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[K. MAHESH KRISHNA](#)*

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Article

Unexpected Uncertainty Principle for Disc Banach Spaces

K. Mahesh Krishna

Statistics and Mathematics Unit, Indian Statistical Institute, Bangalore Centre, Karnataka 560 059, India; kmaheshak@gmail.com

Abstract: Let $(\{f_n\}_{n=1}^\infty, \{\tau_n\}_{n=1}^\infty)$ and $(\{g_n\}_{n=1}^\infty, \{\omega_n\}_{n=1}^\infty)$ be unbounded continuous p -Schauder frames ($0 < p < 1$) for a disc Banach space $\mathcal{X}$. Then for every $x \in (\mathcal{D}(\theta_f) \cap \mathcal{D}(\theta_g)) \setminus \{0\}$, we show that $\|\theta_f x\|_0 \|\theta_g x\|_0 \geq \frac{1}{\left(\sup_{n,m \in \mathbb{N}} |f_n(\omega_m)|\right)^{\frac{1}{p}} \left(\sup_{n,m \in \mathbb{N}} |g_m(\tau_n)|\right)^{\frac{1}{p}}} (1)$, where $\theta_f : \mathcal{D}(\theta_f) \ni x \mapsto \theta_f x := \{f_n(x)\}_{n=1}^\infty \in \ell^p(\mathbb{N})$, $\theta_g : \mathcal{D}(\theta_g) \ni x \mapsto \theta_g x := \{g_n(x)\}_{n=1}^\infty \in \ell^p(\mathbb{N})$. Inequality (1) is unexpectedly different from both bounded uncertainty principle [arXiv:2308.00312v1] and unbounded uncertainty principle [arXiv:2312.00366v1] for Banach spaces.

Keywords: uncertainty principle; frame; Banach space

MSC: 42C15

1. Introduction

Given a finite collection $\{\tau_j\}_{j=1}^n$ in a finite dimensional Hilbert space $\mathcal{H}$ over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), define

$$\theta_\tau : \mathcal{H} \ni h \mapsto \theta_\tau h := (\langle h, \tau_j \rangle)_{j=1}^n \in \mathbb{K}^n.$$

Recall that a collection $\{\tau_j\}_{j=1}^n$ in $\mathcal{H}$ is said to be a Parseval frame [1] for $\mathcal{H}$ if

$$\|h\|^2 = \sum_{j=1}^n |\langle h, \tau_j \rangle|^2, \quad \forall h \in \mathcal{H}.$$

Most general form of discrete uncertainty principle for finite dimensional Hilbert spaces is the following.

Theorem 1.1 (Donoho-Stark-Elad-Bruckstein-Ricaud-Torrésani Uncertainty Principle [2–4]). Let $\{\tau_j\}_{j=1}^n, \{\omega_j\}_{j=1}^n$ be two Parseval frames for a finite dimensional Hilbert space $\mathcal{H}$. Then

$$\frac{\|\theta_\tau h\|_0^2 + \|\theta_\omega h\|_0^2}{2} \geq \left(\frac{\|\theta_\tau h\|_0 + \|\theta_\omega h\|_0}{2} \right)^2 \geq \|\theta_\tau h\|_0 \|\theta_\omega h\|_0 \geq \frac{1}{\max_{1 \leq j, k \leq n} |\langle \tau_j, \omega_k \rangle|^2}, \quad \forall h \in \mathcal{H} \setminus \{0\}.$$

Recently, Theorem 1.1 has been derived for Banach spaces using continuous p -Schauder frames.

Definition 1.2 ([5]). Let (Ω, μ) be a measure space. Let $\{\tau_\alpha\}_{\alpha \in \Omega}$ be a collection in a Banach space $\mathcal{X}$ and $\{f_\alpha\}_{\alpha \in \Omega}$ be a collection in $\mathcal{X}^*$. The pair $(\{f_\alpha\}_{\alpha \in \Omega}, \{\tau_\alpha\}_{\alpha \in \Omega})$ is said to be a **continuous p -Schauder frame** for $\mathcal{X}$ ($1 \leq p < \infty$) if the following holds.

- (i) For every $x \in \mathcal{X}$, the map $\Omega \ni \alpha \mapsto f_\alpha(x) \in \mathbb{K}$ is measurable.
- (ii) For every $x \in \mathcal{X}$,

$$\|x\|^p = \int_{\Omega} |f_\alpha(x)|^p d\mu(\alpha) \quad \text{if } 1 \leq p < \infty,$$

$$\|x\| = \text{ess sup}_{\alpha \in \Omega} |f_\alpha(x)| \quad \text{if } p = \infty.$$

- (iii) For every $x \in \mathcal{X}$, the map $\Omega \ni \alpha \mapsto f_\alpha(x)\tau_\alpha \in \mathcal{X}$ is weakly measurable.
 (iv) For every $x \in \mathcal{X}$,

$$x = \int_{\Omega} f_\alpha(x)\tau_\alpha d\mu(\alpha),$$

where the integral is weak integral.

Given a continuous p -Schauder frame $(\{f_\alpha\}_{\alpha \in \Omega}, \{\tau_\alpha\}_{\alpha \in \Omega})$ for $\mathcal{X}$, define

$$\theta_f : \mathcal{X} \ni x \mapsto \theta_f x \in \mathcal{L}^p(\Omega, \mu); \quad \theta_f x : \Omega \ni \alpha \mapsto (\theta_f x)(\alpha) := f_\alpha(x) \in \mathbb{K}$$

Theorem 1.3 (Functional Continuous Donoho-Stark-Elad-Bruckstein-Ricaud-Torrésani Uncertainty Principle [5–7]). Let $(\Omega, \mu), (\Delta, \nu)$ be measure spaces. Let $(\{f_\alpha\}_{\alpha \in \Omega}, \{\tau_\alpha\}_{\alpha \in \Omega})$ and $(\{g_\beta\}_{\beta \in \Delta}, \{\omega_\beta\}_{\beta \in \Delta})$ be continuous p -Schauder frames for a Banach space $\mathcal{X}$. Then

- (i) for $p > 1$, we have

$$\begin{aligned} \mu(\text{supp}(\theta_f x))^{\frac{1}{p}} \nu(\text{supp}(\theta_g x))^{\frac{1}{q}} &\geq \frac{1}{\sup_{\alpha \in \Omega, \beta \in \Delta} |f_\alpha(\omega_\beta)|}, \quad \forall x \in \mathcal{X} \setminus \{0\}; \\ \nu(\text{supp}(\theta_g x))^{\frac{1}{p}} \mu(\text{supp}(\theta_f x))^{\frac{1}{q}} &\geq \frac{1}{\sup_{\alpha \in \Omega, \beta \in \Delta} |g_\beta(\tau_\alpha)|}, \quad \forall x \in \mathcal{X} \setminus \{0\}. \end{aligned}$$

where q is the conjugate index of p .

- (ii) for $p = 1$, we have

$$\mu(\text{supp}(\theta_f x)) \geq \frac{1}{\sup_{\alpha \in \Omega, \beta \in \Delta} |f_\alpha(\omega_\beta)|}, \quad \nu(\text{supp}(\theta_g x)) \geq \frac{1}{\sup_{\alpha \in \Omega, \beta \in \Delta} |g_\beta(\tau_\alpha)|}, \quad \forall x \in \mathcal{X} \setminus \{0\}.$$

- (iii) for $p = \infty$, we have

$$\nu(\text{supp}(\theta_g x)) \geq \frac{1}{\sup_{\alpha \in \Omega, \beta \in \Delta} |f_\alpha(\omega_\beta)|}, \quad \mu(\text{supp}(\theta_f x)) \geq \frac{1}{\sup_{\alpha \in \Omega, \beta \in \Delta} |g_\beta(\tau_\alpha)|}, \quad \forall x \in \mathcal{X} \setminus \{0\}.$$

An unbounded version of 1.3 has been recently derived for unbounded frames.

Definition 1.4 ([8]). Let (Ω, μ) be a measure space and $1 \leq p \leq \infty$. Let $\{\tau_\alpha\}_{\alpha \in \Omega}$ be a collection in a Banach space $\mathcal{X}$ and $\{f_\alpha\}_{\alpha \in \Omega}$ be a collection of linear functionals on $\mathcal{X}$ (which may not be bounded). The pair $(\{f_\alpha\}_{\alpha \in \Omega}, \{\tau_\alpha\}_{\alpha \in \Omega})$ is called an **unbounded continuous p -Schauder frame** or **continuous semi p -Schauder frame** for $\mathcal{X}$ if the following conditions holds.

- (i) For every $x \in \mathcal{X}$, the map $\Omega \ni \alpha \mapsto f_\alpha(x) \in \mathbb{K}$ is measurable.
 (ii) The map

$$\theta_f : \mathcal{D}(\theta_f) \ni x \mapsto \theta_f x \in \mathcal{L}^p(\Omega, \mu); \quad \theta_f x : \Omega \ni \alpha \mapsto (\theta_f x)(\alpha) := f_\alpha(x) \in \mathbb{K}$$

is well-defined (need not be bounded).

- (iii) For every $x \in \mathcal{X}$, the map $\Omega \ni \alpha \mapsto f_\alpha(x)\tau_\alpha \in \mathcal{X}$ is weakly measurable.
 (iv) For every $x \in \mathcal{D}(\theta_f)$,

$$x = \int_{\Omega} f_\alpha(x)\tau_\alpha d\mu(\alpha),$$

where the integral is weak integral.

Theorem 1.5 (Unbounded Donoho-Stark-Elad-Bruckstein-Ricaud-Torrésani Uncertainty Principles [8]). Let (Ω, μ) , (Δ, ν) be measure spaces and $p = 1$ or $p = \infty$. Let $(\{f_\alpha\}_{\alpha \in \Omega}, \{\tau_\alpha\}_{\alpha \in \Omega})$ and $(\{g_\beta\}_{\beta \in \Delta}, \{\omega_\beta\}_{\beta \in \Delta})$ be unbounded continuous p -Schauder frames for a Banach space $\mathcal{X}$. Then for every $x \in (\mathcal{D}(\theta_f) \cap \mathcal{D}(\theta_g)) \setminus \{0\}$, we have

$$\mu(\text{supp}(\theta_f x)) \nu(\text{supp}(\theta_g x)) \geq \frac{1}{\left(\sup_{\alpha \in \Omega, \beta \in \Delta} |f_\alpha(\omega_\beta)| \right) \left(\sup_{\alpha \in \Omega, \beta \in \Delta} |g_\beta(\tau_\alpha)| \right)}.$$

Above results cover the important Lebesgue spaces for $1 \leq p \leq \infty$. The next natural class of spaces is the Lebesgue spaces for $0 < p < 1$. Recall that for $0 < p < 1$, we define

$$\ell^p(\mathbb{N}) := \left\{ \{a_n\}_{n=1}^\infty : a_n \in \mathbb{K}, \forall n \in \mathbb{N}, \sum_{n=1}^\infty |a_n|^p < \infty \right\}$$

equipped with the inhomogeneous norm

$$\|\{a_n\}_{n=1}^\infty\|_p := \sum_{n=1}^\infty |a_n|^p, \quad \forall \{a_n\}_{n=1}^\infty \in \ell^p(\mathbb{N}).$$

In this paper, we derive a surprising result which is counter intuitive to the feeling we gain from Theorem 1.3. This is why we called the uncertainty principle we obtained as unexpected uncertainty principle.

2. Unexpected Uncertainty Principle

We start by recalling the following definition.

Definition 2.1 ([9]). Let $\mathcal{X}$ be a vector space over $\mathbb{K}$. We say that $\mathcal{X}$ is a **disc Banach space** if there exists a map called as **disc norm** $\|\cdot\| : \mathcal{X} \rightarrow [0, \infty)$ satisfying the following conditions.

- (i) If $x \in \mathcal{X}$ is such that $\|x\| = 0$, then $x = 0$.
- (ii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in \mathcal{X}$.
- (iii) $\|\lambda x\| \leq |\lambda| \|x\|$ for all $x \in \mathcal{X}$ and for all $\lambda \in \mathbb{K}$ with $|\lambda| \geq 1$.
- (iv) $\|\lambda x\| \geq |\lambda| \|x\|$ for all $x \in \mathcal{X}$ and for all $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$.
- (v) $\mathcal{X}$ is complete w.r.t. the metric $d(x, y) := \|x - y\|$ for all $x, y \in \mathcal{X}$.

Banach space frame theory which is modeled on classical Lebesgue sequence spaces [10] and the theory of unbounded frames for Hilbert and Banach spaces [11–17] naturally gives the following definition.

Definition 2.2. Let $\mathcal{X}$ be a disc Banach space. Let $\{\tau_n\}_{n=1}^\infty$ be a collection in $\mathcal{X}$ and $\{f_n\}_{n=1}^\infty$ be a collection of linear functionals on $\mathcal{X}$ (which may not be bounded). The pair $(\{f_n\}_{n=1}^\infty, \{\tau_n\}_{n=1}^\infty)$ is said to be a **unbounded p -Schauder frame** ($0 < p < 1$) or **semi p -Schauder frame** for $\mathcal{X}$ if the following conditions holds.

- (i) The map

$$\theta_f : \mathcal{D}(\theta_f) \ni x \mapsto \theta_f x := \{f_n(x)\}_{n=1}^\infty \in \ell^p(\mathbb{N})$$

is well-defined (need not be bounded).

(ii) For every $x \in \mathcal{D}(\theta_f)$,

$$x = \sum_{n=1}^{\infty} f_n(x) \tau_n$$

We are going to use the following important result.

Theorem 2.3 ([18,19]). For every $0 < p < 1$,

$$\left(\sum_{n=1}^{\infty} |a_n| \right)^p \leq \sum_{n=1}^{\infty} |a_n|^p, \quad \forall \{a_n\}_{n=1}^{\infty} \in \ell^p(\mathbb{N}).$$

Following is the main result of the paper.

Theorem 2.4. Let $(\{f_n\}_{n=1}^{\infty}, \{\tau_n\}_{n=1}^{\infty})$ and $(\{g_n\}_{n=1}^{\infty}, \{\omega_n\}_{n=1}^{\infty})$ be unbounded p -Schauder frames for a disc Banach space $\mathcal{X}$. Then for every $x \in (\mathcal{D}(\theta_f) \cap \mathcal{D}(\theta_g)) \setminus \{0\}$, we have

$$\|\theta_f x\|_0 \|\theta_g x\|_0 \geq \frac{1}{\left(\sup_{n,m \in \mathbb{N}} |f_n(\omega_m)| \right)^p \left(\sup_{n,m \in \mathbb{N}} |g_m(\tau_n)| \right)^p}.$$

Proof. Let $x \in \mathcal{D}(\theta_f) \setminus \{0\}$. Then using Theorem 2.3,

$$\begin{aligned} \|\theta_f x\| &= \sum_{n=1}^{\infty} |f_n(x)|^p = \sum_{n \in \text{supp}(\theta_f x)} |f_n(x)|^p = \sum_{n \in \text{supp}(\theta_f x)} \left| f_n \left(\sum_{m=1}^{\infty} g_m(x) \omega_m \right) \right|^p \\ &= \sum_{n \in \text{supp}(\theta_f x)} \left| \sum_{m \in \text{supp}(\theta_g x)} g_m(x) f_n(\omega_m) \right|^p \leq \sum_{n \in \text{supp}(\theta_f x)} \left(\sum_{m \in \text{supp}(\theta_g x)} |g_m(x) f_n(\omega_m)| \right)^p \\ &\leq \left(\sup_{n,m \in \mathbb{N}} |f_n(\omega_m)| \right)^p \sum_{n \in \text{supp}(\theta_f x)} \left(\sum_{m \in \text{supp}(\theta_g x)} |g_m(x)| \right)^p \\ &= \left(\sup_{n,m \in \mathbb{N}} |f_n(\omega_m)| \right)^p \|\theta_f x\|_0 \left(\sum_{m=1}^{\infty} |g_m(x)| \right)^p \leq \left(\sup_{n,m \in \mathbb{N}} |f_n(\omega_m)| \right)^p \|\theta_f x\|_0 \left(\sum_{m=1}^{\infty} |g_m(x)|^p \right) \\ &= \left(\sup_{n,m \in \mathbb{N}} |f_n(\omega_m)| \right)^p \|\theta_f x\|_0 \|\theta_g x\|. \end{aligned}$$

Therefore

$$\frac{1}{\left(\sup_{n,m \in \mathbb{N}} |f_n(\omega_m)| \right)^p} \|\theta_f x\| \leq \|\theta_f x\|_0 \|\theta_g x\|. \quad (2)$$

On the other hand, let $x \in \mathcal{D}(\theta_g) \setminus \{0\}$. Then again using Theorem 2.3,

$$\begin{aligned}
 \|\theta_g x\| &= \sum_{m=1}^{\infty} |g_m(x)|^p = \sum_{m \in \text{supp}(\theta_g x)} |g_m(x)|^p = \sum_{m \in \text{supp}(\theta_f x)} \left| g_m \left(\sum_{n=1}^{\infty} f_n(x) \tau_n \right) \right|^p \\
 &= \sum_{m \in \text{supp}(\theta_g x)} \left| \sum_{n \in \text{supp}(\theta_f x)} f_n(x) g_m(\tau_n) \right|^p \leq \sum_{m \in \text{supp}(\theta_g x)} \left(\sum_{n \in \text{supp}(\theta_f x)} |f_n(x)(x) g_m(\tau_n)| \right)^p \\
 &\leq \left(\sup_{n, m \in \mathbb{N}} |g_m(\tau_n)| \right)^p \sum_{m \in \text{supp}(\theta_g x)} \left(\sum_{n \in \text{supp}(\theta_f x)} |f_n(x)| \right)^p \\
 &= \left(\sup_{n, m \in \mathbb{N}} |g_m(\tau_n)| \right)^p \|\theta_g x\|_0 \left(\sum_{n=1}^{\infty} |f_n(x)| \right)^p \leq \left(\sup_{n, m \in \mathbb{N}} |g_m(\tau_n)| \right)^p \|\theta_g x\|_0 \left(\sum_{n=1}^{\infty} |f_n(x)|^p \right) \\
 &= \left(\sup_{n, m \in \mathbb{N}} |g_m(\tau_n)| \right)^p \|\theta_g x\|_0 \|\theta_f x\|.
 \end{aligned}$$

Therefore

$$\frac{1}{\left(\sup_{n, m \in \mathbb{N}} |g_m(\tau_n)| \right)^p} \|\theta_g x\| \leq \|\theta_g x\|_0 \|\theta_f x\|. \quad (3)$$

Multiplying Inequalities (2) and (3) we get

$$\frac{1}{\left(\sup_{n, m \in \mathbb{N}} |f_n(\omega_m)| \right)^p \left(\sup_{n, m \in \mathbb{N}} |g_m(\tau_n)| \right)^p} \|\theta_f x\| \|\theta_g x\| \leq \|\theta_f x\|_0 \|\theta_g x\|_0 \|\theta_f x\| \|\theta_g x\|,$$

$$\forall x \in (\mathcal{D}(\theta_f) \cap \mathcal{D}(\theta_g)) \setminus \{0\}.$$

A cancellation of $\|\theta_f x\| \|\theta_g x\|$ gives the inequality. $\square$

As continuous version of Theorem 2.3 fails (even for finite measure spaces) it seems that continuous version of Theorem 2.4 fails.

In view of Tao's uncertainty principle [20] we believe that Theorem 2.3 can be improved in prime dimensions.

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