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The Second Critical Exponent for a Time Fractional Reaction-Diffusion Equation

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Abstract: In this paper, we consider the Cauchy problem of a time fractional nonlinear diffusion equation. According to the Kaplan's first eigenvalue method, we first prove the blow-up of the solutions in finite time for some sufficient conditions. We next give sufficient conditions for the existence of global solutions by using the result of Zhang and Sun. In conclusions, we find the second critical exponent for the existence of global and non-global solutions via the decay rates of the initial data at spatial infinity.

Keywords: time fractional diffusion equation; blow-up; global existence; critical exponent

MSC: 26A33; 35A01; 35B44; 35K15; 35R11

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1. Introduction

We study the Cauchy problem for a time fractional reaction-diffusion equation

$$\begin{cases} \partial_t^\alpha u = \Delta u + |u|^{p-1}u, & x \in \mathbf{R}^n,\ t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbf{R}^n, \end{cases} \tag{1}$$

where $n \geq 1$, $0 < \alpha < 1$, $p > 1$, $u_0 \in C_0(\mathbf{R}^n) := \{f \in C(\mathbf{R}^n); \lim_{|x|\to\infty} f(x) = 0\}$, and ∂_t^α denotes the Caputo time fractional derivative of order α defined by

$$\partial_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\partial u}{\partial s}(x,s)ds, \quad 0 < \alpha < 1. \tag{2}$$

Here, $\Gamma(\cdot)$ is the Gamma function. Moreover, the Caputo time fractional derivative (2) is related to the Riemann-Liouville derivative by

$$\partial_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt}(t^{-\alpha} * [u - u_0])(t) = \frac{\partial}{\partial t}\, {}_0I_t^{1-\alpha}(u(x,t) - u_0(x)), \tag{3}$$

where ${}_0I_t^{1-\alpha}$ denotes left Riemann-Liouville fractional integrals of order $1-\alpha$ and is defined by

$${}_0I_t^{1-\alpha} u = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u(s) ds.$$

For a given initial data u_0, let $T^* = T^*(u_0)$ be the maximal existence time of the solution of (1). If $T^* = \infty$, the solution is global in time. However, if $T^* < \infty$, then the solution is not global in time in the sense that it blows up at $t = T^*$ such as

$$\limsup_{t\to T^*} \|u(\cdot,t)\|_{L^\infty(\mathbf{R}^n)} = \infty.$$

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A lot of significant results on the critical exponents for nonlinear parabolic equations have been obtained during the past decades. Fujita [1] considered the following Cauchy problem:

$$\begin{cases} u_t = \Delta u + u^p, & x \in \mathbf{R}^n,\ t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbf{R}^n. \end{cases} \tag{4}$$

In [1], it is shown that (4) possesses the critical Fujita exponent $p^* = 1 + 2/n$ such that

- If $1 < p < p^*$, then the solution blows up in finite time for any nontrivial initial data.
- If $p > p^*$, then there are both global solutions and nonglobal solutions corresponding to small and large initial data, respectively.

According to Hayakawa [2], Kobayashi et al. [3], and Weissler [4], it has been known that $p = p^* = 1 + 2/n$ belongs to the blow-up case. In some situations, the size of the initial data required by the global and nonglobal solutions can be determined through the so-called second critical exponent with respect to the decay rates of the initial data as $|x| \to \infty$. When $p > p^* = 1 + 2/n$, Lee and Ni [5] established the second critical exponent $a^* = 2/(p-1)$ for (4) with initial data $u_0(x) = \lambda\psi(x)$, where $\lambda > 0$ and $\psi(x)$ is a bounded continuous function in $\mathbf{R}^n$, such that the following conditions hold:

- If $\liminf_{|x|\to\infty} |x|^a \psi(x) > 0$ for some $a \in (0, a^*)$ and any $\lambda > 0$, then the solution blows up in finite time.
- If $\limsup_{|x|\to\infty} |x|^a \psi(x) < \infty$ for some $a \in (a^*, n)$, then there is $\lambda_0 > 0$ such that the solution is global in time whenever $\lambda \in (0, \lambda_0)$.

Lee and Ni [5] proved that $a^* = 2/(p-1)$ belongs to the global case.

The weighted source case

$$\begin{cases} u_t = \Delta u + K(x)u^p, & x \in \mathbf{R}^n,\ t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbf{R}^n. \end{cases}$$

with $K(x) > 0$ of the order $|x|^\sigma$ for $\sigma > -1$ if $n = 1$ or for $\sigma > -2$ if $n \geq 2$ was considered with the critical Fujita exponent $p^* = 1 + (2+\sigma)/n$ by Pinsky [6].

The degenerate case

$$\begin{cases} u_t = \Delta u^m + u^p, & x \in \mathbf{R}^n,\ t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbf{R}^n, \end{cases} \tag{5}$$

with $m > 1$ and $\max(0, 1-2/n) < m < 1$ was thoroughly studied with the critical Fujita exponent $p^* = m + 2/n$ by Galaktionov et al. [7], Qi [8], and Mochizuki and Mukai [9]. Furthermore, Galaktionov [10], Mochizuki and Mukai [9], Kawanago [11], and Mochizuki and Suzuki [12] have shown that $p = p^* = m + 2/n$ belongs to the blow-up case. When $p > p^* = m + 2/n$, Mukai et al. [13], and Guo and Guo [14] obtained the second critical exponent $a^* = 2/(p-m)$ for (5).

The extended case

$$\begin{cases} u_t = \Delta u^m + K(x,t)u^p, & x \in \mathbf{R}^n,\ t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbf{R}^n. \end{cases} \tag{6}$$

with $K(x,t) = t^s|x|^\sigma$ for $s \geq 0$, $m > \max(0, 1-2/n)$, $p > \max(1, m)$, $\sigma > -1$ if $n = 1$, or $\sigma > -2$ if $n \geq 2$ was obtained with the critical Fujita exponent $p^* = m + s(m-1) + (2+2s+\sigma)/n$ by Qi [15]. In the case with $K(x,t) = K(x)$ of the order $|x|^\sigma$ as $|x| \to \infty$ with $\sigma \in \mathbf{R}$ in some cone D and $K(x,t) = 0$ if otherwise, Suzuki [16] considered (6) for $1 \leq m < p$ and obtained the critical Fujita exponent $p^* = m + \{2 + \max(\sigma, -n)\}/n$ and the second critical exponent $a^* = \{2 + \max(\sigma, -n)\}/(p-m)$.

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Winkler [17] considered the nonlinear diffusion equation not in divergence form

$$\begin{cases} u_t = u^p \Delta u + u^q, & x \in \mathbf{R}^n,\ t > 0, \\ u(x,0) = u_0(x) > 0, & x \in \mathbf{R}^n, \end{cases} \tag{7}$$

and obtained the critical Fujita exponent $q^* = p + 1$ for $p \geq 1$ such that

- Suppose that $1 \leq q < q^*$ ($1 \leq q < 3/2$ if $p = 1$). If u_0 decreases sufficiently fast in space, all positive solutions of (7) are global and unbounded.
- Suppose that $q = q^*$. Then, all positive solutions of (7) blow-up in finite time.
- Suppose that $q > q^*$. If u_0 is sufficiently large, then any positive solution of (7) blows up in finite time. If $u_0(x) \leq f(|x|)$ in $\mathbf{R}^n$, then the solutions of (7) are global, where f satisfies for $u(x,t) = (1+t)^{-\alpha} f((1+t)^{-\beta}|x|)$

$$\begin{cases} f'' + \left(\frac{n-1}{r} + \beta r f^{-p}\right) f' + \alpha f^{1-p} + f^{q-p} = 0, & r \in (0,\infty), \\ f(0) = f_0, \quad f'(0) = 0, \end{cases}$$

with $\alpha = 1/(q-1)$, $\beta = (q-p-1)/(2q-2)$, $f_0 > 0$, $r = (1+t)^{-\beta}|x|$.

Furthermore, Li and Mu [18] also considered (7) and obtained the second critical exponent $a^* = 2/(q-p-1)$ for $p > 1$ and $q > p+1+2/n$ with initial data $u_0(x) = \lambda\psi(x)$, where $\lambda > 0$ and $\psi(x)$ is a bounded continuous function in $\mathbf{R}^n$, such that

- Let $n \geq 2$. Assume that $\liminf_{|x|\to\infty} |x|^a \psi(x) > 0$. If $0 < a < a^*$, or $a = a^*$ and λ is large enough, then the solution $u(x,t)$ of (7) blows up in finite time.
- Assume that $\limsup_{|x|\to\infty} |x|^a \psi(x) < \infty$. If $a > a^*$, then there exists $\lambda_0 > 0$ such that the solution $u(x,t)$ of (7) is global in time whenever $\lambda \in (0, \lambda_0)$.

Yang et al. [19] and the author [20] studied the extended case

$$\begin{cases} u_t = u^p \Delta u + K(x) u^q, & x \in \mathbf{R}^n,\ t > 0, \\ u(x,0) = u_0(x) > 0, & x \in \mathbf{R}^n, \end{cases} \tag{8}$$

with the positive weight function $K \in C^0(\mathbf{R}^n)$ satisfying

$$c_1|x|^\sigma \leq K(x) \leq c_2|x|^\sigma, \quad |x| > R_0,\ \sigma > -2$$

for some $R_0, c_1, c_2 > 0$. Then, Yang et al. [19] obtained the critical Fujita exponent $q^* = p + 1$ for $p \geq 1$ such that

- Suppose that $1 < q < q^*$ ($1 < q < 3/2$ if $p = 1$). If u_0 decreases sufficiently fast in space, all positive solutions of (8) are global and unbounded.
- Suppose that $q = q^*$. Then, all positive solutions of (8) blow-up in finite time.
- Suppose that $q > q^*$. If u_0 is sufficiently large, then any positive solution of (8) blows up in finite time. If $u_0(x) \leq f(|x|)$ in $\mathbf{R}^n$, then the solutions of (8) are global, where f satisfies for $u(x,t) \leq (1+t)^{-\alpha} f((1+t)^{-\beta}|x|)$

$$\begin{cases} f'' + \left(\frac{n-1}{r} + \beta r f^{-p}\right) f' + \alpha f^{1-p} + C_1 r^\sigma f^{q-p} = 0, & r \in (0,\infty), \\ f(0) = f_0, \quad f'(0) = 0, \end{cases}$$

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with $-2 < \sigma < 0$, $C_1 = \sup_{\mathbf{R}^n}(K(x)|x|^{-\sigma})$, $\alpha = (\sigma+2)/(2q-2+p\sigma)$, $\beta = (q-p-1)/(2q-2+p\sigma)$, $f_0 > 0$, $r = (1+t)^{-\beta}|x|$, and

$$\begin{cases} f'' + \left(\frac{n-1}{r} + \beta r f^{-p}\right) f' + \alpha f^{1-p} + C_2 \max(r^\sigma, 1) f^{q-p} = 0, & r \in (0,\infty), \\ f(0) = f_0, \quad f'(0) = 0, \end{cases}$$

with $\sigma \geq 0$, $C_2 = \sup_{\mathbf{R}^n}(K(x)\min(|x|^{-\sigma}, 1))$.

Moreover, Yang et al. [19] and the author [20] obtained the second critical exponent $a^* = (2+\sigma)/(q-p-1)$ for $p \geq 1$ and $q > p+1$ with initial data $u_0(x) = \lambda\psi(x)$, where $\lambda > 0$ and $\psi(x)$ is a bounded continuous function in $\mathbf{R}^n$, such that

- Assume that $\liminf_{|x|\to\infty} |x|^a \psi(x) > 0$. If $0 < a < a^*$ with $\sigma > -2$, or $a = a^*$ with $\sigma > -2$ and λ is large enough, then the solution $u(x,t)$ of (8) blows up in finite time.
- Assume that $\limsup_{|x|\to\infty} |x|^a \psi(x) < \infty$. If $a > a^*$ with $\sigma \geq 0$, or $a \geq a^*$ with $-2 < \sigma < 0$, then there exists $\lambda_0 > 0$ such that the solution $u(x,t)$ of (8) is global in time whenever $\lambda \in (0, \lambda_0)$.

By reading the literature of time fractional nonlinear diffusion equations, we found that there are no studies on the second critical exponent to the Cauchy problem (1). In this paper, we give the second critical exponent to the Cauchy problem (1) based on the above mentioned literature.

The rest of this paper is organized as follows. In section 2, we give some preliminaries for the Cauchy problem (1). In section 3, two sufficient conditions for the blow-up of solutions in finite time are given in Theorem 1. In section 4, we state the existence of global solutions according to some conditions in Theorem 2. In Section 5, conclusions is presented.

2. Preliminaries

In this section, we give some preliminaries.

We need the following Wright type function

$$\phi_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\Gamma(-\alpha k + 1 - \alpha)}, \quad 0 < \alpha < 1,\ z \in \mathbf{C}. \tag{9}$$

The function ϕ_α is an entire function and satisfies the following properties:

(a) $\phi_\alpha(\theta) \geq 0$ for $\theta \geq 0$ and $\int_0^\infty \phi_\alpha(\theta) d\theta = 1$.

(b) $\int_0^\infty \phi_\alpha(\theta)\theta^r d\theta = \dfrac{\Gamma(1+r)}{\Gamma(1+\alpha r)}$ for $r > -1$.

The operator $A = \Delta$ generates a semigroup $\{T(t)\}_{t\geq 0}$ on $C_0(\mathbf{R}^n)$ with domain

$$D(A) = \{u \in C_0(\mathbf{R}^n) : \Delta u \in C_0(\mathbf{R}^n)\}.$$

Then $T(t)$ is an analytic and contractive semigroup on $C_0(\mathbf{R}^n)$, and

$$[T(t)u_0](x) = \int_{\mathbf{R}^n} G(x-y,t) u_0(y) dy,$$

where

$$G(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/(4t)}, \quad x \in \mathbf{R}^n,\ t > 0.$$

For $t \geq 0$, we define the operators $P_\alpha(t)$ and $S_\alpha(t)$ as

$$\begin{aligned}[P_\alpha(t)u_0](x) &= \int_0^\infty \phi_\alpha(\theta)[T(t^\alpha\theta)u_0](x)d\theta \\ &= \int_0^\infty \phi_\alpha(\theta) \int_{\mathbf{R}^n} G(x-y, t^\alpha\theta)u_0(y)dyd\theta, \qquad (10)\\ [S_\alpha(t)u_0](x) &= \alpha\int_0^\infty \theta\phi_\alpha(\theta)[T(t^\alpha\theta)u_0](x)d\theta \\ &= \alpha\int_0^\infty \theta\phi_\alpha(\theta) \int_{\mathbf{R}^n} G(x-y, t^\alpha\theta)u_0(y)dyd\theta, \qquad (11)\end{aligned}$$

where $\phi_\alpha(\theta)$ is the function defined by (9). Note that for given $x \in \mathbf{R}^n \backslash \{0\}$ and $t > 0$, $G(x, t^\alpha\theta) \to 0$ as $\theta \to 0$. Hence, $\int_0^\infty \phi_\alpha(\theta)G(x, t^\alpha\theta)d\theta$ is well defined. Since $\int_0^\infty \phi_\alpha(\theta)d\theta = 1$ and $\int_{\mathbf{R}^n} G(x,t)dx = 1$, we know that

$$\int_0^\infty \phi_\alpha(\theta) \int_{\mathbf{R}^n} G(x, t^\alpha\theta)dxd\theta = 1 \quad \text{for } t > 0.$$

Consider the linear equation

$$\begin{cases} \partial_t^\alpha u = \Delta u + f(x,t), & x \in \mathbf{R}^n,\ t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbf{R}^n, \end{cases} \qquad (12)$$

where $u_0 \in C_0(\mathbf{R}^n)$ and $f \in L^1((0,T), C_0(\mathbf{R}^n))$. If u is a solution of (12), then by [21], it satisfies

$$u(x,t) = [P_\alpha(t)u_0](x) + \int_0^t (t-s)^{\alpha-1}[S_\alpha(t-s)f(\cdot,s)](x)ds,$$

where $P_\alpha(t)$ and $S_\alpha(t)$ are given by (10) and (11), respectively.

Zhang and Sun [21] obtained the following lemmas related to the operators $P_\alpha(t)$ and $S_\alpha(t)$.

Lemma 1 ([21]). *If $u_0(x) \geq 0$, $u_0(x) \not\equiv 0$, then $[P_\alpha(t)u_0](x) > 0$, $[S_\alpha(t)u_0](x) > 0$ and*

$$\|P_\alpha(t)u_0\|_{L^1(\mathbf{R}^n)} = \|u_0\|_{L^1(\mathbf{R}^n)}, \quad \|S_\alpha(t)u_0\|_{L^1(\mathbf{R}^n)} = \frac{1}{\Gamma(\alpha)}\|u_0\|_{L^1(\mathbf{R}^n)}.$$

Proof. See Lemmas 2.1 (a) and 2.2 (a) in [21]. □

Lemma 2 ([21]). *Let $1 \leq p \leq q \leq +\infty$ and $1/r = 1/p - 1/q$.*

(a) *If $1/r < 2/n$, then*

$$\|P_\alpha(t)u_0\|_{L^q(\mathbf{R}^n)} \leq (4\pi t^\alpha)^{-\frac{n}{2r}} \frac{\Gamma(1-\frac{n}{2r})}{\Gamma(1-\frac{\alpha n}{2r})}\|u_0\|_{L^p(\mathbf{R}^n)}.$$

(b) *If $1/r < 4/n$, then*

$$\|S_\alpha(t)u_0\|_{L^q(\mathbf{R}^n)} \leq \alpha(4\pi t^\alpha)^{-\frac{n}{2r}} \frac{\Gamma(2-\frac{n}{2r})}{\Gamma(1+\alpha-\frac{\alpha n}{2r})}\|u_0\|_{L^p(\mathbf{R}^n)}.$$

Proof. See Lemmas 2.1 (b) and 2.2 (b) in [21]. □

Next, we give the definition of a mild solution of the Cauchy problem (1).

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Definition 1. *Let $u_0 \in C_0(\mathbf{R}^n)$ and $T > 0$. We call that $u \in C([0,T], C_0(\mathbf{R}^n))$ is a mild solution of the problem (1) if u satisfies the integral equation*

$$u(x,t) = [P_\alpha(t)u_0](x) + \int_0^t (t-s)^{\alpha-1}[S_\alpha(t-s)|u(\cdot,s)|^{p-1}u(\cdot,s)](x)ds, \quad t \in [0,T].$$

For the Cauchy problem (1), Zhang and Sun [21] has established the following local existence result.

Proposition 1 (Theorem 3.2 in [21]). *Let $0 < \alpha < 1$. For given $u_0 \in C_0(\mathbf{R}^n)$, there exists a maximal existence time $T^* > 0$ such that the problem (1) has a unique mild solution $u \in C([0,T^*], C_0(\mathbf{R}^n))$ and either $T^* = +\infty$ or $T^* < +\infty$ and $\|u\|_{L^\infty((0,t),C_0(\mathbf{R}^n))} \to \infty$. In addition, if $u_0(x) \geq 0$ and $u_0(x) \not\equiv 0$, then $u(x,t) \geq [P_\alpha(t)u_0](x) > 0$ for $t \in (0,T^*)$. Moreover, if $u_0 \in L^r(\mathbf{R}^n)$ for some $r \in [1,\infty)$, then $u \in C([0,T^*), L^r(\mathbf{R}^n))$.*

Furthermore, Zhang and Sun [21] also obtained the following blow-up and global existence results.

Proposition 2 (Theorem 4.3 in [21]). *Let $0 < \alpha < 1$, $u_0 \in C_0(\mathbf{R}^n)$ and $u_0(x) \geq 0$. If*

$$\int_{\mathbf{R}^n} u_0(x)\chi(x)dx > 1, \text{ where } \chi(x) = \left(\int_{\mathbf{R}^n} e^{-\sqrt{n^2+|x|^2}}dx\right)^{-1} e^{-\sqrt{n^2+|x|^2}},$$

then the mild solutions of (1) blow up in a finite time.

Proposition 3 (Theorem 4.4 in [21]). *Let $0 < \alpha < 1$, $u_0 \in C_0(\mathbf{R}^n)$, $u_0(x) \geq 0$ and $u_0(x) \not\equiv 0$.*

(a) *If $1 < p < 1 + 2/n$, then the mild solution of (1) blows up in a finite time.*
(b) *If $p \geq 1 + 2/n$ and $\|u_0\|_{L^{q_c}(\mathbf{R}^n)}$ is sufficiently small, where $q_c = n(p-1)/2$, then the mild solution of (1) exists globally.*

3. Blow-Up of Solution

In this section, we shall prove the following blow-up result.

Theorem 1. *Let $n \geq 1$ and $0 < \alpha < 1$. Assume that the initial data $u_0(x) = \lambda\psi(x) \geq 0$, where $\lambda > 0$ and $\psi \in C_0(\mathbf{R}^n)$. Suppose that one of the following two conditions holds:*

(a) *$\lambda > 0$ is large enough;*
(b) *$0 < a < 2/(p-1)$ and*

$$\liminf_{|x|\to\infty} |x|^a \psi(x) > 0. \tag{13}$$

Then, the solution of (1) blows up in finite time.

Proof. We take the similar strategy as Theorem 1 in [20] and Theorem 3.7 in [22], using the Kaplan's first eigenvalue method [23].

Let

$$B_m = \left\{x \in \mathbf{R}^n;\ |x - x_m| < \frac{1}{2}m\right\} \tag{14}$$

for a sequence $\{x_m\}_{m=1}^\infty$ satisfying $|x_m| = m$ for any $m \in \mathbf{N}$.

Remark 1. *The method using the sequence of balls B_m in (14) was used in [20,24–28].*

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Let $\lambda_m > 0$ denote the principal eigenvalue of $-\Delta$ with Dirichlet problem in B_m, and let $\phi_m(x)$ denote the corresponding eigenfunction, normalized by

$$\int_{B_m} \phi_m(x)dx = 1. \tag{15}$$

Let $T \in (0, T^*)$ be arbitrarily fixed. We define

$$F_m(t) = \int_{B_m} u(x,t)\phi_m(x)dx. \tag{16}$$

It follows from (1) that

$$\partial_t^\alpha \int_{B_m} u(x,t)\phi_m(x)dx = \int_{B_m} \Delta u(x,t)\phi_m(x)dx + \int_{B_m} u(x,t)^p \phi_m(x)dx \quad \text{for } t \in (0,T),$$

supplemented with the initial condition

$$F_m(0) = \int_{B_m} u_0(x)\phi_m(x)dx. \tag{17}$$

By integrating by parts, and the fact that $\phi_m(x) = 0$ and $\partial\phi_m/\partial\nu \leq 0$ on ∂B_m, where ν denotes the outward unit normal vector to B_m at $x \in \partial B_m$, and applying Green's formula, we have

$$\partial_t^\alpha \int_{B_m} u(x,t)\phi_m(x)dx \geq \int_{B_m} u(x,t)\Delta\phi_m(x)dx + \int_{B_m} u(x,t)^p \phi_m(x)dx.$$

Since the principal eigenvalue $\lambda_m > 0$ and the eigenfunction $\phi_m(x)$ satisfy

$$\Delta\phi_m(x) = -\lambda_m \phi_m(x),$$

we obtain

$$\partial_t^\alpha \int_{B_m} u(x,t)\phi_m(x)dx \geq -\lambda_m \int_{B_m} u(x,t)\phi_m(x)dx + \int_{B_m} u(x,t)^p \phi_m(x)dx, \tag{18}$$

By (15), (16) and Hölder' s inequality, we have

$$F_m(t) = \int_{B_m} u(x,t)\phi_m(x)^{\frac{1}{p}} \phi_m(x)^{1-\frac{1}{p}} dx \leq \left(\int_{B_m} u(x,t)^p \phi_m(x)dx \right)^{1/p}.$$

So, we obtain

$$\int_{B_m} u(x,t)^p \phi_m(x)dx \geq F_m(t)^p. \tag{19}$$

Using (16) and (19) in (18), it yields

$$\partial_t^\alpha F_m(t) \geq -\lambda_m F_m(t) + F_m(t)^p \quad \text{for } t \in (0,T).$$

Since B_m is an n-dimensional ball of radius $\frac{1}{2}m$, it follows that λ_m satisfies

$$\lambda_m \leq \frac{c_1}{m^2}, \tag{20}$$

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where $c_1 > 0$ depends only on the dimension n. Thus, we have

$$\partial_t^\alpha F_m(t) \geq -c_1 m^{-2} F_m(t) + F_m(t)^p. \tag{21}$$

Setting $H(\zeta) := -c_1 m^{-2}\zeta + \zeta^p$, then the function $H(\zeta)$ is convex in $\zeta > 0$ since $H \in C^2(0,\infty)$ and $H''(\zeta) \geq 0$. By (3), writing $\frac{d}{dt}(k * [F_m - F_m(0)])(t)$ instead of $\partial_t^\alpha F_m(t)$ with $k(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$ in (21), we obtain

$$\frac{d}{dt}(k * [F_m - F_m(0)])(t) \geq H(F_m(t)) \quad \text{for } t \in (0,T). \tag{22}$$

It is clear that $H(\zeta) > 0$ and $H'(\zeta) > 0$ for all $\zeta > (c_1 m^{-2})^{\frac{1}{p-1}}$.

Suppose now that

$$F_m(0) > (c_1 m^{-2})^{\frac{1}{p-1}}. \tag{23}$$

We claim that (22) implies that $F_m(t) > (c_1 m^{-2})^{\frac{1}{p-1}}$ for all $t \in (0,T)$. (The fact is stated in the proof of Theorem 3.7 in [22].) Knowing that $F_m(t) \geq F_m(0) > (c_1 m^{-2})^{\frac{1}{p-1}}$ for all $t \in (0,T)$, it follows from (22) that

$$\partial_t^\alpha F_m(t) = \frac{d}{dt}(k * [F_m - F_m(0)])(t) \geq H(F_m(t)) > 0 \quad \text{for all } t \in (0,T). \tag{24}$$

Therefore the function $F_m(t)$ satisfying (24) is an upper solution of the problem

$$\partial_t^\alpha \zeta = H(\zeta) = -c_1 m^{-2}\zeta + \zeta^p, \quad \zeta(0) = F_m(0), \tag{25}$$

we have by comparison principle $F_m(t) \geq \zeta(t)$ (see Theorem 4.10 in [29]).

On the other hand, since $H(0) = 0$, $H(\zeta) > 0$ and $H'(\zeta) > 0$ for all $\zeta \geq F_m(0) > (c_1 m^{-2})^{\frac{1}{p-1}}$. It then follows from Lemma 3.8 in [22] that $v(t) = w\left(\frac{t^\alpha}{\Gamma(\alpha+1)}\right)$ is a lower solution for (25), where $v(t)$ satisfies

$$\partial_t^\alpha v \leq H(v) = -c_1 m^{-2} v + v^p, \quad v(0) \leq F_m(0),$$

and $w(t)$ solves the ordinary differential equation

$$\frac{dw}{dt} = H(w) = -c_1 m^{-2} w + w^p, \quad w(0) = F_m(0). \tag{26}$$

By the comparison principle (see Theorem 4.10 in [29]), we obtain $\zeta(t) \geq v(t)$. Solving the initial value problem (26), we have the solution

$$w(t) = \left[F_m(0)^{1-p} - \frac{1 - \exp\{(1-p)c_1 m^{-2} t\}}{c_1 m^{-2}}\right]^{\frac{1}{1-p}} \exp\left(-c_1 m^{-2} t\right).$$

By the comparison principle (see Theorem 4.10 in [29]), we conclude that

$$\begin{aligned} F_m(t) \geq v(t) &= w\left(\frac{t^\alpha}{\Gamma(\alpha+1)}\right) \\ &= \left[F_m(0)^{1-p} - \frac{1 - \exp\{(1-p)c_2 t^\alpha\}}{c_2 \Gamma(\alpha+1)}\right]^{\frac{1}{1-p}} \exp(-c_2 t^\alpha). \end{aligned} \tag{27}$$

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with $c_2 = \frac{c_1 m^{-2}}{\Gamma(\alpha+1)}$. Therefore, from (27), we obtain that $v(t) \to \infty$ as

$$t \to \left[\frac{\log(1 - c_2\Gamma(\alpha+1)F_m(0)^{1-p})}{(1-p)c_2}\right]^{\frac{1}{\alpha}},$$

and that $F_m(t) \to \infty$. This implies that the solution $u(x,t)$ blows up in finite time when (23) holds.

As a result of these arguments, we have the following lemma.

Lemma 3. *Let $F_m(t)$ be defined by (16). If $F_m(0)$ as in (17) satisfies (23) for some $m \in \mathbf{N}$, i.e.,*

$$F_m(0) > Am^{-\frac{2}{p-1}} \quad \textit{for some } m \in \mathbf{N},$$

where $A = c_1^{\frac{1}{p-1}}$ with c_1 as in (20), then $u(x,t)$ blows up in finite time.

Here, we shall state the rest of the proof for Theorem 1.

Supposing that $u(x,t)$ is a nontrivial global solution, we prove by reductio ad absurdum.

By Lemma 3, then it follows that for any $m \in \mathbf{N}$

$$F_m(0) \leq Am^{-\frac{2}{p-1}}.$$

Then, by (17) and $u_0(x) = \lambda\psi(x) \geq 0$, we obtain

$$\lambda \int_{B_m} \psi(x)\phi_m(x)dx \leq Am^{-\frac{2}{p-1}}. \tag{28}$$

Here, if we choose λ to be large enough for any $m \in \mathbf{N}$, then the left-hand side of (28) is larger than the right-hand side of (28). Thus, we arrive at a contradiction. This completes the proof for the condition (a).

Next, if $\psi \in C_0(\mathbf{R}^n)$ satisfies (13), then there is a positive constant L such that $\psi(x) \geq L|x|^{-a}$ for sufficiently large $|x|$. Then, we have for sufficiently large m

$$\lambda L \int_{B_m} |x|^{-a}\phi_m(x)dx \leq Am^{-\frac{2}{p-1}}.$$

By noting that $|x| \leq \frac{3}{2}m$ in B_m by (14), we obtain

$$\lambda L \left(\frac{3}{2}m\right)^{-a} \int_{B_m} \phi_m(x)dx \leq Am^{-\frac{2}{p-1}},$$

and then by (15), we have

$$\left(\frac{3}{2}\right)^{-a} \lambda L m^{-a} \leq Am^{-\frac{2}{p-1}}. \tag{29}$$

By multiplying both sides of (29) by m^a, we obtain

$$\left(\frac{3}{2}\right)^{-a} \lambda L \leq Am^{a-\frac{2}{p-1}}. \tag{30}$$

Then, if $0 < a < \dfrac{2}{p-1}$ and m is sufficiently large, then the left-hand side of (30) is larger than the right-hand side of (30). Thus, we arrive at a contradiction. This completes the proof for the condition (b). □

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4. Global Existence

In this section, we state the following global existence result.

Theorem 2. *Let $n \geq 1$ and $0 < \alpha < 1$. Assume that the initial data $u_0(x) = \lambda\psi(x) \geq 0$, where $\lambda > 0$ and $\psi \in C_0(\mathbf{R}^n)$. Suppose that $p \geq 1 + 2/n$, and that $a > 2/(p-1)$ and*

$$\limsup_{|x|\to\infty} |x|^a \psi(x) < \infty. \tag{31}$$

Then, the mild solution of (1) exists globally whenever $\lambda > 0$ is small enough.

Proof. In what follows, by the letter C we denote generic positive constants, and they may have different values also within the same line.

Since $\psi \in C_0(\mathbf{R}^n)$ satisfies (31), there is a constant $C > 0$ such that

$$\psi(x) \leq C(1+|x|)^{-a} \ \text{ for all } \ x \in \mathbf{R}^n. \tag{32}$$

Let $q_c = n(p-1)/2$. First, if $p > 1 + 2/n$ and $a > 2/(p-1)$, then we know $aq_c > n$. Hence, it follows from (32) that

$$\begin{aligned}\|\psi\|^{q_c}_{L^{q_c}(\mathbf{R}^n)} &= \int_{\mathbf{R}^n} |\psi(x)|^{q_c} dx \leq C \int_{\mathbf{R}^n} (1+|x|)^{-aq_c} dx \\ &\leq C \int_0^\infty (1+r)^{-aq_c} r^{n-1} dr \leq C \int_0^\infty (1+r)^{n-aq_c-1} dr \leq C.\end{aligned} \tag{33}$$

Next, if $p = 1 + 2/n$ and $a > 2/(p-1)$, then we have $q_c = 1$ and $a > n = 2/(p-1)$. Hence, it follows from (32) that

$$\begin{aligned}\|\psi\|_{L^1(\mathbf{R}^n)} &= \int_{\mathbf{R}^n} |\psi(x)| dx \leq C \int_{\mathbf{R}^n} (1+|x|)^{-a} dx \\ &\leq C \int_0^\infty (1+r)^{-a} r^{n-1} dr \leq C \int_0^\infty (1+r)^{n-a-1} dr \leq C.\end{aligned} \tag{34}$$

By (33) and (34), if $p \geq 1 + 2/n$ and $a > 2/(p-1)$, then $\|\psi\|_{L^{q_c}(\mathbf{R}^n)} \leq C$. Since $u_0(x) = \lambda\psi(x)$, $\|u_0\|_{L^{q_c}(\mathbf{R}^n)}$ is sufficiently small whenever $\lambda > 0$ is small enough. Therefore, the mild solution of (1) exists globally by Proposition 3 (b).

This completes the proof. □

5. Conclusions

In the present paper, we analyze a reaction–diffusion equation with a Caputo fractional derivative in time and with initial conditions. By comparing the conclusions of Zhang and Sun [21] (Proposition 3) and the author (Theorems 1 and 2), we see that the Cauchy problem (1) possesses the critical Fujita exponent

$$p^* = 1 + \frac{2}{n},$$

and the second critical exponent

$$a^* = \frac{2}{p-1}.$$

Then we may be summarized in Table 1.

Table 1. Critical Fujita exponent p^* and second critical exponent a^*

	$a > a^*$	$0 < a < a^*$
$1 < p < p^*$	BU[1] [Proposition 3 (a)]	BU[1] [Theorem 1 (b)]
$p \geq p^*$	BU[1] for large $\lambda > 0$ GE[2] for small $\lambda > 0$ [Theorems 1 (a) and 2]	BU[1] [Theorem 1 (b)]

[1]BU: Blow-up; [2]GE: Global existence

In the case $a = a^*$, if $1 < p < p^*$ then BU[1] by Zhang and Sun [21] (Proposition 3 (a)), but if $p \geq p^*$ then there are few studies. Therefore, we will consider to study in the case $a = a^*$ and $p \geq p^*$ in the future.

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