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Aziz Yazla * and Muhammed Talat Sarıaydın

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Article

Modeling of (*n*,*m*) Type MPH Curves with Hopf Map and Applications MODELING OF (*n*,*m*) TYPE MINKOWSKI PYTHAGOREAN HODOGRAPH CURVES WITH HOPF MAP AND APPLICATIONS

Muhammed T. SARIAYDIN [1,*] and Aziz YAZLA [2]

[1] Selcuk University, Faculty of Sceince, Department of Mathematics, 42130, Konya, TÜRKİYE, ORCID: 0000-0002-3613-4276; talatsariaydin@gmail.com

[2] Selcuk University, Faculty of Sceince, Department of Mathematics, 42130, Konya, TÜRKİYE, ORCID: 0000-0003-3720-9716

* Correspondence: azizyazla@gmail.com

Abstract: In present paper, spatial Minkowski Pythagorean Hodograph (MPH) curves are characterized with Rational Rotation Minimizing Frames (RRMFs). We define Euler-Rodrigues Frame (ERF) for MPH curves and by means of this concept, we reach the definition of MPH curves of type (n, m). Expressing the conditions provided by these curves in the form of Minkowski-Hopf map that we define, it is aimed to establish a connection with the Lorentz force which occurs during the process of Computer Numerical Control (CNC) type sinker Electronic Discharge Machines (EDMs). This approach is reinforced by split quaternion polynomials. Finally, we give conditons satisfied by MPH curves of low degree to be type (n, m) and construct illustrative examples.
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Keywords: Minkowski Pythagorean Hodograph Curve, Rational Rotation Minimizing Frame, Euler-Rodrigues Frame, Split Quaternion Polynomial, Minkowski-Hopf Map, Type (n, m) Curve.

1. Introduction

Polynomials are symbolic objects that are frequently used, especially in computer science and computational algebra. Consisting of polynomial components, polynomial curves are one of the curves studied extensively in computational geometry. Pythagorean hodograph curves, simply PH curves, are polynomial curves which provide the equality called the Pythagorean condition. This condition is satisfied by the hodograph of these curves and a distinguishing property for them among the polynomial curves. PH curves were defined by Farouki and Sakkalis (1990). Euler-Rodrigues frame (ERF) on spatial PH curves is defined by Choi and Han (2002). Han (2008) gave the necessary and sufficient condition for a spatial PH curve to have a rational rotation minimizing frame (RRMF). Using this, PH curves of type (n, m) is defined by Dospra (2015). For further information on PH curves and applications, see (Farouki, 2008), (Sariaydin, 2019), (Erken et al., 2020). The Pythagorean condition was expressed according to the Minkowski metric and Minkowski Pythagorean hodograph curves, simply MPH curves, were defined by Moon (1999). Also planar MPH curves are characterized in this study. Spatial MPH curves are represented by Choi et al. (2002), using Clifford algebra methods. The characterization of planar MPH curves with hyperbolic polynomials and spatial MPH curves with split quaternion polynomials are given by Ramis (2013).

One of the important application areas of PH curves is on computer numerical control (CNC) machines. The purpose of the real-time interpolator in a CNC machine is to transform tool path and feedrate information into reference points for each interval of the system. Not only linear interpolations are provided by Modern CNC machines, but also parametric interpolations are offered by them. Reduction of errors and shortening machining time of parametric interpolations in comparison with linear interpolations have shown by researchers, (Tsai et al., 2008). For calculating parameter values of

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successive reference points, the general rational B-spline curves rely on Taylor series expansions. By omission of higher-order terms, such schemes inevitably incur truncation errors, (Farouki and Sakkalis, 1994). Describing the tool path in terms of the PH curves overcomes this problem, (Farouki and Sakkalis, 1990). A closed-form reduction of the interpolation integral is easily done due to the algebraic structure of PH curves. This yields real-time computer numerical control interpolator algorithms for constant or variable feedrates which are notably accurate, (Tsai et al., 2001). There are also CNC type electronic discharge machines (EDMs) which are computer-controlled machine tools that shape metal using electrical discharges or sparks. A sinker EDM applies electrical discharges through an insulating liquid (oil or dielectric fluid). The evolution of Lorentz forces due to the external magnetic field along with this plasma pressure acts as value addition in EDM by restricting its expansion. At high spark energy, erosion efficiency improves due to the development of Lorentz force, which results in an increase of the positive erosion volume from the melt pool on the workpiece surface. These machine tools are capable of cutting hard metals to any specified design, which is not achievable with other types of conventional cutting tools. They are capable of shaping exceedingly hard metals in ways that many other cutting tools and equipment cannot. As a consequence of the tool's crucial cutting capabilities, the final product is a metal item with an excellent surface polish, (Singh et al., 2018). One of the aims of this study is to use the magnetic fields generated by MPH curves with RRMFs in the EDM processes mentioned above.

In this paper, our main goal is to characterize spatial MPH curves with RRMFs and to express the conditions provided by such curves using split quaternion polynomials and the Minkowski-Hopf map that we define. With this approach, by using these characterization methods, we open up an avenue for applications of MPH curves on CNC machines. We use symbolic computation methods for the definition and computational geometry of MPH curves of type (n, m).

2. Preliminaries

In this section, we present some basic definitons and theorems for MPH curves, their representations, hyperbolic numbers and split quaternions. We begin with the definition of the Minkowski metric and 3-dimensional Minkowski space. The symmetric bilinear form $\langle , \rangle_L$ defined by

$$\langle , \rangle_L : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}, \langle x, y \rangle_L = x_1 y_1 + x_2 y_2 - x_3 y_3$$

is called Lorentz metric or Minkowski metric, where $\mathbb{R}^3$ is the real vector space and $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{R}^3$. In this case, $(\mathbb{R}^3, \langle , \rangle_L)$ is called 3-dimensional Minkowski space and is denoted by $\mathbb{R}^3_1$. Lorentz norm of x is defined as $\|x\|_L = \sqrt{|\langle x, x \rangle_L|}$, (O'Neill, 1983).

Let $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3_1$ be a differentiable curve, where I is an interval. If $\langle \alpha'(t), \alpha'(t) \rangle_L = 0$ and $\alpha'(t) \neq 0$ for all $t \in I$, then α is said to be a null curve. If $\langle \alpha'(t), \alpha'(t) \rangle_L > 0$ or $\alpha'(t) = 0$ for all $t \in I$, then α is said to be a spacelike curve. If $\langle \alpha'(t), \alpha'(t) \rangle_L < 0$ and for all $t \in I$, then α is said to be a timelike curve. If $\alpha'(t) \neq 0$ for all $t \in I$, then α is said to be a regular curve, (O'Neill, 1983).

Definition 1. *An orthonormal frame $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ on a space curve α in $\mathbb{R}^3_1$ is an orthonormal basis defined at each curve point, where $\mathbf{f}_1$ coincides with the curve tangent $\mathbf{T} = \frac{\alpha'}{\|\alpha'\|_L}$ and $\mathbf{f}_2, \mathbf{f}_3$ span the normal plane, such that $\mathbf{f}_1 \times_L \mathbf{f}_2 = \mathbf{f}_3$. The angular velocity of this frame is defined by*

$$\omega = \omega_1 \mathbf{f}_1 + \omega_2 \mathbf{f}_2 + \omega_3 \mathbf{f}_3,$$

and the following relations are satisfied

$$\mathbf{f}_1' = \sigma\omega \times_L \mathbf{f}_1,\ \mathbf{f}_2' = \sigma\omega \times_L \mathbf{f}_2,\ \mathbf{f}_3' = \sigma\omega \times_L \mathbf{f}_3,$$

where $\sigma = \|\alpha'\|_L$ is the parametric speed of α. $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ is a rotation minimizing frame (RMF) of α if and only if its angular velocity satisfies $\langle \omega, f_1 \rangle_L = 0$, i.e., ω has no component along $\mathbf{f}_1$. If $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ is an RMF of α

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and vector fields $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$ are rational according to the curve parameter, then $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ is said to be a rational rotation minimizing frame (RRMF) of α, (O'Neill, 1983).

Definition 2. *Let $\alpha(t) = (x(t), y(t), z(t))$ be a polynomial curve in $\mathbb{R}^3_1$ whose hodograph $\alpha'(t)$ satisfies*

$$\left[x'(t)\right]^2 + \left[y'(t)\right]^2 - \left[z'(t)\right]^2 = \sigma^2(t) \tag{1}$$

for polynomial $\sigma(t)$, then $\alpha(t)$ is said to be a spatial Minkowski Pythagorean hodograph curve, simply a spatial MPH curve. Condition (1) is called the Minkowski Pythagorean condition, (Moon, 1999).

Note that, all null curves in $\mathbb{R}^3_1$ are MPH curves and there is no timelike MPH curve in $\mathbb{R}^3_1$, (Moon, 1999). In our study, we consider regular spacelike spatial MPH curves. One of the characterization methods for MPH curves is using hyperbolic polynomials. Therefore, we present the definition and basic properties of hyperbolic numbers. Let H be a set which consists of ordered pair of real numbers defined as

$$H = \left\{ z = x + \mathbf{e}y : x, y \in \mathbb{R}, \mathbf{e}^2 = 1, \mathbf{e} \notin \mathbb{R} \right\}.$$

The elements of this 2-dimensional commutative real algebra H are said to be hyperbolic numbers or split complex numbers, (Catoni et al., 2011). For the algebraic properties of hyperbolic numbers, see (Catoni et al., 2011).

The curve $\alpha(t) = (x(t), y(t), z(t))$ is a MPH curve if and only if there exist polynomials $u_1(t), u_2(t), u_3(t), u_4(t)$ with

$$\begin{aligned}
x'(t) &= u_1^2(t) - u_2^2(t) + u_3^2(t) - u_4^2(t), \\
y'(t) &= 2\left[u_1(t)u_4(t) - u_2(t)u_3(t)\right], \\
z'(t) &= 2[u_1(t)u_3(t) - u_2(t)u_4(t)], \\
\sigma(t) &= \pm[u_1^2(t) - u_2^2(t) - u_3^2(t) + u_4^2(t)],
\end{aligned} \tag{2}$$

(Moon, 1999).

In order to characterize MPH curves with split quaternion polynomials, we present the definition of split quaternions. The ring

$$\widetilde{\mathbb{H}} = \left\{ \epsilon = \epsilon_0 + \epsilon_1 \mathbf{i} + \epsilon_2 \mathbf{j} + \epsilon_3 \mathbf{k} : \epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3 \in \mathbb{R}, \mathbf{i}^2 = \mathbf{j}^2 = 1, \mathbf{k}^2 = -1, \mathbf{ijk} = 1 \right\}$$

which is defined in $(-,+,+,-)$ signed $\mathbb{R}^4_2$ semi-Euclidean space is called the ring of split quaternions. Norm of ϵ is defined as $\|\epsilon\| = \sqrt{|\epsilon\epsilon^*|}$ and modulus of ϵ is defined as $|\epsilon| = \epsilon\epsilon^*$, (Inoguchi, 1998). For the algebraic properties of split quaternions, see (Cockle, 1849).

We present the classification of split quaternions according to their semi-Euclidean scalar product with themselves in $\mathbb{R}^4_2$. Let $\epsilon = \epsilon_0 + \epsilon_1 \mathbf{i} + \epsilon_2 \mathbf{j} + \epsilon_3 \mathbf{k} \in \widetilde{\mathbb{H}}$, then $\langle \epsilon, \epsilon \rangle_{\mathbb{R}^4_2} = -\epsilon_0^2 + \epsilon_1^2 + \epsilon_2^2 - \epsilon_3^2$. If this value is positive, negative or zero, then ϵ is called spacelike, timelike or lightlike split quaternion, respectively, (Inoguchi, 1998).

Finally, we present the characterization of MPH curves with split quaternion polynomials. Let $\alpha(t) = (x(t), y(t), z(t))$ is a MPH curve whose hodograph is given by the equalities (2). Then $\alpha'(t)$ is expressed with the split quaternion polynomial $Q(t) = u_1(t) + \mathbf{i}u_2(t) + \mathbf{j}u_3(t) + \mathbf{k}u_4(t)$ as $\alpha'(t) = Q(t)\mathbf{i}Q^*(t)$, where $Q^*(t)$ is conjugate of $Q(t)$. If $\gcd(u_1(t), u_2(t), u_3(t), u_4(t))$ is constant, then $Q(t)$ is said to be a primitive split quaternion polynomial. Similarly, if $h(t) = a_1(t) + \mathbf{e}a_2(t)$ is a hyperbolic polynomial such that $\gcd(a_1(t), a_2(t))$ is constant, then $h(t)$ is said to be a primitive hyperbolic polynomial, (Ramis, 2013).

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3. Characterization of Spatial MPH Curves with RRMFs

In this section, we give a representation of spatial MPH curves in terms of hyperbolic polynomials in Minkowski-Hopf map form. We define ERF for this kind of curves and we get the necessary and sufficient condition for spatial MPH curves to have RRMFs. Then, we define type (n, m) curve for spatial MPH curves. Thus, we aim to achieve results that will increase the efficiency and usefulness of curves in CNC machine processes.

Theorem 1. *Let $\alpha(t)$ be a spatial MPH curve represented by $Q(t)$, where $|Q(t)| \neq 0$, then the set of vectors $\{\mathbf{g}_1(t), \mathbf{g}_2(t), \mathbf{g}_3(t)\}$ defined by*

$$(\mathbf{g}_1(t), \mathbf{g}_2(t), \mathbf{g}_3(t)) = \frac{(Q(t)\mathbf{i}Q^*(t), Q(t)\mathbf{j}Q^*(t), Q(t)\mathbf{k}Q^*(t))}{|Q(t)|},$$

is a rational orthonormal frame for $\alpha(t)$.

Proof. We compute

$$\begin{aligned} Q(t)\mathbf{i}Q^*(t) &= (u_1^2 - u_2^2 + u_3^2 - u_4^2)\mathbf{i} + 2(u_1u_4 - u_2u_3)\mathbf{j} + 2(u_1u_3 - u_2u_4)\mathbf{k}, \\ Q(t)\mathbf{j}Q^*(t) &= -2(u_1u_4 + u_2u_3)\mathbf{i} + (u_1^2 + u_2^2 - u_3^2 - u_4^2)\mathbf{j} - 2(u_1u_2 + u_3u_4)\mathbf{k}, \\ Q(t)\mathbf{k}Q^*(t) &= 2(u_1u_3 + u_2u_4)\mathbf{i} + 2(u_3u_4 - u_1u_2)\mathbf{j} + (u_1^2 + u_2^2 + u_3^2 + u_4^2)\mathbf{k} \end{aligned} \quad (3)$$

and

$$|Q(t)| = u_1^2 - u_2^2 - u_3^2 + u_4^2.$$

Since $|Q(t)| \neq 0$, it is obvious that $\{\mathbf{g}_1(t), \mathbf{g}_2(t), \mathbf{g}_3(t)\}$ is a rational frame for $\alpha(t)$. On the other hand, one can easily see that

$$\langle \mathbf{g}_1(t), \mathbf{g}_2(t) \rangle_L = \langle \mathbf{g}_1(t), \mathbf{g}_3(t) \rangle_L = \langle \mathbf{g}_2(t), \mathbf{g}_3(t) \rangle_L = 0$$

and

$$\|\mathbf{g}_1(t)\|_L = \|\mathbf{g}_2(t)\|_L = \|\mathbf{g}_3(t)\|_L = 1.$$

Thus, these equalities show that $\{\mathbf{g}_1(t), \mathbf{g}_2(t), \mathbf{g}_3(t)\}$ is orthonormal. □

Definition 3. *Let $\alpha(t)$ be a spatial MPH curve represented by $Q(t)$, where $|Q(t)| \neq 0$, then the rational orthonormal frame $\{\mathbf{g}_1(t), \mathbf{g}_2(t), \mathbf{g}_3(t)\}$ defined by*

$$(\mathbf{g}_1(t), \mathbf{g}_2(t), \mathbf{g}_3(t)) = \frac{(Q(t)\mathbf{i}Q^*(t), Q(t)\mathbf{j}Q^*(t), Q(t)\mathbf{k}Q^*(t))}{|Q(t)|},$$

is called Euler-Rodrigues frame, simply ERF for $\alpha(t)$.

Theorem 2. *If $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ is a rational orthonormal frame of a spatial MPH curve $\alpha(t)$ represented by $Q(t)$, then the following statements hold:*

(1) $\mathbf{f}_1(t) = \mathbf{g}_1(t)$,
(2) There exist polynomials $a_1(t), a_2(t)$ such that

$$\begin{bmatrix} \mathbf{f}_2(t) \\ \mathbf{f}_3(t) \end{bmatrix} = \frac{1}{a_1^2(t) - a_2^2(t)} \begin{bmatrix} a_1^2(t) + a_2^2(t) & 2a_1(t)a_2(t) \\ 2a_1(t)a_2(t) & a_1^2(t) + a_2^2(t) \end{bmatrix} \begin{bmatrix} \mathbf{g}_2(t) \\ \mathbf{g}_3(t) \end{bmatrix},$$

where $\gcd(a_1(t), a_2(t))$ is constant.

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Proof. We can write

$$\begin{aligned} \mathbf{f}_1(t) &= \mathbf{g}_1(t), \\ \mathbf{f}_2(t) &= \cosh(\phi(t))\mathbf{g}_2(t) + \sinh(\phi(t))\mathbf{g}_3(t), \\ \mathbf{f}_3(t) &= \sinh(\phi(t))\mathbf{g}_2(t) + \cosh(\phi(t))\mathbf{g}_3(t), \end{aligned}$$

for some $\phi(t)$. Since $\mathbf{f}_i(t)$ and $\mathbf{g}_j(t)$ are all rational, the coefficients $\cosh(\phi(t))$ and $\sinh(\phi(t))$ are rational. Therefore, we write

$$\cosh(\phi(t)) = \frac{\gamma(t)}{\delta(t)} \text{ and } \sinh(\phi(t)) = \frac{\beta(t)}{\delta(t)},$$

for polynomials $\gamma(t), \beta(t), \delta(t)$ with $\gcd(\gamma(t), \beta(t), \delta(t))$ is constant. Since $\cosh^2(\phi(t)) - \sinh^2(\phi(t)) = 1$, the polynomials $\gamma(t), \beta(t), \delta(t)$ satisfy the Minkowski Pythagorean condition in $\mathbb{R}^2_1$, i.e., $\gamma^2(t) - \beta^2(t) = \delta^2(t)$, therefore $\gcd(\gamma(t), \beta(t))$ is constant. Then, there exist polynomials $a_1(t), a_2(t)$ of $\gcd(a_1(t), a_2(t))$ is constant, satisfying

$$\gamma(t) = a_1^2(t) + a_2^2(t),\ \beta(t) = 2a_1(t)a_2(t),\ \delta(t) = a_1^2(t) - a_2^2(t).$$

Thus, one can get the result by making the necessary calculations. □

Theorem 3. *A spatial MPH curve $\alpha(t)$ represented by $Q(t)$ has a RRMF if and only if the following statement holds:*

- There exist polynomials $a_1(t), a_2(t)$ such that

$$\frac{u_1u_2' - u_1'u_2 - u_3u_4' + u_3'u_4}{u_1^2 - u_2^2 - u_3^2 + u_4^2} = \frac{a_1a_2' - a_1'a_2}{a_1^2 - a_2^2}. \tag{4}$$

Proof. A rational orthonormal frame $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ of $\alpha(t)$ is rotation minimizing if and only if either of $\mathbf{f}_2'(t)$ and $\mathbf{f}_3'(t)$ is parallel to $\mathbf{f}_1(t)$, (Bishop, 1975). Equivalently,

$$\langle \mathbf{f}_2'(t), \mathbf{f}_3(t) \rangle_L = 0 \tag{5}$$

is the necessary and sufficient condition for $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ to be rotation minimizing. By Theorem 2, there exist polynomials $a_1(t), a_2(t)$ with $\gcd(a_1(t), a_2(t))$ is constant and

$$\begin{aligned} \mathbf{f}_1(t) &= \mathbf{g}_1(t), \\ \mathbf{f}_2(t) &= \frac{a_1^2(t) + a_2^2(t)}{a_1^2(t) - a_2^2(t)}\mathbf{g}_2(t) + \frac{2a_1(t)a_2(t)}{a_1^2(t) - a_2^2(t)}\mathbf{g}_3(t), \\ \mathbf{f}_3(t) &= \frac{2a_1(t)a_2(t)}{a_1^2(t) - a_2^2(t)}\mathbf{g}_2(t) + \frac{a_1^2(t) + a_2^2(t)}{a_1^2(t) - a_2^2(t)}\mathbf{g}_3(t). \end{aligned}$$

One can get

$$\langle \mathbf{g}_2'(t), \mathbf{g}_3(t) \rangle_L = 2\frac{u_1u_2' - u_1'u_2 - u_3u_4' + u_3'u_4}{u_1^2 - u_2^2 - u_3^2 + u_4^2}.$$

Then, we obtain

$$\langle \mathbf{f}_2'(t), \mathbf{f}_3(t) \rangle_L = 2\frac{a_1'a_2 - a_1a_2'}{a_1^2 - a_2^2} + 2\frac{u_1u_2' - u_1'u_2 - u_3u_4' + u_3'u_4}{u_1^2 - u_2^2 - u_3^2 + u_4^2}.$$

Thus, by the condition (5), the result is clear. □

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In order to define type (n,m) curve for spatial MPH curves, when the spatial MPH curve $\alpha(t)$ which is represented by $Q(t)$ has a RRMF, we must show that the degrees of polynomials $a_1(t), a_2(t)$ which exist by Theorem 3 are uniquely determined. As in Farouki (2010), it is practical to use the notations

$$[Q(t)] = [u_1, u_2, u_3, u_4] = \frac{u_1u_2' - u_1'u_2 - u_3u_4' + u_3'u_4}{u_1^2 - u_2^2 - u_3^2 + u_4^2} \text{ and } [h(t)] = [a_1, a_2] = \frac{a_1a_2' - a_1'a_2}{a_1^2 - a_2^2},$$

where $h(t) = a_1(t) + \mathbf{e}a_2(t)$ is a hyperbolic polynomial.

The following theorem includes some features of these quotients which we need for the next discussions and also it shows that the degrees of polynomials $a_1(t), a_2(t)$ are uniquely determined. Henceforth the split quaternion basis element $\mathbf{i}$ and the hyperbolic number unit $\mathbf{e}$ are considered equivalent. Thus, we can multiply a split quaternion with a hyperbolic number considering a hyperbolic number $z = x + \mathbf{e}y$ as a split quaternion $z = x + \mathbf{i}y + \mathbf{j}0 + \mathbf{k}0$.

Lemma 1. *Let $a_1(t)$,$a_2(t)$, $a_3(t)$, $a_4(t)$, $u_1(t)$,$u_2(t)$, $u_3(t)$, $u_4(t)$ be real polynomials, $s \in \widetilde{\mathbb{H}}$ and $r = \lambda + \mathbf{e}\mu \in H$. Then following assertions hold.*

(1) *If we write $sQ(t)$ in place of $Q(t)$ for any s such that $|s| \neq 0$, condition (4) remains unchanged.*
(2) *$[u_1, u_2, u_3, u_4] \pm [a_1, a_2] = [U_1, U_2, U_3, U_4]$, where $U_1 + U_2\mathbf{i} + U_3\mathbf{j} + U_4\mathbf{k} = (u_1 + u_2\mathbf{i} + u_3\mathbf{j} + u_4\mathbf{k})(a_1 \pm a_2\mathbf{e})$. In particular, $[a_1, a_2] \pm [a_3, a_4] = [A_1, A_2]$, where $A_1 + A_2\mathbf{e} = (a_1 + a_2\mathbf{e})(a_3 \pm a_4\mathbf{e})$. In addition, $[(t-r)Q(t)] = \frac{\mu}{(t-\lambda)^2 - \mu^2} \frac{u_1^2 - u_2^2 + u_3^2 - u_4^2}{u_1^2 - u_2^2 - u_3^2 + u_4^2} + [u_1, u_2, u_3, u_4]$.*
(3) *If $a_3 + a_4\mathbf{e} = (t-r)^m$ for $m \in \mathbb{N}$, then $[a_3, a_4] = m\mu[(t-\lambda)^2 - \mu^2]^{-1}$. Moreover, if $[a_1, a_2] = 0$, then a_1, a_2 are linearly dependent over $\mathbb{R}$.*
(4) *If $a_1 + a_2\mathbf{e}$ and $a_3 + a_4\mathbf{e}$ are primitive hyperbolic polynomials satisfying $[a_1, a_2] = [a_3, a_4]$, then $a_1 + a_2\mathbf{e} = z(a_3 + a_4\mathbf{e})$ for $z \in H$.*

Proof. 1. Observe that $u_1u_2' - u_1'u_2 - u_3u_4' + u_3'u_4$ is the $\mathbf{i}$ component of $-Q'^*(t)Q(t)$. If $Q(t)$ is replaced by $sQ(t)$, then $Q'^*(t)Q(t)$ becomes $Q'^*(t)s^*sQ(t) = |s|\, Q'^*(t)Q(t)$ and $|Q(t)|$ becomes $|s|\, |Q(t)|$. Thus, condition (4) is clearly unaltered when we write $sQ(t)$ in place of $Q(t)$.

2. Let $U_1 + U_2\mathbf{i} + U_3\mathbf{j} + U_4\mathbf{k} = (u_1 + u_2\mathbf{i} + u_3\mathbf{j} + u_4\mathbf{k})(a_1 + a_2\mathbf{e})$. After the multiplication, we get $U_1 = a_1u_1 + a_2u_2, U_2 = a_1u_2 + a_2u_1, U_3 = a_1u_3 + a_2u_4, U_4 = a_1u_4 + a_2u_3$. Thus, we obtain

$$\begin{aligned}
[U_1, U_2, U_3, U_4] &= \frac{U_1U_2' - U_1'U_2 - U_3U_4' + U_3'U_4}{U_1^2 - U_2^2 - U_3^2 + U_4^2} \\
&= \frac{(u_1^2 - u_2^2 - u_3^2 + u_4^2)(a_1a_2' - a_1'a_2) + (u_1u_2' - u_1'u_2 - u_3u_4' + u_3'u_4)(a_1^2 - a_2^2)}{(u_1^2 - u_2^2 - u_3^2 + u_4^2)(a_1^2 - a_2^2)} \\
&= \frac{a_1a_2' - a_1'a_2}{a_1^2 - a_2^2} + \frac{u_1u_2' - u_1'u_2 - u_3u_4' + u_3'u_4}{u_1^2 - u_2^2 - u_3^2 + u_4^2} \\
&= [a_1, a_2] + [u_1, u_2, u_3, u_4].
\end{aligned}$$

When $U_1 + U_2\mathbf{i} + U_3\mathbf{j} + U_4\mathbf{k} = (u_1 + u_2\mathbf{i} + u_3\mathbf{j} + u_4\mathbf{k})(a_1 - a_2\mathbf{e})$, similarly one can get $[U_1, U_2, U_3, U_4] = [u_1, u_2, u_3, u_4] - [a_1, a_2]$. As a result, when $A_1 + A_2\mathbf{e} = (a_1 + a_2\mathbf{e})(a_3 \pm a_4\mathbf{e})$ in particular, $[A_1, A_2] = [a_1, a_2] \pm [a_3, a_4]$ is obtained.

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Let $(t-r)Q(t) = ((t-\lambda)u_1 - \mu u_2) + ((t-\lambda)u_2 - \mu u_1)\mathbf{i} + ((t-\lambda)u_3 + \mu u_4)\mathbf{j} + ((t-\lambda)u_4 + \mu u_3)\mathbf{k} = U_1 + U_2\mathbf{i} + U_3\mathbf{j} + U_4\mathbf{k}$. Then, we obtain

$$\begin{aligned}[(t-r)Q(t)] &= [U_1, U_2, U_3, U_4] \\ &= \frac{U_1U_2' - U_1'U_2 - U_3U_4' + U_3'U_4}{U_1^2 - U_2^2 - U_3^2 + U_4^2} \\ &= \frac{\mu(u_1^2 - u_2^2 + u_3^2 - u_4^2) + ((t-\lambda)^2 - \mu^2)(u_1u_2' - u_1'u_2 - u_3u_4' + u_3'u_4)}{((t-\lambda)^2 - \mu^2)(u_1^2 - u_2^2 - u_3^2 + u_4^2)} \\ &= \frac{\mu}{(t-\lambda)^2 - \mu^2}\frac{u_1^2 - u_2^2 + u_3^2 - u_4^2}{u_1^2 - u_2^2 - u_3^2 + u_4^2} + [u_1, u_2, u_3, u_4].\end{aligned}$$

3. For $m = 1$, it is clear that the equality is satisfied. With the help of the second part of the item 2, the first part is proved by induction on m. Now let $[a_1, a_2] = 0$. We get $a_1a_2' = a_1'a_2$ and so the Wronskian $W(a_1, a_2)$ vanishes, which shows that a_1, a_2 are linearly dependent over $\mathbb{R}$.

4. Suppose that $a_1 + a_2\mathbf{e}$ and $a_3 + a_4\mathbf{e}$ are monic. Hence, $\deg(a_1) > \deg(a_2)$ and $\deg(a_3) > \deg(a_4)$. Since $[a_1, a_2] = [a_3, a_4]$, $(a_1 + a_2\mathbf{e})(a_3 - a_4\mathbf{e}) = (a_1a_3 - a_2a_4) + (a_2a_3 - a_1a_4)\mathbf{e}$, item 2 implies that $[a_1a_3 - a_2a_4, a_2a_3 - a_1a_4] = 0$, and therefore $a_1a_3 - a_2a_4$ and $a_2a_3 - a_1a_4$ are linearly dependent. But $\deg(a_1a_3 - a_2a_4) > \deg(a_2a_3 - a_1a_4)$, thus $a_2a_3 - a_1a_4 = 0$. This shows that $a_1 = a_3, a_2 = a_4$. Now, let $z_1, z_2 \in H$ be such that $z_1(a_1 + a_2\mathbf{e})$ and $z_2(a_3 + a_4\mathbf{e})$ are monic. Item 1 implies that $[z_1(a_1 + a_2\mathbf{e})] = [z_2(a_3 + a_4\mathbf{e})]$ and thus $z_1(a_1 + a_2\mathbf{e}) = z_2(a_3 + a_4\mathbf{e})$. Therefore, $a_1 + a_2\mathbf{e} = z_1^{-1}z_2(a_3 + a_4\mathbf{e})$, as required. □

Definition 4. *Let $Q(t)$ be a primitive split quaternion polynomial of degree n and $h(t)$ be a primitive hyperbolic polynomial of degree m, satisfying (4). Then the MPH curve $\alpha(t)$ with the hodograph $\alpha'(t) = Q(t)\mathbf{i}Q^*(t)$ is called of type (n, m) curve.*

Definition 5. *For all $z, w \in H$, the map*

$$\varphi : H \times H \to \mathbb{R}_1^3$$

defined by

$$\varphi(z, w) = (|z| - |w|, 2Re(z\overline{w}), -2Hyp(z\overline{w})),$$

is called Minkowski-Hopf map.

Let $\alpha(t)$ be a spatial MPH curve which is represented by $Q(t)$ and $h_1(t) = u_1(t) + \mathbf{e}u_2(t), h_2(t) = u_4(t) + \mathbf{e}u_3(t)$ be hyperbolic polynomials. Then, it can be easily shown that the hodograph of $\alpha(t)$ can be given in the Minkowski-Hopf map form as follows,

$$\begin{aligned}\alpha'(t) &= (|h_1(t)| - |h_2(t)|, 2\,Re(h_1(t)\overline{h_2}(t)), -2\mathrm{Hyp}(h_1(t)\overline{h_2}(t))) \\ &= \varphi(h_1(t), h_2(t)).\end{aligned} \tag{6}$$

Using the Minkowski-Hopf map representation (6), one can easily see that the RRMF condition (4) is equivalent to satisfaction of

$$\frac{\mathrm{Hyp}(\overline{h_1}h_1' + \overline{h_2}h_2')}{|h_1| + |h_2|} = \frac{\mathrm{Hyp}(\overline{h}h')}{|h|}. \tag{7}$$

Remark 1. *When $h(t)$ is real polynomial or constant, the angle $\theta(t)$ between the ERF and RRMF is constant. This is equivalent to*

$$Hyp(\overline{h_1}h_1' + \overline{h_2}h_2') = 0. \tag{8}$$

So, we may consider (8) as the condition for ERF to be rotation minimizing. Note that in view of (4) condition (8) is equivalent to

$$scal(Q(t)\mathbf{i}Q'^*(t)) = 0. \tag{9}$$

Lemma 2. *Let $u_1(t)$,$u_2(t)$, $u_3(t)$, $u_4(t)$ are polynomials of degree $m \geq 1$. Then hyperbolic values μ, ν exist such that under the map*

$$\begin{bmatrix} h_1(t) \\ h_2(t) \end{bmatrix} \to \begin{bmatrix} \mu & -\overline{\nu} \\ \nu & \overline{\mu} \end{bmatrix} \begin{bmatrix} h_1(t) \\ h_2(t) \end{bmatrix} \tag{10}$$

the transformed polynomials $u_2(t), u_3(t), u_4(t)$ are of degree $m-1$ at most.

Proof. If we write $h_1(t) = c_m t^m + \cdots + c_1 t + c_0$ and $h_2(t) = d_m t^m + \cdots + d_1 t + d_0$, where $c_i = u_{1i} + \mathbf{e}u_{2i}$ and $d_i = u_{4i} + \mathbf{e}u_{3i}$ for $i = 0, \ldots, m$, the coefficients transform according to

$$\begin{bmatrix} c_i \\ d_i \end{bmatrix} \to \begin{bmatrix} \mu & -\overline{\nu} \\ \nu & \overline{\mu} \end{bmatrix} \begin{bmatrix} c_i \\ d_i \end{bmatrix}$$

for $i = 0, \ldots, m$. In particular, with the choices $\mu = \frac{\overline{c_m}}{|c_m|+|d_m|}$ and $\nu = -\frac{d_m}{|c_m|+|d_m|}$, we obtain $(c_m, d_m) \to (1, 0)$. □

Remark 2. *By Lemma 2, we can take $u_1(t) = t^m + \cdots + u_{11}t + u_{10}$ and $u_2(t), u_3(t), u_4(t)$ are of degree $m-1$ at most. $(u_1(t), u_2(t), u_3(t), u_4(t))$ polynomial quadruple in this form is called normal.*

Lemma 3. *If the RRMF condition (7) is satisfied by hyperbolic polynomials $h_1(t), h_2(t)$ and $h(t)$, also it is satisfied when they are replaced by $\mu h_1(t) - \overline{\nu} h_2(t), \nu h_1(t) + \overline{\mu} h_2(t)$ and $\eta h(t)$ for any hyperbolic numbers $(\mu, \nu) \neq (0, 0)$ and $\eta \neq 0$.*

Proof. For hyperbolic numbers $(\mu, \nu) \neq (0, 0)$, application of the transformation (10) to the polynomials $h_1(t), h_2(t)$ leads to

$$\begin{aligned} |h_1| + |h_2| &\to (|\mu| + |\nu|)(|h_1| + |h_2|), \\ \overline{h_1}h_1' + \overline{h_2}h_2' &\to (|\mu| + |\nu|)(\overline{h_1}h_1' + \overline{h_2}h_2'), \end{aligned}$$

and hence the left-hand side of (7) remains unchanged. Similarly, we have $\text{Hyp}(\overline{h}h') \to |\eta|\text{Hyp}(\overline{h}h')$ and $|h| \to |\eta|\,|h|$ when $h \to \eta h$, and therefore the other side of (7) is unaltered. □

Remark 3. *Lemma 3 shows that the transformation (10) does not influence the RRMF property of a spatial MPH curve.*

Theorem 4. *Let $Q(t)$ be defined by the normal quadruple $(u_1(t), u_2(t), u_3(t), u_4(t))$ and $\alpha(t)$ be a MPH curve with hodograph $\alpha'(t) = Q(t)\mathbf{i}Q^*(t)$. Then,*

(1) *$\alpha(t)$ is planar, other than a straight line, if and only if*

$$(u_3^2 - u_4^2)(u_1 u_2' - u_1' u_2) = (u_1^2 - u_2^2)(u_3 u_4' - u_3' u_4), \tag{11}$$

with $(u_3(t), u_4(t)) \neq (0, 0)$.

(2) *$\alpha(t)$ is a straight line if and only if $(u_3(t), u_4(t)) = (0, 0)$.*

Proof. The necessary and sufficient condition for $\alpha(t)$ to be planar is linearly dependence of $x'(t), y'(t), z'(t)$. Since we consider the normal form, from (3), $x'(t)$ is of degree $2m$, while $y'(t), z'(t)$ are of degree $2m-1$ at most. Hence, $\alpha(t)$ is planar if and only if $y'(t)$ and $z'(t)$ are linearly dependent, i.e., $y'z'' = y''z'$, which is equivalent to (11). On the other hand, when $\alpha(t)$ is a straight line, $x'(t), y'(t)$

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and $x'(t), z'(t)$ are linearly dependent, respectively. Similarly, from the normal form, we derive $y'(t) = z'(t) = 0$, which shows $u_3(t) = u_4(t) = 0$, because of $u_1^2(t) + u_2^2(t) \neq 0$. The converse is trivial. □

4. Type (n, m) Curves of Low Degree

Let $\alpha(t)$ be the MPH curve generated by the quadratic split quaternion polynomial $Q(t)$ which is in normal form. This section is devoted to derivation of the necessary and sufficient conditions for a MPH curve $\alpha(t)$ to be of type $(2,1)$ and $(2,0)$, when $Q(t)$ is expressed in a factorization form

$$Q(t) = (t - C_1)(t - C_2), \tag{12}$$

with

$$C_1 = \gamma_0 + \gamma_1 \mathbf{i} + \gamma_2 \mathbf{j} + \gamma_3 \mathbf{k} \in \widetilde{\mathbb{H}},$$

and

$$C_2 = \beta_0 + \beta_1 \mathbf{i} + \beta_2 \mathbf{j} + \beta_3 \mathbf{k} \in \widetilde{\mathbb{H}}.$$

Let

$$w = scal(Q(t)\mathbf{i}Q'^{*}(t)) = w_2 t^2 + w_1 t + w_0, \tag{13}$$

be the negative of the numerator on the left side in (4) and

$$\sigma = |Q(t)| = t^4 + \sigma_3 t^3 + \sigma_2 t^2 + \sigma_1 t + \sigma_0, \tag{14}$$

be its denominator.

Since split quaternions are not division algebra and contain zero divisors, factorization as (12) is not possible for every quadratic split quaternion polynomial. Now, we present two results which are given in Scharler et al. (2020) and state conditions for the factorizability of quadratic split quaternion polynomials. Let $Q(t) = t^2 + bt + c$ be a quadratic split quaternion polynomial where $b = b_0 + b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, $c = c_0 + c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k} \in \widetilde{\mathbb{H}}$ and $|Q(t)| \neq 0$.

Theorem 5. *If the coefficients $\{1, b, c\}$ are linearly independent, then Q admits a factorization, (Scharler et al., 2020).*

Theorem 6. *Let the coefficients $\{1, b, c\}$ are linearly dependent.*

(1) *If $b = 0$ and $c = c_0 \in \mathbb{R}$, then Q admits infinitely many factorizations.*
(2) *If $b = 0$ and $c \in \widetilde{\mathbb{H}} \setminus \mathbb{R}$, then Q admits a factorization if and only if $vect(c)vect(c)^* > 0$ or $cc^* \geq 0$ and $c_0 < 0$.*
(3) *Let $b \in \widetilde{\mathbb{H}} \setminus \mathbb{R}$, $b_0 = 0$ and $c = \lambda + \mu b$ where $\lambda, \mu \in \mathbb{R}$. Then,*

- *if $bb^* > 0$, then Q admits a factorization.*
- *if $bb^* = 0$, then Q admits a factorization if and only if $\lambda + \mu^2 = 0$ or $\lambda < 0$.*
- *if $bb^* < 0$, then Q admits a factorization if and only if $\lambda + \mu^2 = 0$ or $bb^* + 4\lambda < 0$ and $bb^* + 4\lambda \leq 4\mu\sqrt{-bb^*} \leq -(bb^* + 4\lambda)$*, (Scharler et al., 2020).

We assume that Q satisfies the necessary factorizability conditions and admits a factorization as (12).

4.1. MPH Curves of Type (2,1)

The quintic MPH curve $\alpha(t)$ is of type $(2,1)$ if and only if polynomials $a_1(t), a_2(t)$ exist with $\gcd(a_1(t), a_2(t))$ is constant, $a_1(t) + \mathbf{e}a_2(t)$ is a linear hyperbolic polynomial and

$$-\frac{w}{\sigma} = \frac{a_1(t)a_2'(t) - a_1'(t)a_2(t)}{a_1^2(t) - a_2^2(t)}.$$

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Since $a_1(t)$ or $a_2(t)$ is linear and they are relatively prime, by Lemmas 2 and 3, we can take $a_1(t) = t - \xi, a_2(t) = \eta$ for $\xi, \eta \in \mathbb{R}$ with $\eta \neq 0$. Expanding (14), we obtain

$$\begin{aligned}
\sigma_3 &= -2(\gamma_0 + \beta_0), \sigma_2 = |C_1| + |C_2| + 4\gamma_0\beta_0, \\
\sigma_1 &= -2(\gamma_0 |C_2| + \beta_0 |C_1|), \sigma_0 = |C_1| |C_2| .
\end{aligned}$$

Since

$$\begin{aligned}
Q(t) &= t^2 - (C_1 + C_2)t + C_1C_2 \\
&= t^2 - (\gamma_0 + \beta_0)t + \gamma_0\beta_0 + \gamma_1\beta_1 + \gamma_2\beta_2 - \gamma_3\beta_3 \\
&\quad +\mathbf{i}(-(\gamma_1 + \beta_1)t + \gamma_0\beta_1 + \gamma_1\beta_0 + \gamma_2\beta_3 - \gamma_3\beta_2) \\
&\quad +\mathbf{j}(-(\gamma_2 + \beta_2)t + \gamma_0\beta_2 - \gamma_1\beta_3 + \gamma_2\beta_0 + \gamma_3\beta_1) \\
&\quad +\mathbf{k}(- (\gamma_3 + \beta_3)t + \gamma_0\beta_3 - \gamma_1\beta_2 + \gamma_2\beta_1 + \gamma_3\beta_0)
\end{aligned}$$

and

$$Q'^{*}(t) = 2t - (\gamma_0 + \beta_0) + (\gamma_1 + \beta_1)\mathbf{i} + (\gamma_2 + \beta_2)\mathbf{j} + (\gamma_3 + \beta_3)\mathbf{k},$$

by substituting in (13), we have that $w(t)$ has coefficients

$$\begin{aligned}
w_2 &= -(\gamma_1 + \beta_1), \\
w_1 &= 2(\gamma_0\beta_1 + \gamma_1\beta_0 + \gamma_2\beta_3 - \gamma_3\beta_2), \\
w_0 &= (\gamma_1 + \beta_1)(\gamma_0\beta_0 + \gamma_1\beta_1 + \gamma_2\beta_2 - \gamma_3\beta_3) - (\gamma_0 + \beta_0)(\gamma_0\beta_1 + \gamma_1\beta_0 + \gamma_2\beta_3 - \gamma_3\beta_2) \\
&\quad -(\gamma_3 + \beta_3)(\gamma_0\beta_2 - \gamma_1\beta_3 + \gamma_2\beta_0 + \gamma_3\beta_1) + (\gamma_2 + \beta_2)(\gamma_0\beta_3 - \gamma_1\beta_2 + \gamma_2\beta_1 + \gamma_3\beta_0).
\end{aligned}$$

Thus, the equality

$$-\frac{w}{\sigma} = \frac{-\eta}{t^2 - 2\xi t + \xi^2 - \eta^2}$$

is equivalent to

$$\begin{aligned}
\eta &= w_2, \\
\eta\sigma_3 &= w_1 - 2\xi w_2, \\
\eta\sigma_2 &= w_2(\xi^2 - \eta^2) + w_0 - 2w_1\xi, \\
\eta\sigma_1 &= w_1(\xi^2 - \eta^2) - 2w_0\xi, \\
\eta\sigma_0 &= w_0(\xi^2 - \eta^2).
\end{aligned} \tag{15}$$

Since $\eta = w_2 \neq 0$, we get

$$\xi = \frac{w_1 - w_2\sigma_3}{2w_2},$$

and hence we obtain that curve $\alpha(t)$ is of type $(2, 1)$ if and only if

$$\xi = \frac{w_1 - w_2\sigma_3}{2w_2} \quad \text{and} \quad \eta = w_2,$$

and these values must satisfy the last three equations of system (15). Thus, the following theorem is proved.

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Theorem 7. *Let $Q(t) = (t - C_1)(t - C_2)$ with $C_1 = \gamma_0 + \gamma_1\mathbf{i} + \gamma_2\mathbf{j} + \gamma_3\mathbf{k}$, $C_2 = \beta_0 + \beta_1\mathbf{i} + \beta_2\mathbf{j} + \beta_3\mathbf{k} \in \widetilde{\mathbb{H}}$. Set $w = scal(Q(t)iQ'^*(t)) = w_2t^2 + w_1t + w_0$ and $\sigma = |Q(t)| = t^4 + \sigma_3t^3 + \sigma_2t^2 + \sigma_1t + \sigma_0$. Then the MPH curve generated by the split quaternion polynomial $Q(t)$ is of type $(2,1)$ if and only if the system*

$$\begin{aligned} \eta\sigma_2 &= w_2(\xi^2 - \eta^2) + w_0 - 2w_1\xi, \\ \eta\sigma_1 &= w_1(\xi^2 - \eta^2) - 2w_0\xi, \\ \eta\sigma_0 &= w_0(\xi^2 - \eta^2) \end{aligned}$$

has the solution

$$(\xi, \eta) = (\frac{w_1 - w_2\sigma_3}{2w_2}, w_2).$$

Example 1. *Let $Q(t) = (t - 1 + \mathbf{j} - \mathbf{k})(t - 1 - \mathbf{i} - 2\mathbf{j} + \mathbf{k})$ be a split quaternion polynomial which defines a MPH quintic curve $\alpha(t)$. We can easily see that*

$$w_0 = 1, w_1 = 0, w_2 = -1,$$

$$\sigma_0 = -3, \sigma_1 = 4, \sigma_2 = 2, \sigma_3 = -4,$$

$$\xi = 2, \eta = -1$$

and the system of Theorem 7 is verified by the values of ξ, η. Thus, $Q(t)$ defines a MPH curve of type $(2,1)$. One can easily see that

$$Q(t) = (t^2 - 2t) - \mathbf{i}t + \mathbf{j}(-t + 2) - \mathbf{k},$$

so since $u_1(t) = t^2 - 2t, u_2(t) = -t, u_3(t) = -t + 2, u_4(t) = -1$, according to condition (2) we find

$$\alpha'(t) = (t^4 - 4t^3 + 4t^2 - 4t + 3, -4t^2 + 8t, -2t^3 + 8t^2 - 10t).$$

By integrating $\alpha'(t)$, we obtain the MPH curve $\alpha(t)$ of type $(2,1)$ with the initial condition $\alpha(0) = (0,0,0)$ as follows,

$$\alpha(t) = (\frac{1}{5}t^5 - t^4 + \frac{4}{3}t^3 - 2t^2 + 3t, -\frac{4}{3}t^3 + 4t^2, -\frac{1}{2}t^4 + \frac{8}{3}t^3 - 5t^2).$$

Since $\sigma = |Q(t)| = t^4 - 4t^3 + 2t^2 + 4t - 3 \neq 0$, we have $t \in \mathbb{R} \setminus \{-1, 1, 3\}$. Using Definition 3, one can easily compute the ERF $\{\mathbf{g}_1(t), \mathbf{g}_2(t), \mathbf{g}_3(t)\}$ of $\alpha(t)$ as follows,

$$\begin{aligned} \mathbf{g}_1(t) &= \frac{(t^4 - 4t^3 + 4t^2 - 4t + 3, -4t^2 + 8t, -2t^3 + 8t^2 - 10t)}{t^4 - 4t^3 + 2t^2 + 4t - 3}, \\ \mathbf{g}_2(t) &= \frac{(0, t^4 - 4t^3 + 4t^2 + 4t - 5, 2t^3 - 4t^2 - 2t + 4)}{t^4 - 4t^3 + 2t^2 + 4t - 3}, \\ \mathbf{g}_3(t) &= \frac{(-2t^3 + 8t^2 - 6t, 2t^3 - 4t^2 + 2t - 4, t^4 - 4t^3 + 6t^2 - 4t + 5)}{t^4 - 4t^3 + 2t^2 + 4t - 3}. \end{aligned}$$

Since $a_1(t) = t - 2$ and $a_2(t) = -1$, from Theorem 2, a RRMF $\{\mathbf{f}_1(t), \mathbf{f}_2(t), \mathbf{f}_3(t)\}$ of $\alpha(t)$ is obtained as follows,

$$\begin{aligned} \mathbf{f}_1(t) &= \mathbf{g}_1(t), \\ \mathbf{f}_2(t) &= \frac{t^2 - 4t + 5}{t^2 - 4t + 3}\mathbf{g}_2(t) + \frac{-2t + 4}{t^2 - 4t + 3}\mathbf{g}_3(t), \\ \mathbf{f}_3(t) &= \frac{-2t + 4}{t^2 - 4t + 3}\mathbf{g}_2(t) + \frac{t^2 - 4t + 5}{t^2 - 4t + 3}\mathbf{g}_3(t). \end{aligned}$$

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4.2. MPH Curves of Type (2,0)

The MPH curve $\alpha(t)$ is of type $(2,0)$ i.e., has a rotation minimizing ERF if and only if $w(t)=0$. The last condition is equivalent to

$$\gamma_1+\beta_1=0, \gamma_0\beta_1+\gamma_1\beta_0+\gamma_2\beta_3-\gamma_3\beta_2=0,$$

$$(\gamma_2+\beta_2)(\gamma_0\beta_3-\gamma_1\beta_2+\gamma_2\beta_1+\gamma_3\beta_0)=(\gamma_3+\beta_3)(\gamma_0\beta_2-\gamma_1\beta_3+\gamma_2\beta_0+\gamma_3\beta_1).$$

One can easily see that if $\alpha(t)$ is a MPH curve of type $(2,0)$, then

$$u_1u_2'-u_1'u_2=0 \text{ and } u_3u_4'-u_3'u_4=0,$$

where $Q(t)=u_1(t)+\mathbf{i}u_2(t)+\mathbf{j}u_3(t)+\mathbf{k}u_4(t)$ is the split quaternion polynomial which generates $\alpha(t)$. Hence, condition (11) is satisfied, so we obtain that the only MPH quintics with rotation minimizing ERFs are planar curves.

Suppose that the MPH curve $\alpha(t)$ is a straight line. By Theorem 4, $u_3(t)=u_4(t)=0$. In view of the above, the curve $\alpha(t)$ is a straight line of type $(2,0)$ if and only if

$$\gamma_1+\beta_1=0, \gamma_0\beta_1+\gamma_1\beta_0+\gamma_2\beta_3-\gamma_3\beta_2=0,$$

$$(\gamma_2+\beta_2)(\gamma_0\beta_3-\gamma_1\beta_2+\gamma_2\beta_1+\gamma_3\beta_0)=(\gamma_3+\beta_3)(\gamma_0\beta_2-\gamma_1\beta_3+\gamma_2\beta_0+\gamma_3\beta_1),$$

$$\begin{aligned}\gamma_2+\beta_2 &= 0, \gamma_0\beta_2-\gamma_1\beta_3+\gamma_2\beta_0+\gamma_3\beta_1=0,\\ \gamma_3+\beta_3 &= 0, \gamma_0\beta_3-\gamma_1\beta_2+\gamma_2\beta_1+\gamma_3\beta_0=0.\end{aligned}$$

The last equalities lead to

$$\frac{\gamma_0}{\beta_0}=-\frac{\gamma_1}{\beta_1}=-\frac{\gamma_2}{\beta_2}=-\frac{\gamma_3}{\beta_3},$$

i.e.,

$$C_1=\lambda C_2^*,\ \ \lambda\in\mathbb{R}\setminus\{0\}\,.$$

Note that if $\lambda=1$, $Q(t)$ is a non-primitive polynomial which is not the case. Thus, the following theorem is proved.

Theorem 8. *Let $Q(t)=(t-C_1)(t-C_2)$ with $C_1=\gamma_0+\gamma_1\mathbf{i}+\gamma_2\mathbf{j}+\gamma_3\mathbf{k}$,$C_2=\beta_0+\beta_1\mathbf{i}+\beta_2\mathbf{j}+\beta_3\mathbf{k}\in\widetilde{\mathbb{H}}$. Set $w=scal(Q(t)iQ'^*(t))=w_2t^2+w_1t+w_0$ and $\sigma=|Q(t)|=t^4+\sigma_3t^3+\sigma_2t^2+\sigma_1t+\sigma_0$. Then the MPH curve generated by the split quaternion polynomial $Q(t)$ is of type $(2,0)$ i.e., has a rotation minimizing ERF if and only if the following equalities are satisfied,*

$$\gamma_1+\beta_1=0, \gamma_0\beta_1+\gamma_1\beta_0+\gamma_2\beta_3-\gamma_3\beta_2=0,$$

$$(\gamma_2+\beta_2)(\gamma_0\beta_3-\gamma_1\beta_2+\gamma_2\beta_1+\gamma_3\beta_0)=(\gamma_3+\beta_3)(\gamma_0\beta_2-\gamma_1\beta_3+\gamma_2\beta_0+\gamma_3\beta_1). \tag{16}$$

Moreover, this curve is a straight line if and only if

$$C_1=\lambda C_2^*,\ \ \lambda\in\mathbb{R}\setminus\{0\}\,,\ \lambda\neq 1.$$

Example 2. *Let $Q(t)=(t-\mathbf{i}-3\mathbf{j}+3\mathbf{k})(t+\mathbf{i}-2\mathbf{j}+2\mathbf{k})$ be a split quaternion polynomial which defines a MPH quintic curve $\alpha(t)$. We can easily see that*

$$w_0=0, w_1=0, w_2=0,$$

$$\sigma_0=1, \sigma_1=0, \sigma_2=-2, \sigma_3=0$$

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and the equalities (16) of Theorem 8 are satisfied. Thus, $Q(t)$ defines a MPH curve of type $(2,0)$. One can easily see that

$$Q(t) = (t^2-1) + 5\mathbf{j}(1-t) + 5\mathbf{k}(t-1),$$

so since $u_1(t) = t^2-1, u_2(t) = 0, u_3(t) = 5(1-t), u_4(t) = 5(t-1)$, according to condition (2) we find

$$\alpha'(t) = ((t^2-1)^2, 10(t^2-1)(t-1), 10(t^2-1)(1-t)).$$

By integrating $\alpha'(t)$, we obtain the MPH curve $\alpha(t)$ of type $(2,0)$ with the initial condition $\alpha(0) = (0,0,0)$ as follows,

$$\alpha(t) = (\frac{1}{5}t^5 - \frac{2}{3}t^3 + t, \frac{5}{2}t^4 - \frac{10}{3}t^3 - 5t^2 + 10t, -\frac{5}{2}t^4 + \frac{10}{3}t^3 + 5t^2 - 10t).$$

Since $\sigma = |Q(t)| = t^4 - 2t^2 + 1 = (t^2-1)^2 \neq 0$, we have $t \in \mathbb{R} \setminus \{-1,1\}$. Using Definition 3, one can easily compute the ERF $\{\mathbf{g}_1(t), \mathbf{g}_2(t), \mathbf{g}_3(t)\}$ of $\alpha(t)$ as follows,

$$\begin{aligned} \mathbf{g}_1(t) &= (1, \frac{10}{t+1}, -\frac{10}{t+1}), \\ \mathbf{g}_2(t) &= (-\frac{10}{t+1}, 1 - \frac{50}{(t+1)^2}, \frac{50}{(t+1)^2}), \\ \mathbf{g}_3(t) &= (-\frac{10}{t+1}, -\frac{50}{(t+1)^2}, 1 + \frac{50}{(t+1)^2}). \end{aligned}$$

Since the MPH curve $\alpha(t)$ is of type $(2,0)$, its ERF is a RRMF.

5. Conclusions

Leaving null curves aside, we study regular spacelike spatial MPH curves and their representations with symbolic computation methods. As an alternative to the split quaternion representation, we give a new characterization of spatial MPH curves in terms of hyperbolic polynomials using the Minkowski-Hopf map. We show that spatial MPH curves can be obtained from a hyperbolic polynomial couple using the Minkowski-Hopf map. Then, we prove the necessary and sufficient conditions for a MPH curve to be planar and to be a straight line.

It's aimed to characterize spatial MPH curves with RRMFs. In order to obtain the necessary and sufficient condition for a spatial MPH curve to have a RRMF, we define the ERF for this kind of curves. Then, we prove that this condition is the existence of polynomials $a_1(t), a_2(t)$ such that $\gcd(a_1(t), a_2(t))$ is constant and

$$\frac{u_1u_2' - u_1'u_2 - u_3u_4' + u_3'u_4}{u_1^2 - u_2^2 - u_3^2 + u_4^2} = \frac{a_1a_2' - a_1'a_2}{a_1^2 - a_2^2},$$

when $Q(t) = u_1(t) + \mathbf{i}u_2(t) + \mathbf{j}u_3(t) + \mathbf{k}u_4(t)$ is the split quaternion polynomial which the spatial MPH curve is represented. In order to define the concept of type (n,m) curve for spatial MPH curves, we have to show that the degrees of these polynomials $a_1(t), a_2(t)$ are uniquely determined. Therefore, we prove a theorem which shows this uniqueness. Thus, we define the concept of type (n,m) curve for spatial MPH curves. This concept is a useful tool to characterize spatial MPH curves. We present the results obtained in Scharler et al. (2020) which state the conditions for the factorizability of quadratic split quaternion polynomials. We characterize quintic spatial MPH curves of type $(2,1)$ and $(2,0)$, when the quadratic split quaternion polynomial which generates the curve is in normal form and admits a factorization. We give illustrative examples for these types of quintic spatial MPH curves.

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