

Article

Not peer-reviewed version

Greedoids and Violator Spaces

[Yulia Kempner](#) and [Vadim Levit](#) *

Posted Date: 1 August 2024

doi: 10.20944/preprints202407.2585.v1

Keywords: violator space; linear programming; closure operator; greedoid; antimatroid



Preprints.org is a free multidiscipline platform providing preprint service that is dedicated to making early versions of research outputs permanently available and citable. Preprints posted at Preprints.org appear in Web of Science, Crossref, Google Scholar, Scilit, Europe PMC.

Copyright: This is an open access article distributed under the Creative Commons Attribution License which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Article

Greedoids and Violator Spaces

Yulia Kempner ¹, Vadim E. Levit ^{2*}

¹ Department of Computer Science Holon Institute of Technology, Israel; yuliak@hit.ac.il

² Department of Mathematics Ariel University, Israel

* Correspondence: levitv@ariel.ac.il

Abstract: The primary objective of this paper is to establish connections between two well-known but previously independently developed theories: the theory of violator spaces and the theory of greedoids. Violator spaces were introduced by Matoušek et al. in 2008 as a generalization of linear programming problems. Greedoids were introduced by Korte and Lovász in 1981 in an effort to characterize combinatorial structures where greedy algorithms yield optimal solutions. In this work, we explore the relationships between violator spaces and greedoids, demonstrating that greedoids can be defined using a variant of a violator operator. These interrelations provide a new characterization of antimatroids.

Keywords: violator space; linear programming; closure operator; greedoid; antimatroid

1. Preliminaries

The primary objective of the paper is to establish connections between two combinatorial concepts: violator spaces and greedoids. Violator spaces emerged as a generalization of Linear Programming (LP) problems. LP-type problems were introduced and studied by Matoušek, Sharir, and Welzl [11,13] as a combinatorial framework encompassing linear programming and other computational geometric problems. Further, Matoušek et al. [1] introduced violator spaces as a simpler framework, which generalizes LP-type problems. Every LP-type problem can naturally be converted into a violator space, but the reverse is not necessarily true. Greedoids were introduced as an attempt to characterize combinatorial structures where the greedy algorithm yields optimal solutions. This work explores the relationships between violator spaces and greedoids, demonstrating that greedoids can be defined using a variant of a violator operator. The interrelations between violator spaces and greedoids provide a new characterization of antimatroids.

2. Violator Spaces

Definition 1. [1] An abstract LP-type problem is a quadruple $(E, w, W, \leq)$, where E is a finite set, W is a set linearly ordered by $\leq$, and w is a mapping $2^E \rightarrow W$ such that for all subsets $X, Y \subseteq E$ the following properties are satisfied:

V1: $X \subseteq Y \Rightarrow w(X) \leq w(Y)$ (monotonicity);

V2: for all $X \subseteq Y$ and $a \in E$ with $w(X) = w(Y)$ and $w(Y) < w(Y \cup a)$, we have $w(X) < w(X \cup a)$ (locality).

Definition 2. [13] Consider an abstract LP-type problem $(E, w, W, \leq)$. We say that $B \subseteq E$ is a basis if for all proper subsets $F \subset B$ we have $w(F) \neq w(B)$. For $X \subseteq E$, a basis of X is an inclusion-minimal subset B of X with $w(B) = w(X)$.

Solving an abstract LP-type problem $(E, w, W, \leq)$ means identifying a basis of E .

The set of all constraints violating $X \subseteq E$, denoted as $\nu(X)$, is defined as:

$$\nu(X) = \{a \in E : w(X \cup a) > w(X)\}. \quad (1)$$

This set includes all elements a in the ground set E that are not in X and, when added to X , cause the objective function w to increase.

One can see that the condition $w(X) = w(Y)$ generally does not suffice for $\nu(X) = \nu(Y)$. For example [1], if $w(X) = |X|$ for all $X \subseteq E$, then any X of the same size have the same $w(X)$, however, $\nu(X) = E - X$, and so no distinct sets X share the value of $\nu(X)$. Thus, we can see that w by itself does not capture the combinatorial structure of the problem. To describe the "structure" of an LP-type problem Matoušek et al. [1] introduced the concept of a violator space.

Definition 3. [1] A violator space is a pair (E, ν) , where E is a finite set and ν is a violator operator, i.e., a mapping $2^E \rightarrow 2^E$ such that for all subsets $X, Y \subseteq E$ the following properties are satisfied:

V11: $X \cap \nu(X) = \emptyset$ (consistency);

V22: $(X \subseteq Y \text{ and } Y \cap \nu(X) = \emptyset) \Rightarrow \nu(X) = \nu(Y)$ (locality).

Let (E, ν) be a violator space. Let $\varphi(X) = E - \nu(X)$. So, if ν is defined by (1) then

$$\varphi(X) = \{a \in E : w(X \cup a) = w(X)\}. \quad (2)$$

Theorem 1. [3] Let E be a finite set and $\nu, \varphi : 2^E \rightarrow 2^E$ two mappings such that $\varphi(X) = E - \nu(X)$ for all $X \subseteq E$. Then (E, ν) is a violator space if and only if φ satisfies

C1: $X \subseteq \varphi(X)$ (extensivity);

C2: $(X \subseteq Y \subseteq \varphi(X)) \Rightarrow \varphi(X) = \varphi(Y)$ (self-convexity).

Thus, we have two equivalent approaches to define violator spaces:

- the classic one - (E, ν) , where the ground set E is equipped with the mapping ν satisfying consistency and locality;
- the dual one - (E, φ) , where the ground set E is equipped with the mapping φ satisfying extensivity and self-convexity.

In what follows, a pair (E, φ) , where E is a finite set and φ is a mapping satisfied extensivity and self-convexity, will be considered a violator space as well.

Since a violator operator φ is extensive it may be interpreted as a some type of a "closure" operator. The concept of closure appears in many disciplines, including topology, algebra, logic, geometry, convexity analysis, graph theory etc. It is also known by other names, such as span, hull, and envelope. For instance, the convex hull operator on the Euclidean space is a classic example of a closure operator.

Definition 4. Let E be a finite set. $\tau : 2^E \rightarrow 2^E$ is a closure operator on E if for all subsets $X, Y \subseteq E$ the following properties are satisfied:

CL1: $X \subseteq \tau(X)$ (extensivity);

CL2: $X \subseteq Y \Rightarrow \tau(X) \subseteq \tau(Y)$ (isotonicity);

CL3: $\tau(\tau(X)) = \tau(X)$ (idempotence).

It was proved that every closure operator is a violator operator, since isotonicity and idempotence imply self-convexity, and extensivity with self-convexity imply idempotence [3]. However, not each violator operator is a closure operator, since it does not have to satisfy isotonicity.

Let us consider some additional properties of violator operators.

Lemma 1. Let (E, φ) be a violator space. Then $X \subseteq \varphi(Y) \Leftrightarrow \varphi(Y) = \varphi(Y \cup X)$ for all $X, Y \subseteq E$.

Proof. "If": Let $\varphi(Y) = \varphi(Y \cup X)$. Since extensivity implies $X \subseteq Y \cup X \subseteq \varphi(Y \cup X)$, we conclude with $X \subseteq \varphi(Y)$.

"Only if": If $X \subseteq \varphi(Y)$, then $Y \subseteq Y \cup X \subseteq \varphi(Y)$. Hence, by self-convexity, $\varphi(Y) = \varphi(Y \cup X)$. $\square$

Proposition 1. Let E be a finite set. For each extensive operator $\varphi : 2^E \rightarrow 2^E$ self-convexity is equivalent to the following property:

$$\mathbf{C22}: (X \subseteq \varphi(Y) \wedge Y \subseteq \varphi(X)) \Rightarrow \varphi(X) = \varphi(Y) \text{ for all } X, Y \subseteq E.$$

Proof. 1. $\mathbf{C2} \Rightarrow \mathbf{C22}$: From Lemma 1 $(X \subseteq \varphi(Y) \wedge Y \subseteq \varphi(X)) \Rightarrow \varphi(Y) = \varphi(X \cup Y) = \varphi(X)$. Hence, $\varphi(X) = \varphi(Y)$.

2. $\mathbf{C22} \Rightarrow \mathbf{C2}$: $(X \subseteq Y \subseteq \varphi(X)) \Rightarrow X \subseteq \varphi(Y) \wedge Y \subseteq \varphi(X)$ (from extensivity). Then $\mathbf{C22}$ implies $\varphi(X) = \varphi(Y)$. $\square$

Lemma 2. ([3]) Let (E, φ) be a violator space. Then

$$\varphi(X) = \varphi(Y) \Rightarrow \varphi(X \cup Y) = \varphi(X) = \varphi(Y) \quad (3)$$

and

$$(X \subseteq Y \subseteq Z) \wedge (\varphi(X) = \varphi(Z)) \Rightarrow \varphi(X) = \varphi(Y) = \varphi(Z) \quad (4)$$

for every $X, Y, Z \subseteq E$.

Since the second property addresses all sets that lie between two given sets, we refer to this property as *convexity*, in accordance with Monjardet [8].

2.1. Uniquely generated violator spaces

Let (E, α) be an arbitrary space with the operator $\alpha : 2^E \rightarrow 2^E$. $B \subseteq E$ is a *generator* of $X \subseteq E$ if $\alpha(B) = \alpha(X)$. A *basis* of X is a inclusion-minimal set $B \subseteq E$ (not necessarily included in X) with $\alpha(B) = \alpha(X)$. A space (E, α) is *uniquely generated* if every set $X \subseteq E$ has a unique basis.

Proposition 2. [3] A violator space (E, φ) is uniquely generated if and only if for every $X, Y \subseteq E$

$$\varphi(X) = \varphi(Y) \Rightarrow \varphi(X \cap Y) = \varphi(X) = \varphi(Y). \quad (5)$$

It is known that a closure operator τ is uniquely generated if and only if it satisfies the *anti-exchange property* [7,10,12]:

$$p, q \notin \tau(X) \wedge p \in \tau(X \cup q) \Rightarrow q \notin \tau(X \cup p).$$

We extended this characterization to violator spaces in the following.

Theorem 2. [3] Let (E, φ) be a violator space. Then (E, φ) is uniquely generated if and only if the operator φ satisfies the anti-exchange property.

An element x of a subset $X \subseteq E$ is an *extreme point* of X if $x \notin \alpha(X - x)$. The set of extreme points of X is denoted by $ex(X)$.

Proposition 3. [3] For violator spaces: $x \in ex(X)$ if and only if $\varphi(X) \neq \varphi(X - x)$.

Theorem 3. [3] Let (E, φ) be a violator space. Then (E, φ) is uniquely generated if and only if for every set $X \subseteq E$, $\varphi(X) = \varphi(ex(X))$.

Corollary 1. [3] Let (E, φ) be a uniquely generated violator space. Then for every $X \subseteq E$ the set $ex(X)$ is the unique basis of X .

3. Greedoids

Let us have a greedoid $(E, \mathcal{F})$, i.e.,
(i) $\emptyset \in \mathcal{F}$;

(ii) $X, Y \in \mathcal{F}, |X| > |Y| \Rightarrow \exists x \in X - Y$ such that $Y \cup x \in \mathcal{F}$ (augmentation property).

Elements of $\mathcal{F}$ are called *feasible sets*.

The rank function of a greedoid is defined as follows:

$$r(X) = \max\{|A| : A \subseteq X, A \in \mathcal{F}\}.$$

Theorem 4. [7] A function $r : 2^E \rightarrow Z^+$ is the rank function of a greedoid if and only if for all $X, Y \subseteq E$ and all $x, y \in E$ the following conditions hold:

$$(1.1) r(\emptyset) = 0$$

$$(1.2) r(X) \leq |X| \text{ (subcardinality)}$$

$$(1.3) X \subseteq Y \text{ implies } r(X) \leq r(Y) \text{ (monotonicity)}$$

$$(1.4) r(X) = r(X \cup x) = r(X \cup y) \text{ implies } r(X) = r(X \cup x \cup y) \text{ (local submodularity).}$$

Moreover, the greedoid is uniquely determined by its rank function:

$$\mathcal{F} = \{X : r(X) = |X|\}. \quad (6)$$

So, the pair (E, r) may be considered as a greedoid with $\mathcal{F}$ defined by (6).

3.1. LP-type Problems and Greedoids

The rank function r of a greedoid satisfies the all properties of the mapping w of LP-type problem. It is monotone (Theorem 4) and satisfies the locality.

Proposition 4. Let $(E, \mathcal{F})$ be a greedoid with rank function r . Then $r(X) = r(Y)$ and $r(Y) < r(Y \cup a)$ implies $r(X) < r(X \cup a)$ for all $X \subseteq Y$ and $a \in E$.

Proof. If $X \subseteq Y$ and $r(X) = r(Y)$ there is a maximal feasible set $A \subseteq X$, with $|A| = r(X) = r(Y)$. In addition, there is a feasible set $C \subseteq (Y \cup a)$ with $|C| > |A|$. Since A is a maximal feasible set in Y , $C \not\subseteq Y$, $a \notin Y$ and $a \in C$. Then (from the augmentation property) there exists $x \in C$ for which $A \cup x \in \mathcal{F}$. Since A is a maximal feasible set in Y , $x \notin Y$, and so $x = a$, i.e., $A \cup a$ is a feasible set, and then $r(X) < r(X \cup a)$. $\square$

Thus a greedoid (E, r) may be considered as an abstract LP-type problem $(E, r, Z^+, \leq)$.

Note, that the family $\mathcal{F}$ is the family of all bases of the LP-type problem, since for each proper subset A of a feasible set

$$X \in \mathcal{F}, r(A) \leq |A| < |X| = r(X).$$

However, if the bases of a LP-type problem $(E, w, Z^+, \leq)$ form a greedoid, a function w does not have to be a greedoid's rank function. For example, every monotone function w with different values on each element of 2^E turns all subsets of E to be bases. While the bases constitute a greedoid, a function w does not have to satisfied subcardinality.

Let us build a specific LP-type problem $(E, v, Z^+, \leq)$ where E is a finite set, v is a mapping $2^E \rightarrow Z^+$ such that for all subsets $X, Y \subseteq E$ the following properties are satisfied:

$$(1) v(\emptyset) = 0;$$

$$(2) v(X) \leq |X| \text{ (subcardinality);}$$

$$(3) X \subseteq Y \text{ implies } v(X) \leq v(Y) \text{ (monotonicity);}$$

$$(4) \text{ For all } X \subseteq Y \text{ and } a \in E \text{ with } v(X) = v(Y) \text{ and } v(Y) < v(Y \cup a), \text{ we have } v(X) < v(X \cup a) \text{ (locality).}$$

The property (4) may be reformulated as follows:

$$(5) \text{ For all } X \subseteq Y \text{ and } a \in E \text{ with } v(X) = v(Y) \text{ and } v(X) = v(X \cup a), \text{ we have } v(Y) = v(Y \cup a).$$

Proposition 5. Let v be a monotone mapping on 2^E . Then locality (5) is equivalent to local submodularity (1.4).

Proof. 1. Let v satisfy locality and $v(X) = v(X \cup x) = v(X \cup y)$. Denote $Y = X \cup x$. Then $X \subseteq Y$, $v(X) = v(Y)$ and $v(X) = v(X \cup y)$, then (from (5)) we have $v(X) = v(Y) = v(Y \cup y) = v(X \cup x \cup y)$.

2. Let v satisfy local submodularity, $X \subseteq Y \subseteq E$, $a \in E$, $v(X) = v(Y)$ and $v(X) = v(X \cup a)$. By repeatedly applying local submodularity we prove that $v(Y) = v(Y \cup a)$. It is easy to see that the proposition is correct for $a \in Y$. Let $Y - X = \{y_1, y_2, \dots, y_n\}$. From monotonicity it follows that $v(X) = v(X \cup y_1) = v(Y)$. Hence, $v(X) = v(X \cup a) = v(X \cup y_1)$. Local submodularity implies $v(X) = v(X \cup y_1 \cup a)$. Denote $Y_k = X \cup \{y_1, y_2, \dots, y_k\}$.

Prove by induction on n that $v(X) = v(Y_k \cup a)$ for each $1 \leq k \leq n$. From monotonicity it follows that $v(X) = v(Y_k)$ for each $1 \leq k \leq n$. Then $v(X) = v(Y_k \cup a) = v(Y_k \cup y_{k+1})$ implies $v(X) = v(Y_{k+1} \cup a)$. Thus $v(X) = v(Y) = v(Y_n \cup a) = v(Y \cup a)$. $\square$

In fact, the property of local submodularity may be extended as follows.

Corollary 2. If $X, Y \subseteq E$ such that $v(X) = v(Y) = v(X \cap Y)$, then $v(X \cap Y) = v(X \cup Y)$.

Theorem 4 implies that a LP-type problem with mapping $v(\emptyset) = 0$ and $v(X) \leq |X|$ determines a greedoid (E, v) by $\mathcal{F} = \{X : v(X) = |X|\}$.

3.2. Violator and Rank Closure Operators on Greedoids

Define (rank) closure operator of greedoids ([7]): $\sigma(X) = \{x : r(X \cup x) = r(X)\}$. The operator is extensive (CL1) and idempotent (CL3), but it does not have to be an isotone operator. So σ is not always a closure operator. At the same time, the definition of σ coincides with the definition of a violator operator (2). The following theorem supports the interpretation of greedoids as a subclass of violator spaces since the properties (i) and (ii) define a violator operator.

Theorem 5. [2] A mapping $\sigma : 2^E \rightarrow 2^E$ is the closure operator of some greedoid $(E, \mathcal{F})$ if and only if

(i) $X \subseteq \sigma(X)$

(ii) $(X \subseteq Y \subseteq \sigma(X)) \Rightarrow \sigma(X) = \sigma(Y)$

(iii) if $z \notin \sigma(X \cup x - z)$ for all $z \in X \cup x$, and $x \in \sigma(X \cup y)$ then $y \in \sigma(X \cup x)$.

Moreover, if σ satisfies (i), (ii), and (iii), then

$\mathcal{F} = \{X \subseteq E : \forall x \in X : x \notin \sigma(X - x)\}$.

Based on the definition of extreme points we have

$$\mathcal{F} = \{X \subseteq E : X = ex(X)\}.$$

Then the following condition $z \notin \sigma(X \cup x - z)$ for all $z \in X \cup x$ is equivalent to $X \cup x \in \mathcal{F}$, and so the property (iii) may be rewritten as follows:

$$(X \cup x \in \mathcal{F}) \wedge (x \in \sigma(X \cup y)) \Rightarrow y \in \sigma(X \cup x). \quad (7)$$

Define $\mathcal{B} = \{X \subseteq E : \forall Y \subset X : \sigma(X) \neq \sigma(Y)\}$ - the family of bases w.r.t. operator σ .

Proposition 6. $\mathcal{B} = \mathcal{F}$ for each violator operator σ .

Proof. If $X \in \mathcal{B}$, i.e., for all $Y \subset X : \sigma(X) \neq \sigma(Y)$, then for all $x \in X$ holds $\sigma(X) \neq \sigma(X - x)$, and so (from Proposition 3) $x \notin \sigma(X - x)$. Then $\mathcal{B} \subseteq \mathcal{F}$. If $X \notin \mathcal{B}$ then there exists $Y \subset X$ such that $\sigma(X) = \sigma(Y)$. Hence for each $x \in X - Y$ self-convexity implies $\sigma(X) = \sigma(X - x)$, i.e., $x \in \sigma(X - x)$. This concludes the proof, with $\mathcal{F} \subseteq \mathcal{B}$. $\square$

It is worth mentioning that Property (iii) is not necessary for $\mathcal{F}$ to be a greedoid.

Example 1. Let $E = \{1, 2, 3\}$. Define $\varphi(X) = X$ for each $X \subseteq E$ except $\varphi(\{1\}) = \varphi(\{1, 3\}) = \{1, 3\}$. It is easy to check that the space (E, φ) is a uniquely generated violator space (satisfies both extensivity and convexity), where the family of bases $\mathcal{F} = P(\{1, 2, 3\}) - \{1, 3\}$ forms a greedoid. At the same time, operator φ does not satisfies (7) which is equivalent to (iii). Indeed, if $X = \emptyset, x = 3, y = 1$, then $X \cup x = \{3\} \in \mathcal{F}$, $x \in \varphi(X \cup y) = \varphi(\{1\}) = \{1, 3\}$, but $y \notin \varphi(X \cup x) = \varphi(\{3\})$.

If we consider the rank function of the greedoid, we can see that $r(\{1\}) = r(\{3\}) = r(\{1, 3\})$, and so $\sigma(\{1\}) = \sigma(\{3\}) = \{1, 3\}$. Then for this function σ the property (iii) holds and we have a not uniquely generated violator space.

Thus the same family of bases may be obtained by different mappings.

3.3. Antimatroids - Uniquely Generated Greedoids

Definition 5. An antimatroid is a greedoid closed under union.

Definition 6. An accessible set system $(E, \mathcal{F})$ is a set system in which every nonempty feasible set $X \in \mathcal{F}$ contains an element x such that $X - x$ is feasible.

By definition, the family of feasible sets of a greedoid is an accessible set system.

Lemma 3. [7] For an accessible set system $(E, \mathcal{F})$ the following statements are equivalent:

- (A1) $(E, \mathcal{F})$ is an antimatroid
- (A2) $\mathcal{F}$ is closed under union
- (A3) $A, A \cup x, A \cup y \in \mathcal{F}$ implies $A \cup \{x, y\} \in \mathcal{F}$.

Proposition 7. An antimatroid is a uniquely generated greedoid.

Proof. Let $\mathcal{F}$ be an antimatroid. Since each antimatroid is a greedoid, it remains to prove that the greedoid is uniquely generated. Suppose there are two bases B_1 and B_2 such that $\sigma(B_1) = \sigma(B_2)$. Since $\mathcal{F}$ is a family of bases, then $B_1, B_2 \in \mathcal{F}$. Hence $B_1 \cup B_2 \in \mathcal{F}$, because $\mathcal{F}$ is an antimatroid. But $\sigma(B_1 \cup B_2) = \sigma(B_1)$ (see Lemma 2). Contradiction. $\square$

Since for each greedoid σ is a violator operator, Theorem 2 implies the following.

Corollary 3. The operator σ of each antimatroid satisfies the anti-exchange property.

Theorem 6. The family $\mathcal{F}$ is an antimatroid if and only if $\mathcal{F}$ is a uniquely generated greedoid.

Proof. It remains to prove that each uniquely generated greedoid is an antimatroid. Suppose, $A, A \cup x, A \cup y \in \mathcal{F}$, but $A \cup \{x, y\} \notin \mathcal{F}$. Then $x, y \notin \sigma(A)$, $x \in \sigma(A \cup y)$ and $y \in \sigma(A \cup x)$. Contradiction to anti-exchange property. $\square$

4. Conclusion

We have demonstrated that greedoids can be defined using a variant of the violator operator. The relationships between violator spaces and greedoids enabled us to provide a new characterization of antimatroids.

To gain further structural insights into antimatroids, one can explore cospanning relations. Given an operator σ , two sets X and Y are *cospanning* if $\sigma(X) = \sigma(Y)$. The equivalence relation known as cospanning has been studied for greedoids [7], and it has been shown that these relations uniquely determine greedoids. Similar equivalence relations have also been explored for violator spaces [3,4,9]. Investigating these structures could lead to new characterizations of antimatroids and other combinatorial structures.

References

1. Gärtner, B., J. Matoušek, L. Rüst, and P. Škovroň, *Violator spaces: structure and algorithms*, Discrete Applied Mathematics **156** (2008), 2124–2141.
2. Goecke, O., Korte, B., and Lovász, L. *Examples and algorithmic properties of greedoids* (2006), 113–161, B. Simone (Ed.), Combinatorial Optimization, Springer-Verlag, New York/Berlin (1986).
3. Kempner, Y., and Levit, V. E. *Violator spaces vs closure spaces*, European Journal of Combinatorics **80** (2019), 203–213.
4. Kempner, Y., and Levit, V.E. *Cospanning characterizations of violator and co-violator spaces*. In: Hoffman, F., Holliday, S., Rosen, Z., Shahrokhi, F., Wierman, J. (eds) Combinatorics, Graph Theory and Computing. SEICCGTC 2021. Springer Proceedings in Mathematics & Statistics, **448** (2024), 109–117. Springer, Cham. https://doi.org/10.1007/978-3-031-52969-6_11
5. Kempner, Y., and Levit, V. E. *Cospanning characterizations of antimatroids and convex geometries* (2021), arXiv:2107.08556.
6. Korte, B., Lovász, L. *Mathematical structures underlying greedy algorithms*. in: Fundamentals of Computation Theory, Lecture Notes in Computer Science **117**, Springer (1981), 205–209.
7. Korte, B., Lovász, L. and R. Schrader, “Greedoids,” Springer-Verlag, New York/Berlin (1991).
8. Monjardet, B. and Raderanirina, V. *The duality between the anti-exchange closure operators and the path independent choice operators on a finite set*, Mathematical Social Sciences **41** (2001), 131–150.
9. Y. Brise, B. Gärtner, *Clarkson’s algorithm for violator spaces*, Computational Geometry **44** (2011), 70–81.
10. P. H. Edelman, R. E. Jamison, *The theory of convex geometries*, Geometriae Dedicata **19** (1985), 247–270.
11. J. Matoušek, M. Sharir, E. Welzl, *A subexponential bound for linear programming*. Algorithmica **16** (1996), 498–516.
12. J. Pfaltz, *Closure Lattices*, Discrete Mathematics **154** (1996), 217–236.
13. M. Sharir, E. Welzl, *A combinatorial bound for linear programming and related problems*, Lecture Notes in Computer Science **577** (1992), 569–579.

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.