Short Note

Reducing Newman–Penrose Spin-Coefficient Constraints to Metric Identities: An Example from the Kerr Spacetime

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Reducing Newman–Penrose Spin-Coefficient Constraints to Metric Identities: An Example from the Kerr Spacetime

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Abstract

The aim of this work is to demonstrate that, for spacetimes with sufficient symmetry, spin coefficient conditions may reduce to compact metric-component identities that are useful for practical metric analysis. This idea is illustrated using the symmetries of the Kerr spacetime. For a vacuum Petrov Type D spacetime admitting geodesic, shear-free null congruences, as characterised by the Goldberg–Sachs theorem, the shear spin coefficient, $\lambda = 0$, can be reformulated using a principal-null-aligned (Kinnersley) tetrad. The resulting relation can be expressed solely in terms of metric components and their radial derivatives within a Kerr-like coordinate gauge. To the best of the author's knowledge, this is one of the first explicit, coordinate-dependent metric identities corresponding to the vanishing of a Newman–Penrose spin coefficient. The resulting condition eliminates explicit tetrad dependence, yielding a purely metric-level identity that can be evaluated once a Kerr-like Boyer–Lindquist gauge is fixed. This reformulation provides a practical diagnostic for verifying the shear-free property of principal null congruences directly from the metric, without constructing a tetrad or imposing a specific ansatz. As such, it offers a useful tool for constraining or partially reconstructing stationary, axisymmetric spacetimes under appropriate symmetry and geometric assumptions. The expression has been validated numerically for several Kerr-like spacetimes, including Kerr, Kerr–Newman, Schwarzschild, and static de Sitter metrics. This points toward a bridge between tetrad-based geometric characterisations and coordinate-level analyses of spacetime structure.

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1. Introduction

1.1. Background and Overview

The Kerr spacetime is well-known as a complicated analytical solution of the Einstein Field Equations (EFE); it was discovered by Roy P. Kerr in 1963 [1]. An approach to solving for the Kerr metric is to look at a *null tetrad* suitable for the spacetime using Newman-Penrose formalism [2].

We consider tetrads capable of describing null congruences that are not *hypersurface-orthogonal*, i.e., those with non-zero twist (See Appendix A). The derivation leverages well-known symmetries such as time and axial symmetry. This reformulation is satisfied by Kerr-like, Petrov Type D (and other algebraic) spacetimes that satisfy *Goldberg-Sachs theorem*, i.e., ones that admit *geodesic, shear-free null congruences*. For *Petrov classification*, see Stewart [3], Sec. 2, pp. 76-80. For the Goldberg-Sachs theorem, see Stewart [3], Sec. 2, p. 104.

According to the Goldberg–Sachs theorem [3], vacuum (or Einstein) Petrov Type D spacetimes admit two repeated principal null directions that are geodesic and shear-free. This forms the basis for our coordinate-based formulation derived from the Newman–Penrose spin coefficient $\lambda = 0$. In Section IV, using a principal-null-aligned tetrad, we derive a coordinate-based formulation of the Newman–Penrose spin coefficient condition $\lambda = \bar{\lambda} = 0$, which can subsequently be expressed

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purely in terms of metric components and their derivatives. This explicit expression reformulates the shear-free, geodesic null congruence condition entirely in terms of metric components and their derivatives, providing a practical tool for analysing Kerr-like and related spacetimes without explicit tetrad construction. Note that the resulting metric equation is not guaranteed to hold in a completely general Type D spacetime since it is a coordinate-based expression.

2. Literature Review

In this Section, prior studies, which use the Newman–Penrose (NP) formalism in the context of exact solutions and spacetime classification, will now be explored. The aim is to assess whether coordinate-dependent constraints or metric reconstruction equations, such as the one derived in this work, have previously been discussed in the literature.

Previous work on Petrov Type D spacetimes and the Newman–Penrose formalism provides a foundation for understanding shear-free, geodesic null congruences. Kinnersley [4] classified all Type D vacuum metrics. The Geroch–Held–Penrose (GHP) formalism [5], including higher-dimensional and modified extensions [6,7], simplifies the analysis of spin coefficients such as λ. Debney, Kerr, and Schild [8] study algebraically degenerate Einstein and Einstein–Maxwell solutions, and non-stationary Kerr-type congruences [9] illustrate the flexibility of tetrad approaches. Newman and Penrose [2] present the full formalism for algebraic classification, while Pravdová and Pravda [10] extend it to higher-dimensional spacetimes.

Although these works establish shear-free, geodesic null congruences, none provide an explicit, coordinate-dependent equation for $\lambda = 0$ for a subset of Type D spacetimes in terms of metric components. The reformulation derived here fills this gap, enabling direct verification of such congruences without constructing a tetrad or assuming an ansatz, bridging abstract NP/GHP formalism with practical computations. It also reveals an important connection that spin coefficients can reduce to metric level identities possibly for any spacetime with sufficient symmetries and geometric constraints. This literature review involved manual searches across arXiv, NASA ADS, and Google Scholar using the keywords "Type D," "spinor," "Newman–Penrose," and "metric reconstruction." Many peer-reviewed articles and surveys were checked, including both classic and recent treatments. No publication was found containing an expression equivalent or closely analogous to Eq.7.

2.1. Notation and Conventions

This paper uses the mostly positive metric signature $(-,+,+,+)$ and natural units where $G = c = 1$. Coordinates are Boyer-Lindquist [11] (t, r, θ, ϕ), where t is time, r is radial, θ is polar, and ϕ is azimuthal.

The spacetime metric components $g_{\mu\nu}$ and their derivatives are assumed sufficiently smooth (at least C^2) to allow the use of differential geometric tools and the Newman-Penrose formalism. Standard causality and global hyperbolicity conditions are also assumed to ensure well-defined null congruences.

For Kerr spacetime, only five independent, non-zero components are needed: g_{tt}, $g_{t\phi}$, g_{rr}, $g_{\theta\theta}$, and $g_{\phi\phi}$, reflecting its stationary and axisymmetric structure. Partial derivatives are denoted by a comma, e.g., $g_{\mu\nu,r} = \partial_r g_{\mu\nu}$.

Spinor notation may use o and ι as shorthand for o^A and ι^A, omitting explicit spinor indices.

3. Theoretical Setup

3.1. Spin Coefficient Equations

While Stewart [3] provides the primary reference for the formalism, the derivation and conclusions in this work are primarily original.

Spin coefficients, components of the Ricci rotation coefficients, play a central role in analysing null congruences. In Type D spacetimes, l^μ and n^μ can each be chosen to each be separate geodesic, shear-free null congruence, eliminating four of the spin coefficients. In Kerr, an additional spin coefficient,

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ϵ can be set to zero [12], though this is not required here. These conditions simplify the NP field equations and provide insight into the structure of the null tetrad components. They underlie the derivation of the *Kinnersley tetrad* and thus the Kerr metric itself. However, in this work, we focus on the single condition $\lambda = 0$.

3.2. Choice of Tetrad

The choice of principal-null-aligned tetrad used here is a convenient derivation tool. The final equation, however, can be evaluated without explicitly constructing the tetrad, once the Kerr-like coordinate gauge is fixed. See Appendix A for the full conditions of a null tetrad.

Following the Kinnersley tetrad [13], we set $l^\theta = n^\theta = 0$, and $m^r = 0$, so that

$$l^\mu = (l^t, l^r, 0, l^\phi), \quad n^\mu = (n^t, n^r, 0, n^\phi), \quad m^\mu = (m^t, 0, m^\theta, m^\phi). \tag{1}$$

Kerr spacetime is stationary and axisymmetric, implying invariance under the Killing vector fields ∂_t and ∂_ϕ. Thus, the Kerr metric satisfies $\mathcal{L}_{\partial_t} g_{\mu\nu} = 0$ and $\mathcal{L}_{\partial_\phi} g_{\mu\nu} = 0$ followed by the Killing vector field condition [3].

The metric tensor components written in terms of spinors and null tetrads can then be substituted into the Killing condition:

$$g^{\mu\nu} = \epsilon^{AB}\epsilon^{A'B'} = -2l^{(\mu}n^{\nu)} + 2m^{(\mu}\bar{m}^{\nu)} \implies \mathcal{L}_{\partial_t} g^{\mu\nu} = \mathcal{L}_{\partial_t}(-2l^{(\mu}n^{\nu)} + 2^{(\mu}\bar{m}^{\nu)}) = 0. \tag{2}$$

The same equation will hold for the axial Killing vector field ∂_ϕ. Therefore, a convenient choice is to let all the components of the null tetrad vectors be independent of time, t, and polar angle, ϕ.

Thus, our null tetrad is now

$$l^\mu = l^\mu(r, \theta), \qquad n^\mu = n^\mu(r, \theta), \qquad m^\mu = m^\mu(r, \theta). \tag{3}$$

To calculate the final equation, only spin coefficient $\lambda = 0$ needs to be considered.

4. Derivation Using $\lambda = 0$

For a complete step-by-step derivation, see Appendix B.

σ describes the shear of l^μ and the λ coefficient is analogous to this and describes the shear of n^μ. If n^μ is along a geodesic, shear-free null congruence, which is the desired setup, then $\lambda = 0$. Since $\lambda = 0$, the conjugate $\bar{\lambda} = 0$ holds. Starting with the equation:

$$\bar{\lambda} = m^\mu m^\nu \nabla_\nu n_\mu = 0. \tag{4}$$

We manipulate the expression to eliminate explicit dependence on tetrad components within the chosen principal-null-aligned tetrad and coordinate gauge. This allows determining metric components directly without making an additional ansatz.

It can be calculated that:

$$g_{t\phi}\Gamma^\phi_{tr} + g_{tt}\Gamma^t_{tr} = \tfrac{1}{2} g_{tt,r} \qquad g_{\phi\phi}\Gamma^\phi_{\phi r} + g_{t\phi}\Gamma^t_{\phi r} = \tfrac{1}{2} g_{\phi\phi,r}\,, \tag{5}$$

and

$$g_{t\phi}\Gamma^\phi_{\phi r} + g_{\phi\phi}\Gamma^\phi_{tr} + g_{t\phi}\Gamma^t_{tr} + g_{tt}\Gamma^t_{\phi r} = g_{t\phi,r}\,. \tag{6}$$

Note that $\Gamma^a_{bc} = \Gamma^a_{cb}$ for the Levi-Civita connection due to its torsion-free property.

The next step is to express the final equation purely in *metric components* and their derivatives. Note that $(g_{t\phi})^2 - g_{tt}g_{\phi\phi} = \triangle \sin^2\theta$ and $\triangle = \frac{g_{\theta\theta}}{g_{rr}}$.

Incorporating these into the equation, produces the final equation.

The explicit form of the final equation is given by:

$$g_{\theta\theta,r} - \frac{(g_{t\phi})^2 g_{\theta\theta}}{g_{rr}((g_{t\phi})^2 - g_{tt}g_{\phi\phi})(1+g_{tt})^2} g_{tt,r} + \frac{2g_{t\phi}g_{\theta\theta}}{g_{rr}((g_{t\phi})^2 - g_{tt}g_{\phi\phi})(1+g_{tt})} g_{t\phi,r} - \frac{g_{\theta\theta}}{g_{rr}((g_{t\phi})^2 - g_{tt}g_{\phi\phi})} g_{\phi\phi,r} = 0 \quad (7)$$

where $g_{\mu\nu}$ denotes the general metric components and $g_{\mu\nu,r} \equiv \frac{\partial g_{\mu\nu}}{\partial r}$ denotes the radial derivatives.

This is the main result: a purely metric-level identity corresponding to the Newman–Penrose condition $\lambda = 0$ within the adopted principal-null-aligned tetrad and Boyer–Lindquist–type coordinate gauge. Through the use of the Kinnersley tetrad and metric identities, this condition is expressed entirely in terms of metric components and their radial derivatives, providing a tetrad-free (within the chosen coordinate gauge) test of the shear-free, geodesic null congruence condition.

5. Extension to Other Spacetimes

This equation provides a constraint relating the metric components, which in specific cases (e.g., Schwarzschild) can be used to determine them, which is illustrated below.

5.1. Deriving Schwarzschild Metric Components

The Schwarzschild metric [14] describes the spacetime for a spherical black hole (uncharged and non-rotating). For this spacetime, the middle two terms involving cross terms become zero. This results in a simplified equation:

$$g_{\theta\theta,r} + \frac{g_{\theta\theta}g_{\phi\phi,r}}{g_{rr}g_{tt}g_{\phi\phi}} = 0. \quad (8)$$

Now considering the spherical symmetry of the spacetime, it can be assumed that $r^2 d\Omega^2$ to define the angular components: $g_{\phi\phi} = r^2 \sin^2\theta$ and $g_{\theta\theta} = r^2$. Therefore substituting into the equation:

$$2r + \frac{2r^3 \sin^2\theta}{g_{rr}g_{tt}r^2 \sin^2\theta} = 2r + \frac{2r}{g_{rr}g_{tt}} = 0. \quad (9)$$

This generates the condition:

$$g_{tt} = -\frac{1}{g_{rr}} \quad (10)$$

on the Schwarzschild metric, which is the same result produced as the *ansatz* method. See Appendix C for the full Schwarzschild metric line element [14]. Note that the same approach can also be used for the Static form of the de Sitter Metric.

5.2. Results for Other Spacetimes

The code was written to compute the numerical values of the equation for four well-known metric spacetimes. They showed results that were consistent (not just the Kerr spacetime from which it was derived) with the analytically derived expression being equal to zero. The numerical results for Kerr, Schwarzschild, static de Sitter, and Kerr-Newman spacetimes all evaluate to zero or values on the order of 10^{-15}, confirming the equation holds numerically. These small deviations are within machine precision and are considered effectively zero.

The Friedmann–Lemaître–Robertson–Walker (FLRW) metric describes a homogeneous and isotropic universe undergoing expansion or contraction.

The de Sitter spacetime metric is a specific solution of the FLRW metric. But for all cases of this metric, the equation itself vanishes. The static form of the de Sitter spacetime uses the de Sitter metric as well, but this is no longer an FLRW solution. For this version of the metric, the equation does not vanish but remains consistent.

For the Minkowski and FLRW metric, because $1 + g_{tt} = 0$ the equation results implies that the cross term $g_{t\phi}$ vanishes in these specific Type O (conformal) spacetimes ($g_{t\phi} = 0$). This equation was derived using the mostly positive convention - the mostly negative version will have a similar

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relation resulting in the same version. Overall, the equation serves as a diagnostic condition that is satisfied in Kerr-like, Type D spacetimes, which are characterised by the existence of two geodesic, shear-free null congruences and can be expressed in Boyer-Lindquist coordinates. As such, it provides a practical, coordinate-based criterion for identifying or verifying such spacetimes, without requiring prior knowledge of their full geometric classification.

See Appendix C for the metric, line element equations for static de Sitter [15], Schwarzschild [14], Kerr [1], Kerr-Newman [16], FLRW([17], [18],[19],[20]), and Minkowski [21].

Table 1. Numerical evaluation of the metric-level $\lambda = 0$ condition for various spacetimes.

Spacetime	Result ($\sim$)
Kerr	10^{-15}
Kerr–Newman	10^{-15}
Schwarzschild	0
Static de Sitter	10^{-15}

Note: All "Result" values are either numerical zeros or effectively zero within floating-point precision ($\mathcal{O}(10^{-15})$).

6. Discussion

This example proves that, for spacetimes under sufficient symmetry, their spin coefficient conditions can reduce to purely metric conditions. The reformulation provides a purely metric-component condition for Kerr-like spacetimes. Whilst its validity has been confirmed for Kerr, Kerr–Newman, Schwarzschild, and static de Sitter spacetimes, further work is needed to assess its applicability to other Type D geometries. Crucially, its formulation in terms of metric components alone may facilitate the analysis or construction of exact certain Type D solutions without the need for an *ansatz*, potentially simplifying the exploration of new solutions of Einstein's field equations. The fact that the equation continues to hold for Kerr–Newman, despite $R_{\mu\nu} \neq 0$, shows that its validity is not restricted to vacuum solutions. There may exist analogous generalisations that include additional cross terms beyond $g_{t\phi}$. More broadly, the existence of such a coordinate-level relation raises the possibility that analogous metric-component equations could exist for other spacetimes satisfying the GS theorem, though this remains to be explored. The main utility of this result lies in expressing the covariant geometric condition in a simplified, coordinate-dependent form, illuminating the geometric structure spacetimes with sufficient symmetry. The derivation uses a principal-null-aligned tetrad for convenience, but the resulting metric equation depends only on the metric components and their derivatives. Its validity is limited by the Kerr-like coordinate gauge and not the choice of tetrad itself.

7. Conclusions

To conclude, this work presents an expression that has a reducible spin coefficient condition and opens the possibility for further reductions in different spacetimes. The reformulation in terms of metric components alone provides a practical tool for analysing and, in some cases, reconstructing spacetime metrics without assuming an *ansatz*, based on the examples considered. This could simplify the process of exploring or discovering new solutions to Einstein's field equations. Open questions include whether similarly simplified relations exist in other coordinate systems, or if the observed structure is specific to Boyer–Lindquist coordinates. It also remains to be explored whether analogous simplified equations can be derived for other Newman–Penrose spin coefficient conditions and for spacetimes of different Petrov types.

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This research was conducted independently.

Conflict of Interest

The author declares no conflicts of interest.

Data Availability

No datasets other than the numerical results presented in the manuscript, which were generated by the author, were used in this study.

Ethics Statement

This work does not involve human participants, animal subjects, or sensitive data requiring ethical approval.

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Appendix A: Definitions and Notation

Ansatz An assumed form or trial solution used to facilitate solving differential equations, which this work notably avoids when deriving metric components.

Connection Coefficients Also known as Christoffel symbols Γ^a_{bc}, these coefficients define the covariant derivative in a coordinate basis and describe how basis vectors change from point to point.

Covariant Derivative The derivative operator ∇_a compatible with the metric, which acts on tensor fields. For a contravariant vector with an induced coordinate basis, the covariant derivative is:

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \tag{11}$$

Gauge freedom Mathematically, it means that certain transformations, called gauge transformations, can be performed on the fields without changing observable quantities. This freedom allows one to choose a convenient gauge (or gauge condition) to simplify calculations or to reveal different physical aspects of the system.

Geodesic A curve in spacetime that parallel-transports its own tangent vector, representing the path of a free-falling particle or light ray. Mathematically, it satisfies the geodesic equation $\nabla_X X = 0$, where X is the tangent vector field along the curve/path.

Hypersurface-Orthogonal A vector field is hypersurface-orthogonal if it is orthogonal to a family of hypersurfaces, which for null vectors implies a twist-free null congruence. Being hypersurface-orthogonal means there must exist a scalar field w s.t. $l_a = v\nabla_a w$. This requires the vector field l^a to satisfy the Frobenius condition $l_{[a}\nabla_b l_{c]} = 0$. See Stewart [3], Sec. 2, p. 98.

Killing Vector A vector field ξ^a generating an isometry of the spacetime metric, satisfying Killing's equation $\nabla_{(a}\xi_{b)} = 0$ or equivalently, in coordinate form, $\mathcal{L}_\xi g_{\mu\nu} = 0$. Such vectors represent symmetries like time translation or rotation invariance.

Kinnersley Tetrad A specific null tetrad choice adapted to the Kerr spacetime, aligned with its principal null directions, which simplifies Newman–Penrose calculations.

Levi-Civita Connection The unique torsion-free connection compatible with the metric g_{ab}, used to define covariant derivatives and parallel transport in Riemannian and Lorentzian geometry.

Lie Derivative An operator $\mathcal{L}_X$ measuring the change of a tensor field along the flow generated by a vector field X.

Metric Components The elements $g_{\mu\nu}$ of the metric tensor g that encode the geometric and causal properties of spacetime.

Newman–Penrose (NP) Formalism A tetrad-based method that decomposes Einstein's equations into scalar relations by expressing spacetime geometry relative to a null tetrad. This formalism simplifies handling algebraically special spacetimes.

Null Tetrad A basis consisting of four null vectors $(l^a, n^a, m^a, \bar{m}^a)$, with l^a and n^a real and m^a, $\bar{m}^a$ complex conjugates, adapted to the causal and algebraic structure of spacetime.

Petrov Classification A scheme that categorises the Weyl tensor at a point in spacetime into algebraic types (I, II, III, D, N, O) based on the multiplicity and alignment of its principal null directions. This classification reveals the spacetime's underlying symmetries.

Principal Null Directions (PNDs) Special null directions along which the Weyl tensor simplifies; these directions characterise the algebraic structure of the curvature and are central in the Petrov classification.

Geodesic, Shear-Free Null Congruence A family of null geodesics whose cross-sectional shape remains undistorted during propagation, characterised by zero shear. These congruences are essential in the geometric analysis of spacetimes.

Twist The rotation of a congruence of curves around its own tangent vector; twist-free congruences have no local rotation.

Weyl Tensor The traceless part of the Riemann curvature tensor that represents the free gravitational field and encodes tidal forces and gravitational waves.

Appendix B: Full Derivation of the $\lambda = 0$ Condition

$\lambda = 0$

See the derivations for spin coefficients in Newman and Penrose [2], although these expressions are not always fully expanded in covariant form. σ describes the shear of l^μ and the λ coefficient is analogous to this and describes the shear of n^μ. If n^μ is along a geodesic, shear-free null congruence, which is the desired setup, then $\lambda = 0$. Since $\lambda = 0$, the conjugate $\bar{\lambda} = 0$ holds. The conjugate will be used simply because it allows us to use m^μ instead of the conjugate $\bar{m}^\mu$. Starting with the equation:

$$\bar{\lambda} = m^\mu m^\nu \nabla_\nu n_\mu = 0. \tag{12}$$

We focus on this contraction because it isolates the spin coefficient λ in terms of the metric. The Kinnersley tetrad is a widely known choice of tetrad for the Kerr spacetime because it satisfies the null tetrad conditions: all skew-symmetric products are zero except for $l^\mu n_\mu = 1$ and $m^\mu \bar{m}_\mu = 1$.

Only certain components of the Kinnersley tetrad (Eqs. 1, 3) will be selected without invoking the full tetrad.

See Loutrel [13], Section V, for the un-rotated version of the Kinnersley tetrad.

$$l^\mu = \left(\frac{r^2+a^2}{\Delta}, 1, 0, \frac{a}{\Delta}\right), \quad n^\mu = \frac{1}{2\Sigma}\left(r^2+a^2, -\Delta, 0, a\right), \quad m^\mu = \frac{1}{\sqrt{2}\Gamma}\left(ia\sin\theta, 0, 1, i\csc\theta\right), \tag{13}$$

Where:

- $\Delta = r^2 - 2Mr + a^2, \quad \Sigma = r^2 + a^2\cos^2\theta, \quad \Gamma = r + ia\cos\theta.$

Here:

- M is the mass of the black hole,
- a is the specific angular momentum (angular momentum per unit mass),
- r is the radial Boyer–Lindquist coordinate,
- θ is the polar angle from the symmetry axis.

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Notice that $l^\theta = n^\theta = m^r = 0$, as explained above.

Since $m^r = 0$, the μ contraction becomes:

$$\bar{\lambda} = m^\mu m^\nu \nabla_\nu n_\mu = m^t m^\nu \nabla_\nu n_t + m^\theta m^\nu \nabla_\nu n_\theta + m^\phi m^\nu \nabla_\nu n_\phi = 0. \tag{14}$$

Then, expanding the covariant component to attain a *covariant derivative* acting on a contravariant component. Recalling that $\nabla_a g_{\mu\nu} = 0$ for the *Levi-Civita connection* [3], this gives $\nabla_\mu n_\phi = g_{\phi\phi} \nabla n^\phi + g_{t\phi} \nabla n^t$.

$$\bar{\lambda} = g_{\theta\theta} m^\theta m^\nu \nabla_\nu n^\theta + (m^t g_{t\phi} + m^\phi g_{\phi\phi}) m^\nu \nabla_\nu n^\phi + (m^\phi g_{t\phi} + m^t g_{tt}) m^\nu \nabla_\nu n^t = 0. \tag{15}$$

Then expand the covariant derivative:

$$\begin{aligned}
\bar{\lambda} = \quad & g_{\theta\theta} m^\theta (m^t \Gamma^\theta_{tm} n^m + m^\phi \Gamma^\theta_{\phi m} n^m + m^\theta \Gamma^\theta_{\theta m} n^m) \\
+ \quad & (m^t g_{t\phi} + m^\phi g_{\phi\phi})(m^t \Gamma^\phi_{tm} n^m + m^\theta n^\phi_{,\theta} + m^\theta \Gamma^\phi_{\theta m} n^m + m^\phi \Gamma^\phi_{\phi m} n^m) \\
+ \quad & (m^t g_{tt} + m^\phi g_{t\phi})(m^t \Gamma^t_{tm} n^m + m^\theta n^t_{,\theta} + m^\theta \Gamma^t_{\theta m} n^m + m^\phi \Gamma^t_{\phi m} n^m) = 0.
\end{aligned} \tag{16}$$

Then, consider the m^μ components. The denominators with $\frac{1}{\sqrt{2}\Gamma}$ cancel out as a common factor for all the terms.

This allows us to focus on the m^μ components within the brackets $(ia \sin\theta, 0, 1, i \csc\theta)$ and focus on the real part of the equation. The separation into real and imaginary parts corresponds to the complex nature of λ; either part vanishing is necessary for algebraic speciality. Since $m^t m^t$, $m^t m^\phi$, $m^\phi m^\phi$ and $m^\theta m^\theta$ are all real-valued, the real part of the equation is

$$\begin{aligned}
g_{\theta\theta} m^\theta m^\theta \Gamma^\theta_{\theta m} n^m + m^t m^t (\Gamma^\phi_{tm} n^m + g_{tt} \Gamma^t_{tm} n^m) \quad & + \quad m^\phi m^\phi (g_{\phi\phi} \Gamma^\phi_{\phi m} n^m + g_{t\phi} \Gamma^t_{\phi m} n^m) \\
+ \quad m^t m^\phi (g_{t\phi} \Gamma^\phi_{\phi m} n^m + g_{\phi\phi} \Gamma^\phi_{tm} n^m + g_{t\phi} \Gamma^t_{tm} n^m + g_{tt} \Gamma^t_{\phi m} n^m) = 0. &
\end{aligned} \tag{17}$$

Final Equation Form

Here we focus on manipulating the equation to ensure it is independent of tetrad components or functions, which both guarantees tetrad invariance and facilitates determining metric components without making additional ansatz.

The following non-zero *connection coefficients* that are of interest:

$$\begin{aligned}
\Gamma^t_{tr} &= \tfrac{1}{2} g^{tt} g_{tt,r} + \tfrac{1}{2} g^{t\phi} g_{t\phi,r} & \qquad \Gamma^t_{\phi r} &= \tfrac{1}{2} g^{tt} g_{t\phi,r} + \tfrac{1}{2} g^{t\phi} g_{\phi\phi,r} \\
\Gamma^\phi_{tr} &= \tfrac{1}{2} g^{\phi\phi} g_{t\phi,r} + \tfrac{1}{2} g^{t\phi} g_{tt,r} & \qquad \Gamma^\phi_{\phi r} &= \tfrac{1}{2} g^{\phi\phi} g_{\phi\phi,r} + \tfrac{1}{2} g^{t\phi} g_{t\phi,r}.
\end{aligned} \tag{18}$$

Also, recalling that $g_{t\phi} g^{t\phi} + g_{tt} g^{tt} = 1$ and $g_{t\phi} g^{tt} + g_{\phi\phi} g^{t\phi} = 0$, It can be calculated that:

$$g_{t\phi} \Gamma^\phi_{tr} + g_{tt} \Gamma^t_{tr} = \tfrac{1}{2} g_{tt,r} \qquad g_{\phi\phi} \Gamma^\phi_{\phi r} + g_{t\phi} \Gamma^t_{\phi r} = \tfrac{1}{2} g_{\phi\phi,r}\,, \tag{19}$$

and

$$g_{t\phi} \Gamma^\phi_{\phi r} + g_{\phi\phi} \Gamma^\phi_{tr} + g_{t\phi} \Gamma^t_{tr} + g_{tt} \Gamma^t_{\phi r} = g_{t\phi,r}\,. \tag{20}$$

Note that $\Gamma^a_{bc} = \Gamma^a_{cb}$ for the Levi-Civita connection due to its torsion-free property.

The only non-zero contribution from contraction over index "m" is "r", as the connection coefficient is either zero or $n^\theta = 0$.

$$\begin{aligned}
n^r [g_{\theta\theta} m^\theta m^\theta \Gamma^\theta_{\theta r} \quad + \quad m^t m^t (\Gamma^\phi_{tr} + g_{tt} \Gamma^t_{tr}) \quad & + \quad m^\phi m^\phi (g_{\phi\phi} \Gamma^\phi_{\phi r} + g_{t\phi} \Gamma^t_{\phi r}) \\
+ \quad m^t m^\phi (g_{t\phi} \Gamma^\phi_{\phi r} + g_{\phi\phi} \Gamma^\phi_{tr} + g_{t\phi} \Gamma^t_{tr} + g_{tt} \Gamma^t_{\phi r})] = 0. &
\end{aligned} \tag{21}$$

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So n^r is a common factor.

Note that $\Gamma^{\theta}_{\theta r} = \frac{1}{2} g^{\theta\theta} g_{\theta\theta,r}$.

Recalling Eq. 19, and Eq. 20, these can be substituted into the equation and notice the factor of $\frac{1}{2}$ will cancel out.

$$g_{\theta\theta} m^{\theta} m^{\theta} g^{\theta\theta} g_{\theta\theta,r} + m^t m^t (g_{tt,r}) + m^{\phi} m^{\phi} (g_{\phi\phi,r}) + 2m^t m^{\phi} (g_{t\phi,r}) = 0. \tag{22}$$

Now consider the Kerr metric and the actual components of the m^{μ} vector in the Kinnersley tetrad.

First, recall the denominator is a common factor.

$$\begin{aligned} &m^{\theta} m^{\theta} \to 1, \quad m^t m^t \to -a^2 \sin^2\theta, \quad m^t m^{\phi} \to -a, \quad m^{\phi} m^{\phi} \to -\tfrac{1}{\sin^2\theta}, \\ &g_{\theta\theta} g^{\theta\theta} g_{\theta\theta,r} + -a^2 \sin^2\theta (g_{tt,r}) - \tfrac{1}{\sin^2\theta}(g_{\phi\phi,r}) - 2a(g_{t\phi,r}) = 0. \end{aligned} \tag{23}$$

The next step is to express the final equation purely in *metric components* and their derivatives. Note that $(g_{t\phi})^2 - g_{tt} g_{\phi\phi} = \triangle \sin^2\theta$ and $\triangle = \frac{g_{\theta\theta}}{g_{rr}}$.

Therefore,

$$\sin^2\theta = \tfrac{g_{rr}((g_{t\phi})^2 - g_{tt} g_{\phi\phi})}{g_{\theta\theta}}, \qquad a\sin^2\theta = -\tfrac{g_{t\phi}}{1+g_{tt}}, \qquad a = -\tfrac{g_{t\phi} g_{\theta\theta}}{g_{rr}((g_{t\phi})^2 - g_{tt} g_{\phi\phi})(1+g_{tt})}. \tag{24}$$

Incorporating these into the equation, produces the final equation.

The explicit form of the final equation is given by:

$$g_{\theta\theta,r} - \frac{(g_{t\phi})^2 g_{\theta\theta}}{g_{rr}((g_{t\phi})^2 - g_{tt} g_{\phi\phi})(1+g_{tt})^2} g_{tt,r} + \frac{2 g_{t\phi} g_{\theta\theta}}{g_{rr}((g_{t\phi})^2 - g_{tt} g_{\phi\phi})(1+g_{tt})} g_{t\phi,r} - \frac{g_{\theta\theta}}{g_{rr}((g_{t\phi})^2 - g_{tt} g_{\phi\phi})} g_{\phi\phi,r} = 0 \tag{25}$$

where $g_{\mu\nu}$ denotes the general metric components and $g_{\mu\nu,r} \equiv \frac{\partial g_{\mu\nu}}{\partial r}$ denotes the radial derivatives.

Appendix C: Known Mathematical Formulations

Einstein Field Equations

The Einstein Field Equations (EFE) describe how matter and energy influence the curvature of spacetime. In geometrised units ($G = c = 1$), they are written as:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \quad , \tag{26}$$

where:

- $R_{\mu\nu}$ is the Ricci curvature tensor,
- R is the Ricci scalar,
- $g_{\mu\nu}$ is the metric tensor,
- Λ is the cosmological constant,
- $T_{\mu\nu}$ is the energy-momentum tensor.

In vacuum (i.e., $T_{\mu\nu} = 0$) and with $\Lambda = 0$, the field equations reduce to $R_{\mu\nu} = 0$.

Metric Spacetime Line Elements

Friedmann–Lemaître–Robertson–Walker (FLRW) Metric

The FLRW metric describes a homogeneous, isotropic expanding (or contracting) universe:

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \right]. \tag{27}$$

where:

- $a(t)$ is the scale factor,

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- $k = 0, \pm 1$ indicates flat, closed, or open spatial geometry.

Kerr Metric in Boyer–Lindquist Coordinates

The Kerr metric describes the spacetime around an uncharged, rotating massive object (e.g., a rotating black hole). In Boyer–Lindquist coordinates (t, r, θ, ϕ), the line element is given by:

$$\begin{aligned} ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - \frac{4Mar\sin^2\theta}{\Sigma}\,dt\,d\phi + \frac{\Sigma}{\Delta}\,dr^2 + \Sigma\,d\theta^2 \\ + \left(r^2 + a^2 + \frac{2Ma^2 r\sin^2\theta}{\Sigma}\right)\sin^2\theta\,d\phi^2. \end{aligned} \tag{28}$$

where the functions Σ and Δ are defined as:

$$\Sigma = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 - 2Mr + a^2. \tag{29}$$

Here:

- M is the mass of the black hole,
- $a = \frac{J}{M}$ is the spin parameter (specific angular momentum),
- J is the angular momentum of the black hole.

Kerr–Newman Metric in Boyer–Lindquist Coordinates

The Kerr–Newman metric describes the geometry of spacetime around a rotating, charged mass (e.g., a charged black hole). In Boyer–Lindquist coordinates (t, r, θ, ϕ), the line element is given by:

$$\begin{aligned} ds^2 = -\left(1 - \frac{2Mr - Q^2}{\Sigma}\right)dt^2 - \frac{4Mar\sin^2\theta}{\Sigma}\,dt\,d\phi + \frac{\Sigma}{\Delta}\,dr^2 + \Sigma\,d\theta^2 \\ + \left(r^2 + a^2 + \frac{(2Mr - Q^2)a^2\sin^2\theta}{\Sigma}\right)\sin^2\theta\,d\phi^2. \end{aligned} \tag{30}$$

where:

$$\Sigma = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 - 2Mr + a^2 + Q^2 \tag{31}$$

and:

- M is the mass of the black hole,
- $a = \dfrac{J}{M}$ is the spin parameter (specific angular momentum),
- J is the angular momentum,
- Q is the electric charge of the black hole.

Minkowski Metric

The Minkowski metric describes flat spacetime in special relativity:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{32}$$

or, in spherical coordinates:

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\,d\phi^2. \tag{33}$$

Schwarzschild Metric

The Schwarzschild metric describes the spacetime outside a static, spherically symmetric, uncharged mass:

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$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2, \tag{34}$$

where M is the mass of the object.

Static de Sitter Metric

The static form of the de Sitter metric, describing a universe with a positive cosmological constant Λ, is:

$$ds^2 = -\left(1 - \frac{\Lambda r^2}{3}\right)dt^2 + \left(1 - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2. \tag{35}$$

Spinors

Conditions on a Null Tetrad

A null tetrad $\{l^a, n^a, m^a, \bar{m}^a\}$ satisfies the following conditions:

$$l^a l_a = n^a n_a = m^a m_a = \bar{m}^a \bar{m}_a = 0, \quad \text{(each vector is null)} \tag{36}$$

$$l^a n_a = -1, \tag{37}$$

$$m^a \bar{m}_a = 1, \tag{38}$$

$$l^a m_a = l^a \bar{m}_a = n^a m_a = n^a \bar{m}_a = 0, \quad \text{(orthogonality conditions)} \tag{39}$$

Here, the metric signature is $(-+++)$.

Goldberg-Sachs Theorem

Theorem 1. *For the Goldberg-Sachs theorem, see Stewart [3], Sec. 2, p. 104. "In a vacuum spacetime, any two of the following conditions imply the third:*
A) The Weyl spinor Ψ_{ABCD} is algebraically special with n-fold repeated principal spinor o (n=2,3,4),
B) either spacetime is flat or o generates a geodesic, shear-free congruence,
C) $\nabla^{AA'}\Psi_{ABCD}$ contracted with (5-n) o's vanishes."

Weyl Scalars in Terms of the Weyl Spinor

Let Ψ_{ABCD} be the totally symmetric Weyl spinor and $\{o^A, \iota^A\}$ a normalised spinor dyad (also known as a spin basis). The five complex Weyl scalars are defined as:

$$\Psi_0 = \Psi_{ABCD}\, o^A o^B o^C o^D, \tag{40}$$

$$\Psi_1 = \Psi_{ABCD}\, o^A o^B o^C \iota^D, \tag{41}$$

$$\Psi_2 = \Psi_{ABCD}\, o^A o^B \iota^C \iota^D, \tag{42}$$

$$\Psi_3 = \Psi_{ABCD}\, o^A \iota^B \iota^C \iota^D, \tag{43}$$

$$\Psi_4 = \Psi_{ABCD}\, \iota^A \iota^B \iota^C \iota^D. \tag{44}$$

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Spin Coefficients

Below are the five spin coefficients of interest for the Kerr spacetime and non-rotated Kinnersley tetrad. ([3], [22]).

$$\kappa = -m^\mu D l_\mu = -m^\mu l^\nu \nabla_\nu l_\mu, \tag{45}$$

$$\sigma = -m^\mu \delta l_\mu = -m^\mu m^\nu \nabla_\nu l_\mu, \tag{46}$$

$$\epsilon = \tfrac{1}{2}\left(n^\mu D l_\mu - \bar{m}^\mu D m_\mu\right) = \tfrac{1}{2}\left(n^\mu l^\nu \nabla_\nu l_\mu - \bar{m}^\mu l^\nu \nabla_\nu m_\mu\right), \tag{47}$$

$$\lambda = -\bar{m}^\mu \Delta \bar{m}_\mu = -\bar{m}^\mu n^\nu \nabla_\nu \bar{m}_\mu, \tag{48}$$

$$\nu = -\bar{m}^\mu \Delta n_\mu = -\bar{m}^\mu n^\nu \nabla_\nu n_\mu. \tag{49}$$

Spin Conditions

The spin basis consists of two independent spinors o^A and ι^A, which form a normalised spin frame.

The conditions for a spin basis o^A, ι^A require:

$$[o, o] = o_A o^A = 0, \quad [\iota, \iota] = \iota_A \iota^A = 0, \quad [o, \iota] = o_A \iota^A = 1. \tag{50}$$

Upper and lower indice spinors are related through the antisymmetric spinor metric ϵ_{AB}, which plays the role of the Levi-Civita symbol in spinor space:

$$o^A = \epsilon^{AB} o_B, \qquad o_A = \epsilon_{AB} o^B, \tag{51}$$

where $\epsilon_{AB}\epsilon^{BC} = \delta_A^{\ C}$.

This index raising/lowering is essential for forming scalar contractions such as $o_A \iota^A$, and for maintaining proper spinor transformation laws under Lorentz transformations.

Appendix D: Computational Details

In this work, a symbolic computation was used to define and analyse metric components of various spacetime geometries, such as the static de Sitter metric, using the `SymPy` Python library.

For example, the static de Sitter metric components were defined symbolically as functions of the radial coordinate r, polar angle θ, and the cosmological constant Λ:

$$g_{tt} = -\left(1 - \frac{\Lambda r^2}{3}\right), \qquad g_{t\phi} = 0, \qquad g_{\phi\phi} = r^2 \sin^2\theta, \qquad g_{rr} = \frac{1}{1 - \frac{\Lambda r^2}{3}}, \qquad g_{\theta\theta} = r^2. \tag{52}$$

Using symbolic differentiation, the derivatives of these metric components were computed with respect to r to fully describe the proposed equation.

The symbolic expressions were then evaluated numerically by substituting representative values for the parameters and coordinates. For instance, the parameters were set as $r = 16$, $\theta = \pi/8$, $M = 9$, and $a = 10$ (where applicable). Five sets of parameters were randomly selected for each spacetime metric tested.

Below is a schematic outline of the computational workflow:

- Definition of metric components as symbolic expressions.
- Computation of partial derivatives with respect to coordinates.
- Substitution of numerical parameter values.
- Numerical evaluation and extraction of float values for analysis.

This approach provides a systematic framework to derive and test metric-related quantities for a variety of spacetime models in General Relativity.

For symbolic differentiation and substitution, the author followed standard usage of SymPy (see refs. [23], [24]). —

If desired, the full computational code is available upon request.

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