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Article

Computing a Maximal Independent Set Modulo an Ideal and a Gröbner Basis of the Ideal Simultaneously

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Abstract

To solve problems on a positive-dimensional ideal $I \subset k[X]$, a maximal independent set $U \subset X$ modulo I and a Gröbner basis of I^e, where I^e is the extension of I to $k(U)[V]$ $(V := X \backslash U)$, are widely used. As far as we know, they are usually computed separately, i.e., U is calculated first and the Gröbner basis is computed after U is obtained. In this paper, we present an efficient algorithm for computing a maximal independent set U modulo I and a Gröbner basis of I^e simultaneously. Different from computing them separately, the algorithm takes full advantage of the polynomial information throughout the Gröbner basis computation to obtain U as soon as possible, hence it significantly improves the computing efficiency.

Keywords: maximal independent set; Gröbner basis; positive-dimensional ideal

MSC: 65N06; 65B99

1. Introduction

As a basic task in computer algebra, polynomial system solving plays an important role in many practical problems that can be described through polynomial systems, which can be divided into zero-dimensional and positive-dimensional ideals. Dealing with problems on a positive-dimensional ideal $I \subset k[X]$, such as computing a primary decomposition of I [1] and Noetherian operators of I [2], requires computing a maximal independent set modulo I. Weispfenning presented an algorithm for computing a maximal independent set in 1993 [3], which needs to compute all strongly independent sets from a Gröbner basis of I. Zhang proposed a method for deciding whether an indeterminate set is a maximal independent set by computing a quasi-Gröbner basis of I in 1995 [4]. In 2023, Yang and Tan presented a method to compute maximal independent sets [5], and the complexity is reduced since the method turns a multivariate problem into a single-variable problem.

Gröbner bases of polynomial ideals, first proposed by Buchberger in 1965 [6], are one of the fundamental algorithmic tools for solving polynomial systems, and essential in a lot of computational tasks in polynomial algebra and algebraic geometry. Given the positive- dimensional ideal I, with a maximal independent set $U \subset X$ modulo I known, a Gröbner basis of I^e, where I^e is the extension of I to $k(U)[V]$ $(V := X \backslash U)$, is usually necessary for subsequent calculations [7,8]. According to [4], a Gröbner basis of I w.r.t. a block monomial order $\prec_{U,V}$ $(U \ll V)$ is a Gröbner basis of I^e. So it is equivalent to computing a Gröbner basis G of I w.r.t. $\prec_{U,V}$.

As far as we know, the maximal independent set and the Gröbner basis are generally computed separately. For example, to obtain Rational Representation Sets of the variety of I, Tan [9] computes U by Zhang's method [4] while Shang [10] and Xiao [11] by Weispfenning's method. After U is computed, they determine $\prec_{U,V}$ without clear rules and compute a Gröbner basis G of I w.r.t. $\prec_{U,V}$. In this

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case, the connection between the calculations of U and G is not considered, which leads to a waste of intermediate calculations. Thus we present an algorithm for computing a maximal independent set U modulo I and a Gröbner basis G of I w.r.t. some block monomial order $\prec_{U,V}$ simultaneously, i.e., U, $\prec_{U,V}$ and G are simultaneously obtained. We compare our algorithm with two widely used methods which compute U and G separately. 1) Compute U, determine $\prec_{U,V}$ artificially and directly compute a Gröbner basis of I w.r.t. $\prec_{U,V}$. It is referred to as the direct method; 2) Compute U from a Gröbner basis of I by Weispfenning's algorithm, determine $\prec_{U,V}$ artificially, and convert the Gröbner basis already computed for getting U to a Gröbner basis w.r.t. $\prec_{U,V}$ by the Gröbner Walk [12]. It is referred to as the W-W method. Experiments show advantages of our algorithm compared with the direct and the W-W methods.

The paper is arranged as follows. In Section 2, we introduce relative notations, definitions and conclusions. In Section 3, we describe the subalgorithms and the main algorithm in detail. Experimental results are listed in Section 4.

2. Preliminaries

In this paper, k denotes a field of characteristic 0, $X := \{x_1, x_2, \cdots, x_n\}$ is an indeterminate set, $k[X]$ is the polynomial ring over k in X, and $T(X)$ denotes the set of all monomials in X. $plex(x_1, x_2, \cdots, x_n)$ and $grlex(x_1, x_2, \cdots, x_n)$ denote the pure lexicographic and the graded reverse lexicographic monomial orders on $T(X)$ with $x_1 \succ x_2 \succ \cdots \succ x_n$, respectively. Let $I \subset k[X]$ be an ideal, $F \subset k[X]$ be a polynomial set and $\prec$ be a monomial order on $T(X)$. Then $LM_\prec(I)$ and $LM_\prec(F)$ denote the sets of leading monomials of all polynomials in I and F w.r.t. $\prec$, respectively. For $U \subset X$, $k[U]$ is the polynomial ring over k in U, $k(U)$ denotes the rational function field over k in U, and $T(U)$ denotes the set of all monomials in U. Suppose $V := X \backslash U$. Then $k(U)[V]$ denotes the polynomial ring over $k(U)$ in V, and I^e denotes the extension of I to $k(U)[V]$. Given $F \subset k[X]$ and $f \in k[X]$, F_V and f_V denote F and f as a polynomial set of $k(U)[V]$ and a polynomial in $k(U)[V]$, respectively.

Moreover, the cardinality of a set S is written as $\sharp S$, and the i-th element of a list or an ordered set L is denoted by L_i. $\mathbb{N}$ is the non-negative integer set, $\mathbb{N}_+$ is the positive integer set and $\mathbb{R}_+$ is the non-negative real number set. We denote $\mathbf{e_i}$ the vector of which the i-th element is 1 and the others are 0.

Definition 1 ([3]). *Let $I \subset k[X]$ be a proper ideal. $U \subset X$ is called* **independent** *modulo I if*

$$I \cap k[U] = \{0\}.$$

Otherwise, U is called **dependent** *modulo I. Moreover, U is called* **maximally independent** *modulo I if it is independent modulo I and not properly contained in any other independent set modulo I.*

Definition 2 ([3]). *Let $I \subset k[X]$ be a proper ideal and $\prec$ be a monomial order on $T(X)$. $U \subset X$ is called* **strongly independent** *modulo I w.r.t. $\prec$ if*

$$LM_\prec(I) \cap T(U) = \emptyset.$$

Furthermore, U is called **maximally strongly independent** *modulo I w.r.t. $\prec$ if it is strongly independent modulo I w.r.t. $\prec$ and not properly contained in any other strongly independent set modulo I w.r.t. $\prec$.*

Proposition 1 ([3]). *Let $I \subset k[X]$ be a proper ideal, $\prec$ be a monomial order on $T(X)$ and $G \subset k[X]$ be a Gröbner basis of I w.r.t. $\prec$. Then $U \subset X$ is strongly independent modulo I w.r.t. $\prec$ iff*

$$LM_\prec(G) \cap T(U) = \emptyset.$$

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Definition 3 ([3]). *Let $U \subset X$, and $V := X \backslash U$. Let $\prec_V$ and $\prec_U$ be monomial orders on $T(V)$ and $T(U)$, respectively. Any monomial $t \in T(X)$ can be written uniquely as $t = t_1 t_2$, where $t_1 \in T(V)$, $t_2 \in T(U)$. Then* **a block monomial order** *on $T(X)$ with $U \ll V$, denoted by $\prec_{U,V}$, is defined by $r \prec s$ iff*

$$\begin{aligned} &r_1 \prec_V s_1, \quad or \\ &r_1 = s_1 \quad and \quad r_2 \prec_U s_2. \end{aligned}$$

$\prec_V$ and $\prec_U$ are called **the restrictions** *of $\prec_{U,V}$ to $T(V)$ and $T(U)$, respectively. Particularly, we denote $\prec_{U,V}^{grl}$ the block monomial order of which $\prec_V = grlex(V)$ and $\prec_U = grlex(U)$.*

Given a positive-dimensional ideal I, according to [13], a strongly independent set modulo I is independent modulo I. However, a maximal strongly independent set modulo I is not always maximally independent modulo I. For example, let $I := \langle zx - x^2, zy - xy \rangle \subset k[x,y,z]$ and $\prec := plex(z,y,x)$. Then $G := \{zx - x^2, zy - xy\}$ is a Gröbner basis of I w.r.t. $\prec$. Since $LM_{\prec}(G) = \{zx, zy\}$, $\{z\}$ is a maximal strongly independent set modulo I w.r.t. $\prec$. By a simple calculation, we have $\{y,z\}$ is independent modulo I, thus $\{z\}$ is not maximally independent modulo I. Fortunately, the situation differs for block monomial orders.

Lemma 1. *Let $I \subset k[X]$ be a positive-dimensional ideal, $U \subset X$, $V := X \backslash U$ and $\prec_{U,V}$ be a block monomial order on $T(X)$. If U is a maximal strongly independent set modulo I w.r.t. $\prec_{U,V}$, then U is maximally independent modulo I.*

Proof. Suppose $U := \{u_1, u_2, \cdots, u_r\}$, $V := \{v_1, v_2, \cdots, v_{n-r}\}$, where $1 \leq r < n$, and G is a Gröbner basis of I w.r.t. $\prec_{U,V}$. Since U is maximally strongly independent modulo I w.r.t. $\prec_{U,V}$, by Definition 2, U is strongly independent modulo I w.r.t. $\prec_{U,V}$, and $U \cup \{v_i\}$ is not strongly independent modulo I w.r.t. $\prec_{U,V}$, $i = 1, 2, \cdots, n-r$. So according to Proposition 1, we have $LM_{\prec_{U,V}}(G) \cap T(U) = \emptyset$ and $LM_{\prec_{U,V}}(G) \cap T(U, v_i) \neq \emptyset$ for $i = 1, 2, \cdots, n-r$. These two facts imply that G contains polynomials of the form

$$\begin{aligned} f_1 &:= c_1 \cdot v_1^{a_1} \cdot u_1^{b_{1,1}} u_2^{b_{1,2}} \cdots u_r^{b_{1,r}} + \text{lower terms w.r.t. } \prec_{U,V}, \\ f_2 &:= c_2 \cdot v_2^{a_2} \cdot u_1^{b_{2,1}} u_2^{b_{2,2}} \cdots u_r^{b_{2,r}} + \text{lower terms w.r.t. } \prec_{U,V}, \\ &\vdots \\ f_{n-r} &:= c_{n-r} \cdot v_{n-r}^{a_{n-r}} \cdot u_1^{b_{n-r,1}} u_2^{b_{n-r,2}} \cdots u_r^{b_{n-r,r}} + \text{lower terms w.r.t. } \prec_{U,V}, \end{aligned} \tag{1}$$

where $a_i \in \mathbb{N}_+$, $b_{i,j} \in \mathbb{N}$, and $c_i \neq 0$ is the leading coefficient of f_i w.r.t. $\prec_{U,V}$, $i = 1, 2, \cdots, n-r$, and $j = 1, 2, \cdots, r$. Let $\prec_V$ be the restriction of $\prec_{U,V}$ to $T(V)$. By Lemma 8.93 from [3], $G \subset k(U)[V]$ is a Gröbner basis of I^e w.r.t. $\prec_V$. Denote t_i a non-leading monomial of f_i w.r.t. $\prec_{U,V}$, written as $t_i = t_{i,V} t_{i,U}$, where $t_{i,V} \in T(V)$, $t_{i,U} \in T(U)$ for $i = 1, 2, \cdots, n-r$. According to Definition 3, $t_{i,V} \preceq_V v_i^{a_i}$. So $LM_{\prec_V}((f_i)_V) = v_i^{a_i}$. Let $F := \{f_1, f_2, \cdots, f_{n-r}\}$. Then $LM_{\prec_V}(F_V) = \{v_1^{a_1}, v_2^{a_2}, \cdots, v_{n-r}^{a_{n-r}}\}$. By Finiteness Theorem [14], I^e is zero-dimensional. According to Proposition 1.2.8 from [4], U is maximally independent modulo I. □

It is necessary to generalize the concepts and conclusions modulo an ideal to the case modulo a polynomial set since subsequent calculations are performed on polynomial sets.

Definition 4. *Let $F \subset k[X]$ be a polynomial set and $\prec$ be a monomial order on $T(X)$. $U \subset X$ is called* **strongly quasi-independent** *modulo F w.r.t. $\prec$ if*

$$LM_{\prec}(F) \cap T(U) = \emptyset.$$

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Furthermore, U is called **maximally strongly quasi-independent** *modulo F w.r.t. $\prec$ if it is strongly quasi-independent modulo F w.r.t. $\prec$ and not properly contained in any other strongly quasi-independent set modulo F w.r.t. $\prec$. For simplicity, a (maximal) strongly quasi-independent set is written as an (M)SQIS.*

Particularly, (M)SQIS's U w.r.t. $\prec$ have certain properties if $\prec$ is a block monomial order "with $U \ll V$". Firstly, let us discuss such SQIS's through Propositions 2 and 3.

Proposition 2. *Let $F \subset k[X]$ be a polynomial set, $U \subset X$, $V := X\backslash U$ and $\prec_{U,V}$ be a block monomial order on $T(X)$. Then U is an SQIS modulo F w.r.t. $\prec_{U,V}$, i.e., $LM_{\prec_{U,V}}(F) \cap T(U) = \emptyset$, iff $F \cap k[U] = \emptyset$.*

Proof. Necessity: Since $LM_{\prec_{U,V}}(F) \cap T(U) = \emptyset$, for each $f \in F$, we have $LM_{\prec_{U,V}}(f) \notin T(U)$. So $f \notin k[U]$, and $F \cap k[U] = \emptyset$.
Sufficiency: Aiming for a contradiction, suppose $LM_{\prec_{U,V}}(F) \cap T(U) \neq \emptyset$. Then there exists $f \in F$, s.t. $LM_{\prec_{U,V}}(f) \in T(U)$. Thus by Definition 3, for each non-leading monomial t of f w.r.t. $\prec_{U,V}$, written as $t = t_1 t_2$, where $t_1 \in T(V)$, $t_2 \in T(U)$, we have $t_1 \preceq_V 1$, i.e., $t \in T(U)$. So $f \in k[U]$, and $F \cap k[U] \neq \emptyset$. This contradiction shows that $LM_{\prec_{U,V}}(F) \cap T(U) = \emptyset$. □

Proposition 3. *Let $F \subset k[X]$ be a polynomial set, $U \subset X$, $V := X\backslash U$ and $\prec_{U,V}$ be a block monomial order on $T(X)$. If U is an SQIS modulo F w.r.t. $\prec_{U,V}$, then for each block monomial order $\prec'_{U,V} \neq \prec_{U,V}$ with $U \ll V$, U is also an SQIS modulo F w.r.t. $\prec'_{U,V}$.*

Proof. Since U is an SQIS modulo F w.r.t. $\prec_{U,V}$, by Definition 4, we have $LM_{\prec_{U,V}}(F) \cap T(U) = \emptyset$. According to Proposition 2, $F \cap k[U] = \emptyset$ and moreover, $LM_{\prec'_{U,V}}(F) \cap T(U) = \emptyset$. So U is an SQIS modulo F w.r.t. $\prec'_{U,V}$. □

Secondly, we discuss such MSQIS's in the rest of this section. The following Theorem 1 reveals the relation between an MSQIS U modulo F w.r.t. a block monomial order $\prec_{U,V}$ and the dependent sets modulo the ideal I, where $F \subset I$.

Theorem 1. *Let $I \subset k[X]$ be a positive-dimensional ideal and $F \subset I$ be a polynomial set. Let $U \subset X$, $V := X\backslash U$ and $\prec_{U,V}$ be a block monomial order on $T(X)$. If U is an MSQIS modulo F w.r.t. $\prec_{U,V}$, then for any $\bar{U} \supsetneq U$, $\bar{U}$ is dependent modulo I.*

Proof. On one hand, if U is dependent modulo I, by Definition 1, $I \cap k[U] \neq \{0\}$. Since $\bar{U} \supsetneq U$, $I \cap k[\bar{U}] \neq \{0\}$. So $\bar{U}$ is dependent modulo I.
On the other hand, if U is independent modulo I, $I \cap k[U] = \{0\}$, i.e., for each $f \in I$, $f \notin k[U]$. According to Proposition 2, $LM_{\prec_{U,V}}(f) \notin T(U)$. Thus $LM_{\prec_{U,V}}(I) \cap T(U) = \emptyset$, and U is strongly independent modulo I w.r.t. $\prec_{U,V}$. Since U is an MSQIS modulo F w.r.t. $\prec_{U,V}$, we have $LM_{\prec_{U,V}}(F) \cap T(\bar{U}) \neq \emptyset$. Then $LM_{\prec_{U,V}}(I) \cap T(\bar{U}) \neq \emptyset$ since $F \subset I$, and it means U is maximally strongly independent modulo I w.r.t. $\prec_{U,V}$. By Lemma 1, U is maximally independent modulo I, and $\bar{U}$ is dependent modulo I since $\bar{U} \supsetneq U$. □

Definition 5 and Proposition 4 provide a way to compute an MSQIS.

Definition 5. *Let $F \subset k[X]$ be a polynomial set. The* **label** *of $x \in X$ on F is defined as*

$$lbl_{x,F} := \{f \in F \mid f \text{ contains } x\}.$$

Moreover, the label list $[lbl_{x_1,F}, lbl_{x_2,F}, \cdots, lbl_{x_n,F}]$ is written as $Lbl_{X,F}$.

Remark 1. *We propose a new kind of indeterminate order on $Lbl_{X,F}$: $x_i \succ x_j$ if $\sharp lbl_{x_i,F} < \sharp lbl_{x_j,F}$ for $1 \leq i, j \leq n$. Throughout the paper, as long as F is updated, $Lbl_{X,F}$ will be modified, and the indeterminates*

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of X will be reordered in descending order. Experiments show that such indeterminate order can improve the efficiency of the main algorithm (Algorithm 1).

Proposition 4. *Let $F \subset k[X]$ be a polynomial set, $U \subset X$, and $V := X \backslash U$. If $\bigcup_{v \in V} lbl_{v,F} = F$ and $\bigcup_{\tilde{v} \in \tilde{V}} lbl_{\tilde{v},F} \neq F$ for any $\tilde{V} \subsetneq V$, then for each block monomial order $\prec_{U,V}$ on $T(X)$ with $U \ll V$, U is an MSQIS modulo F w.r.t. $\prec_{U,V}$.*

Proof. Since $\bigcup_{v \in V} lbl_{v,F} = F$, for each $f \in F$, there exists $v \in V$, s.t. $f \in lbl_{v,F}$. So f contains v by Definition 5, i.e., $F \cap k[U] = \emptyset$. According to Proposition 2, $LM_{\prec_{U,V}}(F) \cap T(U) = \emptyset$. Thus U is an SQIS modulo F w.r.t. $\prec_{U,V}$. Moreover, let $\tilde{V} := X \backslash \tilde{U}$ for each $\tilde{U} \supsetneq U$. Then $\tilde{V} \subsetneq V$. Since $\bigcup_{\tilde{v} \in \tilde{V}} lbl_{\tilde{v},F} \neq F$, there exists $f \in F$, s.t. $f \notin lbl_{\tilde{v},F}$ for each $\tilde{v} \in \tilde{V}$, i.e., f does not contain any indeterminate of $\tilde{V}$. So $f \in k[\tilde{U}]$ and $F \cap k[\tilde{U}] \neq \emptyset$. Thus by Proposition 2, $LM_{\prec_{U,V}}(F) \cap T(\tilde{U}) \neq \emptyset$, and U is an MSQIS modulo F w.r.t. $\prec_{U,V}$. □

Proposition 5 ensures the construction of the main algorithm (Algorithm 1).

Proposition 5. *Let $F_1, F_2 \subset k[X]$ be polynomial sets with $F_2 \supsetneq F_1$ and $H := F_2 \backslash F_1$. Let $U \subset X$, $V := X \backslash U$ and $\prec_{U,V}$ be a block monomial order on $T(X)$. Suppose U is an SQIS modulo F_1 w.r.t. $\prec_{U,V}$. Then*

1) U is not an SQIS modulo F_2 w.r.t. $\prec_{U,V}$ iff $H \cap k[U] \neq \emptyset$, which is called **the altering condition**.

2) If the altering condition is true, then for any block monomial order $\prec'_{U,V} \neq \prec_{U,V}$, U is not an MSQIS modulo F_2 w.r.t. $\prec'_{U,V}$.

Proof. 1) **Necessity:** Since U is an SQIS modulo F_1 w.r.t. $\prec_{U,V}$ but not an SQIS modulo F_2 w.r.t. $\prec_{U,V}$, by Definition 4, $LM_{\prec_{U,V}}(F_1) \cap T(U) = \emptyset$ and $LM_{\prec_{U,V}}(F_2) \cap T(U) \neq \emptyset$. Since $H := F_2 \backslash F_1$, we have $LM_{\prec_{U,V}}(H) \cap T(U) \neq \emptyset$. According to Proposition 2, $H \cap k[U] \neq \emptyset$.
Sufficiency: Since $H \cap k[U] \neq \emptyset$, we have $LM_{\prec_{U,V}}(H) \cap T(U) \neq \emptyset$ by Proposition 2, and $LM_{\prec_{U,V}}(F_2) \cap T(U) \neq \emptyset$ since $F_2 \supset H$. Thus U is not an SQIS modulo F_2 w.r.t. $\prec_{U,V}$.

2) Since the altering condition holds, U is not an SQIS modulo F_2 w.r.t. $\prec_{U,V}$ from 1). By Proposition 3, U is not an SQIS modulo F_2 w.r.t. $\prec'_{U,V}$. So according to Definition 4, U is certainly not an MSQIS modulo F_2 w.r.t. $\prec'_{U,V}$. □

3. The Idea and Algorithms

In this section, we present an algorithm for computing a maximal independent set U modulo a positive-dimensional ideal $I := \langle F \rangle \subset k[X]$ and a Gröbner basis of I w.r.t. a block monomial order $\prec_{U,V}$ simultaneously.

3.1. The Idea

Let us describe the procedure of the algorithm briefly.
Step 1: Compute $U \subset X$, s.t. there exists a block monomial order $\prec_{U,V}$, w.r.t. which U is an MSQIS modulo F. Then
a) By Definition 4 and Proposition 3, U is an SQIS modulo F w.r.t. $\prec^{grl}_{U,V}$.
b) By Theorem 1, any $\bar{U} \supsetneq U$ is dependent modulo I.
Step 2: Execute the Buchberger operations on F w.r.t. $\prec^{grl}_{U,V}$. Let $S := \{(f,g) \mid f,g \in F, f \neq g\}$ be the critical pair set of F. Then select a prior subset $\tilde{S}$ of S with a certain strategy and compute the normal form set H of the S-polynomials of all $s \in \tilde{S}$ modulo F w.r.t. $\prec^{grl}_{U,V}$.
Step 3: Let $F' := F \cup H$ and check whether the altering condition $H \cap k[U] \neq \emptyset$ is true. If it is not true, then go to Step 2 and continue to execute the Buchberger operations on F' w.r.t. $\prec^{grl}_{U,V}$. Otherwise, compute $U' \subset X$, s.t. there exists a block monomial order $\prec_{U',V'}$, w.r.t. which U' is an MSQIS modulo F'. In this case, a) and b) hold for U', F' and $\prec^{grl}_{U',V'}$. Then go to Step 2 and switch to execute the

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Buchberger operations on F w.r.t. $\prec^{grl}_{U',V'}$. U and U' are called the "old" and the "new" MSQIS's, respectively.

If the algorithm terminates in finite steps, then we denote by $\hat{U}$, $\hat{F}$ and $\prec^{grl}_{\hat{U},\hat{V}}$ the final indeterminate set, polynomial set and block monomial order, respectively. It can be proved that $\hat{U}$ is a maximal independent set modulo I and $\hat{F}$ is a Gröbner basis of I w.r.t. $\prec^{grl}_{\hat{U},\hat{V}}$. The algorithm is referred to as "the main algorithm".

The above procedure shows that the idea of the main algorithm is adjusting MSQIS's continuously according to the results of the S-polynomial calculations.

Note that the key issue of the main algorithm is computing U $(U') \subset X$, s.t. there exists a block monomial order $\prec_{U,V}$ $(\prec_{U',V'})$, w.r.t. which U (U') is an MSQIS modulo F (F'). Next we give an algorithm for resolving this issue.

3.2. Computing an MSQIS U w.r.t. Each Block Monomial Order with $U \ll V$

According to Proposition 4, we present Subalgorithm 1 for computing U, which is an MSQIS modulo F w.r.t. each block monomial order with $U \ll V$.

Subalgorithm 1. *(MSQIS_Alg 1)*

Input: $F \subset I$ *a polynomial set, where* $I \subset k[X]$ *is a positive-dimensional ideal*
Output: $U \subset X$ *an MSQIS modulo* F *w.r.t. each block monomial order with* $U \ll V$
Line 1: *Compute* $Lbl_{X,F}$*, and reorder* X *in descending order and* $Lbl_{X,F}$ *on the order of* X*.*
Line 2: $V := \emptyset$*, and* $Vlbl := \emptyset$*.*
Line 3: while $Vlbl \neq F$ **do**
Line 4: $V := V \cup \{X_1\}$, $X := X \backslash \{X_1\}$ *and* $Vlbl := Vlbl \cup (Lbl_{X,F})_1$*.*
Line 5: *Remove* $(Lbl_{X,F})_1$ *from* $Lbl_{X,F}$*.*
Line 6: $d := \sharp V$*.*
Line 7: for i **from** $d-1$ **to** 1 **by** -1 **do**
Line 8: **if** $\bigcup_{v\in V, v\neq V_i} lbl_{v,F} = F$ **then**
Line 9: $V := V \backslash \{V_i\}$*, and turn back to Line 6.*
Line 10: $U := X \backslash V$*, and* **return** U*.*

Theorem 2. *Let* $I \subset k[X]$ *be a positive-dimensional ideal and* $F \subset I$ *be a polynomial set. Then Subalgorithm 1 computes an MSQIS* U *modulo* F *w.r.t. each block monomial order with* $U \ll V$*.*

Proof. Termination: Since $\bigcup_{x\in X} lbl_{x,F} = F$, Lines 3-5 terminate. Moreover, the repetitive running of Line 9 gives rise to a strictly descending chain of sets, so the algorithm terminates.
Correctness: If the algorithm terminates, we have $\bigcup_{v\in V} lbl_{v,F} = F$ and $\bigcup_{v\in V, v\neq V_i} lbl_{v,F} \neq F$, $i = 1, 2, \cdots, \sharp V$. We prove $V \neq X$ by contradiction. Suppose $V = X$. Then $\bigcup_{x\in X, x\neq x_i} lbl_{x,F} \neq F$, $i = 1, 2, \cdots, n$. So for each $x_i \in X$, there exists $f_i \in F$, s.t. $f_i \in lbl_{x_i,F}$ and $f_i \notin lbl_{x_j,F}$ with $j \neq i$. It means f_i only contains x_i, i.e., $f_i \in k[x_i]$, $i = 1, 2, \cdots, n$. Then I is zero-dimensional, which contradicts with the condition, thus $V \neq X$, and $U \neq \emptyset$. So according to Proposition 4, U is an MSQIS modulo F w.r.t. each block monomial order with $U \ll V$. □

Although Subalgorithm 1 computes an MSQIS U w.r.t. each block monomial order with $U \ll V$, which certainly means that there exists $\prec_{U,V}$, w.r.t. which U is an MSQIS, experiments show that the cardinality of the "new" MSQIS obtained with Subalgorithm 1 may be not smaller than the "old" one, and it affects the efficiency of the main algorithm: From the procedure of the main algorithm, it contains two aspects of calculations: a series of MSQIS's and corresponding S-polynomial computations. Moreover, experiments show that the cost of the latter is dominant. By Theorem 1, if U is verified to be an MSQIS w.r.t. $\prec_{U,V}$, then we exclude all the dependent sets $\bar{U} \supsetneq U$ modulo I. In this sense, to avoid

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redundant S-polynomial computations corresponding to $\bar{U}$, MSQIS's with smaller cardinalities are preferred. To make up for such defect of Subalgorithm 1, we present another algorithm for computing a new MSQIS with strictly smaller cardinality than the old MSQIS.

3.3. Computing a New MSQIS with Smaller Cardinality Than the Old

Let U_0 be an MSQIS modulo F_0 w.r.t. $\prec_{U_0,V_0}$, H be the normal form set obtained by executing the Buchberger operations on F_0 w.r.t. $\prec^{grl}_{U_0,V_0}$ and $F := F_0 \cup H$. Suppose that the altering condition $H \cap k[U_0] \neq \emptyset$ holds. Now we think about computing a new MSQIS from a different point of view. Let $U_{max} := \{\tilde{U} \subset U_0 \mid \sharp\tilde{U} = \sharp U_0 - 1\}$. Then we try to find U from U_{max}, s.t. there exists $\prec_{U,V}$, w.r.t. which U is an MSQIS modulo F.

Proposition 6. *Let $F \subset k[X]$ be a polynomial set, $U \subset X$, $V := X\backslash U$, and $F \cap k[U] = \emptyset$. There exists $\prec_{U,V}$ on $T(X)$, s.t. U is an MSQIS modulo F w.r.t. $\prec_{U,V}$, iff there exists $\prec_V$ on $T(V)$, s.t. $LM_{\prec_V}(F_V) \cap T(v) \neq \emptyset$ for each $v \in V$.*

Proof. **Necessity:** Since U is an MSQIS modulo F w.r.t. $\prec_{U,V}$, by Definition 4, for each $v \in V$, we have $LM_{\prec_{U,V}}(F) \cap T(U) = \emptyset$ and $LM_{\prec_{U,V}}(F) \cap T(U,v) \neq \emptyset$, i.e., there exists $f \in F$, s.t. $LM_{\prec_{U,V}}(f) \notin T(U)$ and $LM_{\prec_{U,V}}(f) \in T(U,v)$. According to Proposition 2, we have $f \notin k[U]$ and $f \in k[U,v]$. Thus $f_V \in k[v]$. Let $\prec_V$ be the restriction of $\prec_{U,V}$ to $T(V)$. Then $LM_{\prec_V}(f_V) \in T(v)$, i.e., $LM_{\prec_V}(F_V) \cap T(v) \neq \emptyset$ for each $v \in V$.
Sufficiency: Let $\prec_{U,V}$ be a block monomial order of which the restriction to $T(V)$ is $\prec_V$. Since $F \cap k[U] = \emptyset$, $LM_{\prec_{U,V}}(F) \cap T(U) = \emptyset$ by Proposition 2. So U is an SQIS modulo F w.r.t. $\prec_{U,V}$. Since $LM_{\prec_V}(F_V) \cap T(v) \neq \emptyset$, there exists $f_V \in F_V$, s.t. $LM_{\prec_V}(f_V) \in T(v)$ for each $v \in V$. Write $LM_{\prec_{U,V}}(f) = t_V t_U$, where $t_V \in T(V)$, $t_U \in T(U)$. By Definition 3, $t_V = LM_{\prec_V}(f_V)$. Thus $t_V \in T(v)$, and $LM_{\prec_{U,V}}(f) \in T(U,v)$, i.e., $LM_{\prec_{U,V}}(F) \cap T(U,v) \neq \emptyset$ for each $v \in V$. So U is an MSQIS modulo F w.r.t. $\prec_{U,V}$. □

Based on Proposition 6, we transfer the work from $k[X]$ to $k[V]$. We first simplify F_V: For $f_V \in F_V$, we collect similar monomials, assign coefficients to 1 and drop the monomials that are factors of others. Let $V := [v_1, v_2, \cdots, v_{n-r}]$ and $\gamma := (\gamma_1, \gamma_2, \cdots, \gamma_{n-r})$. Then V^γ is defined as $v_1^{\gamma_1} v_2^{\gamma_2} \cdots v_{n-r}^{\gamma_{n-r}}$. We check the existence of $\prec_V$ in two steps. 1) Check whether there exists $\tilde{F} \subset F_V$ (maybe not unique) consisting of $n-r$ polynomials of the form

$$f_{V,i} = V^{\alpha(i)} + \sum_{\beta} V^{\beta}, \tag{2}$$

where $\alpha(i) = m_i \mathbf{e_i}$, $m_i \in \mathbb{N}_+$, and $\beta \in (\mathbb{N})^{n-r}$, $i = 1, 2, \cdots, n-r$. If $\tilde{F}$ does not exist, there does not exist $\prec_V$, s.t. $LM_{\prec_V}(F_V) \cap T(v) \neq \emptyset$ for each $v \in V$. If $\tilde{F}$ exists, we go to the next step. 2) For each $\tilde{F}$, we check whether there exists $\prec_V$, s.t. $LM_{\prec_V}(f_{V,i}) = V^{\alpha(i)}$, $i = 1, 2, \cdots, n-r$. If $\prec_V$ exists, then $LM_{\prec_V}(F_V) \cap T(v) \neq \emptyset$ for each $v \in V$. If $\prec_V$ does not exist for all $\tilde{F}$, then there does not exist $\prec_V$, s.t. $LM_{\prec_V}(F_V) \cap T(v) \neq \emptyset$ for each $v \in V$.

However, it is difficult to check the existence of such a monomial order. According to the relation between a monomial order and a regular matrix in [15], we give a sufficient condition for that.

Proposition 7. *Let $U \subset X$, $V := X\backslash U := [v_1, v_2, \cdots, v_{n-r}]$, $\tilde{F} \subset k(U)[V]$ be a polynomial set of the form (2) and $\mathbf{w} \in (\mathbb{R}_+)^{n-r}$ be a vector. Define*

$$C_{\tilde{F}} := \{\mathbf{w} \in (\mathbb{R}_+)^{n-r} \mid (\alpha(i) - \beta) \cdot \mathbf{w} > 0,\ i = 1, 2, \cdots, n-r\}.$$

If $C_{\tilde{F}} \neq \emptyset$, then there exists $\prec_V$ on $T(V)$, s.t. $LM_{\prec_V}(f_{V,i}) = V^{\alpha(i)}$, $i = 1, 2, \cdots, n-r$.

Proof. According to [15], a regular $n \times n$ matrix A over $\mathbb{R}_+$ determines a monomial order $\prec$ as follows. Each row A_i of A is actually a weight vector, $i = 1, 2, \cdots, n$, and monomials $t_1 \prec t_2$ is determined

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by comparing the A_i-degrees of them until the first inequality is found. So if $C_{\tilde{F}} \neq \emptyset$, there exists $\prec_V$ on $T(V)$ determined by an $(n-r)\times(n-r)$ matrix of which the first row is a vector $\mathbf{w} \in C_{\tilde{F}}$, s.t. $LM_{\prec_V}(f_{V,i}) = V^{\alpha(i)}, i = 1, 2, \cdots, n-r$. □

Moreover, $C_{\tilde{F}}$ is regarded as the feasible region of a linear programming problem, and the work is done at a smallcost. Details are referred to [16].

We next present Subalgorithm 2 for seeking U from U_{max}, s.t. there exists $\prec_{U,V}$, w.r.t. which U is an MSQIS modulo F.

Subalgorithm 2. *(MSQIS_Alg 2)*

Input: $F \subset k[X]$ *a polynomial set*
$U_0 \subset X$ *an indeterminate set*

Output: $U \neq \emptyset$ *there exists* $\prec_{U,V}$, *w.r.t. which* U *is an MSQIS modulo* F *or* $\emptyset$ *if we find none*

Line 1: $U_{max} := \{\tilde{U} \subset U_0 \mid \sharp\tilde{U} = \sharp U_0 - 1\}$.
Line 2: **for** i **from** 1 **to** $\sharp U_{max}$ **do**
Line 3: **if** $F \cap k[(U_{max})_i] = \emptyset$ **then**
Line 4: $V := X \backslash (U_{max})_i$, *and for each* $f_V \in F_V$, *collect similar monomials, assign coefficients to 1 and drop the monomials that are factors of others.*
Line 5: $F_{(2)} := \{\tilde{F} \subset F_V \mid \tilde{F}$ *is a polynomial set of the form (2)*$\}$.
Line 6: **if** $F_{(2)} \neq \emptyset$ **then**
Line 7: $\tilde{F} := (F_{(2)})_1$, *and* $F_{(2)} := F_{(2)} \backslash \{\tilde{F}\}$.
Line 8: **if** $C_{\tilde{F}} \neq \emptyset$ *(through linear programming theory)* **then**
Line 9: **return** $(U_{max})_i$.
Line 10: **else**
Line 11: *Turn back to Line 6.*
Line 12: **return** $\emptyset$.

The case that Subalgorithm 2 fails to obtain an MSQIS will be illustrated by Example 1. Furthermore, the finiteness of U_{max} and $F_{(2)}$ plus the linear programming theory guarantee the termination of Subalgorithm 2, and the correctness of it follows from Proposition 7.

Example 1. *Consider the polynomial set* $F_0 = \{f_1 = xy^2 - xz^2, f_2 = yz - uv, f_3 = y^2w + zw, f_4 = u^2v - uw^2\} \subset k[x, y, z, u, v, w]$.

First, let $\prec_{U_0,V_0} := \prec_{[u,z,y],[x,w,v]}$ *with* $\prec_{U_0} = grlex(u, z, y)$ *and* $\prec_{V_0} = plex(x, w, v)$. *Then* $U_0 = \{u, z, y\}$ *is an MSQIS modulo* F_0 *w.r.t.* $\prec_{U_0,V_0}$.

Second, we execute the Buchberger operations on F_0 *w.r.t.* $\prec^{grl}_{[u,z,y],[x,w,v]}$. *The critical pair set* $S := \{(f_i, f_j) \mid 1 \leq i, j \leq 4, i < j\}$. *Let* $\tilde{S} := \{(f_3, f_4)\}$ *be a prior subset of* S. *The normal form set* H *of the S-polynomial of* (f_3, f_4) *modulo* F_0 *w.r.t.* $\prec^{grl}_{[u,z,y],[x,w,v]}$ *is* $\{f_5 = uzy^3 + uz^2y\}$.

Third, let $F := F_0 \cup H := \{f_1, f_2, \cdots, f_5\}$. *Note that* $H \cap k[U_0] \neq \emptyset$, *we try to find* U *from* U_{max} *with Subalgorithm 2, where* $U_{max} := \{\{z, y\}, \{u, y\}, \{u, z\}\}$, *s.t. there exists* $\prec_{U,V}$, *w.r.t. which* U *is an MSQIS modulo* F. *It requires us to check each* $\tilde{U} \in U_{max}$ *in turn.*

$\{z, y\}$: *simplified* $F_{[x,w,v,u]} := \{x, vu, w, vu^2 + w^2u, u\}$, *thus there does not exist a power of* v *in* $F_{[x,w,v,u]}$, *and there does not exist* $\tilde{F} \subset F_{[x,w,v,u]}$ *consisting of polynomials of the form (2).*

$\{u, y\}$: *simplified* $F_{[x,w,v,z]} := \{xz^2, v + z, wz, w^2 + v, z^2\}$, *thus there does not exist a power of* x *in* $F_{[x,w,v,z]}$, *and there does not exist* $\tilde{F} \subset F_{[x,w,v,z]}$ *consisting of polynomials of the form (2).*

$\{u, z\}$: *simplified* $F_{[x,w,v,y]} := \{xy^2, v + y, wy^2, w^2 + v, y^3\}$, *thus there does not exist a power of* x *in* $F_{[x,w,v,y]}$, *and there does not exist* $\tilde{F} \subset F_{[x,w,v,y]}$ *consisting of polynomials of the form (2).*

In this case, Subalgorithm 2 fails to return $U \neq \emptyset$, *s.t. there exists* $\prec_{U,V}$, *w.r.t. which* U *is an MSQIS modulo* F.

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Example 1 shows that Subalgorithm 2 does not always work, even though it returns a new MSQIS with smaller cardinality than the old if it succeeds. We can now describe the alternative application of Subalgorithms 1 and 2 to obtain a not bad MSQIS. At the start of the main algorithm, since we have no MSQIS, we need to compute one with Subalgorithm 1. In other cases, we will try with Subalgorithm 2 at first if the altering condition holds, and then we can compute an MSQIS with Subalgorithm 1 if Subalgorithm 2 fails. In 96% of more than 100 examples we have experimented, MSQIS's are obtained with Subalgorithm 2. In the rest, Subalgorithm 2 fails, and MSQIS's are computed with Subalgorithm 1.

Furthermore, we need to make some explanation for the choice of U_{max}. First, since $H \cap k[U_0] \neq \emptyset$, U_0 is dependent modulo I. It means that at least one indeterminate of U_0 has to be moved into V_0. So we give priority to $U \subsetneq U_0$ rather than other indeterminate sets with smaller cardinalities than U_0. Second, generally speaking, the greater the cardinality of U is, the greater the possibility of U becoming an MSQIS is, so we only check the proper subsets of maximal cardinality of U_0.

3.4. Strategy for Selecting a Prior Subset of the Critical Pair Set

Weispfenning proposed the normal strategy for selecting prior critical pairs from the critical pair set for the normal form computation in [3]. Similarly, we present a strategy for our problem, in order to compute fewer S-polynomials to arrive at the altering condition, and furthermore, reduce the number of critical pairs calculated before we get a maximal independent set.

From the procedure of the main algorithm, if the altering condition $H \cap k[U] \neq \emptyset$ is true, we will compute a new MSQIS U' and switch to execute the Buchberger operations w.r.t. $\prec^{grl}_{U',V'}$. So the Buchberger operations w.r.t. one block monomial order can be regarded as eliminating the indeterminates of V. To compute the S-polynomial of (f, g), f and g will be multiplied by s and t, respectively. Generally speaking, the smaller the degrees of s and t w.r.t. V are, the sooner the elements of V will be eliminated. To accelerate the elimination, we give priority to such critical pairs. Note that in Subalgorithm 3, for a monomial m and $U \subset X$, $Coeff(m, U)$ returns the coefficient of m w.r.t. U and $Degree(m)$ computes the total degree of m w.r.t. X.

Subalgorithm 3. *(SelCritPairs)*

Input: P *a critical pair set*
$\prec_{U,V}$ *a block monomial order*
Output: P_{pri} *a critical pair set*
Line 1: $DegV := [\,]$.
Line 2: **for** i **from** 1 **to** $\sharp P$ **do**
Line 3: $LcmP := lcm(LM_{\prec_{U,V}}((P_i)_1), LM_{\prec_{U,V}}((P_i)_2))$.
Line 4: $q_1 := LcmP/LM_{\prec_{U,V}}((P_i)_1)$, $q_2 := LcmP/LM_{\prec_{U,V}}((P_i)_2)$.
Line 5: $DegP := Degree(Coeff(q_1, U)) + Degree(Coeff(q_2, U))$.
Line 6: *Append* $DegP$ *to* $DegV$.
Line 7: $P_{pri} := \{P_i \in P \mid (DegV)_i = min(DegV)\}$.
Line 8: **return** P_{pri}.

Termination and correctness of Subalgorithm 3 are obvious, and we omit the proofs.

3.5. The Main Algorithm

Based on Subalgorithms 1, 2 and 3, we present the main algorithm (Algorithm 1). We apply into it the Buchberger Criteria [3] and the F_4 technique [17]. Note that in Algorithm 1, the function $Update(G, P, H, \prec)$ plays a role in updating the ideal basis G and the critical pair set P w.r.t. a polynomial set H and a monomial order $\prec$. It also leads to the fact that Algorithm 1 computes the reduced Gröbner basis of I [3]. The function $F_4Reduce(P, G, \prec)$ computes the normal form set of a critical pair set P modulo G w.r.t. $\prec$.

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Algorithm 1. *(ComputeUandG)*

Input: $I := \langle F_0 \rangle \subset k[X]$ *a positive-dimensional ideal*

Output: U *a maximal independent set modulo* I

G *the reduced Gröbner basis of* I *w.r.t.* $\prec_{U,V}^{grl}$ $(V := X \backslash U)$

Line 1: $F_{all} := F_0$, $U := MSQIS\_Alg\ 1(F_{all})$ *and* $V := X \backslash U$.

Line 2: $(G, P) := Update(\emptyset, \emptyset, F_0, \prec_{U,V}^{grl})$, $P_p := SelCritPairs(P, \prec_{U,V}^{grl})$ *and* $P_r := P \backslash P_p$.

Line 3: while $P_p \neq \emptyset$ **do**

Line 4: $H := F_4Reduce(P_p, G, \prec_{U,V}^{grl})$, *and* $F_{all} := F_{all} \cup H$.

Line 5: **if** $H \cap k[U] \neq \emptyset$ **then**

Line 6: $U := MSQIS\_Alg\ 2(F_{all}, U)$.

Line 7: **if** $U = \emptyset$ **then**

Line 8: $U := MSQIS\_Alg\ 1(F_{all})$, $V := X \backslash U$, *and turn back to Line 2.*

Line 9: **else**

Line 10: $(G, P) := Update(G, P_r, H, \prec_{U,V}^{grl})$, $P_p := SelCritPairs(P, \prec_{U,V}^{grl})$ *and* $P_r := P \backslash P_p$.

Line 11: return U, G.

Theorem 3. *Let* $I \subset k[X]$ *be a positive-dimensional ideal. Then Algorithm 1 computes a maximal independent set* U *modulo* I *and a Gröbner basis of* I *w.r.t.* $\prec_{U,V}^{grl}$.

Proof. Termination: Since Subalgorithms 1, 2 and 3, Functions $Update$ and $F_4Reduce$ all terminate, we only need to prove the termination of the while loop. First we consider Lines 6-8. Let $Y := \{\tilde{X} \subset X \mid \tilde{X} \neq \emptyset\}$. If $H \cap k[U] \neq \emptyset$, then Lines 6-8 run, and they compute U' (distinguishing from U), s.t. there exists $\prec_{U',V'}$, w.r.t. which U' is an MSQIS modulo F_{all}. Furthermore, according to 2) of Proposition 5, U is not an MSQIS modulo F_{all} w.r.t. any block monomial order with $U \ll V$, so $U' \neq U$. Denote $Y_1 := Y \backslash \{U\}$. The repetitive running of Lines 6-8 gives rise to a strictly descending chain of sets: $Y \supsetneq Y_1 \supsetneq Y_2 \supsetneq \cdots$. Since Y is finite, Lines 6-8 terminate. Suppose Lines 6-8 terminate with $U := \hat{U}$ and $V := \hat{V}$. Then after they terminate, the while loop are just the execution of the Buchberger operations on F_0 w.r.t. $\prec_{\hat{U},\hat{V}}^{grl}$. Algorithm 1 terminates since the Buchberger's Algorithm terminates.

Correctness: If the algorithm terminates, it returns a Gröbner basis G of I w.r.t. $\prec_{\hat{U},\hat{V}}^{grl}$, where $\hat{U}$ is an SQIS modulo F_{all} w.r.t. $\prec_{\hat{U},\hat{V}}^{grl}$ by Proposition 3. Since $F_{all} \supset G$, F_{all} is also a Gröbner basis of I w.r.t. $\prec_{\hat{U},\hat{V}}^{grl}$. By Proposition 1, $\hat{U}$ is strongly independent modulo I w.r.t. $\prec_{\hat{U},\hat{V}}^{grl}$. Thus $\hat{U}$ is independent modulo I. Furthermore, according to Theorem 1, any $\bar{U} \supsetneq \hat{U}$ is dependent modulo I, so $\hat{U}$ is maximally independent modulo I. □

The cyclic 4 problem illustrates Algorithm 1 and the advantage of Subalgorithm 2.

Example 2. *Consider the ideal* $I := \langle F_0 \rangle \subset k[a, b, c, d]$, *where* $F_0 = \{f_1 = a + b + c + d, f_2 = ab + bc + ad + cd, f_3 = abc + abd + acd + bcd, f_4 = abcd - 1\}$.

Initialization:

$F_{all} = \{f_1, f_2, f_3, f_4\}$, $U = [c, b, a]$ *and* $V = [d]$.

$G = \{f_1\}$, $P_p = P = \{\{f_1, f_2\}, \{f_1, f_3\}, \{f_1, f_4\}\}$ *and* $P_r = \emptyset$.

The while loop runs for the 1st time:

$H = \{f_5 = a^2bc + ab^2c + abc^2 + 1, f_6 = a^2b + a^2c + ab^2 + 2abc + ac^2 + b^2c + bc^2, f_7 = a^2 + 2ac + c^2\}$, *and* $F_{all} = \{f_1, f_2, \cdots, f_7\}$. *Since* $H \cap k[c, b, a] \neq \emptyset$, *Lines 6-8 run, and Function* $MSQIS\_Alg\ 2(F_{all}, [c, b, a])$ *returns* $U = [b, a]$. *Then* $V = [d, c]$, *and back to Line 2,* $G = \{f_1\}$, $P_p = P = \{\{f_1, f_2\}, \{f_1, f_3\}, \{f_1, f_4\}\}$ *and* $P_r = \emptyset$.

If we remove Subalgorithm 2, $MSQIS\_Alg\ 1$ *will return* $U = [d, b, a]$, $V = [c]$. *However, since* $\{b, a\}$ *is an MSQIS modulo* F_{all} *w.r.t.* $\prec_{[b,a],[d,c]}^{grl}$, *by Theorem 1,* $\{d, b, a\}$ *is dependent modulo* I, *and the computation of*

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the S-polynomials w.r.t. $\prec^{grl}_{[d,b,a],[c]}$ *will be redundant.*

The while loop runs for the 2nd time:

$H = \{f_5, f_6, f_7\}$ *and* F_{all} *remains unchanged. Since* $H \cap k[b,a] = \emptyset$*, Line 10 runs, and it returns* $G = \{f_1, f_7\}$, $P_p = P = \{\{f_5, f_6\}, \{f_6, f_7\}\}$ *and* $P_r = \emptyset$.

The while loop runs for the 3rd time:

$H = \{f_8 = -a^4 + a^3b - a^3c + a^2b^2 + a^2bc - 1, f_9 = -a^3 - a^2c + ab^2 + b^2c\}$, *and* $F_{all} = \{f_1, f_2, \cdots, f_9\}$. *Since* $H \cap k[b,a] = \emptyset$*, Line 10 runs and returns* $G = \{f_1, f_7, f_8, f_9\}$, $P = \{\{f_7, f_8\}, \{f_7, f_9\}, \{f_8, f_9\}\}$, $P_p = \{\{f_8, f_9\}\}$ *and* $P_r = \{\{f_7, f_8\}, \{f_7, f_9\}\}$.

The while loop runs for the 4th time:

$H = \{f_{10} = a^3b^2 + a^2b^3 - a - b\}$, *and* $F_{all} = \{f_1, f_2, \cdots, f_{10}\}$. *Since* $H \cap k[b,a] \neq \emptyset$*, Lines 6-8 run, and MSQIS_Alg* $2(F_{all}, [b,a])$ *returns* $U = [a]$*, then* $V = [d,b,c]$. *Back to Line 2,* $G = \{f_1\}$, $P = \{\{f_1, f_2\}, \{f_1, f_3\}, \{f_1, f_4\}\}$, $P_p = \{\{f_1, f_2\}\}$ *and* $P_r = \{\{f_1, f_3\}, \{f_1, f_4\}\}$.

In fact, $\{a\}$ *is a maximal independent set modulo I. We skip remaining calculations and give the final results. After the while loop runs for the 10th time,* $P_p = \emptyset$ *and the algorithm terminates. It returns* $U = \{a\}$ *as a maximal independent set modulo I and* $G = \{a + b + c + d, a^2 + 2ac + c^2, -a^4 + a^3b - a^3c + a^2b^2 + a^2bc - 1, -a^3 - a^2c + ab^2 + b^2c, a^5 + a^4c - a - c\}$ *as the reduced Gröbner basis of I w.r.t.* $\prec^{grl}_{[a],[d,b,c]}$.

4. Experiments

We have compared Algorithm 1 with the direct method and the W-W method, and both of them use Weispfenning's algorithm to compute a maximal independent set from a Gröbner basis w.r.t. a graded reverse lexicographic monomial order. It should be mentioned that the Buchberger Criteria and the F_4 technique are also applied to these two methods. Table 1 shows the final maximal independent set (U), the numbers of critical pairs (Number) and the CPU times (Time) of 18 examples, which are referred to [18]. Experiments were conducted on a Dell notebook, which is configured with Intel Core i5-8250U, 8G memory. The development environment is Windows 10 Professional, Maple 2020.

Table 1. Comparison with the direct method and the W-W method.

Example	U	Number	Time (sec)
	Direct/W-W, Ours	Direct, W-W, Ours	Direct, W-W, Ours
Chemistry	$\{j,h,e\}, \{b,f,g\}$	168, 202, 81	**7542.234**, 354.578, **180.375**
Sturmfels and Eisenbud	$\{z,y,x\}, \{b,u,t\}$	104, 158, 62	1.484, 2.906, 0.359
Schimoyama/Yokoyama$_1$	$\{z,y\}, \{x\}$	35, 37, 17	0.203, 0.219, 0.125
Schimoyama/Yokoyama$_2$	$\{x,t,z,b,w,u\}, \{y,w,u,s,v\}$	48, 72, 26	0.313, 0.641, 0.109
Horrocks	$\{d,e,f\}, \{e,f,c\}$	**513**, **1057**, **285**	78.140, 191.250, 18.281
Schwarz	$\{c\}, \{c\}$	80, 119, 90	**63.735**, **60.985**, **6.125**
Cyclic roots 4	$\{b\}, \{a\}$	27, 33, 17	0.141, 0.172, 0.078
Cyclic roots 5 homog	$\{h\}, \{a\}$	228, 334, 167	44.360, 65.453, 22.016
De Jong	$\{c,b,g,f,l,k,h,d\}$, $\{l,k,d,a,e,j\}$	**326**, **449**, **62**	**191.406**, **415.000**, **5.266**
mat3^2	$\{h,g,e,a\}, \{c,b,e,a\}$	131, 173, 100	7.687, 10.547, 5.937
Schimoyama/Yokoyama$_3$	$\{b,c,d,e,f,g,h,j,l\}$, $\{b,c,d,e,f,g,h,j,l\}$	72, 108, 36	51.844, 45.719, 7.047
Riemenschneider	$\{y,p,q,z,s,w,t,v\}$, $\{p,w,t,x\}$	92, 97, 44	0.984, 0.765, 0.187
Mikro	$\{b,g,f,e,c\}, \{c,d\}$	**848**, **1114**, **319**	**178650.985**, 625.438, **111.828**
Buchberger	$\{x,c,t,d\}, \{c,a,t,d\}$	7, 11, 7	0.094, 0.172, 0.078
Lanconelli	$\{l,k,j,h,e,g,c\}$, $\{j,h,f,d,e,g,c\}$	5, 8, 2	0.719, 0.610, 0.078
Wang2	$\{y\}, \{z\}$	2, 9, 5	0.016, 0.079, 0.032
Siebert	$\{y,x\}, \{z\}$	239, 306, 398	101.640, 116.172, 34.422
Macaulay	$\{d,c\}, \{c,b\}$	55, 84, 40	0.656, 1.203, 0.297

As is evident from Table 1, our algorithm generally computes fewer critical pairs and our CPU times are usually less than the other two methods. Precisely speaking, in 14 of 18 examples, our algorithm computes the fewest critical pairs, and especially in Examples Horrocks, De Jong and Mikro, our algorithm computes critical pairs several times fewer than the other two methods. In 17 of 18 examples except Wang2, our CPU times are all less than the other two approaches. Especially in

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Example Mikro, our algorithm is thousands of times faster than the direct method. In Example De Jong, our CPU time is dozens of times less than the other two methods.

5. Conclusions

In this paper, we have presented an efficient algorithm to compute a maximal independent set U modulo a positive-dimensional ideal I and a Gröbner basis of I w.r.t. a block monomial order $\prec_{U,V}$ simultaneously. It is based on a new concept we proposed: the maximal strongly quasi-independent set modulo a polynomial set. As a "local" concept, it reveals some information about the independent sets modulo I in advance, thus our algorithm considerably reduces the number of critical pairs calculated compared with existing methods which compute the maximal independent set and the Gröbner basis separately. Experiments verify the good performance of our algorithm.

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