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Manuel D. Ortigueira *

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Article

A Factory of Fractional Derivatives

Manuel D. Ortigueira

CTS-UNINOVA and LASI, NOVA School of Science and Technology, NOVA University of Lisbon, Portugal; mdo@fct.unl.pt

Abstract: This paper aims to demonstrate that, beyond the small world of Riemann-Liouville and Caputo derivatives, there is a vast and rich world with many derivatives suitable for specific problems and various theoretical frameworks to develop, corresponding to the different paths taken. Notions of time and scale sequences are introduced and general associated basic derivatives, namely right/stretching and left/shrinking, are defined. A general framework for fractional derivative definitions is reviewed and applied to get both known and new fractional order derivatives. Several fractional derivatives are considered, mainly: Liouville, Hadamard, Euler, bilinear, tempered, q-derivative, and Hahn.

Keywords: shift-invariant; scale-invariant; nabla derivative; fractional q-derivative; fractional hahn derivative

MSC: 26A33

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1. Introduction

During this century, fractional calculus has been increasingly adopted as a model for many natural and artificial phenomena, since traditional tools are not suitable for making fair descriptions of their behavior [1–5]. This happens, for example , in Physics [6–13], Electrochemstry [14–20], Electromagnetics [21–25], Viscoelasticity [26–28], and Biomedical Engineering [29–35]. Other interesting applications are found in the long range processes, $1/f$ noise, fractional chaos, fractional Gaussian noise, and fractional Brownian motion (fBm) [36–42].

Four centuries after the first reference to the possibility of non-integer order derivatives [43], we have arrived at a very confusing situation in which we find

- Multiple definitions of derivatives without clear application domains;
- A great confusion between the notions of derivative and system;
- Mixtures of different types of derivatives without stating reasons and validity;
- Incorrect use of initial conditions;
- False derivatives.

This situation is causing huge problems that make life difficult for those who want to only make science and engineering applications.

Since the first formal introduction of the concept of fractional derivative (FD) by Liouville (1832) [44,45], a long journey has been traveled by mathematicians, physicists, and engineers. Although Liouville's inability to impose his vision, the main definitions we find today are based on the formulæ he presented, mainly the Riemann–Liouville (RL) [46], (Dzherbashian)-Caputo (C) [47,48], and Grünwald-Letnikov (GL) [46] definitions. However, and based on these derivatives, new ones have been proposed alongside these, such as Hadamard's [48] or Marchaud's [46]. The success of the "fractional derivative" designation motivated and gave rise to the proposal of "new fractional derivatives". In the last 20 years, many modifications or combinations of the above derivatives have been introduced, along with other operators claiming to be "fractional derivatives". Some are mere high-pass filters and others are disguised order 1 derivatives. If we also consider pseudo-derivatives, we conclude that the situation is really messy and confusing, as shown by Oliveira and Machado, first, and Teodoro et al., more recently [49,50], who listed such operators and introduced a classification according to some specified criteria. Since 2015, there has been a great deal of discussion in forums, conferences and articles, where we found that most people are unaware of the distinction between the concepts of system and derivative. This situation was the main reason for the attempt, described

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in [51], to provide a methodology for testing whether a given operator is suitable to be a fractional derivative. Such a goal was continued with the formalizations introduced in [52,53]. However, new generalizations appeared recently having as base the Abel's algorithm for the tautochrone problem. This was done first by Kochubei [54,55], and after, by Luchko [56,57], Tarasov [58] and Fernandez [59]. Hanyga [60] used Abel approach in the discussion of the concept of fractional derivative.

However, it is important to note that these developments only deal with shift-invariant operators in continuous-time, without considering operators in discrete-time and scale-invariant. On the other hand, we see the existence of what we can call the RL/C dictatorship that tries to impose the formats of RL and C everywhere, even in discrete-time cases. In fact, Liouville found that the definition of fractional anti-derivative could not be used directly to define derivative, since the involved integral became singular. He cleverly decided to transfer the singularity to a derivative of integer order, giving rise to two procedures that originated the RL and C derivatives. Consequently, such procedures are only necessary in the presence of a singularity, and are unnecessary and sometimes useless when there is no singularity, as occours in discrete-time, since they involve an increase in the number of operations, which can be very important in practical applications. What we can observe are attempts to formulate RL and C derivatives in many situations, mainly discrete, where a single formula serves for all derivative orders, positive and negative. This work shows how we should proceed and allows us to easily detect when a singularity transfer is necessary.

Here we recover the framework introduced in [61] and apply it to fractionalize various derivatives. We consider the general case of the nabla and delta derivatives introduced by Aulbach and Hilger [62,63] and several interesting particular cases. These derivatives are simple linear operators that deserve to be generalized. To do it, we calculate their eigenfunctions (generalized exponentials) and their eigenvalues (transfer functions). From these, we introduce the corresponding higher-order and fractional-order derivatives. We do not intend to make a detailed study of the different fractionalizations that exist for each derivative, but to show that the path introduced is coherent and leads to general and simple solutions. This is a practical point of view that do not want to dismiss other pure mathematical approaches. In particular, we do not treat existence problems.

The paper outlines as follows. A revision of the notion of time sequence and the introduction of the concept of scale sequence are done in Section 2. Some mathematical tools are listed in Appendix A. In Section 3 we recover the discussion about the concept of FD we introduced in [51]. We generalize the Hilger's Definitions of nabla/delta derivatives [62,63] by introducing the corresponding scale, tempered, and bilinear derivatives (3.2). The general framework we proposed in [61] will be revised in Appendix B. With this framework and the derivatives introduced in sub-section 3.2, a general form of fractional derivatives is proposed in Section 4. For it, we will use the notion of generalized exponentials that are the eigenfunctions of the derivatives (sub-section 4.1) and serve to define generalized Laplace transforms (sub-section 4.2). In Section 5 we describe the derivatives defined in time sequences, namely continuous-time Liouville (sub-section 5.2), discrete-time Euler (sub-section 5.4), and discrete-time bilinear (sub-section 5.5) derivatives. The corresponding tempered derivatives are also considered. The main derivatives on scale sequences, chiefly the scale-invariant derivatives are described in Section 6. In particular, the Hadamard (sub-section 6.1), tempered Hadamard (sub-section 6.2) and bilinear (sub-section 6.3) derivatives are introduced. The q–derivative and Hahn derivative are suggested in Section 6.4. The bilateral derivatives are considered in Section 7. Finally, some conclusions are presented in Section 8.

2. Time and Scale Sequences

The Hilger's approach to the continuous/discrete unification was based in the concept of *time scale* [62,63]. This refers to any non-empty closed subset $\mathbb{T}$ of the set $\mathbb{R}$ of real numbers. The designation "time scale" is misleading, since the word "scale" is associated to a notion of stretching or shrinking, frequently having a relation to a speed or a rate. We prefer the designation *time sequence*.

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Let $t \in \mathbb{T}$ be the current instant. The previous next instant is denoted by $\rho(t)$. Similarly, the next following point on the time sequence $\mathbb{T}$ is denoted by $\sigma(t)$. One has

$$\rho(t) = t - \nu(t), \quad \sigma(t) = t + \mu(t),$$

where $\nu(t)$ and $\mu(t)$ are called the *graininess* functions. These functions can be used to construct any time sequences. Let us define

$$\nu^0(t) := 0, \quad \nu^n(t) := \nu^{n-1}(t) + \nu\Big(t - \nu^{n-1}(t)\Big), \quad n \in \mathbb{N},$$

and $\rho^0(t) := t$, $\rho^n(t) := \rho\big(\rho^{n-1}(t)\big)$, $n \in \mathbb{N}$. Note that $\nu^1(t) = \nu(t)$ and $\rho^1(t) = \rho(t)$. When moving into the past, we have

$$\begin{aligned}
&\rho^0(t) = t = t - \nu^0(t),\\
&\rho^1(t) = \rho(t) = t - \nu(t) = t - \nu^1(t),\\
&\rho^2(t) = \rho(\rho(t)) = \rho(t) - \nu(\rho(t)) = t - \nu(t) - \nu(t - \nu(t)) = t - \nu^2(t),\\
&\qquad \vdots\\
&\rho^n(t) = t - \nu^n(t).
\end{aligned}$$

Moving into the future, the definitions and results are similar:

$$\begin{aligned}
\mu^0(t) := 0, \quad \mu^n(t) := \mu^{n-1}(t) + \mu\Big(t + \mu^{n-1}(t)\Big),\\
\sigma^0(t) := t, \quad \sigma^n(t) := \sigma\Big(\sigma^{n-1}(t)\Big),
\end{aligned}$$

$n \in \mathbb{N}$, and we have $\sigma^n(t) = t + \mu^n(t)$.

The time sequences $\mathbb{T}$ defined by a set of discrete instants t_n, $n \in \mathbb{Z}$, and by the corresponding *graininess* functions are very interesting. We define a direct graininess [35,64]

$$t_n = t_{n-1} + \nu_n, \quad n \in \mathbb{Z},$$

and reverse graininess

$$t_n = t_{n+1} - \mu_n, \quad n \in \mathbb{Z},$$

where we avoid representing any reference instant t_0. These Definitions of "irregular" sequences allow us to introduce the most interesting time sequences we find in practice. However, we have some difficulties in doing some kind of manipulations that are also very common. Let us consider a time sequence defined on $\mathbb{R}$ and unbounded when $t \to \pm\infty$. For example , a time sequence defined by

$$t_n = nT + \vartheta_n, \quad n \in \mathbb{Z}, \quad T > 0, \quad |\vartheta_n| < \frac{T}{2},$$

which we can call an "almost linear sequence" [65]. However, in the most interesting engineering applications, we consider regular (uniform) sequences having constant graininess, h,

$$\mathbb{T} = h\mathbb{Z} = \{\ldots, -3h, -2h, -h, 0, h, 2h, 3h, \ldots\},$$

with $h \in \mathbb{R}^+$.

This way of contructing sequences, that we can call additive, is not the unique way of obtaining working domains. There is another way, multiplicative, that can be obtained by introducing log-grainінesses, $\theta, \zeta > 1$. To understand the idea assume that we have

$$\sigma(t) = t\zeta(t), \quad \rho(t) = t/\theta(t),$$

so that the logarithm of the sequence is a time sequence with grainінesses, $\log\theta_n$ and $\log\zeta_n$ giving rise to

$$\log\tau_n = \log\tau_{n-1} + \log\theta_n, \quad n \in \mathbb{Z},$$

and

$$\log\tau_n = \log\tau_{n+1} - \log\zeta_n, \quad n \in \mathbb{Z}.$$

Therefore, we obtain what we can call "scale sequences," Θ, *progressive*

$$\tau_n = \tau_{n-1}\theta_n, \quad n \in \mathbb{Z},$$

and *regressive*

$$\tau_n = \tau_{n+1}/\zeta_n, \quad n \in \mathbb{Z}.$$

A simple example is

$$\tau_n = \tau_0 q^n \in \Theta, \quad n \in \mathbb{Z}, \tau_0 \in \mathbb{R}.$$

We will use these sequences when dealing with scale-invariant derivatives.

3. What is a Derivative?

3.1. The Classic Derivatives

The notion of *derivative*, in classic sense, comes from the works of Leibniz, Newton and Lagrange and is defined for functions of continuous variable (frequently, time), usually in an interval over $\mathbb{R}$, using a presentation that comes from Cauchy formulation. In a simple way, we can say that derivative is the instantaneous rate of change of a function that can be stated, for $t \in \mathbb{R}$, as

$$Df(t) = \frac{df}{dt} = f'(t) = \lim_{h\to 0}\frac{f(t) - f(t-h)}{h}. \quad (1)$$

This is the nowadays referred as causal derivative formulation, although mathematics usually prefer the anti-causal formula

$$Df(t) = \frac{df}{dt} = f'(t) = \lim_{h\to 0}\frac{f(t+h) - f(t)}{h}. \quad (2)$$

There are other alternative approaches namely the one of Carathéodory [66]. However, in practical applications we deal with functions defined in domains that are not necessarily continuous, requiring more general approaches.

3.2. Nabla/Stretching and Delta/Shrinking Derivatives on General Time/Scale Sequences

Modernly, the notions of derivatives came from the works of Hilger trying to make a discrete/continuous unification [62,63,67].

Definition 1. *Consider a given time sequence, $\mathbb{T}$. According to such an approach, we define the nabla derivative by*

$$D_\nabla f(t) = \begin{cases} \frac{f(t)-f(\rho(t))}{\nu(t)} & \text{if } \nu(t) \neq 0 \\ \lim\limits_{h\to 0}\frac{f(t)-f(t-h)}{h} & \text{if } \nu(t) = 0 \end{cases} \quad (3)$$

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and the delta derivative by

$$D_\Delta f(t) = \begin{cases} \frac{f(\sigma(t))-f(t)}{\mu(t)} & \text{if } \mu(t) \neq 0 \\ \lim\limits_{h\to 0} \frac{f(t+h)-f(t)}{h} & \text{if } \mu(t) = 0. \end{cases} \tag{4}$$

We can give another form to these derivatives, in agreement with our comments done before about time sequences. According to it, the nabla derivative reads

$$D_\nabla f(t_n) = \begin{cases} \frac{f(t_n)-f(t_{n-1})}{\nu_n} & \text{for any } t_n \text{ that is not left dense,} \\ \lim\limits_{h\to 0^+} \frac{f(t_n)-f(t_n-h)}{h} & \text{for } t_n \text{ left dense,} \end{cases} \tag{5}$$

where $\nu_n = t_n - t_{n-1}$, and the delta derivative

$$D_\Delta f(t_n) = \begin{cases} \frac{f(t_{n+1})-f(t_n)}{\mu_n} & \text{for any } t_n \text{ that is not right dense,} \\ \lim\limits_{h\to 0^+} \frac{f(t_n+h)-f(t_n)}{h} & \text{for } t_n \text{ right dense,} \end{cases} \tag{6}$$

where $\mu_n = t_{n+1} - t_n$ is the graininess.

From these basic derivatives we can formulate others that we consider in the nabla version (the delta is similar). Let $a, b \in \mathbb{R}^+$. We define

1. Tempered nabla derivative

$$D_\nabla f(t) = \begin{cases} \frac{f(t)-a^{-b}f(\rho(t))}{\nu(t)} & \text{if } \nu(t) \neq 0 \\ \lim\limits_{h\to 0} \frac{f(t)-a^{-b}f(t-h)}{h} & \text{if } \nu(t) = 0; \end{cases} \tag{7}$$

2. Bilinear nabla derivative

$$\frac{D_\nabla f(t) + D_\nabla f(\rho(t))}{2} = \begin{cases} \frac{f(t)-f(\rho(t))}{\nu(t)} & \text{if } \nu(t) \neq 0 \\ \lim\limits_{h\to 0} \frac{f(t)-f(t-h)}{h} & \text{if } \nu(t) = 0; \end{cases} \tag{8}$$

3. Tempered bilinear nabla derivative

$$\frac{D_\nabla f(t) + a^{-b}D_\nabla f(\rho(t))}{2} = \begin{cases} \frac{f(t)-a^{-b}f(\rho(t))}{\nu(t)} & \text{if } \nu(t) \neq 0 \\ \lim\limits_{h\to 0} \frac{f(t)-a^{-b}f(t-h)}{h} & \text{if } \nu(t) = 0. \end{cases} \tag{9}$$

Besides these derivatives, we can define others on scale sequences.

Definition 2. *Let Θ be a scale sequence. For any $t \in \Theta$ we define the stretching derivative*

$$D_\nabla f(t) = \begin{cases} \frac{f(t)-f(\rho(t))}{\log\theta(t)} & \text{if } \theta(t) \neq 1 \\ \lim\limits_{q\to 1} \frac{f(t)-f(t/q)}{\log q} & \text{if } \theta(t) = 1, \end{cases} \tag{10}$$

where $\rho(t) = t\theta(t)$ and $q > 1$. Similarly, we define the shrinking derivative

$$D_\nabla f(t_n) = \begin{cases} \frac{f(t_n)-f(t_{n-1})}{\log\theta_n} & \text{for any } t_n \text{ that is not left dense,} \\ \lim\limits_{q\to 1^+} \frac{f(t_n)-f(t_n/q)}{\log q} & \text{for } t_n \text{ left dense,} \end{cases} \tag{11}$$

where $\theta_n = t_n/t_{n-1}$.

We do not present the other versions corresponding to the delta, tempered, and bilinear cases. They are obtained in a similar way.

In any time/scale sequence, $\mathbb{T}$ we define the Heaviside unit step by:

$$\varepsilon(t) = \begin{cases} 1 & t \geq t_0 \\ 0 & t < t_0. \end{cases}$$

Any derivative of this function

$$\delta_f(t - t_0) = D\varepsilon(t - t_0), \quad t, t_0 \in \mathbb{T}, \tag{12}$$

will be called *impulse*.

3.3. What Is a Fractional Derivative?

Some years ago, we gave answers to this question based on two criteria [51]. These criteria expressed the characteristics we desired for fractional continuous-time shift-invariant derivatives. The *wide sense criterion* reads as follows. An operator is considered as a FD in this criterion if it enjoys the properties **P** defined as:

^{1}**P1** Linearity
The operator is linear.

^{1}**P2** Identity
The zero order derivative of a function returns the function itself.

^{1}**P3** Backward compatibility
When the order is integer, FD gives the same result as the ordinary derivative.

^{1}**P4** The index law holds

$$D^\alpha D^\beta f(t) = D^{\alpha+\beta} f(t) \tag{13}$$

for $\alpha < 0$ and $\beta < 0$.

^{1}**P5** Generalised Leibniz rule

$$D^\alpha [f(t)g(t)] = \sum_{i=0}^{\infty} \binom{\alpha}{i} D^i f(t) D^{\alpha - i} g(t) \tag{14}$$

The index law property can be modified to include positive orders. This leads to the *strict sense criterion*. Therefore, criterion ^{2}**P** has five conditions, where ^{1}**P4** is modified to:

^{2}**P4** The index law

$$D^\alpha D^\beta f(t) = D^{\alpha+\beta} f(t), \tag{15}$$

for any α and β.

Remark 1.

- *The strict sense criterion in the form ^{2}**P4**, presented by (15) opens a discussion: a given property is a requirement for the operator itself (fractional derivative) or for the domain of the operator, that, the space of functions? The "philosophical controversy" if the properties are from the operators, or of the functions, is very important in applications. If we consider that we should not use relation (15), then an electrical circuit RLC is different from an RCL. But this is not considered in practice: they constitute the same circuit. The situation is more involved if we have two coils associated in series. Therefore, the commutativity has to be assumed. Similarly, the additivity is important in practice. Without it, the realization of a shift-invariant system would become shift-variant.*
- *Another important question concerns with the so-called "starting-point". Should a given definition include such point? If such a point is included, we have a definition"à la carte", one for each function,*

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when we expected to have one definitionfor all the functions (this may pose existence problems, but this is another question). Nobody questions the definition of Fourier transform: it is unique for all the functions. No starting point.

4. General Fractional Derivatives

4.1. Generalized Exponentials

In sub-section 3.2 we introduced a general definition of order 1 derivatives. With the application of the previous framework we are led to generalized fractional derivatives. This procedure was done in [64] for the Euler type derivatives and in [68] for the bilinear cases. Here, we recover those results and present a general expression for the fractional nabla derivative (the delta case is treated similarly).

Theorem 1. *Consider that our domain is a time sequence, $\mathbb{T}$ and take $t_0 \in \mathbb{T}$ as reference instant (as we did above, this parameter will be removed when it is not important). The generalized nabla exponential is given by*

$$e_\nabla(t, t_0; s) = \begin{cases} \prod_{k=1}^{n} [1 - s\mu_k(t_0)]^{-1} & \text{if} \quad t = t_n > t_0 \\ 1 & \text{if} \quad t = t_0 \\ \prod_{k=1}^{m} [1 - s\nu_k(t_0)] & \text{if} \quad t = t_m < t_0 \end{cases} \tag{16}$$

and the delta exponential by

$$e_\Delta(t, t_0; s) = \begin{cases} \prod_{k=1}^{n} [1 + s\mu_k(t_0)] & \text{if} \quad t = t_n > t_0 \\ 1 & \text{if} \quad t = t_0 \\ \prod_{k=1}^{m} [1 + s\nu_k(t_0)]^{-1} & \text{if} \quad t = t_m < t_0. \end{cases} \tag{17}$$

For proof, see [35,64]. These exponentials have several interesting properties that we will not consider here [64]. Those we are interested here read:

1. *Relation between nabla and delta exponentials.*
   The function defined in (17) with the substitution of $-s$ for s is the inverse of (16):
$$e_\Delta(t, t_0; s) = 1/e_\nabla(t, t_0; -s). \tag{18}$$
2. Let $h_M = \max(\nu_k, \mu_l)$ and $h_m = \min(\nu_k, \mu_l)$, $k, l \in \mathbb{Z}$. The nabla exponential $e_\nabla(t, t_0; s)$ [64]
   - is a real number for any real s;
   - is positive for any real number s such that $s < \frac{1}{h_M}$;
   - oscillates for any real number s such that $s > \frac{1}{h_m}$;
   - is bounded for values of s inside the inner Hilger circle $|1 - sh_m| = 1$;
   - has an absolute value that increases as $|s|$ increases outside the outer Hilger circle $|1 - sh_M| = 1$, going to infinite as $|s| \to \infty$.

   This means that each graininess defines a circle centred at its inverse and passing at $s = 0$. In general, we have infinite circles that reduce to one when the graininess is constant. When it is null, it degenerates into the imaginary axis.
3. *Product of exponentials*. The following relations hold:
$$\begin{aligned} e_\nabla(t, t_0; s) \cdot e_\Delta(\tau, t_0; -s) &= e_\nabla(t, \tau; s) \\ &= e_\Delta(\tau, t; -s). \end{aligned} \tag{19}$$

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Now let $t_n > t_m + \tau$. We have

$$e_\nabla(t_n, t_0; s) = e_\nabla(t_n - \tau, t_0; s) \cdot e_\nabla(t_n, t_n - \tau; s) \tag{20}$$

and

$$e_\nabla(t_n - \tau, t_0; s) = e_\nabla(t_n, t_0; s) \cdot e_\nabla(t_n, t_n - \tau; -s). \tag{21}$$

Relation (21) states the *translation or shift property* of an exponential.

4. *Scale change*. Let a be a positive real number. We have:

$$e_\nabla(at, t_0; s) = e_\nabla(t, t_0; as).$$

If $a < 0$, a similar relation is obtained but involving the delta exponential.

We introduced above the nabla and delta exponentials. Now, we consider the stretching/shrinking exponentials. defined on scale sequences. In agreement with our study done in 3.2 we obtain such exponentials through simple substitutions of the grainінesses. We obtain the stretching exponential

$$e_\nabla(t, t_0; s) = \begin{cases} \prod_{k=1}^{n} [1 - s\log(\theta_k(t_0))]^{-1} & \text{if} \quad t = t_n > t_0 \\ 1 & \text{if} \quad t = t_0 \\ \prod_{k=1}^{m} [1 - s\log(\zeta_k(t_0))] & \text{if} \quad t = t_m < t_0 \end{cases} \tag{22}$$

and the shrinking exponential

$$e_\Delta(t, t_0; s) = \begin{cases} \prod_{k=1}^{n} [1 + s\log(\theta_k(t_0))] & \text{if} \quad t = t_n > t_0 \\ 1 & \text{if} \quad t = t_0 \\ \prod_{k=1}^{m} [1 + s\log(\zeta_k(t_0))]^{-1} & \text{if} \quad t = t_m < t_0. \end{cases} \tag{23}$$

The example s we will treat below will help understanding the roles of the different parameters.

4.2. Suitable General Laplace Transforms and Corresponding Derivatives

4.2.1. Nabla Case

To define a general Laplace transform, we express a given function in terms of the exponentials above deduced. This defines the inverse transform.

Definition 3. *Assume that we have a function, $f(t)$, defined on a given time sequence, having a transform $F_\nabla(s)$ (undefined for now; defined later by (26)). The inverse transform will serve to synthesize it from a continuous set of elemental exponentials:*

$$f(t) = -\frac{1}{2\pi i} \oint_\gamma F_\nabla(s) e_\nabla(t + \mu(t), t_0; s) ds, \tag{24}$$

where the integration path, γ, is a simple closed contour in a region of analyticity of $F_\nabla(s)$ and encircling the poles introduced by the exponential.

Consider the Hilger circle with radius $R > \frac{1}{h_m} = \frac{1}{\min(\nu_k, \mu_l)}$, centred at $\frac{1}{R}$, and described in the counterclockwise direction. We call this region the *fundamental region*, and we represent it by R_γ. In the continuous-time case, the circle degenerates into a vertical straight line. The expression (24) states our definition of inverse nabla LT . It is a simple task to show that the inverse nabla LT of $F_\nabla(s) \equiv 1$ is the

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impulse $\delta(t_n - t_0) = \frac{1}{\nu_1(t_0)}$. With this result we can prove one of the most important properties of the nabla LT :

$$\mathcal{L}_\nabla[D_\nabla f(t)] = sF_\nabla(s), \tag{25}$$

where $F_\nabla(s)$ is the nabla LT of $f(t)$, i.e., $F_\nabla(s) = \mathcal{L}_\nabla[f(t)]$. We deduce several properties of the nabla LT from the inversion integral. Precisely, the following properties hold (we assume s to be inside the region of convergence of the nabla LT):

- *Linearity*
$$\mathcal{L}_\nabla[f(t) + g(t)] = \mathcal{L}_\nabla[f(t)] + \mathcal{L}_\nabla[g(t)].$$
- *Transform of the derivative* Attending to the equality (25), we easily deduce, by repeated application of (25), that if $N \in \mathbb{N}_0$, then
$$\mathcal{L}_\nabla\left[D_\nabla^N f(t)\right] = s^N F_\nabla(s),$$
restating a well known result in the context of the LT .
- *Time scaling* Let a be a positive real number.[1] We have
$$\mathcal{L}_\nabla[f(at)] = \frac{1}{a} F_\nabla\left(\frac{s}{a}\right).$$

The shift property (21) gives a suggestion of how to define the nabla LT . In fact it must be defined in terms of the delta exponential, due to property (19) particularised to $t \to t + \mu(t)$ and $\tau = t$.

Definition 4. *We define the generalized nabla LT by*

$$F_\nabla(s) = \sum_{n=-\infty}^{+\infty} \nu_n f(t_n) e_\Delta(t_n, t_0; -s) = \sum_{n=-\infty}^{+\infty} \nu_n f(t_n) e_\nabla^{-1}(t_n, t_0; s) \tag{26}$$

with a suitable "region of convergence".

Example 1. *Unit steps. The nabla LT of the unit step is* $1/s$*:*

$$\mathcal{L}_\nabla[\epsilon(t)] = \frac{1}{s}$$

for s inside R_γ and

$$\mathcal{L}_\nabla[-\epsilon(-t - \mu(t))] = \frac{1}{s}$$

for s outside R_γ.

According to our framework we define the fractional nabla derivative by

$$D_\nabla^\alpha f(t) = \mathcal{L}^{-1}[s^\alpha F_\nabla(s)]. \tag{27}$$

Using (24)

$$D_\nabla^\alpha f(t) = \begin{cases} -\frac{1}{2\pi i} \oint_\gamma s^\alpha F_\nabla(s) \prod_{k=1}^{n+1} [1 - s\mu_k(t_0)]^{-1} ds = \frac{(-1)^n}{\prod_{k=1}^{n+1} \mu_k(t_0)} \oint_\gamma s^\alpha F_\nabla(s) \prod_{k=1}^{n+1} \left[s - \frac{1}{\mu_k(t_0)}\right]^{-1} ds & t = t_n \geq t_0 \\ -\frac{1}{2\pi i} \oint_\gamma s^\alpha F_\nabla(s) \prod_{k=1}^{m+1} [1 - s\nu_k(t_0)] ds = \frac{(-1)^n}{\prod_{k=1}^{m+1} \nu_k(t_0)} \oint_\gamma s^\alpha F_\nabla(s) \prod_{k=1}^{m+1} \left[s - \frac{1}{\nu_k(t_0)}\right] ds & t = t_m < t_0. \end{cases} \tag{28}$$

[1] It could be negative, but it is not interesting.

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To express this relation directly in terms of the values of $f(t)$ we need to introduce the convolution that is a somehow involved process [64]. We will do it in each particular case we will treat.

4.2.2. Stretching Case

The treatment of this case is similar to the previous. We only have to remind the need to use the log-grainisses. The rest of the procedure is equal, as we will see in the example s.

5. Main Derivatives on Time Sequences

5.1. Liouville Derivatives

Consider that the time sequence is the real line, $\mathbb{T} = \mathbb{R}$. Therefore, the graininess is null so that the basic derivative was introduced above

$$D_f f(t) = \frac{df(t)}{dt} = \lim_{h \to 0^+} \frac{f(t) - f(t-h)}{h}. \tag{29}$$

The application of the above framework gives:

1. The eigenfunction is the usual exponential

$$e_f(t) = e^{st}, \quad t \in \mathbb{R}, s \in \mathbb{C}; \tag{30}$$

2. The associated transform is the classic (bilateral) LT (A1)
3. The FD was introduced by Liouville (1832) through [45]

$$\frac{d^\alpha e^{st}}{dt^\alpha} = s^\alpha e^{st}, \tag{31}$$

with suitable ROC. Therefore, from (A2) we have

$$\frac{d^\alpha x(t)}{dt^\alpha} = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} s^\alpha X(s) e^{st}\, \mathrm{d}s, \tag{32}$$

with $t \in \mathbb{R}$ and $a > 0$;
4. The (causal) differintegrator has TF $H(s) = s^\alpha$, being valid for $Re(s) > 0$ or $s = \pm j\omega, \omega \in \mathbb{R}^+$. The anti-causal differintegrator has the left complex half-plane as ROC [69].
5. (Liouville–)Grünwald-Letnikov derivative reads [70]

$$\frac{d^\alpha x(t)}{dt^\alpha} = \lim_{h \to 0^+} h^{-\alpha} \sum_{k=0} \frac{(-\alpha)_k}{k!} x(t - kh), \tag{33}$$

6. Impulse response
The impulse response is given by [70]

$$\mathcal{L}^{-1} s^\alpha = \frac{t^{-\alpha-1}}{\Gamma(-\alpha)} \varepsilon(t); \tag{34}$$

7. Liouville (anti-)derivative
The impulse response and the convolution give another representation for the fractional anti-derivative:

$$\frac{d^\alpha x(t)}{dt^\alpha} = \int_0^\infty x(t-\tau) \frac{\tau^{-\alpha}}{\Gamma(-\alpha)} \mathrm{d}\tau, \tag{35}$$

that is the causal version of Liouville's first integral formula. He noted that this formula must not be used for positive values of α (derivative case), since the integral may be singular. He

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introduced two procedures stated as $s^\alpha = s^{\alpha-N}s^N = s^N s^{\alpha-N}, \alpha < N \in \mathbb{N}$. Alernatively, (35) can be regularized [70].

5.2. Tempered Shift-Invariant Derivatives

Definition 5. *Let $\lambda > 0$. In agreement with (7), we define the tempered shift-invariant derivative by [71]*

$$D^1_{\lambda,f} f(t) = \lim_{h\to 0^+} \frac{f(t) - e^{-\lambda h} f(t-h)}{h}. \tag{36}$$

For this derivative, the following steps result from the above framework.

1. The eigenfunction of this derivative is

$$e_{\nabla,\lambda}(t,s) = e^{(s-\lambda)t}, \quad t \in \mathbb{R}, s \in \mathbb{C}. \tag{37}$$

   We could define a modified LT . However, this result expresses only a translation, suggesting the we can use the simple exponential e^{st}, modifying the eigenvalue

$$D^1_{\lambda,f} e^{st} = (s+\lambda)e^{st}.$$

   Therefore, the usual LT is suitable to our objectives.
2. We define the tempered Liouville FD through

$$D^\alpha_\nabla e^{st} = (s+\lambda)^\alpha e^{st}, \quad t \in \mathbb{R}, s \in \mathbb{C}^+; \tag{38}$$

3. The differintegrator, $H(s) = (s+\lambda)^\alpha$, has $Re(s) > -\lambda$ as ROC. As

$$(s+\lambda)^\alpha = \lim_{h\to 0^+} \left(\frac{1 - e^{-(s+\lambda)}}{h} \right)^\alpha, \quad Re(s) > -\lambda,$$

   then

$$D^\alpha_{\lambda,f} f(t) = \lim_{h\to 0^+} h^{-\alpha} \sum_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} e^{-n\lambda h} f(t-nh); \tag{39}$$

4. The impulse response corresponding to $H(s)$ is

$$h(t) = e^{-\lambda t} \frac{t^{-\alpha-1}}{\Gamma(-\alpha)} \varepsilon(t), \tag{40}$$

   from where it is not difficult to obtain the integral versions of the tempered Liouville derivatives. The tempered Liouville anti-derivative reads

$$D^\alpha_{\lambda,f} f(t) = \int_0^\infty f(t-\tau) e^{-\lambda\tau} \frac{\tau^{-\alpha-1}}{\Gamma(-\alpha)} d\tau = e^{-\lambda t} \int_{-\infty}^t f(\tau) e^{\lambda\tau} \frac{(t-\tau)^{-\alpha-1}}{\Gamma(-\alpha)} d\tau. \tag{41}$$

   From this relation it is possible to obtain the regularized, Riemann-Liouville, or Liouville-Caputo derivatives [71].

5.3. Discrete-Time Euler Derivatives

Assume that our domain is the discrete-time sequence $\mathbb{T} = h\mathbb{Z}$ where h is the graininess or sampling interval of the time sequence.
Set $t = nh,\ n \in \mathbb{Z}$. From (3) we obtain the Euler *nabla derivative* by:

$$D_\nabla f(t) = \frac{f(t) - f(t-h)}{h}. \tag{42}$$

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For it, we have:

1. The nabla exponential is given by (A8)

$$e_{\nabla}(nh,s) = (1-sh)^{-n}, \quad n \in \mathbb{Z}, s \in \mathbb{C}.$$

The corresponding complex sinusoid is obtained when s is over the circle $[1-sh] = 1$. This defines the so-called right Hilger circle [63,67].

2. The nabla LT was introduced above and is given by [72]

$$\mathcal{L}_{\nabla}[f(nh)] = F_{\nabla}(s) = h\sum_{n=-\infty}^{+\infty} f(nh)e_{\nabla}(-nh,s),$$

while its inverse transform is

$$f(nh) = -\frac{1}{2\pi j}\oint_{\gamma} F_{\nabla}(s)e_{\nabla}((n+1)h,s)ds,$$

where the integration path, γ, is any simple closed contour in a region of analyticity of the integrand that includes the point $s = \frac{1}{h}$. The simplest path is a circle with centre at $s = \frac{1}{h}$. With the substitution $s = \frac{1-z^{-1}}{h}$ we recover the usual Z transform. Therefore the associated convolution is (A7).

3. We define the nabla FD through

$$D^{\alpha}_{\nabla}e_{\nabla}(nh,s) = s^{\alpha}e_{\nabla}(nh,s). \tag{43}$$

For non integer orders we have to consider a branchcut line starting at $s = 0$ and lying in the left complex half-plane.

4. Nabla fractional derivative
If $|z| > 1$ that implies $|1-hs| < 1$ (s inside the Hilger circle) then, from (43) and (A7).

$$D^{\alpha}_{\nabla}x(nh) = h^{-\alpha}\sum_{k=0}\frac{(-\alpha)_k}{k!}x(nh-kh), \tag{44}$$

that is the discrete-time nabla FD valid for any real (or complex) order.

5. The differintegrator, $H(s) = s^{\alpha}$, has the Hilger disk as ROC.

5.4. Tempered Euler Derivatives

These derivatives result from the Euler's by introducing a tempering factor. For the nabla case, we have

$$D^{1}_{\nabla,\lambda}f(nh) = \frac{f(nh) - e^{-\lambda h}f(nh-h)}{h}, \quad n \in \mathbb{Z}. \tag{45}$$

It is a simple matter to see that the eigenfunction reads

$$e_{\nabla,\lambda}(nh,s) = e^{-n\lambda h}(1-sh)^{-n}, \quad n \in \mathbb{Z}, s \in \mathbb{C}. \tag{46}$$

From it, the procedure for the corresponding transform is easy. It is interesting to state the expression for the FD that is (from (39))

$$D^{\alpha}_{\nabla,\lambda}f(nh) = h^{-\alpha}\sum_{m=0}^{\infty}\frac{(-\alpha)_m}{m!}e^{-m\lambda h}f(nh-mh). \tag{47}$$

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5.5. Bilinear Shift-Invariant Derivatives

Definition 6. *Let us assume again that our domain is the time scale $t \in \mathbb{T} = h\mathbb{Z}$. We define the order 1 forward (nabla) bilinear derivative $\nabla_b x(nh)$ of $x(nh)$ as the solution of the difference equation [73,74]*

$$\nabla_b f(nh) + \nabla_b f(nh-h) = \frac{2}{h}[f(nh) - f(nh-h)]. \tag{48}$$

With this Definition, we can deduce the following results.

1. The bilinear exponential is [73]

$$e_b(nh, s) = \left(\frac{2+hs}{2-hs}\right)^n, \quad n \in \mathbb{Z},\ s \in \mathbb{C}. \tag{49}$$

This relation suggests us to consider the discrete-time exponential function, $z^n,\ n \in \mathbb{Z}$. We define the forward bilinear derivative (∇_b) as an elemental DT system such that

$$\nabla_b z^n = \frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}} z^n. \tag{50}$$

Remark 2. *This change shows that, with this formulation, it is indifferent to work in the plane of the variable s or in the usual "discrete time" plane of the variable z.*

2. Fractional derivative
In agreement with our scheme, the bilinear FD is defined through

$$\nabla_b^\alpha e_b(nh, s) = s^\alpha e_b(nh, s), \tag{51}$$

with suitable ROC. In terms of the variable z, we can state

$$\nabla_b^\alpha z^n = \left(\frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}\right)^\alpha z^n, \qquad |z| > 1, \tag{52}$$

a differintegrator with TF

$$H_b(z) = \left(\frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}\right)^\alpha, \qquad |z| > 1. \tag{53}$$

For non-integer orders, we again have to consider a branchcut line. In this case, it is any line that lies inside the unit disk, joining the points $-1, +1$.
3. Z transform
As $z = \frac{2+hs}{2-hs}$ leads to the ZT and such transformation sets the unit circle $|z| = 1$ as the image of the imaginary axis in s, independently of the value of h, we will use the ZT. Once we have defined the derivative of an exponential, we can obtain the derivative of any signal that has a ZT. From (51) and (53) we conclude that, if $x(n)$ is a function with ZT $X(z)$, analytic in the ROC defined by $z \in \mathbb{C} : |z| > a,\ \ a < 1$, then

$$\nabla_b^\alpha x(n) = \frac{1}{2\pi i j}\oint_\gamma \left(\frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}\right)^\alpha X(z) z^{n-1} \mathrm{d}z, \tag{54}$$

with the integration path outside the unit disk.
4. The *bilinear differintegrator* is given by

$$H_b(z) = \left(\frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}\right)^\alpha.$$

5. GL type derivative
   It can be shown [73] that, letting ψ_k^α, $k = 0, 1, \cdots$, be the inverse ZT of $\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^\alpha$, we define the $\alpha-$order bilinear nabla derivative as

$$\nabla_b^\alpha x(n) = \left(\frac{2}{h}\right)^\alpha \sum_{k=0}^{\infty} \psi_k^\alpha x(n-k), \tag{55}$$

   analog to the GL derivative. The impulse response is obtained as follows. If $\alpha \in \mathbb{R}$ but $\alpha \notin \mathbb{Z}^-$, then

$$\psi_k^\alpha = (-1)^k \frac{(\alpha)_k}{k!} \sum_{m=0}^{k} \frac{(-\alpha)_m(-k)_m}{(-\alpha-k+1)_m} \frac{(-1)^m}{m!}, \quad k \in \mathbb{Z}_0^+, \tag{56}$$

   and

$$\psi_k^{-N} = \frac{(N)_k}{k!} \sum_{m=0}^{min(k,N)} \frac{(-N)_m(-k)_m}{(-N-k+1)_m} \frac{(-1)^m}{m!}, \quad k \in \mathbb{Z}_0^+, \tag{57}$$

   when $\alpha = -N$, $N \in \mathbb{Z}^+$.

5.6. Tempered Bilinear Shift-Invariant Derivatives

Definition 7. *In the conditions stated in the previous sub-section, we define the order* 1 *tempered nabla bilinear derivative* $\nabla_{\lambda,b} x(nh)$ *of* $x(nh)$ *as the solution of the difference equation [73,74]*

$$\nabla_{\lambda,b} f(nh) + e^{-\lambda h} \nabla_{\lambda,b} f(nh-h) = \frac{2}{h}\left[f(nh) - e^{-\lambda h} f(nh-h)\right]. \tag{58}$$

The application of our framework is similar to the previous. We will not do it here.

6. Main Derivatives on Scale Sequences

6.1. Hadamard Derivatives

Another interesting example of our framework is given by the scale-invariant derivatives. Retake the time sequence of the previous sub-section. Let $q \in \mathbb{R}^+$. As seen in Section 2, for any $t \in \mathbb{R}$, if we set $\tau = q^t \in \mathbb{R}^+$ we obtain a scale sequence Θ.

Definition 8. *We define a scale invariant derivative by*

$$\mathfrak{D}_s x(\tau) = \lim_{q \to 1} \frac{x(\tau) - x(\tau q^{-1})}{\log(q)}. \tag{59}$$

If $q > 1$*, we call it stretching derivative and if* $q < 1$ *it is a shrinking derivative.*

This definition agrees with the general formulae introduced in section 4 with $\rho(\tau) = \tau q^{-1}$ and $\theta(\tau) = q$.

Our framework gives

1. The eigenfunction is the usual power function [75]

$$e_s(\tau, s) = \tau^s, \quad s \in \mathbb{C}. \tag{60}$$

   The condition $\pm Re(s) > 0$ defines the stretching (+) and shrinking (-) derivatives.
2. The scale FD was introduced by Hadamard (1892) through [46,75]

$$\mathfrak{D}_s^\alpha \tau^s = s^\alpha \tau^s, \tag{61}$$

   with suitable ROC.

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3. The transform implied by the relation (60) is the usual Mellin transform (MT) (A3). Then, by (A4)

$$\mathfrak{D}_s^{\alpha} x(\tau) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} s^{\alpha} X(s) \tau^s \, \mathrm{d}s, \tag{62}$$

with $\tau \in \mathbb{R}^+$ and $a > 0$.
4. Differintegrator
Again, the differintegrator has TF $H(s) = s^{\alpha}$. We shall be considering the stretching case in what follows. Therefore, $Re(s) > 0$ or $s = \pm j\omega, \omega \in \mathbb{R}^+$ (we must note the similarity with the Liouville case).
5. The stretching GL-type derivative [75] is given by

$$\mathfrak{D}_s^{\alpha} x(\tau) = \lim_{h \to 1} \ln^{-\alpha}(q) \sum_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} x(\tau q^{-n}). \tag{63}$$

6. Impulse response [75]
The impulse response is the derivative of an impulse at $\tau = 1$, $\delta(\tau - 1)$. If $Re(s) > 0$, we have

$$\mathcal{M}^{-1} s^{\alpha} = \frac{\ln^{-\alpha-1}(\tau)}{\Gamma(-\alpha)} u(\tau - 1). \tag{64}$$

7. Hadamard (anti-)derivatives
The impulse response and the convolution give another representation for the fractional anti-derivative. Let $\alpha < 0$ and $\tau \in \mathbb{R}^+$. The relation stated in (64) used in the Mellin convolution leads to the scale anti-derivative given by

$$\begin{aligned} \mathfrak{D}_s^{\alpha} x(\tau) &= \frac{1}{\Gamma(-\alpha)} \int_1^{\infty} x(\tau/\eta) \ln^{-\alpha-1}(\eta) \frac{\mathrm{d}\eta}{\eta} \\ &= \frac{1}{\Gamma(-\alpha)} \int_0^{\tau} x(\eta) \ln^{-\alpha-1}(\tau/\eta) \frac{\mathrm{d}\eta}{\eta}. \end{aligned} \tag{65}$$

These relations express the usually called Hadamard integrals [46,48]. As in the Liouville derivative case, these integrals become singular when the derivative order is positive. To avoid the problem, we can regularize them or adopt Liouville procedures based on the relations ($s^{\alpha} = s^{\alpha-N} s^N = s^N s^{\alpha-N}, \alpha < N \in \mathbb{N}$) [75].

6.2. Tempered Scale-Invariant Derivatives

Definition 9. *As in the previous, let $\mu > 0$. Define a tempered stretching derivative by [75,76]*

$$\mathfrak{D}_{\mu,s} x(\tau) = \lim_{q \to 1^+} \frac{x(\tau) - q^{-\mu} x(\tau q^{-1})}{\ln q}. \tag{66}$$

This case is analogous to the previous. With a similar procedure, we obtain the eigenfunction correspnding to the eigenvalue v

$$e_{\lambda}(\tau, v) = \tau^{-\mu} \tau^{v}.$$

However, with a translation, we can use τ^v, by moving the eigenvalue to $v + \mu$, and using the MT [75]. Therefore, the results obtained in (6.1) are easily adaptable. The situation is analogue to the tempered shift-invariant case [75].

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6.3. Bilinear Scale-Invariant Derivatives

Definition 10. *We can follow the development in sub-section 5.5 by introducing a bilinear scale derivative as the solution of the equation*

$$\mathfrak{D}_{bi,s}x(\tau) + \mathfrak{D}_{bi,s}x(\tau/q) = \frac{2}{\ln(q)}[x(\tau) - x(\tau/q)]. \tag{67}$$

It is not very difficult to show that the bilinear exponential is

$$e_{b,s}(q^n, v) = \left[\frac{2 + v\log(q)}{2 - vlog(q)}\right]^n, \quad n \in \mathbb{Z}, \tag{68}$$

where v is again the eigenvalue. From here, we go on as in the case treated behind (5.5). The larger difference lies in the use of an exponential sampling instead of a linear. We can go further and introduce the tempered bilinear derivatives.

Definition 11. *We define a tempered bilinear scale derivative as the solution of*

$$\mathfrak{D}_{bi,\mu,s}x(\tau) + q^{-\mu}\mathfrak{D}_{bi,\mu,s}x(\tau/q) = \frac{2}{\ln(q)}\left[x(\tau) - q^{-\mu}x(\tau/q)\right]. \tag{69}$$

The procedure for obtaining the corresponding FD is the same as above.

6.4. Other Fractional Derivatives

6.4.1. q-Derivatives

Besides the anti-causal/shrinking cases, not treated, the above framework can be used to introduce other fractional derivatives, provided suitable order–1 derivatives are defined. These can be obtained by mixing different concepts. For instance, the approach above used to introduce the scale-invariant derivatives can give rise to other alternatives like:

- Use of (59) with a removal of the limit;
- With the substitution of $\ln(q)$ by $(1-q^{-1})\tau$ with or without limit (quantum derivative) [77,78];
- Introduction of a tempering term, as in (6.2).

Here, we will treat the second case, which is known by quantum derivative (q-derivative) [77,79].

Definition 12. *Let $q > 1$. The nabla q-derivative reads*

$$\mathfrak{D}_{q^{-1}}x(t) = \frac{x(t) - x(tq^{-1})}{(1-q^{-1})t}. \tag{70}$$

Substituting $q \to q^{-1}$ we obtain the delta q-derivative

$$\mathfrak{D}_q x(t) = \frac{x(qt) - x(t)}{(q-1)t}. \tag{71}$$

According to 4, we set $\rho(t) = t/q, \sigma(t) = qt, \mu(t) = (q-1)t,\ \nu(t) = (1-q^{-1})t$. These functions originate the following recursions:

1. (a) $\nu_0(t) = 0$;
   (b) $\nu_1(t) = (1-q^{-1})t$;
   (c) $\nu_n(t) = \nu_{n-1}(t) + \nu(t - \nu_{n-1}(t)) = q^{-1}\nu_{n-1}(t) + (1-q^{-1})t,\ n = 2,3,\cdots$
   - $\nu_2(t) = (1-q^{-2})t$;
   - $\nu_3(t) = (1-q^{-3})t$;
   - $\cdots$

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- $\nu_n(t) = (1 - q^{-n})t,\ n = 0, 1, 2, 3, \cdots$

2. (a) $\mu_0(t) = 0;$
   (b) $\mu_1(t) = (q - 1)t;$
   (c) $\mu_n(t) = \mu_{n-1}(t) + \mu(t + \mu_{n-1}(t)) = q\mu_{n-1}(t) + (q - 1)t,\ n = 2, 3, \cdots$
   - $\mu_2(t) = (q^2 - 1)t;$
   - $\mu_3(t) = (q^3 - 1)t;$
   - $\cdots$
   - $\mu_n(t) = (q^n - 1)t,\ n = 0, 1, 2, 3, \cdots$

With these granularity functions and using the exponentials and transforms defined in Section 4, we obtain the fractional q–derivatives.
Analogously, we can introduce other scale q–derivatives. We set again $\rho(t) = t/q$ and $\sigma(t) = qt$, but now $\mu(t) = \nu(t) = \log(q)$.

Definition 13. *Let $q > 1$. The stretching q-derivative reads*

$$\mathfrak{D}_{q^{-1}}x(t) = \frac{x(t) - x(tq^{-1})}{\log(q)}, \tag{72}$$

while the shrinking q-derivative is

$$\mathfrak{D}_q x(t) = \frac{x(qt) - x(t)}{\log(q)}. \tag{73}$$

Therefore, $\mu_n(t) = \nu_n(t) = \log^n(q)$. We must note the relation with the development introduced in subsection 6.1.

6.4.2. (q,h)–Derivatives

Definition 14. *Let $h \geq 0, q \geq 1$ and $t \in \mathbb{R}$. We define the nabla and delta Hahn derivatives [80,81] through*

$$\nabla_{q,h} f(t) = \frac{f(t) - f(t/q - h)}{(1 - q^{-1})t + h}, \quad (1 - q^{-1})t + h \neq 0 \tag{74}$$

and

$$\Delta_{q,h} f(t) = \frac{f(qt + h) - f(t)}{(q - 1)t + h}, \quad (q - 1)t + h \neq 0. \tag{75}$$

According to Section 4, we set $\rho(t) = t/q - h,\ \sigma(t) = qt + h,\ \mu(t) = (q - 1)t + h,\ \nu(t) = (1 - q^{-1})t + h.$

As above, we obtain the following recursions:

1. (a) $\nu_0(t) = 0;$
   (b) $\nu_1(t) = (1 - q^{-1})t + h.;$
   (c) $\nu_n(t) = \nu_{n-1}(t) + \nu(t - \nu_{n-1}(t)) = q^{-1}\nu_{n-1}(t) + (1 - q^{-1})t,\ n = 2, 3, \cdots$
   - $\nu_2(t) = (1 - q^{-2})t + q^{-1}h;$
   - $\nu_3(t) = (1 - q^{-3})t + q^{-2}h;$
   - $\cdots$
   - $\nu_n(t) = (1 - q^{-n})t + q^{-n+1}h,\ n = 1, 2, 3, \cdots$
2. (a) $\mu_0(t) = 0;$
   (b) $\mu_1(t) = (q - 1)t + h;$
   (c) $\mu_n(t) = \mu_{n-1}(t) + \mu(t + \mu_{n-1}(t)) = q\mu_{n-1}(t) + (q - 1)t,\ n = 1, 2, 3, \cdots$
   - $\mu_2(t) = (q^2 - 1)t + qh;$
   - $\mu_3(t) = (q^3 - 1)t + q^2h;$
   - $\cdots$
   - $\mu_n(t) = (q^n - 1)t + q^{n-1}h,\ n = 1, 2, 3, \cdots$

Again, with these granularity functions and using the exponentials and transforms defined in Section 4 we obtain the Hahn fractional derivatives.

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Remark 3. *We must note that*

- *With $q = 1$, we obtain the usual Euler nabla and delta derivatives above introduced [74];*
- *Letting $h = 0$, we obtain the quantum derivatives considered in the previous subsection;*
- *$\mu(t) = 0$ implies that $q = 1$ and $h = 0$ simultaneously. In such case, we compute the corresponding limits. we will not do it since it is similar to the Liouville type derivative.*
- *We can define scale (q,h)–derivatives by setting $\mu(t) = \log((q-1) + h/t)$ and $\nu(t) = \log((1-q^{-1}) + h/h)$.*

7. Fractional Two-Sided Derivatives

In the previous sections, we dealt with one-sided derivatives. In particular we studied the causal and stretching derivatives. However, two-sided derivatives are useful in many applications [13,28,82,83]. The shift invariant bilateral derivatives have been already studied with generality [84,85]. This did not happen with the scale counterpart. We will not do it, but the following definitionhas a wide validity.

Definition 15. *Let $\alpha, \beta \in \mathbb{R}$. As shown in [61] for the shift-invariant case, a two-sided derivative D^{γ}_{θ} can be defined by*

$$D^{\alpha+\beta}_{\alpha-\beta} f(t) = \mathcal{L}^{-1}\left[s^{\alpha}(-s)^{\beta}F(s)\right], \tag{76}$$

where $\gamma = \alpha + \beta$ is the derivative order, and $\theta = \alpha - \beta$ is an asymmetry parameter and stating a unified formulation that includes the one-sided derivatives.

An interesting particular case is the following. Let $f(t),\ t \in \mathbb{R}$, be a real function and $\gamma, \theta \in \mathbb{R}$ two real parameters. The two-sided Grünwald-Letnikov type bilateral derivative of $f(t)$ is given by

$$D^{\gamma}_{\theta} f(t) = \lim_{h \to 0^+} h^{-\gamma} \sum_{n=-\infty}^{+\infty} \frac{(-1)^n \Gamma(\gamma+1) f(t-nh)}{\Gamma(\frac{\gamma+\theta}{2} - n + 1)\Gamma(\frac{\gamma-\theta}{2} + n + 1)}, \tag{77}$$

For the scale-invariant case, it is a simple task to define a similar derivative by

$$D^{\gamma}_{\theta} f(t) = \lim_{q \to 1} \log^{-\gamma}(q) \sum_{n=-\infty}^{+\infty} \frac{(-1)^n \Gamma(\gamma+1) f(t/q^n)}{\Gamma(\frac{\gamma+\theta}{2} - n + 1)\Gamma(\frac{\gamma-\theta}{2} + n + 1)}, \tag{78}$$

The two-sided derivatives are useful in representing the space variation in partial differential equations.

8. Conclusions

The basic principles underlying the construction of fractional derivatives were introduced and examined. The underlying idea was the formalization and development of schemes based on the eigenfunctions of simple order 1 derivatives. We showed that the proposed scheme allows us to obtain a unified framework for dealing with continuous/discrete-time shift-invariant or scale-invariant derivatives. This opens many doors for applications. On the other hand, our previous criteria for FD definition appeared as consequence of the scheme.

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Abbreviations

The following abbreviations are used in this manuscript:

C	Caputo
FD	Fractional derivative
GL	Grünwald-Letnikov
LT	Laplace transform
MT	Mellin transform
RL	Riemann-Liouville
ROC	Region of convergence

Appendix A. Mathematical Tools

Some interesting mathematical tools are:

1. Usual (D'Alembert's or Duhamel's) convolution [86]

$$y(t) = x(t) * g(t) = \int_{-\infty}^{+\infty} x(t-\eta)g(\eta)\mathrm{d}\eta.$$

that is suitable for expressing the input/output relation in shift-invariant systems [87]

2. Mellin convolution [75,88]

$$y(\tau) = x(\tau) \star g(\tau) = \int_0^{\infty} x\left(\frac{\tau}{\eta}\right) g(\eta)\frac{d\eta}{\eta},$$

used for expressing the input/output relation in scale-invariant systems [75]

3. The (bilateral) Laplace transform (LT) is given [73]:

$$\mathcal{L}[h(t)] = H(s) = \int\limits_{-\infty}^{\infty} h(t)e^{-st}\,\mathrm{d}t,\ s \in \mathbb{C}, \tag{A1}$$

that is assumed to converge in some non-void region (region of convergence—ROC) which may degenerate into the imaginary axis, givig rise to the Fourier transform (with $s = i\omega$.)

4. We define the inverse LT by the Bromwich integral

$$h(t) = \mathcal{L}^{-1}F(s) = \frac{1}{2\pi i}\int\limits_{a-i\infty}^{a+i\infty} H(s)e^{st}\,\mathrm{d}s,\ t \in \mathbb{R}, \tag{A2}$$

where $a \in \mathbb{R}$ is called abscissa of convergence. Frequently, we denote by γ the integration path.

5. In a similar way, we define the Mellin transform (MT) by [75]

$$\mathcal{M}[h(t)] = H(v) = \int\limits_{0}^{\infty} h(t)t^{-v-1}\,\mathrm{d}t,\ v \in \mathbb{C}, \tag{A3}$$

with an inverse similar to (A2)

$$h(t) = \mathcal{M}^{-1}H(v) = \frac{1}{2\pi i}\int\limits_{\gamma} H(v)t^{v}\,\mathrm{d}v,\ t \in \mathbb{R}^{+}. \tag{A4}$$

The MT in (A3) has a parameter sign change $-v \to v$ relatively to the usual MT [88]. However, this definition establishes a better parallelism with the LT, concerning the region of convergence.

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6. We define the Z transform [69,89] by

$$\mathcal{Z}[f(n)] = F(z) = \sum_{n=-\infty}^{+\infty} f(n)z^{-n}, \quad z \in \mathbb{C}, \tag{A5}$$

with the inverse given by the Cauchy integral

$$f(n) = \frac{1}{2\pi\, i} \oint_c F(z)z^{n-1}\mathrm{d}z, \tag{A6}$$

where c is the unit circle. With the change of variable $z = e^{i\omega}$, $-\pi < \omega \leq \pi$, we obtain the discrete-time Fourier transform.

7. The convolution associated with the Z transform reads

$$x(n) * y(n) = h \sum_{k=-\infty}^{\infty} x(kh)y(nh - kh). \tag{A7}$$

8. Nabla exponential
This exponential reads

$$e_\nabla(nh, s) = (1 - sh)^{-n}, \quad n \in \mathbb{Z}, h \in \mathbb{R}^+, s \in \mathbb{C}. \tag{A8}$$

9. Nabla LT (NLT)
The analysis equation for the NLT is given by

$$\mathcal{N}[f(nh)] = F_\nabla(s) = h \sum_{n=-\infty}^{+\infty} f(nh)e_\nabla(-nh, s) \quad h \in \mathbb{R}^+, s \in \mathbb{C}. \tag{A9}$$

Its inverse transform (synthesis equation) is given by

$$f(nh) = -\frac{1}{2\pi i} \oint_\gamma F_\nabla(s)e_\nabla((n+1)h, s)ds, \quad n \in \mathbb{Z}, h \in \mathbb{R}^+. \tag{A10}$$

where the integration path, γ, is any simple closed contour in a region of analyticity of $F_\nabla(s)$ that includes the point $s = \frac{1}{h}$. The simplest path is a circle with centre at $s = \frac{1}{h}$.
With the substitution $s = \frac{1-z^{-1}}{h}$ we recover the usual Z transform. Therefore the associated convolution is (A7).

10. The binomial Theorem
Let $\alpha \in \mathbb{R} - \mathbb{Z}^-$. Then,

$$(1 - z)^\alpha = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} z^k, \quad |z| < 1. \tag{A11}$$

We can extend it for negative integer values of α through the Pochhammer symbol for the rising factorial

$$(a)_n = \prod_{k=0}^{n-1} (a + k) = \frac{\Gamma(a+n)}{\Gamma(a)}, \text{ with } (a)_0 = 1,$$

obtaining

$$(1 - z)^\alpha = \sum_{k=0}^{\infty} \frac{(-\alpha)_k}{k!} z^k, \quad |z| < 1. \tag{A12}$$

Appendix B. Framework for Fractionalizing Derivatives

In [61] a framework the fractionalization of derivatives was proposed, intended to deduce those derivatives suitable for signal processing, namely to define linear systems. It consists of the following steps:

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1. Let $\mathbb{T}$ denote any time or scale sequence and define there a generically called *derivative* (see subsection 3.2), $\mathcal{D}_f$. We may consider the types: right or stretching, left or shrinking, and bilateral. The treatment of the left/shrinking is similar to the right/stretching. Therefore, we will not consider them, unless there is a particular interest. The bilateral will have a different approach that will be described later.
2. For such a derivative, compute the pair eigenvalue/eigenfunction from

$$\mathcal{D}_f e_f(t,s) = s\, e_f(t,s), \tag{A13}$$

where $t \in \mathbb{T}$ and $s \in \mathbb{C}$. The eigenfunction $e_f(t,s)$ will be called *generalized exponential*.
3. Define the corresponding fractional derivative, $\mathcal{D}_f^\alpha$, through

$$\mathcal{D}_f^\alpha e_f(t,s) = s^\alpha\, e_f(t,s), \tag{A14}$$

where $\alpha \in \mathbb{R}$ is the derivative order. The elemental system characterized by the transfer function (TF), $H(s) = s^\alpha$, with a suitable region of convergence, is called *differintegrator*.
4. With the generalized exponential, above introduced, define a (generalized Laplace) transform,

$$X(s) = \mathcal{L}_f[x(t)],$$

so that

   (a) Any function, $x(t)$, can be expressed in terms of the eigenfunction (synthesis equation) by

$$x(t) = \mathcal{L}_f^{-1} X(s) = \frac{1}{2\pi j}\int_\gamma X(s) e_f(t,s) \mathrm{d}s, \tag{A15}$$

   where γ is in general a closed simple integration path in a region of the complex plane where $X(s)$ is analytic. It may happen we have to make an adjustment in (A15) so that

$$\mathcal{L}_f\left[\delta_f(t)\right] = 1,$$

   in the whole complex plane. The relation (A15) defines the inverse transform.

   (b) From (A15) and using (A14), the classic relation

$$\mathcal{L}_f[\mathcal{D}^\alpha x(t)] = s^\alpha X(s) \tag{A16}$$

   emerges and is valid in a suitable ROC.

   (c) Let $g(t) = \mathcal{L}_f^{-1} G(s)$ and $x(t) = \mathcal{L}_f^{-1} X(s)$. From (A15) we define a convolution operation through

$$y(t) = g(t) * x(t) = \mathcal{L}_f^{-1}[G(s)X(s)]. \tag{A17}$$

   (d) From (A17), we conclude that the convolution is commutative and, as $s^\alpha G(s)X(s) = G(s)s^\alpha X(s)$, then

$$D_f^\alpha[g(t) * x(t)] = D_f^\alpha g(t) * x(t) = g(t) * D_f^\alpha x(t). \tag{A18}$$

   (e) The analysis equation (direct transform) is defined, for each case, in agreement with the properties of the eigenfunction.

Remark A1. *The values $w \in \mathbb{R}$ for which $s = \phi(w)$ lead to $|e_f(t,s)| = 1$ are called frequencies and $H(w)$ is the frequency response of the differintegrator. In such cases, the generalized LT degenerates into a Fourier transform and the FD of a sinusoid is a sinusoid. This is very important in applications.*

The above definitionof FD operator as *an elemental system having TF equal to $H(s) = s^\alpha$*, brings some properties that are fundamental in applications and are easily deduced from (A15) [70]:

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1. Associativity of the orders
$$\mathcal{D}_f^{\alpha}\mathcal{D}_f^{\beta}x(t) = \mathcal{D}_f^{\alpha+\beta}x(t); \tag{A19}$$
2. Identity
$$\mathcal{D}_f^{0}x(t) = x(t); \tag{A20}$$
3. Inverse
$$\mathcal{D}_f^{\alpha}\mathcal{D}_f^{-\alpha}x(t) = \mathcal{D}_f^{-\alpha}\mathcal{D}_f^{\alpha}x(t) = x(t). \tag{A21}$$

Remark A2. *These properties, together with the generalized Leibniz rule, constitute the SSC for deciding if a given elemental system can be considered as a fractional derivative, as seen above (3.3) [51]. The main difference lies in the fact that the criterion emerges here as a consequence, without being axiomatically imposed. With the transform abve introduced we can obtain the generalized Leibniz rule, but we will not do it.*

We are going to apply the framework for several useful systems.

Remark A3. *It is important to highlight some interesting facts:*

1. *Although from a purely mathematical point of view, the derivative order can be any complex number, this cannot be the case from applications point of view, since a complex order leads to non-Hermitian derivatives and systems [69,90].*
2. *For a non integer order, s^{α} represents a multivalued expression. To define a function we need to introduce a branchcut line, starting at $s = 0$. The complete location is specified in agreement with the ROC.*

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