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Article

Equation of State Parameters for Stringy Particles in Cosmology with Dark Energy

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Abstract

We construct the strong energy conditions (SECs) for both massive and massless stringy particles in the higher dimensional cosmology (HDC) with dark energy Λ . Exploiting these conditions we find the equation of state (EOS) parameters $w \geq -(D-4)/D$ for both the massive and massless stringy particles in D dimensional cosmology. We elucidate the relations between the EOS parameter in the HDC with dark energy and that of Hawking-Penrose limit for the massive and massless point particles in the $D = 3 + 1$ dimensions. We evaluate the EOS parameters in terms of the contributions from the point particle property, dark energy, and extended object degrees of freedom, respectively. We also investigate the weak energy condition (WEC) and dominant energy condition for the massive and massless stringy particles in the HDC, and those for the massive and massless point particles in the $D = 3 + 1$ dimensions, respectively. We show that the Dark Energy Spectroscopic Instrument (DESI) data satisfies the WEC in the HDC. The high temperature asymptotic behavior of a dilaton-Einstein-Gauss-Bonnet scenario is also shown to be consistent with the corresponding WECs obtained in the HDC.

Keywords: cosmology with dark energy; energy conditions; equation of state parameter; DESI; dilaton-Einstein-Gauss-Bonnet

1. Introduction

The observational data for the accelerating expansion of the universe has suggested a positive vacuum expectation value of the dark energy (or cosmological constant) [1]. In the inflationary cosmology, it is believed that, after the Big Bang explosion, the radiation dominated phase occurred, followed by the matter dominated one, even though there was a hot thermalization period of radiation and matter immediately after the Big Bang. In the inflationary cosmology, the equation of state (EOS) of the massive particle is different from that of the massless particle, and thus the massive particle phase is not the same as the massless particle one. As a result, a phase transition exists between massive particle and massless particle phases in the universe. The inflationary cosmology is based heavily on the Hawking-Penrose singularity theorem [2] which states that once collapse approaches a certain point, evolution to a singularity is unavoidable. The singularity theorem thus implies that there was a singularity at the beginning of our universe which is believed to have expanded. In the inflationary cosmology, the expansion of the universe from a single point, namely the Big Bang is assumed to have occurred at this singularity. Recently there have been attempts to measure the EOS parameter by exploiting the Dark Energy Spectroscopic Instrument (DESI) [3,4]. One of the main purposes of the DESI is to measure the history of the universe expansion. To do this, the DESI has exploited the multi-object spectroscopy. These observational efforts are closely related with the corresponding theoretical predictions in our higher dimensional cosmology (HDC).

On the other hand, in the $D = 3 + 1$ dimensional point particle cosmology, a perfect fluid [5–9] is introduced to describe a continuous distribution of matter with energy-momentum tensor T_{ab} in terms

of the mass-energy density and the pressure. This fluid is called perfect because of the absence of heat conduction terms and stress terms corresponding to viscosity [6]. The D dimensional perfect fluid property has been generalized to the HDC without dark energy [8,9]. In the HDC, the stringy universe evolves without any phase transition because there is only one EOS both for the massless and massive particle eras. Note that in the point particle inflationary cosmology we have two EOSs for the massless and massive particle eras, respectively. In the HDC, as the early universe evolves with the expansion rate, this rate increases and the twist of the stringy congruence decreases exponentially, and the initial twist should be extremely large enough to support the whole rotation of the ensuing universe. The effect of the shear has been shown to be negligible and thus the universe is isotropic and homogeneous [8,9].

Since Hubble discovered the expansion of our universe, the inflationary Big Bang cosmology has been developed into a precision science by cosmological observations including supernova data [1] and measurements of cosmic microwave background radiation [10]. These observations triggered an explosion of recent interest in the origin of dark energy [11]. As far as increasing Hubble constant H_0 is concerned, the late universe may not be able to accommodate H_0 , baryon acoustic oscillation (BAO) and dark energy described by effective field theory [12]. Recently, the holographic Friedman-Robertson-Walker (FRW) universe and the ensuing observational constraints have been investigated on the $D = 3 + 1$ dimensional membrane embedded in the $D = 4 + 1$ dimensional bulk spacetime [13–15]. In the HDC without background of dark energy, the singularities in geodesic surface congruence for the time-like and null stringy particles have been investigated to yield the Raychaudhuri type equations possessing correction terms related with the characteristics owing to the stringy particles [8,9]. Assuming the stringy strong energy condition (SEC) in the HDC without the dark energy, the Hawking-Penrose type EOS inequality equations have been obtained. To be specific, the stringy SECs of both the time-like and null string congruences produce the same EOS inequality in the HDC without the dark energy. This EOS inequality has been shown to be not equivalent to the Hawking-Penrose EOS inequality equations in the $D = 3 + 1$ dimensional FRW inflationary cosmology [8,9]. Recently the null energy condition¹ has been applied to the models of dynamical black holes [17,18]. In the case of dark energy, the smeared null energy condition has been shown to be consistent with the recent constraints from the DESI [18,19]. Moreover, the violation of the null energy condition [18–21] may play an important role in the early universe, and the null energy condition violation during inflation and pulsar timing array (PTA) observations has been studied [21]. Next, exploiting black hole solutions, whether the third law of thermodynamics of the black hole is a result of the weak energy condition (WEC) has been studied [22]. The WEC has been also investigated in the regular black hole manifold [23]. Next, it has been shown that the violation of null energy condition is restricted to some regions in the vicinity of the throat of a wormhole [24].

In this review, we will investigate the SECs, WECs and dominant energy conditions (DECs) and the ensuing EOS in the HDC in the background of dark energy by extending the results of the works [8,9]. To do this, we will study singularities associated with the Raychaudhuri type equations and construct the SECs and the ensuing EOSs in geodesic surface congruence in the HDC. We will then discuss the dark energy in the HDC. Moreover we will elucidate the origin of the relations between the Hawking-Penrose type EOS inequality equations in the $D = 3 + 1$ dimensional limits in the HDC, which are missing in Refs. [8,9], and the Hawking-Penrose EOS inequality in the $D = 3 + 1$ dimensions in the FRW inflationary cosmology. To this end, we will split the SEC in the HDC into three pieces: the point particle property degrees of freedom (DOF), dark energy and extended object DOF only. We will discuss the EOS parameters in the HDC and the corresponding EOS parameters in the DESI data. Next, we will explicitly construct the SECs for the stringy particles in the HDC having the dark energy contributions, to show that the EOS inequalities for the massive stringy particles are the same as those for the massless ones, even though their corresponding formulas are different to each other. Moreover, our study will evaluate the energy conditions for the point particles (or $(p = 0)$ -branes) possessing

¹ In this work we call the null energy condition and null dominant energy condition [16], the weak and dominant energy conditions for the massless point particles, respectively.

the mass-energy density ρ_0 and the pressure P_0 described in the $D = 3 + 1$ dimensional inflationary cosmology. We also will investigate the SEC and WEC for the DESI data, and the high temperature asymptotic behaviors of a dilaton-Einstein-Gauss-Bonnet (dEGB) theory [25,26] in the framework of the HDC.

In Section 2, we will construct the geometry and Raychaudhuri type equations in the HDC. We will briefly sketch the HDC with dark energy. In Section 3, we will formulate the SECs, WECs, DEC and EOS parameters for massive stringy particles and discuss the dark energy in the HDC. In Section 4, we will formulate the SECs and WECs, DEC and EOS parameters for massless stringy particles and study the dark energy in the HDC. We will study the EOS parameters in the DESI observational data and dEGB scenario. Section 5 includes conclusions. In Appendix A, we will pedagogically study $D = 3 + 1$ dimensional point particle cosmology. In Appendix B, we will study the details of the Raychaudhuri type equation in the HDC. In particular, we will list the definitions of the variables employed in Section 2.

2. Geometry and Raychaudhuri Type Equations in HDC

In this section, we investigate the geometry of the HDC and the ensuing Raychaudhuri type equations. To do this, we first introduce a fibration: $\pi : M \rightarrow N$ over a base manifold N associated with a fiber space manifold F [9,27,28]. In this work the base manifold N is fixed to reside on the $D = 3 + 1$ dimensional spacetime. In analogy of the relativistic action of a point particle in N , the action for a string is proportional to the area of the surface spanned in total manifold M by the evolution along time direction of the string in the fiber manifold F . Note that in the HDC the extended manifold associated with the extended p -brane ($p \geq 1$) including the $(p = 1)$ -string resides on the fiber space manifold F . Note also that in the HDC the $(p = 0)$ -brane denotes the point particle which lives on the fiber manifold F . In order to define the action on the curved manifold, let (M, g_{ab}) be a D dimensional manifold associated with the metric g_{ab} . Given g_{ab} , we can have a unique covariant derivative ∇_a satisfying [6] $\nabla_a g_{bc} = 0$, $\nabla_a \omega^b = \partial_a \omega^b + \Gamma_{ac}^b \omega^c$ and $(\nabla_a \nabla_b - \nabla_b \nabla_a) \omega_c = R_{abc}^d \omega_d$ ($a = 0, 1, \dots, D - 1$). Next we will investigate the Raychaudhuri type equations which will be studied in terms of the SECs for the massive and massless stringy and point particles with dark energy. Moreover the Raychaudhuri type equations will be discussed to clarify the attractive gravitational force.

We investigate explicitly the action and the ensuing geometry in the HDC. To do this, in Appendix A we will first study the total action, which consists of the relativistic action for the $(p = 0)$ -brane (or the point particle) and actions of gravity and perfect fluid, for the $D = 3 + 1$ dimensional point particle cosmology having the dark energy Λ in the curved manifold. Keeping the total action for the point particle in mind, we investigate the stringy particle in the curved manifold. To this end, we need to introduce the gravity action in addition to the extended $(p = 1)$ -brane action (or Nambu-Goto action) [29,30]. In this work we investigate the HDC possessing the dark energy Λ and the ensuing EOS. The case without dark energy has been analyzed in Refs. [8,9,31]. We start with D dimensional gravity related with total action of the form

$$\begin{aligned} S &= S_{p=1} + S_{gr} + S_{pf}, \\ S_{p=1} &= -\kappa \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma f(\tau, \sigma), \\ S_{gr} &= \frac{1}{16\pi} \int d^D x \sqrt{-g} (R - 2\Lambda) \end{aligned} \quad (2.1)$$

where $S_{p=1}$, S_{gr} and S_{pf} are the extended $(p = 1)$ -brane action, gravity and perfect fluid actions, respectively, and $\kappa = \frac{1}{2\pi\alpha'}$ with α' being the universal slope of Regge trajectory and Λ is a dark energy. Here $f(\tau, \sigma)$ is defined as

$$f(\tau, \sigma) = (-\det h_{MN})^{1/2} = [(\xi \cdot \xi)^2 - (\xi \cdot \xi)(\xi \cdot \xi)]^{1/2}, \quad (2.2)$$

where we have used the relation $h_{MN} = g_{ab} \frac{\partial x^a}{\partial \sigma^M} \frac{\partial x^b}{\partial \sigma^N}$ with $\sigma^M = (\tau, \sigma)$ ($M = 0, 1$), and the notations $\xi^a = (\partial/\partial\tau)^a$ and $\zeta^a = (\partial/\partial\sigma)^a$ ($a = 0, 1, \dots, D-1$) with metric $(-, +, \dots, +)$. Note that in this work, for completeness, in addition to $S_{p=1}$ we have included the actions S_{gr} and S_{pf} , which were not treated in the Refs. [8,9,31]. To be more specific, these actions S_{gr} possessing the dark energy Λ and S_{pf} generate the Einstein field equation of the form: $R_{ab} - \frac{1}{2}g_{ab}R + g_{ab}\Lambda = 8\pi T_{ab}$ for a perfect fluid. Here R_{ab} and R are the Ricci curvature tensor and the Ricci scalar curvature, respectively.

We consider a smooth congruence of time-like geodesic surfaces in M . We parameterize the surface generated by the evolution of a time-like string by two world sheet coordinates τ and σ , and then we have the corresponding vector fields $\xi^a = (\partial/\partial\tau)^a$ and $\zeta^a = (\partial/\partial\sigma)^a$ as shown in (2.2). Note that ξ^a is a time-like vector field and ζ^a is a space-like one. Since we have gauge DOF, we can choose the orthonormal gauge: $\xi \cdot \xi = -\zeta \cdot \zeta = -1$ and $\xi \cdot \zeta = 0$ [32].

We introduce the deviation vector field $\eta^a = (\partial/\partial\alpha)^a$ which represents the displacement to an infinitesimally nearby world sheet, and let Σ denote the three-dimensional submanifold spanned by the world sheets $\gamma_\alpha(\tau, \sigma)$. We then may choose τ, σ and α as coordinates of Σ to yield the commutator relations,

$$\mathcal{L}_\xi \eta^a = \xi^b \nabla_b \eta^a - \eta^b \nabla_b \xi^a = 0, \quad \mathcal{L}_\zeta \eta^a = \zeta^b \nabla_b \eta^a - \eta^b \nabla_b \zeta^a = 0. \quad (2.3)$$

Exploiting the above commutators, we arrive at

$$\frac{dS_{p=1}}{d\alpha} = - \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma (P_\tau^b \xi^a \nabla_a \eta_b + P_\sigma^b \zeta^a \nabla_a \eta_b) = \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \eta_b (\xi^a \nabla_a P_\tau^b + \zeta^a \nabla_a P_\sigma^b) + \text{boundary terms}, \quad (2.4)$$

where we have introduced the energy-momentum currents: $P_\tau^a = \frac{\kappa}{f} [(\xi \cdot \xi) \zeta^a - (\zeta \cdot \xi) \xi^a]$ and $P_\sigma^a = \frac{\kappa}{f} [(\xi \cdot \xi) \xi^a - (\xi \cdot \zeta) \zeta^a]$. Here the boundary terms vanish if we exploit the boundary conditions: $\eta^a(\tau = \tau_1; \sigma) = \eta^a(\tau = \tau_2; \sigma) = 0$ and $P_\sigma^a(\tau; \sigma = 0) = P_\sigma^a(\tau; \sigma = \pi) = 0$ [32,33] to produce a stringy geodesic equation: $\xi^a \nabla_a P_\tau^b + \zeta^a \nabla_a P_\sigma^b = 0$. Applying the orthonormal gauge conditions to the above stringy geodesic equation, we obtain the stringy geodesic equation of the form

$$-\xi^a \nabla_a \xi^b + \zeta^a \nabla_a \zeta^b = 0. \quad (2.5)$$

Exploiting the first equation in (2.4), we are left with the second derivative of $S_{p=1}$ to yield a stringy geodesic deviation equation [31]²

$$\frac{d^2 S_{p=1}}{d\alpha^2} = \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \eta_a (\chi \eta)^a, \quad (2.6)$$

where

$$(\chi \eta)^a = \xi^b \nabla_b (\eta^c \nabla_c P_\tau^a) + \zeta^b \nabla_b (\eta^c \nabla_c P_\sigma^a) + R_{bcd}^a (\xi^b P_\tau^d + \zeta^b P_\sigma^d) \eta^c. \quad (2.7)$$

On the other hand, the variation of $S_{gr} + S_{pf}$ under δg^{ab} ($a, b = 0, 1, \dots, D-1$) yields the Einstein field equation

$$R_{ab} - \frac{1}{2}g_{ab}R + g_{ab}\Lambda = 8\pi T_{ab}, \quad (2.8)$$

where we have exploited the definition $R_{ab} = R_{acb}^c$ with R_{bcd}^a being defined in (2.7). Here we have exploited the relation

$$\frac{\delta S_{pf}}{\delta g^{ab}} = -\frac{1}{2}\sqrt{-g}T_{ab}. \quad (2.9)$$

Note that R_{ab} in (2.8) is affected by the dark energy Λ . Note also that in this work the dark energy Λ is introduced to investigate the EOS associated with the non-zero and positive Λ in Section 3.

² For more algebraic details associated with outline derivations of $\frac{dS_{p=1}}{d\alpha}$ and $\frac{d^2 S_{p=1}}{d\alpha^2}$, see Appendix A where these two quantities are pedagogically explained in the ($p=0$)-brane case.

We briefly investigate the HDC associated with the Raychaudhuri type equation for the massive stringy congruence [8,9]

$$\frac{d\theta}{d\tau} - \frac{d\bar{\theta}}{d\sigma} = -\frac{1}{D-1}(\theta^2 - \bar{\theta}^2) - \sigma_{ab}\sigma^{ab} + \bar{\sigma}_{ab}\bar{\sigma}^{ab} + \omega_{ab}\omega^{ab} - \bar{\omega}_{ab}\bar{\omega}^{ab} - R_{ab}(\bar{\zeta}^a\bar{\zeta}^b - \zeta^a\zeta^b). \quad (2.10)$$

Here $(\theta, \sigma_{ab}, \omega_{ab})$ are the expansion, shear and twist of the universe and $(\bar{\theta}, \bar{\sigma}_{ab}, \bar{\omega}_{ab})$ are those of the string, respectively. The more details of these variables will be defined in Appendix B. To be specific, in the point particle case, we have only DOF of $(\theta, \sigma_{ab}, \omega_{ab})$ and ζ^a . In contrast, in the HDC we possess new DOF related with the physical variables $(\bar{\theta}, \bar{\sigma}_{ab}, \bar{\omega}_{ab})$ and $\bar{\zeta}^a$. In particular, in the HDC, we have the DOF of the twist (or rotation) of the string associated with $\bar{\omega}_{ab}$. This feature implies that the string performs the dynamic rotational motion with respect to the $D = 3 + 1$ dimensional base manifold N . For more derivation details of the Raychaudhuri type equation in (2.10), see Appendix B. We assume that $\sigma_{ab} = \bar{\sigma}_{ab}$ and $\omega_{ab} = \bar{\omega}_{ab}$. Taking an ansatz that the expansion $\bar{\theta}$ is constant along the σ direction, we end up with

$$\frac{d\theta}{d\tau} = -\frac{1}{D-1}(\theta^2 - \bar{\theta}^2) - R_{ab}(\bar{\zeta}^a\bar{\zeta}^b - \zeta^a\zeta^b). \quad (2.11)$$

We assume a SEC for a massive stringy particle

$$R_{ab}(\bar{\zeta}^a\bar{\zeta}^b - \zeta^a\zeta^b) \geq 0, \quad (2.12)$$

then the Raychaudhuri type equation in (2.11) has a solution of the form

$$\frac{1}{\theta(\tau)} \geq \frac{1}{\theta(0)} + \frac{1}{D-1} \left(\tau - \int_0^\tau d\tau \left(\frac{\bar{\theta}}{\theta} \right)^2 \right). \quad (2.13)$$

We assume that $\theta(0)$ is negative so that the congruence is initially converging as in the point particle case [2]. The EOS inequality in (2.13) then implies that $\theta(\tau)$ should pass through the singularity within a proper time

$$\tau \leq \frac{D-1}{|\theta(0)|} + \int_0^\tau d\tau \left(\frac{\bar{\theta}}{\theta} \right)^2. \quad (2.14)$$

Similar to the massive stringy particle case in (2.11), we consider the Raychaudhuri type equation associated with expansion of the massless stringy congruence related with the corresponding null vector k^a [8,9]

$$\frac{d\theta}{d\lambda} = -\frac{1}{D-2}\theta^2 + \frac{1}{D-1}\bar{\theta}^2 - R_{ab}(k^ak^b - \bar{\zeta}^a\bar{\zeta}^b), \quad (2.15)$$

where λ is an affine parameter. Note that, differently from the factor $D-1$ in (2.11), we have the factor $D-2$ in the first term in (2.15) which originates from the fact that the massless particle has only two transverse DOF without a longitudinal DOF. We assume a SEC for a massless stringy particle

$$R_{ab}(k^ak^b - \bar{\zeta}^a\bar{\zeta}^b) \geq 0, \quad (2.16)$$

then the Raychaudhuri type equation in (2.15) has a solution

$$\frac{1}{\theta(\lambda)} \geq \frac{1}{\theta(0)} + \frac{1}{D-2} \left(\lambda - \frac{D-2}{D-1} \int_0^\lambda d\lambda \left(\frac{\bar{\theta}}{\theta} \right)^2 \right). \quad (2.17)$$

Note that the SECs in (2.12) and (2.16) have been analyzed in the HDC without the dark energy in Refs. [8,9]. In the next subsection these SECs will be also exploited in the HDC possessing the dark energy. We assume that $\theta(0)$ is negative so that the congruence can be initially converging. The

EOS inequality in (2.17) then implies that $\theta(\tau)$ should pass through the singularity within an affine length [8,9]

$$\lambda \leq \frac{D-2}{|\theta(0)|} + \frac{D-2}{D-1} \int_0^\lambda d\lambda \left(\frac{\bar{\theta}}{\theta} \right)^2. \quad (2.18)$$

Note that the SECs in (2.12) and in (2.16), and the Raychaudhuri type equations in (2.11) and (2.15) associated with the attractive gravitational forces will be discussed in Section 3 and Section 4.

3. Energy Conditions and EOS Parameters for Massive Stringy Particles in Background with Dark Energy

In this section, we investigate the SECs and the ensuing EOS parameters in the HDC with the dark energy. In the results [8,9], we have found the SECs for both a massive and massless stringy particles. Note that these SECs have produced the same EOS inequalities in the HDC without the background of dark energy. In this section, we extend these SECs to the case possessing the dark energy. To this end, we first study the stringy SECs in (2.12) and (2.16) by considering the energy-momentum tensor T_{ab} for a perfect fluid in the HDC possessing the dark energy

$$T_{ab} = \rho u_a u_b + P(g_{ab} + u_a u_b), \quad (3.1)$$

where $u^a = (-1, 0, \dots, 0)$ is a time-like D velocity in the rest frame. Combining (2.8) and (3.1), we find

$$R_{ab} - \frac{1}{2} g_{ab} R = 8\pi \text{diag}(\rho + \rho_\Lambda, P + P_\Lambda, P + P_\Lambda, P + P_\Lambda, \dots), \quad (3.2)$$

where ρ_Λ and P_Λ are the contributions from the dark energy, defined as

$$\rho_\Lambda \equiv \frac{\Lambda}{8\pi}, \quad P_\Lambda \equiv -\frac{\Lambda}{8\pi}, \quad (3.3)$$

and the ellipsis denotes the higher extended p -brane ($p \geq 1$) contributions.

On the other hand, taking the trace of T_{ab} in (3.1), we find

$$T \equiv g^{ab} T_{ab} = -\rho + (D-1)P \quad (3.4)$$

which, together with (2.8), yields

$$R \equiv g^{ab} R_{ab} = -\frac{2}{D-2} (8\pi T - D\Lambda). \quad (3.5)$$

Inserting T in (3.4) and R in (3.5) into (2.8), we obtain

$$R_{ab} = 8\pi \left(T_{ab} - \frac{1}{D-2} g_{ab} T \right) + \frac{2}{D-2} g_{ab} \Lambda. \quad (3.6)$$

For the massive stringy particle, we find $T_{ab} \zeta^a \zeta^b = \rho$ and $T_{ab} \tilde{\zeta}^a \tilde{\zeta}^b = P$ to yield

$$T_{ab} (\tilde{\zeta}^a \tilde{\zeta}^b - \zeta^a \zeta^b) = \rho - P. \quad (3.7)$$

Exploiting the SEC for a massive stringy particle in (2.12) and the Einstein field equation having the dark energy in (2.8), we find that the SEC produces

$$R_{ab} (\tilde{\zeta}^a \tilde{\zeta}^b - \zeta^a \zeta^b) = 8\pi \left(\frac{D-4}{D-2} \rho + \frac{D}{D-2} P - \frac{1}{2\pi(D-2)} \Lambda \right) \geq 0, \quad (3.8)$$

where we have used (3.4), (3.6) and (3.7). Note that in (3.8) the dark energy Λ is incorporated into the results in [8,9]. Using ρ_Λ and P_Λ in (3.3), we rewrite the EOS inequality in (3.8) for the massive stringy particle with dark energy as

$$R_{ab}(\bar{\zeta}^a \bar{\zeta}^b - \zeta^a \zeta^b) = 8\pi \left(\frac{D-4}{D-2}(\rho + \rho_\Lambda) + \frac{D}{D-2}(P + P_\Lambda) \right) \geq 0, \quad (3.9)$$

which is listed in Table 1 as shown above.³ Note that the EOS inequality in (3.9) yields the corresponding EOS parameter

$$w = \frac{P + P_\Lambda}{\rho + \rho_\Lambda} \geq -\frac{D-4}{D}. \quad (3.10)$$

Note that the EOS parameter in (3.10) is not the same as that for the massive stringy particle without the dark energy [8,9], since (3.10) has the dark energy contribution. For the massive stringy particle in the $D = 5$ dimensions, the EOS parameter is given by $w \geq -\frac{1}{5}$ for instance.

Table 1. The energy conditions for the stringy particles in the HDC defined in $D \geq 5$ dimensional manifold. Here we have the dark energy contributions $\rho_\Lambda = \frac{\Lambda}{8\pi}$ and $P_\Lambda = -\frac{\Lambda}{8\pi}$ in the strong energy conditions. Note that the EOS inequalities for the massive stringy particles are the same as those for the massless ones as shown in the last column, even though their corresponding formulas are different to each other as shown in the second and third columns.

Energy condition	Massive particle	Massless particle	EOS inequality
Strong	$R_{ab}(\bar{\zeta}^a \bar{\zeta}^b - \zeta^a \zeta^b) \geq 0$	$R_{ab}(k^a k^b - \zeta^a \zeta^b) \geq 0$	$\frac{D-4}{D-2}(\rho + \rho_\Lambda) + \frac{D}{D-2}(P + P_\Lambda) \geq 0$
Weak	$T_{ab}(\bar{\zeta}^a \bar{\zeta}^b - \zeta^a \zeta^b) \geq 0$	$T_{ab}(k^a k^b - \zeta^a \zeta^b) \geq 0$	$\rho - P \geq 0$
Dominant	$T_{ab}T_c^b(\bar{\zeta}^a \bar{\zeta}^c - \zeta^a \zeta^c) \leq 0$	$T_{ab}T_c^b(k^a k^c - \zeta^a \zeta^c) \leq 0$	$\rho^2 + P^2 \geq 0$

We investigate the relations between the EOS parameter w and the Hawking-Penrose limit in the massive point particle in the $D = 3 + 1$ dimensions. These relations have not been elucidated even in the case without the dark energy analyzed in Refs. [8,9]. First we split the SEC related with $R_{ab}(\bar{\zeta}^a \bar{\zeta}^b - \zeta^a \zeta^b)$ in (3.9) into three parts: the massive point particle (or $(p = 0)$ -brane) property DOF related with $R_{ab}\bar{\zeta}^a \bar{\zeta}^b$ only, dark energy Λ , and extended object DOF associated with $R_{ab}\zeta^a \zeta^b$ only. First, for the contribution from the massive point particle DOF related with $\bar{\zeta}^a$ in the background with dark energy, we construct

$$R_{ab}\bar{\zeta}^a \bar{\zeta}^b = 8\pi \left(\frac{D-3}{D-2}(\rho_0 + \rho_\Lambda) + \frac{D-1}{D-2}(P_0 + P_\Lambda) \right), \quad (3.11)$$

where ρ_0 and P_0 are the contributions from the massive point particle (or $(p = 0)$ -brane) property associated with $R_{ab}\bar{\zeta}^a \bar{\zeta}^b$ without the dark energy, and ρ_Λ and P_Λ are given in (3.3). Here the subscript 0 denotes the point particle.

Second, we obtain the extended object DOF contribution of the massive particle associated with the vector field ζ^a as follows

$$R_{ab}\zeta^a \zeta^b = 8\pi \left(\frac{1}{D-2}(\rho_0 + \rho_\Lambda) - \frac{1}{D-2}(P_0 + P_\Lambda) \right), \quad (3.12)$$

³ In Table 1 we list the EOS inequalities for the massive and massless *stringy* particles in the D dimensions. In contrast, in Table 2 we list the EOS inequalities for the massive and massless *point* particles in the $D = 3 + 1$ dimensions. Note that the massive and *massless* particles are studied in Section 3 and Section 4, respectively.

which, together with $R_{ab}\tilde{\zeta}^a\tilde{\zeta}^b$ in (3.11), reproduces (3.9). Next, we define $R_{ab}\zeta^a\zeta^b$ in (3.12) as

$$R_{ab}\zeta^a\zeta^b \equiv -8\pi\left(\frac{D-3}{D-2}\rho_{ext} + \frac{D-1}{D-2}P_{ext}\right), \quad (3.13)$$

where ρ_{ext} and P_{ext} are the contributions from the extended object DOF for the massive stringy particle, defined as

$$\rho_{ext} \equiv -\frac{1}{D-3}(\rho_0 + \rho_\Lambda), \quad P_{ext} \equiv \frac{1}{D-1}(P_0 + P_\Lambda). \quad (3.14)$$

Combining (3.11) and (3.13), we obtain

$$R_{ab}(\tilde{\zeta}^a\tilde{\zeta}^b - \zeta^a\zeta^b) = 8\pi\left(\frac{D-3}{D-2}(\rho_0 + \rho_\Lambda + \rho_{ext}) + \frac{D-1}{D-2}(P_0 + P_\Lambda + P_{ext})\right) \geq 0, \quad (3.15)$$

which yields the EOS parameter w for the total system consisting of the massive point particle property associated with $R_{ab}\tilde{\zeta}^a\tilde{\zeta}^b$, dark energy Λ , and extended object DOF related with $R_{ab}\zeta^a\zeta^b$ as follows

$$w_0^{\Lambda,ext} = \frac{P_0 + P_\Lambda + P_{ext}}{\rho_0 + \rho_\Lambda + \rho_{ext}} \geq -\frac{D-3}{D-1}. \quad (3.16)$$

This result is the most general form for the massive stringy particle in the background with dark energy. Note that in (3.16), (ρ_0, P_0) , $(\rho_\Lambda, P_\Lambda)$ and (ρ_{ext}, P_{ext}) are contributions from the massive point particle DOF property associated with $R_{ab}\tilde{\zeta}^a\tilde{\zeta}^b$, dark energy Λ , and extended object DOF related with $R_{ab}\zeta^a\zeta^b$, respectively.

Third, we study the $D = 3 + 1$ dimensional point particle limit of the stringy massive particle. Since the point particle has no extended object DOF contribution, we have $\rho_{ext} = P_{ext} = 0$ in (3.16) to yield

$$w_0^\Lambda = \frac{P_0 + P_\Lambda}{\rho_0 + \rho_\Lambda} \geq -\frac{1}{3}, \quad (3.17)$$

for the massive point particle in the background with dark energy. Here ρ_Λ and P_Λ are given in (3.3). Note that, in the $D = 3 + 1$ dimensional inflationary cosmology, (3.17) produces the EOS inequality for the SEC for massive particle

$$\rho_0 + \rho_\Lambda + 3(P_0 + P_\Lambda) \geq 0, \quad (3.18)$$

which is listed in Table 2. In the point particle limit with $\Lambda = 0$, the SEC $R_{ab}\tilde{\zeta}^a\tilde{\zeta}^b \geq 0$ for massive particle produces the equivalent Hawking-Penrose EOS inequality defined in the $D = 3 + 1$ dimensions [2,5–9,16]

$$\rho_0 + 3P_0 \geq 0. \quad (3.19)$$

Table 2. The energy conditions for the point particles (or $(p = 0)$ -branes) possessing the mass-energy density ρ_0 and the pressure P_0 described in the $D = 3 + 1$ dimensions. Here the subscript 0 denotes the point particle, and we have the dark energy contributions $\rho_\Lambda = \frac{\Lambda}{8\pi}$ and $P_\Lambda = -\frac{\Lambda}{8\pi}$ in the strong energy conditions. Note that in this table we call the null energy condition and null dominant energy condition [16], the weak and dominant energy conditions for the massless point particles, respectively.

Energy condition	Massive particle	EOS inequality	Massless particle	EOS inequality
Strong	$R_{ab}\tilde{\zeta}^a\tilde{\zeta}^b \geq 0$	$\frac{\rho_0 + \rho_\Lambda}{3(P_0 + P_\Lambda)} \geq 0$	$R_{ab}k^ak^b \geq 0$	$\frac{\rho_0 + \rho_\Lambda + P_0 + P_\Lambda}{P_\Lambda} \geq 0$
Weak	$T_{ab}\tilde{\zeta}^a\tilde{\zeta}^b \geq 0$	$\rho_0 \geq 0$	$T_{ab}k^ak^b \geq 0$	$\rho_0 + P_0 \geq 0$
Dominant	$T_{ab}T_c^b\tilde{\zeta}^a\tilde{\zeta}^c \leq 0$	$\rho_0^2 \geq 0$	$T_{ab}T_c^bk^ak^c \leq 0$	$\rho_0^2 - P_0^2 \geq 0$

We can apply our formalism to a WEC for a massive stringy particle in the HDC defined in $D \geq 5$ dimensional manifold. To do this, we assume a WEC for a massive stringy particle:

$$T_{ab}(\xi^a \xi^b - \zeta^a \zeta^b) \geq 0. \quad (3.20)$$

Exploiting $T_{ab}\xi^a\xi^b = \rho$ and $T_{ab}\zeta^a\zeta^b = P$ in (3.20), we arrive at the WEC for the massive stringy particle in the HDC

$$\rho - P \geq 0, \quad (3.21)$$

which is listed in Table 1. Note that (3.21) yields the corresponding EOS parameter for the massive stringy particle in the HDC

$$w \leq 1. \quad (3.22)$$

Note also that the WEC in (3.22) contains the extended object DOF contribution associated with ξ^a .

Keeping $T_{ab}\xi^a\xi^b = P$ in mind, we obtain the WEC for the massive point particle in the $D = 3 + 1$ dimensions

$$T_{ab}\xi^a\xi^b \geq 0, \quad (3.23)$$

to yield the EOS inequality for the WEC for the massive point particle [2,5–7,16]

$$\rho_0 \geq 0, \quad (3.24)$$

which is listed in Table 2 and is consistent with the outcomes [2,5–7,16]. This result suggests that the assumption in (3.20) is physically well defined. Note that the EOS inequality for the massive point particle in (3.24) is different from that for the massive stringy particle in (3.21).

Similar to the WEC analyzed above, we apply our formalism to a DEC⁴ for a massive stringy particle in the HDC. To this end, we assume a DEC for a massive stringy particle [16]

$$T_{ab}T_c^b(\xi^a\xi^c - \zeta^a\zeta^c) \leq 0. \quad (3.25)$$

Using $T_{ab}T_c^b\xi^a\xi^c = -\rho^2$ and $T_{ab}T_c^b\zeta^a\zeta^c = P^2$ in (3.25), we find the DEC for the massive stringy particle in the HDC

$$\rho^2 + P^2 \geq 0, \quad (3.26)$$

which is listed in Table 1. Keeping $T_{ab}T_c^b\zeta^a\zeta^c = P^2$ in mind, we find the DEC for the massive point particle in the $D = 3 + 1$ dimensions

$$T_{ab}T_c^b\xi^a\xi^c \leq 0, \quad (3.27)$$

to produce the EOS inequality for the corresponding DEC [2,5–7,16]

$$\rho_0^2 \geq 0, \quad (3.28)$$

which is listed in Table 2.

Next, in the $D = 3 + 1$ dimensional cosmology without the dark energy, we assume that $\theta = \bar{\theta} = \zeta^a = 0$ in (2.11) to produce

$$\frac{d\theta}{d\tau} = -R_{ab}\xi^a\xi^b = -4\pi(\rho_0 + 3P_0) \quad (3.29)$$

which, using the SEC in (3.19), becomes negative. The SEC for the massive point particle thus suggests that gravitation is attractive [16]. Up to now we have investigated the massive stringy and point particles. In the next section we will study the massless stringy and point particles.

⁴ For the definition of the DEC, we introduce (3.25) only, differently from Ref. [16] where the WEC in (3.20) is additionally combined to the DEC for a massive stringy particle case in (3.25). Similar statement will be apply to the DEC for the massless stringy particle in (4.19).

4. Energy Conditions and EOS Parameters for Massless Stringy Particles in Background with Dark Energy

In this section we investigate the energy conditions for the massless stringy and point particles, and the corresponding EOS parameters in the background of dark energy in the HDC. To do this, we first define the null vector k^a and its auxiliary null vector l^a as $k^a \equiv \frac{1}{\sqrt{2}}(u^a + v^a)$ and $l^a \equiv \frac{1}{\sqrt{2}}(u^a - v^a)$. Here v^a is a vector residing on the subspace $N_{\perp}^{D-2} = \{v^a | v^a k_a = 0, v^a l_a = 0\}$ which is a $(D-2)$ dimensional manifold perpendicular to the light-cone originated from k^a and l^a . Note that the null vector field $k^a = (\partial/\partial\lambda)^a$ is given in terms of the affine parameter λ and l^a points in the opposite spatial direction to k^a . To be more specific, k^a and l^a are normalized as follows [9,16]

$$k^a k_b = l^a l_b = 0, k^a l_a = -1. \quad (4.1)$$

For the massless stringy particle, exploiting T_{ab} in (3.1) we find $T_{ab} k^a k^b = \frac{1}{2}(\rho + P)$ and $T_{ab} \zeta^a \zeta^b = P$ to yield the EOS inequality for the WEC

$$T_{ab}(k^a k^b - \zeta^a \zeta^b) = \frac{1}{2}(\rho - P), \quad (4.2)$$

which is listed in Table 1. Note that the WEC in (4.2) is firstly defined for the massless stringy particle in this work. Making use of the SEC for a massless stringy particle in (2.16) and the Einstein field equation in (2.8) possessing the dark energy, we find that the SEC associated with the corresponding null vector k^a yields

$$R_{ab}(k^a k^b - \zeta^a \zeta^b) = 4\pi \left(\frac{D-4}{D-2} \rho + \frac{D}{D-2} P - \frac{1}{2\pi(D-2)} \Lambda \right) \geq 0. \quad (4.3)$$

where we have exploited (3.4), (3.6) and (4.2). Note that even though we include the dark energy Λ in both the massive and massless stringy particle cases, the EOS inequality condition associated with the corresponding SECs in (4.3) is the same as that in (3.8). Exploiting ρ_{Λ} and P_{Λ} in (3.3), we rewrite the EOS inequality in (4.3) for the massless stringy particle in the dark energy background as

$$R_{ab}(k^a k^b - \zeta^a \zeta^b) = 4\pi \left(\frac{D-4}{D-2} (\rho + \rho_{\Lambda}) + \frac{D}{D-2} (P + P_{\Lambda}) \right) \geq 0, \quad (4.4)$$

which is listed in Table 1. Note that the EOS inequality in (4.4) for the massless stringy particle is equivalent to that for the massive stringy particle in (3.9). Next the EOS inequality in (4.4) produces the EOS parameter w for the massless stringy particle in the background with dark energy

$$w = \frac{P + P_{\Lambda}}{\rho + \rho_{\Lambda}} \geq -\frac{D-4}{D}. \quad (4.5)$$

Note that the EOS parameter w in (4.5) is the same as that in (3.10) for the massive stringy particle. Note also that w in (4.5) is equivalent to that of the case for the massive stringy particle without the dark energy [8,9].

We investigate the relations between the EOS parameter w in (4.5) and the Hawking-Penrose limit in the massless point particle in the $D = 3 + 1$ dimensions. Similar to the massive stringy particle case discussed in Section 3, we split the SEC related with $R_{ab}(k^a k^b - \zeta^a \zeta^b)$ in (4.4) into three pieces: the massless point particle property DOF related with $R_{ab} k^a k^b$ only, dark energy Λ , and extended object DOF associated with $R_{ab} \zeta^a \zeta^b$ only. First, similar to (3.11), for the contribution from the massless point particle DOF related with k^a in the background with dark energy, we obtain

$$R_{ab} k^a k^b = 4\pi(\rho_0 + \rho_{\Lambda} + P_0 + P_{\Lambda}), \quad (4.6)$$

where ρ_0 and P_0 are the contributions from the massless point particle property associated with $R_{ab}k^ak^b$ without the dark energy, and ρ_Λ and P_Λ are defined in (3.3). Second, we find the extended object DOF contribution of the massless particle associated with the vector field ζ^a as follows

$$R_{ab}\zeta^a\zeta^b = 8\pi\left(\frac{1}{D-2}(\rho_0 + \rho_\Lambda) - \frac{1}{D-2}(P_0 + P_\Lambda)\right), \quad (4.7)$$

which, together with $R_{ab}k^ak^b$ in (4.6), reproduces (4.4). Third, we define $R_{ab}\zeta^a\zeta^b$ in (4.7) as

$$R_{ab}\zeta^a\zeta^b \equiv -4\pi(\rho_{ext} + P_{ext}), \quad (4.8)$$

where ρ_{ext} and P_{ext} are the contributions from the extended object DOF for the massless stringy particle

$$\rho_{ext} \equiv -\frac{2}{D-2}(\rho_0 + \rho_\Lambda), \quad P_{ext} \equiv \frac{2}{D-2}(P_0 + P_\Lambda). \quad (4.9)$$

Note that the above ρ_{ext} and P_{ext} for the massless stringy particle are different from those for the massive case in (3.14).

Combining (4.6) and (4.8), we find

$$R_{ab}(k^ak^b - \zeta^a\zeta^b) = 4\pi(\rho_0 + \rho_\Lambda + \rho_{ext} + P_0 + P_\Lambda + P_{ext}) \geq 0, \quad (4.10)$$

which yields the EOS parameter w for the total system consisting of the massless point particle property related with $R_{ab}k^ak^b$, dark energy Λ , and extended object DOF associated with $R_{ab}\zeta^a\zeta^b$

$$w_0^{\Lambda,ext} = \frac{P_0 + P_\Lambda + P_{ext}}{\rho_0 + \rho_\Lambda + \rho_{ext}} \geq -1. \quad (4.11)$$

This EOS parameter is the most general form for the massless stringy particle in the background with dark energy.

We investigate the $D = 3 + 1$ dimensional point particle limit of the stringy massless particle. Since the point particle has no extended object DOF contribution, we have $\rho_{ext} = P_{ext} = 0$ in (4.11) to yield

$$w_0^\Lambda = \frac{P_0 + P_\Lambda}{\rho_0 + \rho_\Lambda} \geq -1, \quad (4.12)$$

for the massless point particle in the background with dark energy. Here ρ_Λ and P_Λ are given in (3.3). Note that (4.12) yields the EOS inequality for the SEC for massless point particle in the background with dark energy [2,5–7,16]

$$\rho_0 + \rho_\Lambda + P_0 + P_\Lambda \geq 0, \quad (4.13)$$

which is listed in Table 2. Moreover, in the point particle limit with $\Lambda = 0$, the SEC $R_{ab}k^ak^b \geq 0$ for massless particle yields the EOS inequality defined in the $D = 3 + 1$ dimensional inflationary cosmology [2,5–7,16]

$$\rho_0 + P_0 \geq 0. \quad (4.14)$$

We now apply our formalism to a WEC for a massless stringy particle in the HDC. To this end, we assume a WEC for a massless stringy particle:

$$T_{ab}(k^ak^b - \zeta^a\zeta^b) \geq 0. \quad (4.15)$$

Using $T_{ab}k^ak^b = \frac{1}{2}(\rho + P)$ and $T_{ab}\zeta^a\zeta^b = P$ in (4.15), we end up with the WEC for the massless stringy particle in the HDC

$$\rho - P \geq 0, \quad (4.16)$$

which is listed in Table 1. Next we define the WEC for the massless point particle (or null energy condition) in the $D = 3 + 1$ dimensional inflationary cosmology

$$T_{ab}k^ak^b \geq 0. \quad (4.17)$$

The WEC in (4.17) produces [2,5–7,16]

$$\rho_0 + P_0 \geq 0, \quad (4.18)$$

which is listed in Table 2. Note that, in the HDC the WEC for the massless stringy particle in (4.16) is the same as that for the massive stringy particle in (3.21). Note that, in the WECs, we do not have the effect of the background with dark energy since the definitions in (3.20) and (4.15) do not possess the corresponding dark energy terms. Similar to (3.24) for the WEC inequality for the massive point particle, in the $D = 3 + 1$ dimensional inflationary cosmology, we find that for the massless point particle in (4.18) which is consistent with the results [2,5–7,16]. This outcome also suggests that the assumption in (4.15) is physically well defined.

Similar to the WEC discussed above, we apply our formalism to a DEC for a massless stringy particle in the HDC. To do this, we assume a DEC for a massless stringy particle

$$T_{ab}T_c^b(k^ak^c - \zeta^a\zeta^c) \leq 0. \quad (4.19)$$

Making use of $T_{ab}T_c^bk^ak^c = -\frac{1}{2}(\rho^2 - P^2)$ and $T_{ab}T_c^b\zeta^a\zeta^c = P^2$ in (4.19), we find the DEC for the massless stringy particle in the HDC

$$\rho^2 + P^2 \geq 0, \quad (4.20)$$

which is listed in Table 1. Keeping $T_{ab}T_c^b\zeta^a\zeta^c = P^2$ in mind, we find the DEC for the massless point particle (or null DEC) in the $D = 3 + 1$ dimensional inflationary cosmology given by

$$T_{ab}T_c^bk^ak^c \leq 0, \quad (4.21)$$

to yield the EOS inequality for the DEC for the massless point particle [16]

$$\rho_0^2 - P_0^2 \geq 0, \quad (4.22)$$

which is listed in Table 2. Note in (3.10) and (4.5) that, in the radiation-dominated and matter-dominated eras in the background with dark energy, we have the same EOS parameter in the SEC in the HDC. In other words, the conditions in (3.10) and (4.5) suggest that there exists no phase transition between the radiation-dominated and matter-dominated eras in the background with dark energy. This feature is the same as the case without the dark energy [8,9]. To be specific, there is only transition between the radiation-matter-dominated era and Λ -dominated one. This feature is contradictory to the results [6,7,16,34] obtained in the $D = 3 + 1$ dimensional FRW model.

It is appropriate to comment on the EOS parameters of the dust and radiation, quintessence, phantom, DESI, and dEGB scenario, in terms of the corresponding EOS parameters of the HDC as shown in Table 3. First, it is known that for the dust (or matter) we have $w(\text{dust}) = 0$, and for the radiation we obtain $w(\text{radiation}) = \frac{1}{3}$. Since the dust consists of the massive stringy particles, $w(\text{dust}) = 0$ satisfies $w \leq 1$ in (3.22) of the WEC and $w \geq -\frac{D-4}{D}$ in (3.10) and (4.5) of the SEC defined on $D \geq 5$ dimensional manifold in the HDC. Since the radiation is made up of the massless stringy particles, $w(\text{radiation}) = \frac{1}{3}$ also fulfills the SEC and the WEC in the HDC.

Table 3. The satisfaction and violation of the energy conditions determined in the stringy particle analysis in the HDC defined on $D \geq 5$ dimensional manifold. Here DESI and dEGB denote the Dark Energy Spectroscopic Instrument and dilaton-Einstein-Gauss-Bonnet, respectively. Next the satisfaction and violation in the energy conditions are given by the abbreviations S and V, respectively. Note that for the case of partial satisfaction we use the the abbreviation SP.

Type	SEC	WEC	Type	SEC	WEC
dust	S	S	DESI	SP	S
radiation	S	S	$w(\text{dEGB}) = 1$	S	S
quintessence	SP	SP	$w(\text{dEGB}) = -\frac{1}{3}$	SP	S
phantom	V	S	$1 < w(\text{dEGB}) < \frac{7}{3}$	S	V

Second, the EOS parameter of the quintessence $w(\text{quintessence}) \geq -1$ satisfies partially both the WEC and the SEC in the HDC. Moreover, $w(\text{phantom}) < -1$ [35] for the phantom energy violates the SEC, but satisfies the WEC. Next, the DESI has detected the time dependence of $w(\text{DESI})$ [3,4]. To be specific, the DESI has measured the effect of dark energy on the expansion of the universe. The DESI data indicate that $w(\text{DESI})$ evolves to yield $-1 < w(\text{DESI}) < 0$ [3] which satisfies partially the SEC, but satisfies the WEC in the HDC.

Third, it is known that in the high temperature asymptotic behavior of cosmology in a dEGB scenario of modified gravity having vanishing scalar potential, the regions of the dEGB parameter spaces are given in terms of three attractors of a set of autonomous differential equations: $w(\text{dEGB}) = -\frac{1}{3}$ (slow era), $w(\text{dEGB}) = 1$ (kination era) and $1 < w(\text{dEGB}) < \frac{7}{3}$ (fast roll era) [25,26]. Note that the reheating temperature of the universe during kination era has been studied in terms of gravitational particle creation during expansion of the universe [36] and cold dark matter abundance [37]. Note also that the condition $1 < w(\text{dEGB}) < \frac{7}{3}$ for the fast roll era is related with decelerated expansion. In this work we find that $w(\text{dEGB}) = 1$ satisfies both the WEC and SEC in the HDC. In contrast, $w(\text{dEGB}) = -\frac{1}{3}$ satisfies partially the SEC, but satisfies the WEC, but in the HDC. Next, $1 < w(\text{dEGB}) < \frac{7}{3}$ satisfies the SEC, but violates the WEC in the HDC. The summary of violations of the above conditions is listed in Table 3.

One of the simplest models to explain the dark energy is the Λ -cold dark matter model where the dark energy is static and positive. This constant accelerates the expansion of the universe. To be more specific, the dark energy Λ in (2.1) is non-zero and positive. This feature implies that the expansion of the universe is accelerating, which is consistent with the corresponding discovery in 1998 [1]. Note that the dark energy does not need to be constant. For instance, the quintessence model describes dynamical dark energy [38–40] having time dependent scalar field. Moreover, in the $f(R)$ theories of the gravity, the integrand in the action S_{gr} in (2.1) is replaced as follows: $\sqrt{-g}(R - 2\Lambda) \rightarrow \sqrt{-g}f(R)$ with $f(R)$ being a function of R . We study explicitly the case that the dark energy is static in the HDC. In this case the energy density is given as $\rho_\Lambda = \frac{\Lambda}{8\pi}$, for both the massive and massless stringy particles as shown in (3.3). In order to explain the accelerating expansion of the universe, we need to find an energy having negative pressure $P_\Lambda = -\frac{\Lambda}{8\pi}$ with the positive dark energy Λ . This means that the accelerating universe in the HDC possesses a positive dark energy.

Finally, in the $D = 3 + 1$ dimensional cosmology without the background with dark energy, we assume that $\theta = \bar{\theta} = \zeta^a = 0$ in (2.15) to produce

$$\frac{d\theta}{d\lambda} = -R_{ab}k^ak^b = -4\pi(\rho_0 + P_0) \quad (4.23)$$

which, using the SEC in (4.18), becomes negative. As a result, the SEC for the massless point particle implies that gravitation is attractive [16]. Note that exploiting $\frac{d\theta}{d\tau} = -R_{ab}(\zeta^a\zeta^b - \zeta^a\zeta^b) \leq 0$ and $\frac{d\theta}{d\lambda} = -R_{ab}(k^ak^b - \zeta^a\zeta^b) \leq 0$ in the HDC having $\theta = \bar{\theta} = 0$ and $\zeta^a \neq 0$, the SECs for the massive and massless stringy particles suggest that gravitation is attractive.

5. Conclusions

In summary, we have investigated the energy conditions in the HDC having a dark energy Λ . As shown in Table 1 in Section 3, the EOS inequalities for the SECs, WECs and DEC for the massive stringy particles are equivalent to those for the massless stringy particles in the HDC. In contrast, in the $D = 3 + 1$ dimensional inflationary cosmology in Section 3, we have evaluated the EOS inequalities for the SECs, WECs and DEC for the massive and massless point particles as listed in Table 2. Next we have studied the EOS parameter w in the HDC with the dark energy. To be specific, we have constructed $w \geq -(D - 4)/D$ for both the massive and massless stringy particles in the HDC in the background with dark energy. We thus have concluded that the HDC has the phase transition between the radiation-matter-dominated era and Λ -dominated one. Note that in [8,9] without Λ the universe also has two phases: radiation-dominated era and matter-dominated one and there exists no phase transition between these two eras.

Moreover, we have split the EOS parameter for the stringy particle in the background with dark energy as $w_0^{\Lambda, ext} = (P_0 + P_\Lambda + P_{ext})/(\rho_0 + \rho_\Lambda + \rho_{ext})$ where (ρ_0, P_0) , $(\rho_\Lambda, P_\Lambda)$ and (ρ_{ext}, P_{ext}) are contributions from the point particle property, dark energy Λ , and extended object DOF, respectively. Since the point particle has no extended object DOF contribution, we have $\rho_{ext} = P_{ext} = 0$. In the point particle limit with $\Lambda = 0$, the SEC produces the EOS parameter related with the Hawking-Penrose type EOS inequality. Moreover, the effect of the background with dark energy has been shown not to exist in the WECs and DEC.

We also have shown that the EOS parameter of the quintessence $w(\text{quintessence}) \geq -1$ satisfies partially both the WEC and the SEC in the HDC. Next, $w(\text{phantom}) < -1$ [35] for the phantom energy violates the SEC, but satisfies the WEC. The DESI has detected the time dependence of $w(\text{DESI})$ [3,4]. To be specific, the DESI has measured the effect of dark energy on the expansion of the universe. The DESI data indicate that $w(\text{DESI})$ evolves to yield $-1 < w(\text{DESI}) < 0$ [3] which satisfies partially the SEC, but satisfies the WEC in the HDC. It is known that in the high temperature asymptotic behavior of cosmology in a dEGB scenario of modified gravity possessing vanishing scalar potential, the regions of the dEGB parameter spaces are given in terms of three attractors: $w(\text{dEGB}) = -\frac{1}{3}$ (slow era), $w(\text{dEGB}) = 1$ (kination era) and $1 < w(\text{dEGB}) < \frac{7}{3}$ (fast roll era) [25,26]. Note also that the condition $1 < w(\text{dEGB}) < \frac{7}{3}$ for the fast roll era is related with decelerated expansion. In this work we have found that $w(\text{dEGB}) = 1$ satisfies both the WEC and SEC in the HDC. In contrast, $w(\text{dEGB}) = -\frac{1}{3}$ satisfies partially the SEC, but satisfies the WEC in the HDC. Next, $1 < w(\text{dEGB}) < \frac{7}{3}$ satisfies the SEC, but violates the WEC in the HDC. The violations of the above conditions in addition to those of the dust and radiation cases are summarized in Table 3.

Next, we have investigated the DEC for the massive and massless stringy particles in the HDC. In the HDC, the DEC for the massless stringy particle is the same as that for the massive stringy particle to yield $\rho^2 + P^2 \geq 0$. Moreover, the effect of the background with dark energy has been shown not to exist in the DEC, as in the cases of the SECs and WECs. This is one of the main points of this work. It will be interesting to investigate the $f(R)$ gravity theory having the Gauss-Bonnet term $R_{GB}^2 = R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}$ [25,26] in the HDC. The next interesting topic will be the FRW model in the HDC in which we investigate the higher dimensional metric g_{ab} ($a, b = 0, 1, 2, 3, 4, \dots$) of the form: $g_{ab} = \text{diag}(-1, a^2, a^2, a^2, b^2, \dots)$ where a is a scale factor defined in the $D = 3 + 1$ dimensional base manifold and b is a scale factor residing on the compact extended ($p = 1$)-brane, and the ellipsis stands for the higher extended p -brane ($p \geq 2$) contributions, respectively. On the other hand, it will be interesting to investigate the wormhole geometry [7,24,41] and the corresponding energy conditions and EOS parameters in the HDC. The next interesting topic will be to study the black hole and the corresponding energy conditions [17,22,23] and EOS parameters in the HDC.

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Appendix A. $D = 3 + 1$ Dimensional Point Particle Cosmology with Dark Energy

In this section, we study an action for a $D = 3 + 1$ dimensional gravity with the ($p = 0$)-brane (or point particle) which is applied to the HDC in Section 2. The action for the $D = 3 + 1$ dimensional gravity with a point particle in the background with dark energy is described as

$$\begin{aligned} S &= S_{p=0} + S_{gr} + S_{pf}, \\ S_{p=0} &= -m \int_{\tau_1}^{\tau_2} d\tau f(\tau), \\ S_{gr} &= \frac{1}{16\pi} \int d^3x \sqrt{-g} (R - 2\Lambda) \end{aligned} \quad (\text{A.1})$$

where $S_{p=0}$, S_{gr} and S_{pf} are the relativistic action for the ($p = 0$)-brane and actions of gravity and perfect fluid associated with the energy-momentum tensor of the form T_{ab} in (3.1). Here m is a test point particle mass and $f(\tau)$ is defined as [32]⁵: $f(\tau) = \left(-g_{ab} \frac{\partial x^a}{\partial \tau} \frac{\partial x^b}{\partial \tau}\right)^{1/2} = (-g_{ab} \dot{\zeta}^a \dot{\zeta}^b)^{1/2}$ where we have used the notation $\dot{\zeta}^a = (\partial/\partial \tau)^a$ ($a = 0, 1, 2, 3$) with metric $(-, +, +, +)$. Here Λ is the dark energy. In order to define the action on the curved manifold, let (M, g_{ab}) be a $D = 3 + 1$ -dimensional manifold associated with the metric g_{ab} . Given g_{ab} , we can have a unique covariant derivative ∇_a satisfying [6] $\nabla_a g_{bc} = 0$, $\nabla_a \omega^b = \partial_a \omega^b + \Gamma_{ac}^b \omega^c$ and $(\nabla_a \nabla_b - \nabla_b \nabla_a) \omega_c = R_{abc}^d \omega_d$ ($a = 0, 1, 2, 3$).

We introduce the deviation vector field $\eta^a = (\partial/\partial \alpha)^a$ which represents the displacement to an infinitesimally nearby world line, and let Σ denote the two-dimensional submanifold spanned by the world lines $\gamma_\alpha(\tau)$ [6]. We then may choose τ and α as coordinates of Σ to yield the commutator relation,

$$\mathcal{L}_{\dot{\zeta}} \eta^a = \dot{\zeta}^b \nabla_b \eta^a - \eta^b \nabla_b \dot{\zeta}^a = 0.$$

We construct the first derivative of S_p with respect to α to produce

$$\frac{dS_p}{d\alpha} = -m \int_{\tau_1}^{\tau_2} d\tau \eta^a \nabla_a f = -m \int_{\tau_1}^{\tau_2} d\tau \eta^a \frac{1}{f} (-\dot{\zeta}^b \nabla_a \dot{\zeta}^b) = - \int_{\tau_1}^{\tau_2} d\tau P_\tau^b \eta^a \nabla_a \dot{\zeta}_b, \quad (\text{A.2})$$

where we have used the energy-momentum current $P_\tau^a = -\frac{m}{f} \dot{\zeta}^a$. Exploiting the commutator in (A.2), we arrive at

$$\frac{dS_p}{d\alpha} = - \int_{\tau_1}^{\tau_2} d\tau P_\tau^b \dot{\zeta}^a \nabla_a \eta_b = \int_{\tau_1}^{\tau_2} d\tau \eta_b \dot{\zeta}^a \nabla_a P_\tau^b - P_\tau^a \eta_a \Big|_{\tau=\tau_1}^{\tau=\tau_2}, \quad (\text{A.3})$$

and the boundary term vanishes if we use the end point condition:

$$\eta^a(\tau = \tau_1) = \eta^a(\tau = \tau_2) = 0, \quad (\text{A.4})$$

to yield a geodesic equation:

$$\dot{\zeta}^a \nabla_a P_\tau^b = 0. \quad (\text{A.5})$$

Applying the time-like condition $\dot{\zeta} \cdot \dot{\zeta} = -1$ to (A.5), we obtain the geodesic equation of the form

$$\dot{\zeta}^a \nabla_a \dot{\zeta}^b = 0. \quad (\text{A.6})$$

⁵ In this work, we restrict ourselves to the case that the variation of $f(\tau)$ under δg^{ab} is neglected, for simplicity. This restriction also will be applied to the string particle in Section 2.

Using the first equation in (A.3) we find the second derivative of S_p as follows

$$\frac{d^2 S_p}{d\alpha^2} = - \int_{\tau_1}^{\tau_2} d\tau \eta^c \nabla_c (P_\tau^b \zeta^a \nabla_a \eta_b) = - \int_{\tau_1}^{\tau_2} d\tau \eta^c [(\nabla_c P_\tau^b) \zeta^a \nabla_a \eta_b + P_\tau^b \{(\nabla_c \zeta^a) \nabla_a \eta_b + \zeta^a \nabla_c \nabla_a \eta_b\}]. \quad (\text{A.7})$$

Making use of the relation $\nabla_c \nabla_a \eta_b = \nabla_a \nabla_c \eta_b + R_{cab}^d \eta_d$ and the commutator in (A.2), we find

$$\begin{aligned} \frac{d^2 S_p}{d\alpha^2} &= - \int_{\tau_1}^{\tau_2} d\tau \left[(\eta^c \nabla_c P_\tau^b) \zeta^a \nabla_a \eta_b + P_\tau^b ((\zeta^c \nabla_c \eta^a) \nabla_a \eta_b + \zeta^a \eta^c \nabla_a \nabla_c \eta_b + R_{cab}^d \eta_d \zeta^a) \right] \\ &= - \int_{\tau_1}^{\tau_2} d\tau \left[(\eta^c \nabla_c P_\tau^b) \zeta^a \nabla_a \eta_b + P_\tau^b ((\zeta^c \nabla_c \eta^a) \nabla_a \eta_b + \zeta^c \eta^a \nabla_c \nabla_a \eta_b) - R_{acb}^d \eta_d \zeta^a P_\tau^b \right] \\ &= - \int_{\tau_1}^{\tau_2} d\tau \left[(\eta^c \nabla_c P_\tau^b) \zeta^a \nabla_a \eta_b + P_\tau^b (\zeta^c \nabla_c) (\eta^a \nabla_a \eta_b) - R_{bcd}^a \eta^c \eta_a \zeta^b P_\tau^d \right] \\ &= - \int_{\tau_1}^{\tau_2} d\tau \left[\zeta^a \nabla_a (\eta_b \eta^c \nabla_c P_\tau^b) - \eta_b \zeta^a \nabla_a (\eta^c \nabla_c P_\tau^b) + \zeta^c \nabla_c (P_\tau^b \eta^a \nabla_a \eta_b) - (\eta^a \nabla_a \eta_b) \zeta^c \nabla_c P_\tau^b - R_{bcd}^a \eta^c \eta_a \zeta^b P_\tau^d \right] \\ &= - \eta_b \eta^c \nabla_c P_\tau^b \Big|_{\tau=\tau_1}^{\tau=\tau_2} - P_\tau^b \eta^a \nabla_a \eta_b \Big|_{\tau=\tau_1}^{\tau=\tau_2} + \int_{\tau_1}^{\tau_2} d\tau \left[\eta_b \zeta^a \nabla_a (\eta^c \nabla_c P_\tau^b) + R_{bcd}^a \eta^c \eta_a \zeta^b P_\tau^d \right] \end{aligned} \quad (\text{A.8})$$

Inserting the end point condition in (A.4) and the geodesic equation in (A.5) into (A.8), we end up with a geodesic deviation equation of the form

$$\frac{d^2 S_p}{d\alpha^2} = \int_{\tau_1}^{\tau_2} d\tau \eta_a (\chi \eta)^a, \quad (\text{A.9})$$

where

$$(\chi \eta)^a = \zeta^b \nabla_b (\eta^c \nabla_c P_\tau^a) + R_{bcd}^a \zeta^b \eta^c P_\tau^d. \quad (\text{A.10})$$

On the other hand, the variation of $S_{gr} + S_{pf}$ under δg^{ab} ($a, b = 0, 1, 2, 3$) produces the Einstein field equation

$$R_{ab} - \frac{1}{2} g_{ab} R + g_{ab} \Lambda = 8\pi T_{ab}, \quad (\text{A.11})$$

where we have exploited the definition $R_{ab} = R_{acb}^c$ with R_{bcd}^a being defined in (A.10). Here we have used the relation $\delta S_{pf} / \delta g^{ab} = -\frac{1}{2} \sqrt{-g} T_{ab}$.

Appendix B. Raychaudhuri Type Equation in HDC

In this section, we construct the details of the Raychaudhuri type equation in (2.10) more completely and rigorously in the HDC, since in Refs. [8,9] we have stated the sketch of the Raychaudhuri type equation. To do this, in the orthonormal gauge, we introduce tensor fields B_{ab} and $\bar{B}_{ab}$ defined as

$$B_{ab} = \nabla_b \zeta_a, \quad \bar{B}_{ab} = \nabla_b \bar{\zeta}_a, \quad (\text{B.1})$$

which fulfill the identities

$$B_{ab} \zeta^a = 0, \quad \bar{B}_{ab} \bar{\zeta}^a = 0, \quad B_{ab} \zeta^b - \bar{B}_{ab} \bar{\zeta}^b = 0. \quad (\text{B.2})$$

Exploiting the commutator relations in (2.3), we find

$$\zeta^a \nabla_a \eta^b - \bar{\zeta}^a \nabla_a \eta^b = (B_a^b - \bar{B}_a^b) \eta^a. \quad (\text{B.3})$$

We define the metrics h_{ab} and $\bar{h}_{ab}$ as

$$h_{ab} = g_{ab} + \zeta_a \zeta_b, \quad \bar{h}_{ab} = g_{ab} - \zeta_a \zeta_b, \quad (\text{B.4})$$

to produce the identities

$$h^{ab}h_{ab} = \bar{h}^{ab}\bar{h}_{ab} = D - 1. \quad (\text{B.5})$$

Using h_{ab} and $\bar{h}_{ab}$ in (B.4), we decompose B_{ab} into three parts

$$B_{ab} = \frac{1}{D-1}\theta h_{ab} + \sigma_{ab} + \omega_{ab}, \quad (\text{B.6})$$

where

$$\theta = B^{ab}h_{ab}, \quad \sigma_{ab} = B_{(ab)} - \frac{1}{D-1}\theta h_{ab}, \quad \omega_{ab} = B_{[ab]}, \quad (\text{B.7})$$

and B_{ab} into three pieces

$$\bar{B}_{ab} = \frac{1}{D-1}\bar{\theta}\bar{h}_{ab} + \bar{\sigma}_{ab} + \bar{\omega}_{ab}, \quad (\text{B.8})$$

where

$$\bar{\theta} = \bar{B}^{ab}\bar{h}_{ab}, \quad \bar{\sigma}_{ab} = \bar{B}_{(ab)} - \frac{1}{D-1}\bar{\theta}\bar{h}_{ab}, \quad \bar{\omega}_{ab} = \bar{B}_{[ab]}. \quad (\text{B.9})$$

We find the identities

$$\begin{aligned} \sigma_{ab}h^{ab} &= \omega_{ab}h^{ab} = 0, & \sigma_{ab}\bar{\zeta}^b &= \omega_{ab}\bar{\zeta}^b = \frac{1}{2}B_{ab}\bar{\zeta}^b, \\ \bar{\sigma}_{ab}\bar{h}^{ab} &= \bar{\omega}_{ab}\bar{h}^{ab} = 0, & \bar{\sigma}_{ab}\bar{\zeta}^b &= \bar{\omega}_{ab}\bar{\zeta}^b = \frac{1}{2}\bar{B}_{ab}\bar{\zeta}^b \end{aligned} \quad (\text{B.10})$$

to yield $-\sigma_{ab}\bar{\zeta}^b + \bar{\sigma}_{ab}\bar{\zeta}^b = 0$ where we have used the stringy geodesic equation in (2.5).

We evaluate the following quantity

$$\begin{aligned} \zeta^c \nabla_c B_{ab} &= \zeta^c \nabla_c \nabla_b \zeta_a = \zeta^c (\nabla_b \nabla_c \zeta_a + R_{cba}^d \zeta_d) = \nabla_b (\zeta^c \nabla_c \zeta_a) - (\nabla_b \zeta^c) (\nabla_c \zeta_a) + R_{cba}^d \zeta^c \zeta_d \\ &= \nabla_b (\zeta^c \nabla_c \zeta_a) - B_b^c B_{ac} + R_{cba}^d \zeta^c \zeta_d. \end{aligned} \quad (\text{B.11})$$

Similarly we find

$$\zeta^c \nabla_c \bar{B}_{ab} = \nabla_b (\zeta^c \nabla_c \bar{\zeta}_a) - \bar{B}_b^c \bar{B}_{ac} + R_{cba}^d \zeta^c \zeta_d. \quad (\text{B.12})$$

Combining the identities in (B.11) and (B.12), we arrive at

$$-\zeta^c \nabla_c B_{ab} + \zeta^c \nabla_c \bar{B}_{ab} = B_b^c B_{ac} - \bar{B}_b^c \bar{B}_{ac} - R_{cba}^d (\zeta^c \zeta^d - \bar{\zeta}^c \bar{\zeta}^d), \quad (\text{B.13})$$

where we have again exploited the stringy geodesic equation in (2.5). We calculate the trace of the above equation by multiplying both sides of (B.13) by g^{ab} . Firstly, we rewrite the trace of left hand side of (B.13) as

$$-\zeta^c \nabla_c (B_{ab}g^{ab}) + \zeta^c \nabla_c (\bar{B}_{ab}g^{ab}) = -\zeta^c \nabla_c (B_{ab}(h^{ab} - \zeta^a \bar{\zeta}^b)) + \zeta^c \nabla_c (\bar{B}_{ab}(\bar{h}^{ab} + \zeta^a \bar{\zeta}^b)) = -\zeta^c \nabla_c \theta + \zeta^c \nabla_c \bar{\theta}, \quad (\text{B.14})$$

where we have used (B.2), (B.4), (B.7) and (B.9). Secondly, we rewrite the trace of right hand side of (B.13) as

$$\begin{aligned} &B^{ca}B_{ac} - \bar{B}^{ca}\bar{B}_{ac} + R_{cd}(\zeta^c \zeta^d - \bar{\zeta}^c \bar{\zeta}^d) \\ &= \left(\frac{1}{D-1}\theta h^{ca} + \sigma^{ca} + \omega^{ca} \right) \left(\frac{1}{D-1}\theta h_{ac} + \sigma_{ac} + \omega_{ac} \right) \\ &\quad - \left(\frac{1}{D-1}\bar{\theta}\bar{h}^{ca} + \bar{\sigma}^{ca} + \bar{\omega}^{ca} \right) \left(\frac{1}{D-1}\bar{\theta}\bar{h}_{ac} + \bar{\sigma}_{ac} + \bar{\omega}_{ac} \right) + R_{cd}(\zeta^c \zeta^d - \bar{\zeta}^c \bar{\zeta}^d) \\ &= \frac{1}{D-1}(\theta^2 - \bar{\theta}^2) + \sigma_{ab}\sigma^{ab} - \bar{\sigma}_{ab}\bar{\sigma}^{ab} - \omega_{ab}\omega^{ab} + \bar{\omega}_{ab}\bar{\omega}^{ab} + R_{ab}(\zeta^a \bar{\zeta}^b - \bar{\zeta}^a \zeta^b) \end{aligned} \quad (\text{B.15})$$

where we have exploited (B.5), (B.6), (B.8) and (B.10).

Combining the identities in (B.14) and (B.15), we are left with

$$-\zeta^c \nabla_c \theta + \zeta^c \nabla_c \bar{\theta} = \frac{1}{D-1} (\theta^2 - \bar{\theta}^2) + \sigma_{ab} \sigma^{ab} - \bar{\sigma}_{ab} \bar{\sigma}^{ab} - \omega_{ab} \omega^{ab} + \bar{\omega}_{ab} \bar{\omega}^{ab} + R_{ab} (\zeta^a \zeta^b - \bar{\zeta}^a \bar{\zeta}^b). \quad (\text{B.16})$$

Exploiting $-\zeta^c \nabla_c \theta + \zeta^c \nabla_c \bar{\theta} = -\frac{d\theta}{d\tau} + \frac{d\bar{\theta}}{d\sigma}$, we find that (B.16) reproduces (2.10).

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