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Paradoxes of Infinity as Reductio ad Absurdum

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Abstract

From Cantor's diagonal argument and Russell's antinomy to Gödel's incompleteness and Turing's undecidability—classical results about infinity are usually read as limitations to be endured within an infinitary foundation. This paper proposes a different diagnosis. We show that the standard derivations uniformly instantiate *reductio* patterns once one isolates the specific infinitary postulate each result requires. We organise the landscape into three normal forms by premise: (AT) *absolute totalities* ("the set of all sets/ordinals"), (IR) a *single actually infinite registry* (global listings of formulas/proofs or machines/inputs), and (AC∞) *unrestricted Choice* over uncountable families. In each case, baseline coherence assumptions (consistency, and where relevant completeness or basic measure) combined with the corresponding infinitary premise yield contradiction or impossibility. We recast Gödel I as an *inconsistent triad* (Consistency + Completeness + IR), so that rejecting IR—rather than completeness—is a coherent resolution. Crucially, the *finite/periodic foundation* is taken as ontologically primary: it *recovers* the practical functionality often attributed to infinitary methods (global choice becomes definable; periodic/equivariant choice handles repeated families; AC-dependent pathologies cannot arise; and diagonal "escape" fails without a global infinite registry). The *infinitary framework* is retained only as an *idealisation*, a convenient approximation language layered on top of the finite/periodic reality. The philosophical payoff is a conservative, practice-preserving foundation in which mathematics is finite, relational, and paradox-free.

Keywords: infinity paradoxes; reductio ad absurdum; finite/periodic foundation; infinitary idealisation; diagonalization; Gödel's incompleteness; Turing halting problem; Russell's paradox; Axiom of Choice; equivariant choice; Banach-Tarski paradox; non-measurable sets; infinite registry; absolute totalities; foundations of mathematics

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1. Introduction

Motivation. Since the late nineteenth century, *infinity* has been treated as foundational in mathematics and logic. Cantor's transfinite arithmetic and diagonal method, Russell's antinomy, Gödel's incompleteness, and Turing's undecidability are now canonical touchstones [1–4]. Their power is undeniable; yet each also exhibits a characteristic kind of breakdown that manifests only when *actual* infinity is admitted: absolute totalities ("the set of all sets" or "all ordinals"), global infinite listings ("all formulas and proofs," "all machines"), and unrestricted selections over uncountable families (as in Vitali sets and Banach-Tarski decompositions [5,6]). The usual moral is that mathematics must live with these limitations by refining axioms and restricting constructions.

Thesis. This paper advances a different reading: many classical paradoxes—Cantor, Russell, Gödel, and Turing among them—admit a uniform interpretation *by reductio*. Properly displayed, their derivations show that the contradiction or impossibility arises from adding a *specific infinitary postulate* to otherwise benign coherence assumptions. On this view the results do not merely reveal "limitations of formal systems" while leaving actual infinity intact; rather, they function as *reductio proofs against the corresponding assumption of actual infinity itself*. This aligns with Lev's proposal that finite mathematics is ontologically primary, while infinitary formalisms are degenerative approximations of the finite foundation [7]. In brief: abandon the offending infinitary postulate, not the coherence principles.

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Contribution. We make this claim precise in three ways.

1. **A systematic reductio interpretation.** We organise the classical results into three *normal forms* according to the infinitary premise they require: (i) *absolute totalities* (AT) for set-theoretic antinomies, (ii) an *infinite registry* (IR) for diagonal arguments, and (iii) *unrestricted Choice* over uncountable families (AC∞) for measure/geometric paradoxes. For each family we separate baseline coherence assumptions (consistency, completeness where relevant, basic measure-theoretic constraints) from the single infinitary premise that drives the paradox, and we state an explicit *reductio principle* recommending rejection of that premise.
2. **Gödel as reductio.** We recast incompleteness as an *inconsistent triad*: (Consistency) + (Completeness) + (IR). The classical stance rejects completeness; the reductio stance rejects IR, i.e., the postulate of a single, actually infinite registry supporting global diagonal self-reference. The proof-theoretic core (fixed-point lemma and standard derivation) is unchanged; only the diagnostic conclusion differs.
3. **Retaining functionality without paradox.** We show that *finite/periodic* structures recover the practical roles often attributed to infinitary assumptions while blocking their paradoxical uses. In particular, the *functionality* of the Axiom of Choice is derivable on a fixed finite universe via canonical order, and extends along periodic or equivariant families via a *Periodic Choice* principle; AC-dependent pathologies (Vitali nonmeasurable sets, Banach–Tarski decompositions) are thereby precluded. Diagonal "escape" steps likewise fail when no global infinite registry is admitted.

Scope and method. We work in classical logic and familiar mathematical practice. The proposal is *diagnostic* and *foundational*: the *finite/periodic foundation* is taken as ontologically primary; the *infinitary framework* functions as a *utilitarian idealisation*——a limit-style shorthand for stable behaviours of the finite/periodic foundations. The paradoxes are then read as guides to which infinitary premises to avoid or regulate. Thus, the benefits of classical methods are retained, while the sources of contradiction are identified and replaced by finite formulations, such as the one detailed in [8].

Roadmap. Section 2 surveys the classical paradoxes by family. Section 3 recasts Gödel's argument as reductio and isolates the infinitary premise. Section 4 states the reductio principle in normal-form templates. Section 5 revisits the Axiom of Choice, formulates periodic/equivariant choice on finite bases, and explains why AC-dependent paradoxes disappear. Section 6 draws the philosophical implications; Section 7 concludes.

2. Classical Paradoxes of Infinity

This section surveys the number of well-known paradoxes and limit phenomena that arise when *actual infinity* is admitted into the foundations of mathematics and logic. We group them into four families:

(i) set-theoretic antinomies driven by unrestricted totalities,

(ii) diagonal constructions (self-reference across infinite listings),

(iii) measure/geometric paradoxes that crucially employ strong forms of the Axiom of Choice (AC), and

(iv) operational thought experiments that dramatise tensions between finite intuition and infinite idealisation. In §3–§4 we recast these patterns in a common *reductio* form; AC-dependent cases are revisited in §5.

2.1. Set-Theoretic Antinomies

Russell's paradox (naïve comprehension). Let $R = \{x \mid x \notin x\}$. Then $R \in R \Leftrightarrow R \notin R$, a contradiction, obtained by combining unrestricted comprehension with an absolute totality of sets [2].

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Burali–Forti (the ordinal of all ordinals). If the totality Ord of all ordinals were a set, its order type would be an ordinal $\alpha \in \mathsf{Ord}$ with $\alpha < \alpha$, contradiction [9]. As with Russell, a global "all-of-a-kind" set collides with routine set formation.

Cantor's paradox (the set of all sets). If a universal set V existed, Cantor's theorem yields $|V| < |\mathcal{P}(V)| \leq |V|$, contradiction.

Moral. These antinomies arise from treating *absolute totalities* (all sets, all ordinals) as sets; modern axiomatics blocks the offending comprehension rather than abandoning actual infinity.

2.2. Diagonal Constructions (Self-Reference Over Infinite Listings)

Cantor's uncountability. Given any enumeration $s_1, s_2, \ldots$ of infinite binary sequences, the diagonal sequence d with $d(n) = 1 - s_n(n)$ differs from every s_n; no listing is complete [1]. The key is a *single actually infinite registry* against which to diagonalise.

Gödel's incompleteness. In any consistent, effectively axiomatized theory of arithmetic with *infinitely many* sentences/proofs, the diagonal lemma yields a sentence G asserting its own unprovability. If G is provable the theory is inconsistent; if unprovable the theory is incomplete [3]. See §3 for a *reductio* reading.

Turing's halting problem. Assuming an effective enumeration $\{M_n\}$ of all machines, a diagonal machine D diverges on n iff M_n halts on n. No total decider exists for all inputs [4]. Again the construction relies on a global, actually infinite listing.

Skolem's phenomenon ("paradox"). First-order set theory has countable models that *internally* satisfy "there exists an uncountable set" [10]. There is no formal contradiction, but a striking mismatch between internal and external cardinality talk in an infinite setting.

Moral. Diagonal arguments uniformly require an actually infinite registry (sequences, theorems, machines). The pivotal "outside-the-list" step disappears in strictly finitary/periodic universes.

2.3. Measure-Theoretic And Geometric Paradoxes (Choice-dependent)

Vitali non-measurable sets. Using AC, choose one representative from each equivalence class of $[0,1]$ under $x \sim y$ iff $x - y \in \mathbb{Q}$; the result is non-measurable [5]. In models with restricted Choice (and strong regularity), no such set exists.

Banach–Tarski decomposition. In $\mathbb{R}^3$, AC permits partitioning a ball into finitely many pieces and reassembling them (via isometries) into two balls congruent to the original [6]. The construction exploits non-amenability of the rotation group and Choice to form wild subsets.

Hausdorff's sphere paradox. Earlier paradoxical decompositions on the 2-sphere (with rotations) foreshadow Banach–Tarski [11].

Moral. These are structural consequences of combining actual infinity with strong selection principles over rich symmetry groups. In finite/periodic domains, canonical orders yield definable choice and finite/amenable actions, precluding such effects (cf. §5).

2.4. Operational And Heuristic Paradoxes

Hilbert's hotel. A full hotel with countably many rooms accommodates countably many new guests by shifting guest n to room $n + 1$; an intuition pump for infinite arithmetic (popularised in [12]).

Zeno's supertasks. Achilles traverses infinitely many shrinking intervals; the arrow is at rest at each instant. Calculus resolves with convergent series, yet the supertask idealisation highlights the strength of infinite divisibility [13].

Moral. These are not formal contradictions but illustrate how infinite idealisations strain operational coherence and physical intuition.

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2.5. Synthesis: A Recurring Pattern

Across these families we see a shared logical shape:

- **Assumption.** Admit actual infinity—absolute totalities, unbounded listings, or unrestricted Choice.
- **Construction.** Form a self-referential/diagonal or Choice-driven object insensitive to measure/structure.
- **Outcome.** Derive antinomy, undecidability/incompleteness, or paradoxical decompositions.

The classical response retains infinity and quarantines trouble via axiomatic refinements (e.g., ZFC, regularity hypotheses, determinacy in lieu of full AC). The *reductio* perspective pursued later reads the contradictions as evidence that the antecedent infinitary assumption should be replaced by finite/periodic structure, preserving practice (including a periodic choice principle) without the paradoxes (§4, §5).

3. Gödel's Incompleteness as *Reductio*

The incompleteness theorems are usually read as internal *limitations* of strong, consistent, effectively axiomatized arithmetical theories: no such theory can be both consistent and complete. Here we present a formulation that isolates the explicitly *infinitary* premise used in the proof and recasts the conclusion in *reductio* form: keep the baseline coherence assumptions and treat the contradiction as a refutation of the infinitary premise (cf. the "IR" premise in §4).

Set-up and standing hypotheses. Let T be a first-order theory such that:

1. **(Eff)** T is effectively axiomatized (its axioms are recursively enumerable).
2. **(Arith)** T interprets a sufficient fragment of arithmetic (e.g., Q or $I\Sigma_1$) to carry out Gödel coding.
3. **(Cns)** T is consistent; when needed we assume the usual mild strengthening (e.g., ω-consistency or Σ^0_1-soundness).

Write $\mathsf{Prov}_T(x)$ for the arithmetized provability predicate and $\ulcorner\varphi\urcorner$ for the Gödel number of φ.

Lemma 1 (Diagonal / Fixed-Point Lemma)**.** *For every formula $\phi(x)$ of T with one free variable there is a sentence G such that*

$$T \vdash G \leftrightarrow \phi(\ulcorner G\urcorner).$$

Taking $\phi(x) \equiv \neg\mathsf{Prov}_T(x)$ yields the usual Gödel sentence G that asserts its own unprovability.

The classical derivation.

Theorem 1 (Gödel I, classical form)**.** *If T satisfies* (Eff) *and* (Arith) *and is consistent and ω-consistent (or at least Σ^0_1-sound), then $T \nvdash G$ and $T \nvdash \neg G$. Hence T is incomplete [3].*

Sketch. If $T \vdash G$, then from $T \vdash G \leftrightarrow \neg\mathsf{Prov}_T(\ulcorner G\urcorner)$ we obtain $T \vdash \neg\mathsf{Prov}_T(\ulcorner G\urcorner)$, but $T \vdash G$ yields $T \vdash \mathsf{Prov}_T(\ulcorner G\urcorner)$, contradicting consistency. If $T \vdash \neg G$, then $T \vdash \mathsf{Prov}_T(\ulcorner G\urcorner)$; under ω-consistency or Σ^0_1-soundness this implies $T \vdash G$, contradicting the first part. □

Theorem 2 (Rosser improvement)**.** *If T satisfies* (Eff) *and* (Arith) *and is merely consistent, then T is incomplete [14].*

The inconsistent triad and the *reductio* reading. We make explicit the infinitary postulate that powers the diagonal step.

(IR) Infinite Registry. *There exists a single, uniform, actually infinite listing of all formulas and all proofs of T (supporting arithmetization and diagonal/fixed-point constructions over the entire list).*

Proposition 1 (Inconsistent triad)**.** *The following cannot all hold simultaneously:*

1. (Cns) *T is (sufficiently) consistent (as above).*

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2. (Cmp) *T is complete (every sentence is decided).*
3. (IR) *The infinite registry premise holds.*

Sketch. Under (IR) and Lemma 1, form G with $T \vdash G \leftrightarrow \neg\mathrm{Prov}_T(\ulcorner G \urcorner)$. If (Cmp) holds then either $T \vdash G$ (contradicting (Cns)) or $T \vdash \neg G$ (which, given the mild soundness assumption bundled into (Cns), again contradicts (Cns)). Hence the triad is inconsistent. □

Corollary 1 (Two coherent resolutions)**.** *From Proposition 1 one may:*

1. Reject completeness (Cmp)*—the classical reading: consistent, effectively axiomatized, sufficiently strong theories over an actually infinite registry are inevitably incomplete [3,14].*
2. Reject the infinitary premise (IR)*—the* reductio *reading: the contradiction shows that postulating a single, actually infinite registry that sustains global diagonal escape is incoherent; without* (IR) *the derivation cannot go through (cf. §4).*

Where the infinity enters. The proof uses actual infinity at three distinct junctures:

1. **Syntactic infinity.** A completed, global listing of formulas/proofs (the *registry*) to which the diagonal construction applies.
2. **Semantic infinity.** The appeal to standard-$\mathbb{N}$ truth for Σ^0_1 sentences (in the ω-consistency / Σ^0_1-soundness clauses).
3. **Diagonal escape.** The fixed-point/diagonal step that defines an object disagreeing with *every* entry of an actually infinite list.

In settings where only finite or periodic registries exist, (1)–(3) fail in their classical form and no "outside-the-list" sentence can be forced; the diagonal returns an element already inside the bounded universe rather than escaping it.

3.1. Allied Meta-Mathematical Results

Tarski's undefinability of truth. A uniform truth predicate for all sentences of an actually infinite language leads, via diagonalisation, to the liar; one rejects *internal* truth (classical stance) or the global infinitary registry premise [15].

Turing's halting problem. Given an actually infinite enumeration of machines and inputs, the diagonal machine disagrees on the diagonal; either accept undecidability (classical) or treat the result as a *reductio* on the global registry premise [4].

Summary. The technical core of incompleteness (Lemma 1, Theorems 1–2) is uncontested. Proposition 1 isolates the precise point at which actual infinity enters and offers a clean *reductio* alternative: keep consistency (and even completeness, if desired) and reject the global infinitary registry (IR). This matches the general *reductio* schema developed in §4.

4. The Reductio Principle for Paradoxes

This section abstracts the common logical *shape* underlying the paradoxes surveyed in §2 and the incompleteness analysis in §3. We isolate a small set of *infinitary premises* that, when combined with basic coherence assumptions, yield contradiction or other impossibility results. The guiding idea is that one may coherently treat such outcomes as *reductio ad absurdum* arguments against the corresponding infinitary premise.

4.1. Infinitary Premises And Baseline Constraints

We distinguish three schematic infinitary premises.

AT (Absolute Totality). There exists a set that collects *all* objects of a given kind (e.g., the set of all sets, the set of all ordinals).

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IR (Infinite Registry). There exists a single, uniform, actually infinite listing of syntactic or algorithmic objects (e.g., all formulas and proofs; all Turing machines and inputs).

AC∞ (Unrestricted Choice). One may choose a representative from *every* member of an arbitrary family of nonempty sets, including uncountable families with no canonical structure.

We also use the following baseline coherence constraints.

Cns. Classical consistency (or minimal soundness such as Σ^0_1-soundness).

Cmp. Completeness (every sentence is decidable) when this is the target assumption.

Meas. Basic measure/coherence principles appropriate to the setting (e.g., countable additivity, isometry invariance, no paradoxical decompositions) when geometric measure is in view.

4.2. Normal Forms Of Paradox

We record three *normal forms* that capture the paradox templates discussed in §2.

Proposition 2 (Antinomy normal form)**.** *The triad (Cns) + (AT) + (unrestricted comprehension/power set) is inconsistent. In particular, Russell's construction and the Burali–Forti argument instantiate this normal form: asserting a universal set or the set of* all *ordinals, together with routine set formation, yields contradiction [2,9].*

Sketch. Assuming AT for sets of a kind K, form the set $R = \{x \in K : x \notin x\}$ (Russell) or the putative set Ord of all ordinals (Burali–Forti). Standard reasoning produces $R \in R \leftrightarrow R \notin R$ and $\alpha < \alpha$ respectively, contradicting Cns. □

Proposition 3 (Diagonal normal form)**.** *The triad (Cns) + (Cmp) + (IR) is inconsistent. Cantor's diagonalisation against a list of infinite sequences [1], Gödel's self-referential fixed point inside a recursively axiomatized arithmetic [3], and Turing's diagonal machine against an effective enumeration of procedures [4] all realize this pattern.*

Sketch. Under IR, construct along the diagonal an object that disagrees with each entry of the registry: a new sequence (Cantor), a sentence G with $G \leftrightarrow \neg\mathsf{Prov}(\ulcorner G \urcorner)$ (Gödel), or a machine D that diverges exactly when M_n halts on n (Turing). Cmp forces a decision on the diagonal object, contradicting Cns as in §3. □

Proposition 4 (Choice/geometry normal form)**.** *The bundle (Meas) + (AC∞) + (non-amenable infinite symmetry) over uncountable domains entails paradoxical decompositions or non-measurability. Vitali sets [5] and the Banach–Tarski paradox [6] are canonical instances.*

Sketch. AC∞ selects representatives without regard to measure or definability, producing non-measurable sets in $\mathbb{R}$ (Vitali). In $\mathbb{R}^3$ the action of a free subgroup of rotations is non-amenable; AC∞ allows selecting orbits to assemble a paradoxical decomposition, violating Meas (Banach–Tarski). □

4.3. The Reductio Principle

Theorem 3 (Reductio principle for paradoxes)**.** *For each normal form in Propositions 2–4, there is a finite set B of baseline coherence constraints (taken as fixed) and a single infinitary premise $I \in \{AT, IR, AC\infty\}$ such that $B + I$ yields contradiction or impossibility. Hence,* by reductio*, one may coherently reject I rather than weakening B.*

Sketch. In the antinomy case, take $B = \{\text{Cns}\}$ and $I = \text{AT}$; contradiction follows by Proposition 2. In the diagonal case, take $B = \{\text{Cns}, \text{Cmp}\}$ and $I = \text{IR}$; contradiction follows by Proposition 3. In the choice/geometry case, take $B = \{\text{Meas}\}$ and $I = \text{AC}\infty$ together with the ambient non-amenable symmetry; contradiction follows by Proposition 4. □

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Application map and finite replacements. The reductio reading recommends replacing each infinitary premise by a finite or periodic construct that preserves useful practice while blocking the paradox.

- Replace AT by *bounded comprehension*: only subsets definable over fixed finite or periodic domains are admitted; global totalities are not.
- Replace IR by *local registries*: only finite or periodic listings exist within a frame, preventing diagonal "escape" outside the list (cf. §3).
- Replace AC∞ by a *periodic choice principle* (see §5): definable choice on each finite orbit with periodic or equivariant extension.

Decision table. For ease of reference we summarise the options.

Class	Infinitary premise	Baseline kept	Reductio option
Antinomies	AT	Cns	Reject AT; use bounded comprehension
Diagonal	IR	Cns, Cmp	Reject IR; keep Cns (and optionally Cmp)
Choice/geometry	AC∞	Meas	Reject AC∞; use periodic choice

Remarks. The reductio stance does not deny the correctness of the classical derivations (e.g., [3,4]); it reassigns the blamed premise from the coherence constraints to the specific infinitary assumption that drives the construction. In §5 we show that much of the practical functionality attributed to AC can be recovered in finite or periodic settings without reintroducing the paradoxes, while §3 already exhibited the same pattern for diagonal arguments.

5. The Axiom of Choice Revisited

Classically, the Axiom of Choice (AC) asserts that for every family of nonempty sets $\{A_i\}_{i\in I}$ there exists a choice function $f : I \to \bigsqcup_i A_i$ with $f(i) \in A_i$ for all i [16]. AC enables powerful constructions but also underwrites measure/geometric paradoxes in the presence of actual infinity (e.g., Vitali, Banach–Tarski; cf. §2.3). In this section we show that the *functionality* typically provided by AC is retained in finite/periodic settings without invoking actual infinity, and that the classical AC-dependent paradoxes disappear as a consequence.

5.1. Global Choice On A Finite Universe

Fix a finite, nonempty "universe" U together with a *canonical* total order $<$ on U (e.g., the numerical order when $U = \mathbb{F}_p$, or a lexicographic order on U^k).

Definition 1 (Definable global choice on U)**.** *Define* $\mathrm{ch} : \mathcal{P}(U) \setminus \{\varnothing\} \to U$ *by*

$$\mathrm{ch}(A) \;=\; \min\nolimits_< A.$$

For a family $\{A_i\}_{i\in I}$ *with each* $A_i \subseteq U$ *nonempty, put* $f(i) = \mathrm{ch}(A_i)$.

Proposition 5 (Finite global choice is derivable)**.** *For any index set I (finite or infinite), Definition 1 yields a choice function $f : I \to U$ with $f(i) \in A_i$ for all $i \in I$. Hence, over a fixed finite base U, AC is* not *an independent axiom: it is a definable theorem.*

Proof. Every nonempty $A \subseteq U$ has a $<$-least element because U is finite and $<$ is total. Thus ch is well-defined and $f(i) = \mathrm{ch}(A_i) \in A_i$ for all i. No comprehension over infinite totalities is required. □

Remark 1 (Products and compactness in the finite case). *Nonemptiness of finite products $\prod_{j=1}^{n} A_j$ is provable by simple induction (no AC). In topological form, finite Tychonoff products of compact spaces are compact* without *AC. Only genuinely infinite products require Choice.*[1]

5.2. Periodic Families: Choice On One Period, Extend By Repetition

Many families in discrete/finite frameworks arise by exact repetition (periodicity) or by the action of a finite symmetry group. We isolate two useful templates.

Definition 2 (Equality-periodic family). *A family $\{A_i\}_{i\in\mathbb{Z}}$ of nonempty subsets of a finite universe U is* periodic of period $N \geq 1$ *if $A_{i+N} = A_i$ for all $i \in \mathbb{Z}$. A choice function $f : \mathbb{Z} \to U$ is* periodic of period N *if $f(i+N) = f(i)$ for all i.*

Lemma 2 (Periodic Choice (equality form)). *If $\{A_i\}_{i\in\mathbb{Z}}$ is periodic of period N, then there exists a choice function f of the same period N.*

Proof. Define $f(i) = \mathrm{ch}(A_i)$ using the fixed order from §5.1. Since $A_{i+N} = A_i$, we have $f(i+N) = \mathrm{ch}(A_{i+N}) = \mathrm{ch}(A_i) = f(i)$. □

Definition 3 (Group-periodic family (functor viewpoint)). *Let G be a finite group. A G-*family of sets *over a finite G-set I is a G-equivariant surjection $\pi : A \to I$ where $A = \bigsqcup_{i\in I} A_i$ and the action is compatible with fibers: $g \cdot A_i = A_{g\cdot i}$. A G-*equivariant choice function *(equivariant section) is a map $s : I \to A$ with $\pi \circ s = \mathrm{id}_I$ and $s(g \cdot i) = g \cdot s(i)$ for all $g \in G$, $i \in I$.*

Theorem 4 (Equivariant Periodic Choice (sufficient and necessary condition)). *Let $\pi : A \to I$ be a finite G-family as in Definition 3, with I having finitely many G-orbits. For a representative i of each orbit, write $G_i = \{g \in G : g \cdot i = i\}$ and view A_i as a G_i-set. Then a G-equivariant section s exists* iff *each stabilizer action $G_i \curvearrowright A_i$ has a fixed point.*

Proof sketch. (⇒) If s is equivariant then $s(i) \in A_i$ satisfies $g \cdot s(i) = s(g \cdot i) = s(i)$ for all $g \in G_i$, hence is a G_i-fixed point. (⇐) Choose, for each orbit representative i, a G_i-fixed point $a_i \in A_i$. Define $s(g \cdot i) = g \cdot a_i$ and check that s is well-defined and equivariant (stabilizer fixed points ensure independence of the chosen representative). □

Corollary 2 (Practical periodic choice). *In the common* equality-periodic *situation ($G = \mathbb{Z}/N$ acting on $I = \mathbb{Z}$ by shifts, with trivial action on fibers so $A_{i+N} = A_i$), Lemma 2 produces a periodic choice function of period N by restricting to one period and repeating.*

5.3. Consequences: Why AC-Based Paradoxes Disappear

Vitali non-measurable sets. On a finite universe U with counting measure, *every* subset is measurable. Families of the form "one representative per $\mathbb{Q}$-coset" do not arise: there is no uncountable domain and no use for unrestricted AC. In periodic variants (e.g., quotients by finite subgroups), selection is definable and preserves measurability.

Banach–Tarski-type decompositions. Paradoxical decompositions exploit non-amenable actions on *infinite* sets with AC [6]. For any action of a finite group on a finite set X, cardinality is invariant and finitely additive counting measure is complete. One cannot partition X into pieces that, after group motions, reproduce two disjoint copies of X—cardinality forbids it. In periodic settings, one works on a fundamental period (finite), so the same obstruction applies.

[1] In the finite/periodic setting below, the product task reduces to a finite index set (one period) together with a periodicity constraint.

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Ultrafilters. On a finite set, all ultrafilters are principal; there are no nonprincipal ultrafilters to fuel AC-driven pathologies. Periodic replication does not change this fact at the level of one period.

5.4. What We Keep, What We Drop

- **Kept (derivable).** Choice on any family of nonempty subsets of a fixed finite universe (Def. 1–Prop. 5); periodic choice on equality-periodic families (Lemma 2); equivariant choice when stabilizers have fixed points (Thm. 4).
- **Dropped (not needed).** Full AC over arbitrary infinite families; constructions that require nonprincipal ultrafilters or nonmeasurable sets; paradoxical decompositions relying on infinite, non-amenable group actions.
- **Independent phenomena.** Diagonal arguments (Cantor, Gödel, Turing) do not depend on AC; their *reductio* treatment is handled separately (§3, §4).

Summary. In finite and periodic frameworks, the *useful practice* enabled by AC is recovered by definable, frame-internal mechanisms: global minima on a finite universe and repetition over one period. The infamous AC-dependent paradoxes vanish because their infinitary preconditions are absent. This preserves mathematical utility while avoiding the pathological consequences associated with unrestricted Choice over actual infinities (cf. §2.3).

6. Philosophical Implications

This section draws out the conceptual consequences of the preceding analysis. The governing claim is that the classical paradoxes surveyed in §2 and the incompleteness pattern in §3 admit a uniform *reductio* reading (§4): when contradictions (or impossibility results) arise from the bundle of baseline coherence assumptions together with a *specific* infinitary premise (absolute totalities, infinite registries, or unrestricted Choice), it is methodologically coherent to reject the infinitary premise rather than the coherence clauses. In this way, the practice-driven virtues of classical mathematics are retained, while its paradoxes are neutralised by finite or periodic constructs (§5).

Ontology: actual infinity as idealisation. On the reductio reading, *actual infinity* is not a mind-independent constituent of mathematical ontology but a powerful and convenient *idealisation* whose systematic use may outstrip coherence. This stance is compatible with mathematical realism about *structures*—one may take the objective content of mathematics to reside in structural relations rather than in the existence of completed infinite totalities [17,18]. The reductio principle thereby furnishes an *ontological moderation*: use infinite talk where it tracks stable practice, but treat paradoxes as diagnostics that the idealisation has exceeded its remit.

Conservativeness for ordinary practice. A core desideratum is conservativeness relative to ordinary practice. Three points are central:

1. **Arithmetic and algebra.** Finite/periodic frameworks recover ordinary algebraic laws and arithmetic manipulations without invoking an actual infinite domain; theorems stated and proved for finitely presented objects remain intact. A concrete realisation of this strategy is a finite/periodic reconstruction of familiar number systems and continuum-like behaviour over a finite base, yielding frame-internal completeness while avoiding infinitary paradoxes, as detailed in [8].
2. **Analysis as approximation theory.** Continuum methods are treated instrumentally as approximation schemes over finite grids with controllable error, rather than ontological commitments to uncountable sets. This preserves the calculational efficacy of analysis in science and engineering.
3. **Choice in practice.** The *functionality* often supplied by AC is derivable on a finite universe and extendable along periodic families (§5); AC-dependent paradoxes rely on genuinely uncountable selection and do not arise.

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6.1. Positioning Among Foundational Programs

The reductio stance is neither a mere reprise of finitism nor an endorsement of full-blown constructivism; it triangulates as follows:

Finitism and predicativism. Hilbert's finitism and predicative programs seek proofs with restricted means and avoid impredicative definitions [19,20]. The present view is sympathetic in spirit, but focuses on *which assumptions to reject* when paradoxes surface (AT/IR/AC∞), rather than prescribing a particular proof calculus.

Intuitionism and constructivism. Brouwer's intuitionism rejects classical logic in favour of constructive reasoning [21]. The reductio stance is ecumenical about logic (classical reasoning remains available) and targets, instead, the *ontological* use of actual infinity in paradox-producing contexts.

Ultrafinitism and feasibility. Ultrafinitists (e.g., Yessenin-Volpin, Parikh) reject very large numbers on feasibility grounds [22,23]. By contrast, the present view preserves arbitrarily large finite structures and emphasises periodic/finite constructs for infinitary premises; it is not a theory of human feasibility but of *foundational parsimony*.

Structuralism and naturalism. The stance aligns with structuralism's emphasis on relations over objects [17] and with a naturalistic sensitivity to mathematical practice [24]: we keep what practice demonstrably needs and excise the specific infinitary premises that generate paradox.

6.2. Methodological Moral: A Decision Rule

The discussion yields a simple normative rule for foundations:

> **Reductio Rule.** When a paradox (or impossibility) is derivable from a set of baseline coherence assumptions B plus an infinitary premise $I \in \{\mathrm{AT}, \mathrm{IR}, \mathrm{AC}\infty\}$, prefer to *reject* I and keep B, unless there is decisive independent evidence that I is indispensable to well-confirmed mathematical practice.

Reduction Rule harmonises with scientific methodology: infinities that appear in physics (e.g., singularities, ultraviolet divergences) are typically read as signals of *theory breakdown* and addressed by reframing (renormalisation, effective theories) rather than by declaring nature inconsistent [25,26]. The mathematical analogue is to replace infinitary premises by finite or periodic constructs that preserve calculational virtues.

6.3. Objections and Replies

(O1) "But ZFC appears consistent; why not keep it and live with paradox boundaries?" The reductio stance is compatible with relative consistency results. Its point is not to *disprove* ZFC, but to explain why paradoxes track specific infinitary premises and to offer principled finite/periodic replacements that avoid them while conserving practice.

(O2) "Doesn't this jettison large swaths of higher set theory?" Yes, it sidelines enterprises that rely essentially on strong infinitary commitments (e.g., large cardinals). The claim is pragmatic: such commitments are not required for the bulk of mathematics in use.

(O3) "What about Skolem phenomena and model-theoretic pluralism?" The reductio stance accepts pluralism about infinite models as a symptom of the same idealisation: internal/external cardinal talk diverges when infinite language and models are admitted. Finite/periodic constructs sidestep the phenomenon rather than attempting to domesticate it.

(O4) "Isn't diagonalisation independent of AC?" Yes; diagonal arguments inhabit the IR-premise column, not the AC-column. The reductio strategy treats *each* paradox in its normal form, rejecting the *relevant* infinitary premise (IR for diagonal results; AT for antinomies; AC∞ for measure/geometric paradoxes).

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(O5) "Do we lose completeness theorems?" Global completeness for arithmetic in the presence of IR is exactly what triggers Gödel's result. Within finite/periodic frames, one may regain *frame-relative* completeness without the diagonal escape (§3); nothing here asserts absolute completeness across frames.

Epistemology and informational limits. If mathematical representation is constrained by physical information bounds, an ontology that eschews actual infinity is not merely conservative but *realistic*. On such a view, continuum talk summarises stable regularities of finite computations/measurements; its success reflects robustness, not metaphysical commitment [27]. The reductio stance is thus both methodological and epistemic: it treats paradoxes as guidance about the limits of idealisation relative to bounded observers.

Summary. The philosophical upshot is threefold: (i) actual infinity functions best as a regulated *idealisation*, not as ontological bedrock; (ii) foundational paradoxes are best read as *reductio* arguments targeting specific infinitary premises (AT/IR/AC∞); and (iii) finite/periodic constructs preserve the mathematics that practice demonstrably needs—computational tractability, structural reasoning, and choice in the small— while ruling out the paradoxes that have historically troubled the subject (§5).

7. Conclusions

Thesis restated. The central claim of this paper is that the celebrated "paradoxes of infinity" are *not* mere limitations to be endured within mathematics; rather, in each case they can be read as *reductio ad absurdum* arguments against a specific infinitary postulate. Antinomy-style results (§2) indict *absolute totalities* (AT); diagonal arguments (§3, §4) indict the *infinite registry* (IR); and measure/geometric paradoxes indict *unrestricted Choice* over uncountable domains (AC∞). In short: *the paradoxes function as reductio proofs of the impossibility of actual infinity in the relevant guise*. The result is a coherent alternative foundation in which mathematics is *finite*, *relational*, and *paradox-free*. The *finite/periodic framework* is ontologically basic; the *infinitary framework* is a convenient *idealisation* used for exposition and approximation.

From paradox to principle. Abstracting the proofs into normal forms (§4) reveals inconsistent triads of the shape

$$\text{(baseline coherence)} + \text{(infinitary postulate)} \implies \bot.$$

The methodological resolution we advocate is conservative: retain the baseline coherence assumptions (consistency, completeness where appropriate, basic measure) and reject the offending infinitary postulate. This reassigns the explanatory burden from "mathematics is intrinsically paradoxical" to "certain infinitary idealisations outstrip coherence."

Recovery of key principles in finite/periodic guise. The finite/periodic perspective retains the working *functionality* of classical tools while blocking their paradoxical uses:

- **Choice without AC∞.** On a fixed finite universe, global choice is definable by a canonical order (Prop. 5); for equality-periodic families one chooses on a single period and repeats (Lemma 2); for group-periodic families, equivariant choice exists exactly when stabilisers have fixed points (Thm. 4). Thus the constructive roles of Choice are preserved (§5), while AC-dependent paradoxes (Vitali, Banach–Tarski) cannot arise in finite/amenable settings.
- **No diagonal "escape."** Without a global, actually infinite registry, the pivotal diagonal step cannot produce an object "outside the list"; the inconsistent triad of Prop. 1 is resolved by rejecting IR rather than completeness.
- **Continuum practices as finite approximation.** Analysis proceeds as controlled approximation on finite/periodic grids, retaining calculational efficacy without ontological commitment to uncountable totalities.

Philosophical payoff. The result is a coherent alternative foundation in which mathematics is *finite*, *relational*, and *paradox-free*. Infinity is treated as a regulated idealisation rather than ontological bedrock;

core mathematical practice is conserved by finite/periodic constructs; and the classical paradoxes are explained rather than merely cordoned off. This reconciles methodological conservatism with conceptual clarity: keep what practice demonstrably needs, and replace the specific infinitary premises that drive contradiction. In doing so, the framework offers a stable platform for further technical development without reintroducing the very paradoxes that motivated foundational reflection in the first place.

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