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Article

From Bateman-Horn to Chowla

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Abstract

In this paper we first give a new formula of the Liouville function and then by using the method for proving the Bateman-Horn conjecture, we give a parallel proof of the Chowla conjecture.

Keywords: Chowla conjecture; the Liouville function

1. Introduction

1.1. The Chowla Conjecture

Let n be a positive integer and let p denote a prime number. By (a, b) we mean the greatest common divisor of positive integers a and b . Let $\omega(n)$ denote the number of distinct prime factors of n and $\Omega(n)$ the total number of prime factors of n counting multiplicity for each prime factor. Let

$$\lambda(n) = (-1)^{\Omega(n)} \tag{1.1}$$

be the Liouville function. The function $\Omega(n)$ is completely additive and $\lambda(n)$ is completely multiplicative. Recall that an arithmetic function $f(n)$ is completely additive (resp. completely multiplicative) if $f(ab) = f(a) + f(b)$ (resp. $f(ab) = f(a)f(b)$) for any positive integers a and b . It is well known that the prime number theorem is equivalent to the estimate

$$\sum_{n \leq x} \lambda(n) = o(x) \tag{1.2}$$

as $x \rightarrow \infty$. In 1965 Chowla [2] proposed the following generalization of (1.2), which is known as the Chowla conjecture.

Conjecture 1. Let $k \geq 1$ and let $h_1 < h_2 < \dots < h_k$ be distinct nonnegative integers. Then as $x \rightarrow \infty$,

$$\sum_{n \leq x} \lambda(n + h_1) \lambda(n + h_2) \cdots \lambda(n + h_k) = o(x). \tag{1.3}$$

The case $k = 1$ is the prime number theorem (1.2), for $k \geq 2$ the Chowla conjecture is open. There are extensive studies in the literature, we refer to the recent works by T. Tao, J. Teräväinen, K. Matomäki and M. Radziwiłł [7,8,10–13] and the references therein. These works make important progress towards the Chowla conjecture. For example Tao [13] proved that

$$\sum_{n \leq x} \frac{\lambda(n) \lambda(n + 1)}{n} = o(\log x), \tag{1.4}$$

and in [8] Matomäki, Radziwiłł, and Tao proved that all eight possible sign patterns for $(\lambda(n), \lambda(n + 1), \lambda(n + 2))$ occur with positive lower natural density.

In this paper we first give a new formula of the Liouville function and then by using the method for proving the Bateman-Horn conjecture [15], we give a parallel proof of the Chowla conjecture.

1.2. Main Results

The first result is the following.

Theorem 2 (=Theorem 5). *Let $n \geq 1$ be an integer, then*

$$\lambda(n) = \sum_{d|n} \lambda(d) 2^{\omega(d)}.$$

Note that the Chowla conjecture 1 is equivalent to

$$\lim_{z \rightarrow 1^-} (1-z) \sum_{n=1}^\infty \lambda(n+h_1) \lambda(n+h_2) \cdots \lambda(n+h_k) z^n = 0. \tag{1.5}$$

Using the method for proving the Bateman-Horn conjecture [15] we prove the following result.

Theorem 3. *Let*

$$f(n) = (n+h_1)(n+h_2) \cdots (n+h_k)$$

where $h_1 < h_2 < \cdots < h_k$ are distinct nonnegative integers and let $N_f(d)$ be the number of solutions of the congruence $f(n) \equiv 0 \pmod{d}$. Then

$$\begin{aligned} & \lim_{z \rightarrow 1^-} (1-z) \sum_{n=1}^\infty \lambda(n+h_1) \lambda(n+h_2) \cdots \lambda(n+h_k) z^n \\ &= \lim_{z \rightarrow 1^-} (1-z) \sum_{d=1}^\infty \frac{\lambda(d) 2^{\omega(d)}}{1-z^d} \sum_{\substack{n=1 \\ d|f(n)}}^d z^n \\ &= \sum_{d=1}^\infty \lambda(d) 2^{\omega(d)} \frac{N_f(d)}{d}. \end{aligned}$$

To complete the proof of Conjecture 1 we shall prove

Theorem 4. *Let notations be as in Theorem 3.*

$$\sum_{d=1}^\infty \lambda(d) 2^{\omega(d)} \frac{N_f(d)}{d} = 0. \tag{1.6}$$

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2. Proof of Theorem 2

In this section we give a formula of the Liouville function, which is new to my knowledge. Let

$$\lambda(n) = \sum_{d|n} f(d), \tag{2.1}$$

then by the Möbius inversion the Dirichlet series of $\lambda(n)$ and $f(n)$ have following relation

$$\zeta(s) \sum_{n=1}^\infty \frac{f(n)}{n^s} = \sum_{n=1}^\infty \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)} \tag{2.2}$$

where $\zeta(s)$ is the Riemann zeta function [14]. The two series above converge for $\Re(s) > 1$. By (2.2) we have

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \frac{\zeta(2s)}{\zeta^2(s)}. \quad (2.3)$$

It follows from [3, D-21] that

$$\frac{\zeta(2s)}{\zeta^2(s)} = \sum_{n=1}^{\infty} \frac{\lambda(n)2^{\omega(n)}}{n^s}, \quad (2.4)$$

and thus

$$f(n) = \lambda(n)2^{\omega(n)}. \quad (2.5)$$

To summarise we have the following formula of Liouville's function.

Theorem 5. *Let $n \geq 1$ be an integer, then*

$$\lambda(n) = \sum_{d|n} \lambda(d)2^{\omega(d)}. \quad (2.6)$$

We apply this formula to the prime number theorem and the Chowla conjecture.

3. Review of the Prime Number Theorem

In the following we denote by $[x]$ the integral part and $\{x\}$ the fractional part of a positive number. The prime number theorem is the estimate

$$\sum_{n \leq x} \lambda(n) = a_0 x + E_0(x) \quad (3.1)$$

where $a_0 = 0$ and the error term $E_0(x) = o(x)$. We now use (2.6) to compute the sum:

$$\sum_{n \leq x} \lambda(n) = \sum_{n \leq x} \sum_{d|n} \lambda(d)2^{\omega(d)} = \sum_{d \leq x} \lambda(d)2^{\omega(d)} \sum_{\substack{n \leq x \\ d|n}} 1 \quad (3.2)$$

$$= \sum_{d \leq x} \lambda(d)2^{\omega(d)} \left[\frac{x}{d} \right] \quad (3.3)$$

$$= \sum_{d \leq x} \lambda(d)2^{\omega(d)} \left(\frac{x}{d} - \left\{ \frac{x}{d} \right\} \right) \quad (3.4)$$

$$= x \sum_{d \leq x} \frac{\lambda(d)2^{\omega(d)}}{d} - \sum_{d \leq x} \lambda(d)2^{\omega(d)} \left\{ \frac{x}{d} \right\}. \quad (3.5)$$

By (2.3) and (2.5) we have, for $\Re(s) > 1$,

$$\sum_{d=1}^{\infty} \frac{\lambda(d)2^{\omega(d)}}{d^s} = \frac{\zeta(2s)}{\zeta^2(s)}. \quad (3.6)$$

Another well known form of the prime number theorem is

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n} = 0. \quad (3.7)$$

In view of (2.2),

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^1} = \frac{\zeta(2)}{\zeta(1)} = 0, \quad (3.8)$$

and this is to say the series $\sum_{n=1}^{\infty} \lambda(n)n^{-s}$ extends to $s = 1$ and thus (2.3) also extends to $s = 1$. Together with (3.6) we have

$$\sum_{d=1}^{\infty} \frac{\lambda(d)2^{\omega(d)}}{d} = \frac{\zeta(2)}{\zeta^2(1)} = 0. \tag{3.9}$$

Compared with (3.1) we have

Proposition 6.

$$a_0 = \sum_{d=1}^{\infty} \frac{\lambda(d)2^{\omega(d)}}{d} = 0, \quad E_0(x) = \sum_{d \leq x} \lambda(d)2^{\omega(d)} \left\{ \frac{x}{d} \right\} = o(x). \tag{3.10}$$

There is a product representation of (3.6). It is well known [14] for a multiplicative function $f(n)$ there is an Euler product:

$$\sum_{n=1}^{\infty} f(n) = \prod_p \left(1 + \sum_{k=1}^{\infty} f(p^k) \right). \tag{3.11}$$

For the summand in (3.6) we have

$$\sum_{k=1}^{\infty} \frac{\lambda(p^k)2^{\omega(p^k)}}{p^{ks}} = \frac{-2p^{-s}}{1+p^{-s}} = \frac{-2}{p^s+1}, \tag{3.12}$$

and combining with (3.11) gives

$$\sum_{d=1}^{\infty} \frac{\lambda(d)2^{\omega(d)}}{d^s} = \prod_p \left(1 - \frac{2}{p^s+1} \right). \tag{3.13}$$

By the prime number theorem the left side of (3.13) extends to $s = 1$ and when $s = 1$,

$$a_0 = \sum_{d=1}^{\infty} \frac{\lambda(d)2^{\omega(d)}}{d} = \prod_p \left(1 - \frac{2}{p+1} \right) = 0. \tag{3.14}$$

4. General Form of Chowla’s Conjecture

There is a general form of Chowla’s conjecture [2] that as follows.

Conjecture 7. *Let $k \geq 1$, let $a_1, \dots, a_k$ be positive integers and let $b_1, \dots, b_k$ be distinct nonnegative integers such that $(a_i, b_i) = 1$ and $\frac{a_i}{a_j} \neq \frac{b_i}{b_j}$ for $1 \leq i < j \leq k$. Then as $x \rightarrow \infty$,*

$$\sum_{n \leq x} \lambda(a_1n + b_1) \cdots \lambda(a_kn + b_k) = o(x). \tag{4.1}$$

Since $\lambda(n)$ is completely multiplicative (7) is equivalent to

$$\sum_{n \leq x} \lambda((a_1n + b_1) \cdots (a_kn + b_k)) = o(x). \tag{4.2}$$

Let

$$f(n) = (a_1n + b_1) \cdots (a_kn + b_k) \tag{4.3}$$

and proceeding as (3.2) we have

$$\sum_{n \leq x} \lambda(f(n)) = \sum_{n \leq x} \sum_{d|f(n)} \lambda(d)2^{\omega(d)} = \sum_{d \leq f(x)} \lambda(d)2^{\omega(d)} \sum_{\substack{n \leq x \\ d|f(n)}} 1 \tag{4.4}$$

The inner sum of the counting function

$$\sum_{\substack{n \leq x \\ d|f(n)}} 1$$

is a well known and important object in sieve theory. We recall some properties of it from [4, p.17, Example 3] and [4, p.18, Example 4]. We have

$$\sum_{\substack{n \leq x \\ d|f(n)}} 1 = N_f(d) \left[\frac{x}{d} \right] = N_f(d) \left(\frac{x}{d} + O_d(1) \right) = \frac{N_f(d)}{d} x + O(N_f(d)) \quad (4.5)$$

where $N_f(d)$ is the number of solutions of the congruence

$$f(n) \equiv 0 \pmod{d}, \quad 0 \leq n \leq d-1$$

that are incongruent modulo d .

Remark 8. In the book [4] the quantity $N_f(d)$ is written as $\rho(d)$.

The function $N_f(d)$ is multiplicative. We denote the discriminant

$$D := \prod_{i=1}^k a_i \prod_{1 \leq i < j \leq k} (a_i b_j - a_j b_i), \quad (4.6)$$

and by the condition $\frac{a_i}{a_j} \neq \frac{b_i}{b_j}$ for $1 \leq i < j \leq k$ in Conjecture 7 we see $D \neq 0$. By [4, p.18, Example 4],

$$N_f(p) = k \text{ if } p \nmid D \quad (4.7)$$

$$N_f(p) < k \text{ if } p | D. \quad (4.8)$$

Since $N_f(d)$ is multiplicative it follows that if d is squarefree and coprime to D , then $N_f(d) = k^{\omega(d)}$.

5. Proof of Theorem 3

In [15] we use the Golomb method to prove the Bateman-Horn conjecture and it amounts to the study of the power series associated with the von Mangoldt function of polynomials. Here we use the same method to prove the Chowla conjecture. For simplicity we shall consider 1.3 but not 7. Thus in the following we let

$$f(n) = (n + h_1)(n + h_2) \cdots (n + h_k) \quad (5.1)$$

where $h_1 < h_2 < \cdots < h_k$ are distinct nonnegative integers. Since $\lambda(n)$ is completely multiplicative we have

$$\lambda(n + h_1) \lambda(n + h_2) \cdots \lambda(n + h_k) = \lambda(f(n)) = \sum_{d|f(n)} \lambda(d) 2^{\omega(d)} \quad (5.2)$$

where the last step follows from Theorem 2. Consider the absolutely convergent series for $|z| < 1$:

$$G_f(z) = \sum_{n \geq 1} \lambda(n + h_1) \lambda(n + h_2) \cdots \lambda(n + h_k) z^n, \quad (5.3)$$

then as in Hindry and Rivoal [5] we have

$$G_f(z) = \sum_{n=1}^{\infty} \left(\sum_{d|f(n)} \lambda(d) 2^{\omega(d)} \right) z^n = \sum_{d=1}^{\infty} \lambda(d) 2^{\omega(d)} \sum_{\substack{n=1 \\ d|f(n)}}^{\infty} z^n \quad (5.4)$$

$$= \sum_{d=1}^{\infty} \lambda(d) 2^{\omega(d)} \sum_{\ell=0}^{\infty} \sum_{\substack{n=1 \\ d|f(n)}}^d z^{n+\ell d} = \sum_{d=1}^{\infty} \frac{\lambda(d) 2^{\omega(d)}}{1 - z^d} \sum_{\substack{n=1 \\ d|f(n)}}^d z^n. \quad (5.5)$$

Let

$$N_f(d) := \sum_{\substack{n=1 \\ d|f(n)}}^d 1 \quad (5.6)$$

be the number of solutions of the congruence $f(n) \equiv 0 \pmod{d}$.

Remark 9. As we shall see in our context the quantity $N_f(d) \neq 0$ for every $d \geq 1$.

As we did for the Bateman-Horn conjecture in [15], to prove Theorem 3 we need to show

$$\lim_{z \rightarrow 1^-} \sum_{d=1}^{\infty} \left(\frac{d(1-z)}{N_f(d)(1-z^d)} \sum_{\substack{n=1 \\ d|f(n)}}^d z^n \right) \lambda(d) 2^{\omega(d)} \frac{N_f(d)}{d} = \sum_{d=1}^{\infty} \lambda(d) 2^{\omega(d)} \frac{N_f(d)}{d}, \quad (5.7)$$

or equivalently

$$\begin{aligned} & \lim_{z \rightarrow 1^-} \sum_{d=1}^{\infty} \left[\left(\frac{d(1-z)}{N_f(d)(1-z^d)} \sum_{\substack{n=1 \\ d|f(n)}}^d z^n - 1 \right) \lambda(d) 2^{\omega(d)} \frac{N_f(d)}{d} + \lambda(d) 2^{\omega(d)} \frac{N_f(d)}{d} \right] \\ &= \sum_{d=1}^{\infty} \lambda(d) 2^{\omega(d)} \frac{N_f(d)}{d}. \end{aligned}$$

To prove the above identity it is equivalent to proving

$$\lim_{z \rightarrow 1^-} \sum_{d=1}^{\infty} \left(\frac{d(1-z)}{N_f(d)(1-z^d)} \sum_{\substack{n=1 \\ d|f(n)}}^d z^n - 1 \right) \lambda(d) 2^{\omega(d)} \frac{N_f(d)}{d} = 0. \quad (5.8)$$

Now we proceed to the proof of Theorem 3.

Lemma 10. We have

$$\lim_{z \rightarrow 1^-} \sum_{d=1}^{\infty} \left| \frac{d(1-z)}{N_f(d)(1-z^d)} \sum_{\substack{n=1 \\ d|f(n)}}^d z^n - 1 \right| = 0. \quad (5.9)$$

Proof. It follows from the proof of [15, Theorem 1.2]. $\square$

The following result is due to Agnew.

Lemma 11. Suppose s_n is a bounded real or complex sequence. Let $c_d(x)$ be a sequence of functions defined over $0 < x < 1$ and satisfying

$$\lim_{x \rightarrow 1^-} c_d(x) = 0, \quad d = 1, 2, 3, \dots \quad (5.10)$$

$$\limsup_{x \rightarrow 1^-} \sum_{d=1}^{\infty} |c_d(x)| = M < +\infty. \quad (5.11)$$

Then

$$\limsup_{x \rightarrow 1^-} \left| \sum_{d=1}^{\infty} c_d(x) s_d \right| \leq M \limsup_{n \rightarrow \infty} |s_n|. \quad (5.12)$$

Proof. See [6, Lemma 5] or Agnew [1, Lemma 3.1]. $\square$

Theorem 12. We have

$$\lim_{z \rightarrow 1^-} \left| \sum_{d=1}^{\infty} \left(\frac{d(1-z)}{N_f(d)(1-z^d)} \sum_{\substack{n=1 \\ d|f(n)}}^d z^n - 1 \right) \lambda(d) 2^{\omega(d)} \frac{N_f(d)}{d} \right| = 0. \quad (5.13)$$

Proof. The proof is analogous to that of [15, Theorem 4.1]. We take

$$s_n := \lambda(n) 2^{\omega(n)} \frac{N_f(n)}{n}, \quad c_d(z) := \left(\frac{d(1-z)}{N_f(d)(1-z^d)} \sum_{\substack{n=1 \\ d|f(n)}}^d z^n - 1 \right) \quad (5.14)$$

in Lemma 11. Since $2^{\omega(n)}N_f(n) = o(n)$ it is clear s_n is bounded. The condition (5.10) is clearly met when $z \rightarrow 1^-$. By Lemma 10,

$$\lim_{z \rightarrow 1^-} \sum_{d=1}^{\infty} \left| \frac{d(1-z)}{N_f(d)(1-z^d)} \sum_{\substack{n=1 \\ d|f(n)}}^d z^n - 1 \right| = 0,$$

and thus

$$\limsup_{z \rightarrow 1^-} \sum_{d=1}^{\infty} \left| \frac{d(1-z)}{N_f(d)(1-z^d)} \sum_{\substack{n=1 \\ d|f(n)}}^d z^n - 1 \right| = 0 := M,$$

hence the condition (5.11) is met. By Lemma 11 and by noting that $\limsup_{n \rightarrow \infty} |s_n| < \infty$ since s_n is bounded, we deduce that

$$\limsup_{z \rightarrow 1^-} \left| \sum_{d=1}^{\infty} \left(\frac{d(1-z)}{N_f(d)(1-z^d)} \sum_{\substack{n=1 \\ d|f(n)}}^d z^n - 1 \right) \lambda(d) 2^{\omega(d)} \frac{N_f(d)}{d} \right| \leq 0. \quad (5.15)$$

On the other hand we have

$$0 \leq \liminf_{z \rightarrow 1^-} \left| \sum_{d=1}^{\infty} \left(\frac{d(1-z)}{N_f(d)(1-z^d)} \sum_{\substack{n=1 \\ d|f(n)}}^d z^n - 1 \right) \lambda(d) 2^{\omega(d)} \frac{N_f(d)}{d} \right|,$$

and thus we conclude (5.13). $\square$

The above implies (5.8) and thus Theorem 3.

Corollary 13. *The relation (5.8) holds:*

$$\lim_{z \rightarrow 1^-} \sum_{d=1}^{\infty} \left(\frac{d(1-z)}{N_f(d)(1-z^d)} \sum_{\substack{n=1 \\ d|f(n)}}^d z^n - 1 \right) \lambda(d) 2^{\omega(d)} \frac{N_f(d)}{d} = 0.$$

Proof. This follows from (5.13) and [15, Lemma 11]. $\square$

6. Examples

Before the proof of Theorem 4 let us see some examples.

6.1. Example 1

Let $f(n) = n(n+1)$, then its discriminant is $D = 1$ and thus for all primes p we have $N_f(p) = 2$ by (4.7). Indeed for every prime power we have

$$N_f(p^\ell) = 2, \quad \ell \geq 1 \quad (6.1)$$

since $(n, n+1) = 1$, there are always exactly two solutions of the congruence

$$n(n+1) \equiv 0 \pmod{p^\ell}, \quad 0 \leq n \leq p^\ell - 1. \quad (6.2)$$

Now it follows from (6.1) that for all $d \geq 1$,

$$N_f(d) = 2^{\omega(d)}. \quad (6.3)$$

By Theorem 3,

$$\lim_{z \rightarrow 1^-} (1-z) \sum_{n=1}^{\infty} \lambda(n) \lambda(n+1) z^n = \sum_{d=1}^{\infty} \frac{\lambda(d) 4^{\omega(d)}}{d}. \quad (6.4)$$

By the Euler product (3.11),

$$\sum_{d=1}^{\infty} \frac{\lambda(d) 4^{\omega(d)}}{d} = \prod_p \left(1 - \frac{4}{p+1} \right) = 0, \quad (6.5)$$

and therefore

$$\sum_{n \leq x} \lambda(n(n+1)) = o(x). \quad (6.6)$$

6.2. Example 2

Let $f(n) = n(n+2)$, then its discriminant is $D = 2$. One may directly find that

$$N_f(2) = 1, \quad N_f(4) = 2, \quad N_f(2^m) = 4, \quad m \geq 3 \quad (6.7)$$

and for odd prime p ,

$$N_f(p^\ell) = 2, \quad \ell \geq 1. \quad (6.8)$$

By Theorem 3,

$$\lim_{z \rightarrow 1^-} (1-z) \sum_{n=1}^{\infty} \lambda(n) \lambda(n+2) z^n = \sum_{d=1}^{\infty} \frac{\lambda(d) 2^{\omega(d)} N_f(d)}{d}. \quad (6.9)$$

By the Euler product (3.11),

$$\sum_{d=1}^{\infty} \frac{\lambda(d) 2^{\omega(d)} N_f(d)}{d} = \prod_p \left(1 + \sum_{k=1}^{\infty} \frac{\lambda(p^k) N_f(p^k) 2^{\omega(p^k)}}{p^k} \right) = \frac{1}{3} \prod_{p>2} \left(1 - \frac{4}{p+1} \right) = 0, \quad (6.10)$$

and therefore

$$\sum_{n \leq x} \lambda(n(n+2)) = o(x). \quad (6.11)$$

7. Proof of Theorem 4

Let $h_1 < h_2 < \dots < h_k$ be distinct nonnegative integers and let

$$f(n) = (n+h_1) \cdots (n+h_k).$$

Then its discriminant is

$$D = \prod_{1 \leq i < j \leq k} (h_j - h_i),$$

which is a finite number. For odd prime $p > D$ we have

$$N_f(p^\ell) = \#\{0 \leq n < p^\ell : (n+h_1) \cdots (n+h_k) \equiv 0 \pmod{p^\ell}\} = k, \quad \ell \geq 1. \quad (7.1)$$

Indeed the solutions of the above congruence are those n for which $n+h_i = p^\ell$, thus the solutions are $n = p^\ell - h_1, \dots, p^\ell - h_k$ and we get (7.1).

By the Euler product (3.11) we have

$$\sum_{d=1}^{\infty} \lambda(d) 2^{\omega(d)} \frac{N_f(d)}{d} = \prod_p \left(1 + \sum_{m=1}^{\infty} \frac{\lambda(p^m) N_f(p^m) 2^{\omega(p^m)}}{p^m} \right) = c_D \prod_{p>D} \left(1 - \frac{2k}{p+1} \right) \quad (7.2)$$

where c_D is the component of the Euler product corresponding to primes $p \leq D$. It follows from [9, Theorem 7] that

$$\prod_{p>D} \left(1 - \frac{2k}{p+1}\right) = 0 \quad (7.3)$$

and therefore

$$\sum_{d=1}^{\infty} \lambda(d) 2^{\omega(d)} \frac{N_f(d)}{d} = 0.$$

The proof is complete.

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