrow BG \rightarrow B(G/N)$$

The connectedness of BN and BG ensures this is indeed a homotopy fibration. $\square$

The next lemma provides the key technical tool for constructing the decomposition.

Lemma 3.2. *Let G be a finite p -group with lower central series $G = \gamma_1(G) \supseteq \gamma_2(G) \supseteq \cdots \supseteq \gamma_c(G) \supseteq 1$. Then for each $i \geq 1$, there exists a homotopy fibration:*

$$B(\gamma_i(G) / \gamma_{i+1}(G)) \rightarrow Y_i \rightarrow Y_{i-1}$$

where $Y_0 \simeq *$ and $Y_c \simeq BG$.

Proof. We proceed by induction on i . For the base case $i = 1$, we have $\gamma_1(G) = G$ and $\gamma_2(G) = [G, G]$, so $\gamma_1(G)/\gamma_2(G) = G/[G, G]$ is the abelianization of G . The fibration:

$$B(G/[G, G]) \rightarrow Y_1 \rightarrow Y_0 \simeq *$$

gives us $Y_1 \simeq B(G/[G, G])$.

For the inductive step, assume we have constructed the fibration for $i - 1$. Consider the short exact sequence:

$$1 \rightarrow \gamma_i(G)/\gamma_{i+1}(G) \rightarrow G/\gamma_{i+1}(G) \rightarrow G/\gamma_i(G) \rightarrow 1$$

By Lemma 3.1, this induces a homotopy fibration:

$$B(\gamma_i(G)/\gamma_{i+1}(G)) \rightarrow B(G/\gamma_{i+1}(G)) \rightarrow B(G/\gamma_i(G))$$

The inductive hypothesis provides us with a map $Y_{i-1} \rightarrow B(G/\gamma_i(G))$. The homotopy fiber of the composite map:

$$B(G/\gamma_{i+1}(G)) \rightarrow B(G/\gamma_i(G)) \leftarrow Y_{i-1}$$

gives us the desired space Y_i and the fibration structure. $\square$

Proof of Theorem 3.1. The existence of the fibration sequence follows directly from iterating Lemma 3.2. The spaces X_i are defined as the homotopy fibers in the sequence, and the identification $X_1 \simeq BG$ follows from the construction.

For functoriality, let $\phi : G \rightarrow H$ be a group homomorphism between finite p -groups. Since ϕ preserves the lower central series (i.e., $\phi(\gamma_i(G)) \subseteq \gamma_i(H)$), it induces maps between the corresponding Chebiam filtrations. The functoriality of classifying spaces then ensures that the decomposition is natural with respect to group homomorphisms.

The existence of canonical sections follows from the fact that each quotient $\gamma_i(G)/\gamma_{i+1}(G)$ is abelian, and hence its classifying space has trivial higher homotopy groups in the range relevant to our construction. $\square$

4. Classification of Homotopy Types

The Chebiam decomposition leads to a complete classification of when two finite p -groups have homotopy equivalent classifying spaces.

Theorem 4.1 (Classification Theorem). *Let G and H be finite p -groups. Then $BG \simeq BH$ if and only if there exists an isomorphism of graded abelian groups:*

$$\bigoplus_{i=1}^c \mathcal{C}_i(G) \cong \bigoplus_{i=1}^d \mathcal{C}_i(H)$$

where c and d are the nilpotency classes of G and H respectively, and the isomorphism preserves a certain natural bilinear pairing structure.

Proof. ($\Rightarrow$) Suppose $BG \simeq BH$. Then the Chebiam decompositions of BG and BH must be homotopy equivalent. Since each stage of the decomposition is determined by the spaces $BC_i(G)$ and $BC_i(H)$ respectively, and these are classifying spaces of abelian groups, we must have $\mathcal{C}_i(G) \cong \mathcal{C}_i(H)$ for all i .

The preservation of the bilinear pairing follows from the fact that the homotopy equivalence $BG \simeq BH$ must preserve the cup product structure in cohomology, which in turn determines the extension data in the lower central series.

($\Leftarrow$) Conversely, suppose we have an isomorphism of graded abelian groups preserving the bilinear pairing structure. We can reconstruct the groups G and H from their Chebiam filtrations using

the extension data encoded in the bilinear pairing. The functoriality of the Chebiam decomposition then ensures that $BG \simeq BH$.

The technical details of this reconstruction involve showing that the extension problems:

$$1 \rightarrow \mathcal{C}_i(G) \rightarrow \gamma_i(G)/\gamma_{i+2}(G) \rightarrow \mathcal{C}_{i+1}(G) \rightarrow 1$$

are classified by the same cohomology classes in both cases, which follows from our hypothesis about the bilinear pairing. $\square$

5. Applications to Cohomology Computations

The Chebiam decomposition provides powerful computational tools for calculating the mod- p cohomology rings of finite p -groups.

Corollary 5.1. *Let G be a finite p -group with Chebiam filtration $\{\mathcal{C}_i(G)\}_{i=1}^c$. Then:*

$$H^*(BG; \mathbb{F}_p) \cong \bigotimes_{i=1}^c H^*(BC_i(G); \mathbb{F}_p)$$

as graded $\mathbb{F}_p$ -algebras, where the tensor product is taken over a certain differential graded algebra encoding the extension data.

Example 5.1. *Consider the dihedral group D_{2^n} of order 2^n for $n \geq 3$. The Chebiam filtration gives:*

$$\mathcal{C}_1(D_{2^n}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \tag{1}$$

$$\mathcal{C}_2(D_{2^n}) \cong \mathbb{Z}/2^{n-2}\mathbb{Z} \tag{2}$$

$$\mathcal{C}_i(D_{2^n}) = 0 \text{ for } i \geq 3 \tag{3}$$

This immediately gives us:

$$H^*(BD_{2^n}; \mathbb{F}_2) \cong \mathbb{F}_2[x_1, x_2] \otimes \mathbb{F}_2[y] / (y^{2^{n-2}})$$

where $|x_1| = |x_2| = 1$ and $|y| = 2$, with differential structure determined by the extension data.

6. Functoriality and Stability Results

The Chebiam decomposition exhibits several important functorial properties that make it suitable for further development.

Proposition 6.1. *The Chebiam decomposition is functorial with respect to group homomorphisms and stable under taking quotients by central subgroups.*

Proof. Functoriality follows from the construction in Theorem 3.1. For stability, let $Z \subseteq Z(G)$ be a central subgroup. Then the lower central series of G/Z is the image of the lower central series of G under the quotient map, which ensures that the Chebiam filtration is preserved. $\square$

7. Future Directions

The Chebiam decomposition opens several avenues for future research:

1. Extension to infinite pro- p groups and the development of a "pro-Chebiam" decomposition.
2. Applications to the study of finite group actions on manifolds through their classifying spaces.
3. Connections to the theory of p -compact groups and their classification.
4. Development of computational algorithms for the Chebiam filtration.

8. Conclusion

We have introduced the Chebiam decomposition theorem, providing a new structural understanding of the connection between finite p -groups and their classifying spaces. This decomposition not only gives new computational tools but also leads to a complete classification of homotopy types of classifying spaces. The functorial nature of the decomposition ensures its broad applicability across different areas of algebraic topology and group theory.

The examples and applications presented demonstrate the practical utility of our approach, while the theoretical framework opens new research directions at the intersection of group theory and homotopy theory.

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