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Article

Resolving the Open Problem by Proving a Conjecture on the Inverse Mostar Index for *c*-Cyclic Graph

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Abstract: Inverse topological index problems involve determining whether a graph exists with a given integer as its topological index. One such index, the **Mostar index**, $Mo(G)$, is defined as: $Mo(G) = \sum_{uv\in E(G)} \left|n_u(e|G) - n_v(e|G)\right|$, where $n_u(e|G)$ and $n_v(e|G)$ represent the number of vertices closer to vertex u than v, and closer to v than u, respectively, for an edge $e = uv$. The inverse Mostar index problem has gained significant attention recently. In their work, Alizadeh et al. [Solving the Mostar index inverse problem, J. Math. Chem. 62 (5) (2024) 1079–1093], proposed the following open problem: "Which nonnegative integers can be realized as Mostar indices of c-cyclic graphs, for a given positive integer c?" Subsequently, one of the present authors [On the inverse Mostar index problem for molecular graphs, Trans. Comb. 14 (1) (2024) 65–77] conjectured that, except for finitely many positive integers, all other positive integers can be realized as the Mostar index of a c-cyclic graph, where $c \geq 3$. In this paper, we fully resolve this open problem by proving the aforementioned conjecture.

Keywords: Mostar index; inverse problem; $c-$cyclic graph

MSC: 92E10, 05C92,05C09, 05C35

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1. Introduction

Let $G = (V, E)$ be a simple, connected graph with vertex set $V(G)$ $(|V(G)| = n)$ and edge set $E(G)$ $(|E(G)| = m)$. For u, $v \in V(G)$, the length of the shortest path between the vertices u and v is their distance, $d_G(u, v)$. Let $e = uv$ be an edge of the graph G (which may contain cycles or be acyclic), connecting the vertices u and v. Define two sets $N_u(e|G)$ and $N_v(e|G)$ as

$$N_u(e|G) = \{w \in V(G) \,|\, d_G(w, u) < d_G(w, v)\},$$
$$N_v(e|G) = \{w \in V(G) \,|\, d_G(w, v) < d_G(w, u)\}.$$

The number of elements of $N_u(e|G)$ and $N_v(e|G)$ are denoted by $n_u(e|G)$ and $n_v(e|G)$, respectively. Thus, $n_u(e|G)$ counts the vertices of G lying closer to the vertex u than to vertex v. The meaning of $n_v(e|G)$ is analogous. Vertices equidistant from both ends of the edge uv belong neither to $N_u(e|G)$ nor

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to $N_v(e|G)$. Note that for any edge $e = uv$ of G, $n_u(e|G) \geq 1$ and $n_v(e|G) \geq 1$, because $u \in N_u(e|G)$ and $v \in N_v(e|G)$.

Topological indices are numerical quantities associated with graphs that remain invariant under graph isomorphisms. The Mostar index, introduced in [8], is a recently developed bond-additive topological index. The Mostar index $Mo(G)$ is defined as:

$$Mo(G) = \sum_{e=uv \in E(G)} \Big| n_u(e|G) - n_v(e|G) \Big|. \tag{1.1}$$

We use the notation $\phi(e|G) = \Big| n_u(e|G) - n_v(e|G) \Big|$ to represent the contribution of the edge $e = uv$ to the Mostar index. Thus we have

$$Mo(G) = \sum_{e=uv \in E(G)} \phi(e|G). \tag{1.2}$$

In recent years, numerous studies have explored the Mostar index and its associated variants. For further literature on the subject, see [6,8,13].

The inverse topological index problem involves the realization of integers as topological indices within various classes of graphs. The first version of this problem was introduced by Gutman in [10], and since then, several variants have been investigated for different topological indices [1,2,7,9,11,15]. Recently, the inverse Mostar index problem for various classes of graphs has been studied extensively [1,2,5]. In [5], Alizadeh *et al.* proposed the following problem on inverse Mostar index for c-cyclic graphs.

Problem 1. *Which nonnegative integers can be realized as Mostar indices of c-cyclic graphs, for a given positive integer c?*

The problem was thoroughly resolved for c-cyclic graphs when $c \in \{0, 1, 2, 3\}$ in [1–3]. Building on these findings and after an in-depth study of Problem 1 as discussed in [2], one of the present authors proposed the following conjecture concerning the inverse problem of the Mostar index for c-cyclic graphs.

Conjecture 1. *For any fixed $c \geq 3$, there exists a c-cyclic graph with Mostar index $Mo(G) = p$, where $p \in \mathbb{N} \setminus A$, A is a finite set.*

In this paper, we fully resolve Problem 1 by proving Conjecture 1.

2. Main Results

In this section, we provide a complete resolution of Problem 1 by proving Conjecture 1. We use the following constructions to prove the result. In each case, we present multiple constructions to address both Conjecture 1 and Open Problem 1. We begin by examining the case of c-cyclic graphs when c is even.

Construction I

Consider a cycle $C = u_1u_2 \ldots u_{2c+2}u_1$ of length $2c + 2$. Starting from $i = 1$, for each path $u_iu_{i+1}u_{i+2}$ of length 2 in C, take a vertex $v_{\frac{i+1}{2}}$ and form a $4-$cycle by attaching the edges $u_iv_{\frac{i+1}{2}}$ and $u_{i+2}v_{\frac{i+1}{2}}$ for $i = 1, 3, 5, \ldots c - 1$. Next, starting from $i = c + 4$, take the path $u_iu_{i+1}u_{i+2}$ and a new vertex $v_{\frac{i-2}{2}}$,

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forming a $4-$cycle by attaching the edges $u_i v_{\frac{i-2}{2}}$ and $u_{i+2} v_{\frac{i-2}{2}}$ for $i = c+4, c+6 \ldots 2c$. Let the resultant graph be denoted by $G_{E,E}$ (see, Figure 1). In $G_{E,E}$, let C_i denote the 4-cycle containing the vertex v_i. Now, let $G^1_{E,E}$ (see, Figure 1) be the graph obtained by inserting a new edge between the cycles $C_{\frac{c}{2}}$ & $C_{\frac{c}{2}+1}$, and C_{c-1} & C_1 in C (or by subdividing the edges $u_{c+1}u_{c+2}$ and $u_{2c+2}u_1$). Relabel the vertices of the cycle C as $u_1 u_2 \ldots u_{2c+3} u_{2c+4}$. Proceeding in this manner, let $G^k_{E,E}$ denote the graph obtained from $G^{k-1}_{E,E}$ by inserting a new edge between the cycles $C_{\frac{c}{2}}$ & $C_{\frac{c}{2}+1}$, and C_{c-1} & C_1 in C (or by subdividing the edges $u_{c+1}u_{c+2}$ and $u_{2c+2}u_1$ k-times in $G_{E,E}$). Clearly, $G_{E,E}$ is a c-cyclic graph with $3c+1$ vertices and $4c$ edges, while $G^k_{E,E}$ is a c-cyclic graph with $3c+2k+1$ vertices and $4c+2k$ edges. In all the graphs considered, let $e_{i,j}$ denote the edge connecting the vertices u_i and u_j, where $i \le j$. Similarly, let $e'_{i,j}$ denote the edge connecting the vertices u_i and v_j, where $i \ge j$.

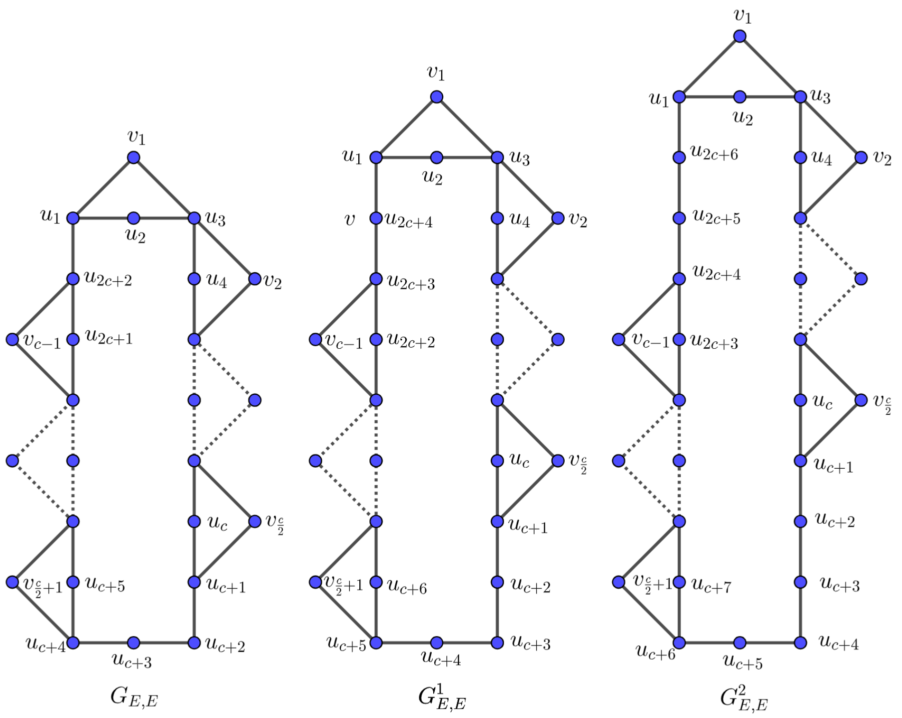

Figure 1. The graphs mentioned in Construction I.

Proposition 2.1. *For even* $c \ge 4$, (a) $Mo(G_{E,E}) = 8c-8$, (b) $Mo(G^1_{E,E}) = Mo(G_{E,E}) + 2$ *and* $Mo(G^k_{E,E}) = Mo(G^{k-1}_{E,E}) + 2$, $k = 2, 3, \ldots$.

Proof. (a) In $G_{E,E}$, consider any 4-cycle C_i $(i = 2, 3, \ldots, \frac{c}{2})$. For the edges $e'_{2i-1,i} = u_{2i-1}v_i$ and $e_{2i-1,2i} = u_{2i-1}u_{2i}$, we obtain

$$n_{u_{2i-1}}(e'_{2i-1,i}|G_{E,E}) = \frac{3c+4}{2} \quad \text{and} \quad n_{v_i}(e'_{2i-1,i}|G_{E,E}) = \frac{3c-2}{2},$$

$$n_{u_{2i-1}}(e_{2i-1,2i}|G_{E,E}) = \frac{3c+4}{2} \quad \text{and} \quad n_{u_{2i}}(e_{2i-1,2i}|G_{E,E}) = \frac{3c-2}{2}.$$

Therefore, there are exactly two edges with contribution:

$$\phi(e_{2i-1,2i}|G_{E,E}) = \phi(e'_{2i-1,i}|G_{E,E}) = \frac{3c+4}{2} - \frac{3c-2}{2} = 3.$$

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For the other two edges $e'_{2i+1,i} = u_{2i+1}v_i$ and $e_{2i,2i+1} = u_{2i}u_{2i+1}$, the following holds:

$$n_{u_{2i+1}}(e'_{2i+1,i}|G_{E,E}) = \frac{3c+2}{2} \text{ and } n_{v_i}(e'_{2i+1,i}|G_{E,E}) = \frac{3c}{2},$$

$$n_{u_{2i+1}}(e_{2i,2i+1}|G_{E,E}) = \frac{3c+2}{2} \text{ and } n_{u_{2i}}(e_{2i,2i+1}|G_{E,E}) = \frac{3c}{2}.$$

Therefore, these two edges have contribution

$$\phi(e'_{2i+1,i}|G_{E,E}) = \phi(e_{2i,2i+1}|G_{E,E}) = \frac{3c+2}{2} - \frac{3c}{2} = 1.$$

Next we consider any 4-cycle C_i $(i = \frac{c}{2}+1,\ldots,c-1)$. For the edges $e'_{2i+2,i} = u_{2i+2}v_i$ and $e_{2i+2,2i+3} = u_{2i+2}u_{2i+3}$, we obtain

$$n_{u_{2i+2}}(e'_{2i+2,i}|G_{E,E}) = \frac{3c+2}{2} \text{ and } n_{v_i}(e'_{2i+2,i}|G_{E,E}) = \frac{3c}{2},$$

$$n_{u_{2i+2}}(e_{2i+2,2i+3}|G_{E,E}) = \frac{3c+2}{2} \text{ and } n_{u_{2i+3}}(e_{2i+2,2i+3}|G_{E,E}) = \frac{3c}{2}.$$

Therefore, there are exactly two edges with contribution:

$$\phi(e_{2i+2,2i+3}|G_{E,E}) = \phi(e'_{2i+2,i}|G_{E,E}) = \frac{3c+2}{2} - \frac{3c}{2} = 1.$$

For the other two edges $e'_{2i+4,i} = u_{2i+4}v_i$ and $e_{2i+3,2i+4} = u_{2i+3}u_{2i+4}$, the following holds:

$$n_{u_{2i+4}}(e'_{2i+4,i}|G_{E,E}) = \frac{3c+4}{2}, \text{ and } n_{v_i}(e'_{2i+4,i}|G_{E,E}) = \frac{3c-2}{2},$$

$$n_{u_{2i+4}}(e_{2i+3,2i+4}|G_{E,E}) = \frac{3c+4}{2}, \text{ and } n_{u_{2i+3}}(e_{2i+3,2i+4}|G_{E,E}) = \frac{3c-2}{2}.$$

Therefore, these two edges have contribution

$$\phi(e'_{2i+4,i}|G_{E,E}) = \phi(e_{2i+3,2i+4}|G_{E,E}) = \frac{3c+4}{2} - \frac{3c-2}{2} = 3.$$

For all the remaining edges $e = uv$ of $G_{E,E}$,

$$n_u(e|G_{E,E}) = \frac{3c+2}{2},\ n_v(e|G_{E,E}) = \frac{3c}{2}.$$

Therefore, these edges have a contribution:

$$\phi(e|G_{E,E}) = \Big|n_u(e|G_{E,E}) - n_v(e|G_{E,E})\Big| = \frac{3c+2}{2} - \frac{3c}{2} = 1.$$

Using the above results, from (1.2), we obtain:

$$Mo(G_{E,E}) = 8(c-2) + 8 = 8c - 8.$$

(b) In $G^1_{E,E}$, the contribution of a new edge $e = uv$ is calculated as follows:

$$\phi(e|G^1_{E,E}) = \Big|n_u(e|G^1_{E,E}) - n_v(e|G^1_{E,E})\Big| = \frac{3c+4}{2} - \frac{3c+2}{2} = 1.$$

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For each of the other edges $e = uv \in E(G^1_{E,E})$, the contribution is:

$$\phi(e|G^1_{E,E}) = \Big|n_u(e|G_{E,E}) + 1 - n_v(e|G_{E,E}) - 1\Big| = \Big|n_u(e|G_{E,E}) - n_v(e|G_{E,E})\Big|.$$

this means that all the other edges have the same contribution as in $G_{E,E}$. Therefore:

$$Mo(G^1_{E,E}) = Mo(G_{E,E}) + 2.$$

Similarly, in $G^k_{E,E}$ and $G^{k-1}_{E,E}$, all the edges except the new edges have the same contribution. The two new edges in $G^k_{E,E}$ each have a contribution $\phi(e|G^k_{E,E}) = 1$. Thus, it follows that:

$$Mo(G^1_{E,E}) = Mo(G_{E,E}) + 2.$$

On a similar manner, we can establish

$$Mo(G^k_{E,E}) = Mo(G^{k-1}_{E,E}) + 2 \text{ for } k = 2, 3, \ldots$$

■

We can also provide several classes of constructions to obtain the same value of the Mostar index as in Proposition 2.1. A general version of the construction is as follows.

General Construction I

Consider the base graph $G_{E,E}$ as in Construction I. For $j = 3, 5, \ldots, c-1$, let u_j and u_{c+j+1} be vertices positioned between the 4-cycles $C_{\frac{j+1}{2}}, C_{\frac{j-1}{2}}$ and $C_{\frac{c+j-1}{2}}, C_{\frac{c+j-3}{2}}$, respectively. Define $G^{j,1}_{E,E}$ as the graph obtained from $G_{E,E}$ by deleting the edges $u_j v_{\frac{j+1}{2}}$ and $u_{c+j+1} v_{\frac{c+j-1}{2}}$, subdividing the edges $u_j u_{j+1}$ and $u_{c+j+1} u_{c+j+2}$, relabeling the vertices of the cycle C as $u_1, u_2, \ldots, u_{2c+4}$, and adding an edge between $u_{j+1} v_{\frac{j+1}{2}}$ and $u_{c+j+3} v_{\frac{c+j-1}{2}}$. This effectively inserts an edge between the cycles $C_{\frac{j+1}{2}}, C_{\frac{j-1}{2}}$ and $C_{\frac{c+j-1}{2}}, C_{\frac{c+j-3}{2}}$ in C. Proceeding iteratively, define $G^{j,k}_{E,E}$ (see, Figure 2) as the graph obtained from $G^{j,k-1}_{E,E}$ by subdividing the edges $u_{j+k-1} u_{j+k}$ and $u_{c+j+2k-1} u_{c+j+2k}$, thereby inserting k edges between the cycles $C_{\frac{j+1}{2}}, C_{\frac{j-1}{2}}$ and $C_{\frac{c+j-1}{2}}, C_{\frac{c+j-3}{2}}$. At each step, relabel the vertices of C in $G^{j,k}_{E,E}$. Clearly, $G^{j,k}_{E,E}$ is a c-cyclic graph with $3c + 2k + 1$ vertices and $4c + 2k$ edges for each $j = 3, 5, \ldots, c-1$.

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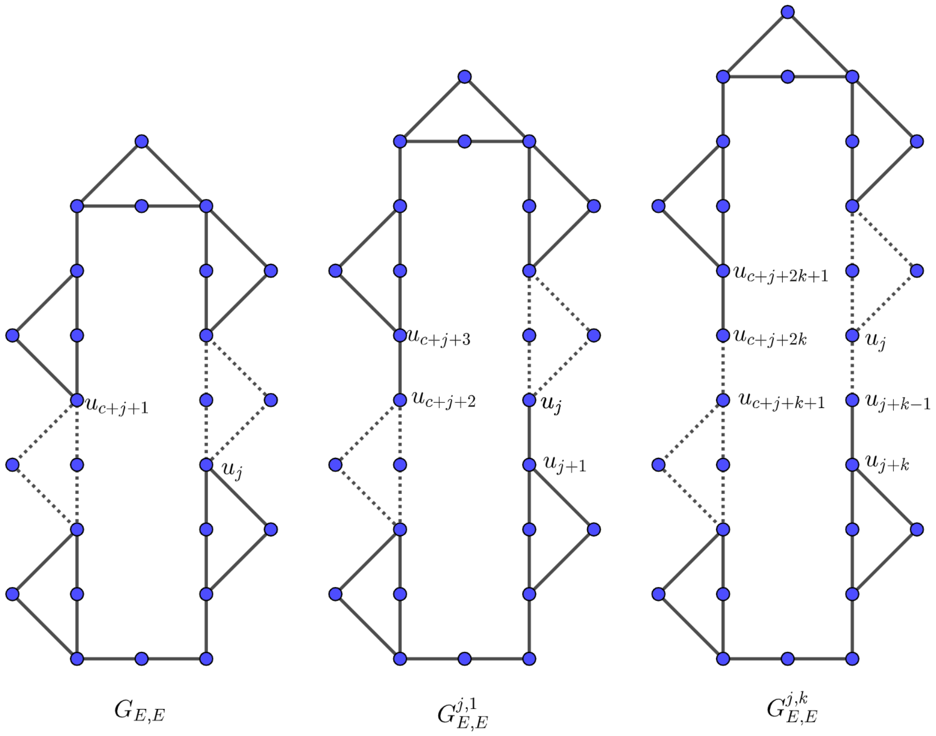

Figure 2. The graphs mentioned in general Construction I.

Proposition 2.2. *For each fixed even integer* $c \geq 4$, $Mo(G^{j,k}_{E,E}) = Mo(G^{j,k-1}_{E,E}) + 2$, *where* $j = 3, 5, \ldots, c-1$ *and* $k = 2, 3 \ldots$.

Proof. From direct computation in $G^{j,k}_{E,E}$, the contribution of the new edges is 1 each, and the contribution of every other edge remains the same as in $G^{j,k-1}_{E,E}$. Therefore,

$$Mo(G^{j,k}_{E,E}) = Mo(G^{j,k-1}_{E,E}) + 2 \text{ and } Mo(G^{j,1}_{E,E}) = Mo(G_{E,E}) + 2 \text{ for } j = 3, 5, \ldots, c-1 \text{ and}$$
$$k = 2, 3, \ldots.$$

■

Construction II

Consider a cycle $C = u_1u_2 \ldots u_{2c+1}u_1$ of length $2c+1$. Select the vertex u_2 and its diametrically opposite edge $u_{c+2}u_{c+3}$ in C. From the path $u_1u_2u_3$ in C, attach the end vertices u_1 and u_3 to a new vertex v_1, forming a 4-cycle C_1. Additionally, attach a pendant edge v_1w at v_1. Starting from $i = 4$, for each path $u_iu_{i+1}u_{i+2}$ of length 2 with $i = 4, 6, \ldots, c$, take a new vertex $v_{\frac{i}{2}}$ and attach u_i and u_{i+2} to $v_{\frac{i}{2}}$ to form a $4-$cycle. Similarly, starting from $i = c+3$, for each path $u_iu_{i+1}u_{i+2}$ of length 2 with $i = c+3, c+5, \ldots, 2c-1$, take a vertex $v_{\frac{i-1}{2}}$ and attach u_i and u_{i+2} to $v_{\frac{i-1}{2}}$ to form a $4-$cycle. The resulting graph is denoted by $G_{E,O}$, which is a c-cyclic graph with order $3c+1$ and size $4c$ (see, Figure 3). Let the 4-cycle in $G_{E,O}$ containing the vertex v_i be denoted as C_i, for $i = 1, 2, \ldots, c-1$. Define $G^1_{E,O}$ (see, Figure 3) as the graph obtained from $G_{E,O}$ by subdividing the edge u_1u_{2c+1} and attaching a new edge between the cycles $C_{\frac{c}{2}}$ and $C_{\frac{c}{2}-1}$, relabeling the vertices of C as $u_1u_2 \ldots u_{2c+3}$. Next, let $G^2_{E,O}$ (see, Figure 3) be the graph obtained from $G^1_{E,O}$ by inserting a new edge between the cycles C_{c-1} & C_1, and another edge between $C_{\frac{c}{2}}$ & $C_{\frac{c}{2}-1}$. Proceeding iteratively, define $G^k_{E,O}$ as the graph obtained from $G^{k-1}_{E,O}$ by inserting edges between the cycles C_{c-1} & C_1, and between $C_{\frac{c}{2}}$ & $C_{\frac{c}{2}-1}$. Clearly, $G^k_{E,O}$ is a c-cyclic graph with $3c+1+2k$ vertices and $4c+2k$ edges for each $k = 1, 2, \ldots$. In all these graphs, let $e_{i,j}$ denote the edge connecting the vertex u_i and u_j, where $i \leq j$ and $e'_{i,j}$ denote the edge connecting the vertex u_i and v_j, where $i \geq j$.

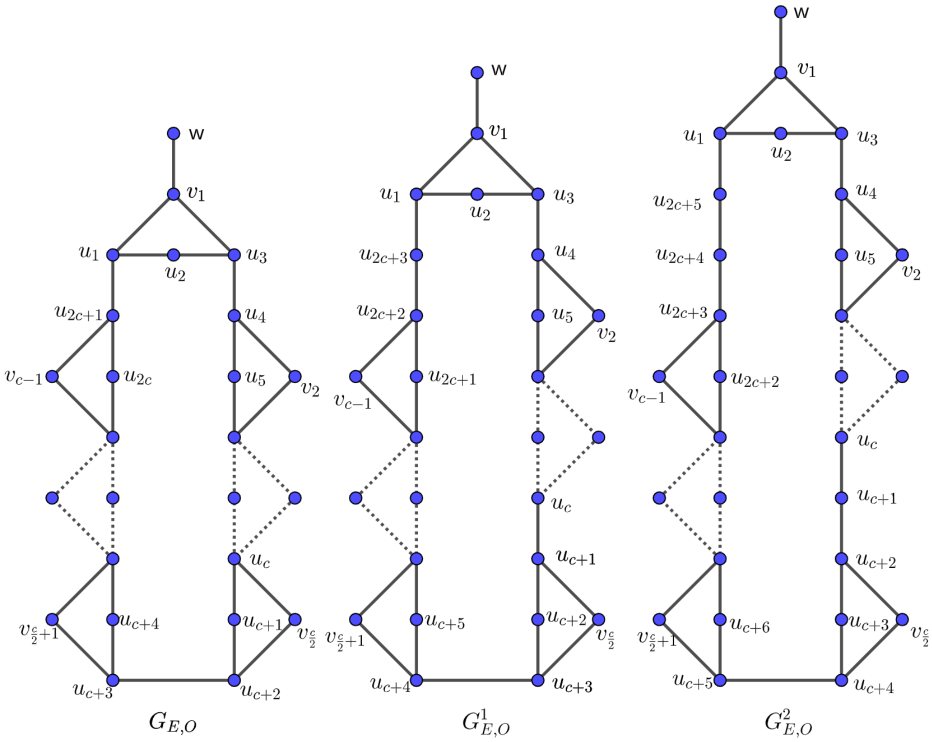

Figure 3. The graphs mentioned in Construction II.

Proposition 2.3. *For even* $c \geq 4$, (a) $Mo(G_{E,O}) = 9c - 11$, (b) $Mo(G^1_{E,O}) = Mo(G_{E,O}) + 2$, *and* $Mo(G^k_{E,O}) = Mo(G^{k-1}_{E,O}) + 2$, $j = 2, 3, \ldots$.

Proof. (a) Since $G_{E,O}$ has $3c+1$ vertices, the pendant edge $e = v_1 w$ has the contribution $\phi(e|G_{E,O}) = 3c - 1$. Now, in the cycle C_1, for the edges $e'_{1,1} = u_1 v_1$ and $e'_{3,1} = u_3 v_1$, we have

$$n_{u_1}(e'_{1,1}|G_{E,O}) = n_{v_1}(e'_{1,1}|G_{E,O}) = \frac{3c}{2}, \text{ and } n_{u_3}(e'_{3,1}|G_{E,O}) = n_{v_1}(e'_{3,1}|G_{E,O}) = \frac{3c}{2}.$$

There are exactly two edges with contributions $\phi(e'_{1,1}|G_{E,O}) = \phi(e'_{3,1}|G_{E,O}) = 0$, and for the other two edges $e_{1,2} = u_1 u_2$, and $e_{2,3} = u_2 u_3$, we have

$$n_{u_1}(e_{1,2}|G_{E,O}) = \frac{3c+2}{2}, \quad n_{u_2}(e_{1,2}|G_{E,O}) = \frac{3c-2}{2},$$
$$\text{and } n_{u_3}(e_{2,3}|G_{E,O}) = \frac{3c+2}{2}, \quad n_{u_2}(e_{2,3}|G_{E,O}) = \frac{3c-2}{2}.$$

Therefore,

$$\phi(e_{1,2}|G_{E,O}) = \phi(e_{2,3}|G_{E,O}) = \frac{3c+2}{2} - \frac{3c-2}{2} = 2.$$

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Similarly, in $C_{\frac{c}{2}}$, there are two edges $e'_{c+2,\frac{c}{2}} = u_{c+2}v_{\frac{c}{2}}$ and $e_{c+1,c+2} = u_{c+1}u_{c+2}$, with

$$n_{u_{c+2}}(e'_{c+2,\frac{c}{2}}|G_{E,O}) = \frac{3c}{2}, \;\; n_{v_{\frac{c}{2}}}(e'_{c+2,\frac{c}{2}}|G_{E,O}) = \frac{3c}{2},$$
$$\text{and } n_{u_{c+2}}(e_{c+1,c+2}|G_{E,O}) = \frac{3c}{2}, \;\; n_{u_{c+1}}(e_{c+1,c+2}|G_{E,O}) = \frac{3c}{2}.$$

Therefore the contribution is

$$\phi(e'_{c+2,\frac{c}{2}}|G_{E,O}) = \phi(e_{c+1,c+2}|G_{E,O}) = \frac{3c}{2} - \frac{3c}{2} = 0,$$

and for the other two edges, $e'_{c,\frac{c}{2}} = u_c v_{\frac{c}{2}}$, and $e_{c,c+1} = u_c u_{c+1}$, we have

$$n_{u_c}(e'_{c,\frac{c}{2}}|G_{E,O}) = \frac{3c+2}{2}, \;\; n_{v_{\frac{c}{2}}}(e'_{c,\frac{c}{2}}|G_{E,O}) = \frac{3c-2}{2},$$

$$n_{u_c}(e_{c,c+1}|G_{E,O}) = \frac{3c+2}{2}, \;\; n_{u_{c+1}}(e_{c,c+1}|G_{E,O}) = \frac{3c-2}{2}.$$

Thus, the contribution is

$$\phi(e'_{c,\frac{c}{2}}|G_{E,O}) = \phi(e_{c,c+1}|G_{E,O}) = \Big|n_u(e_{c,c+1}|G_{E,O}) - n_v(e_{c,c+1}|G_{E,O})\Big| = \frac{3c+2}{2} - \frac{3c-2}{2} = 2.$$

Similarly, in $C_{\frac{c}{2}+1}$, there are two edges, each incident at u_{c+5}, with contribution $\phi(e|G_{E,O}) = 2$ and the other two edges incident at u_{c+3} have contribution $\phi(e|G_{E,O}) = 0$. For the remaining cycles $C_i, i = 2, \ldots \frac{c}{2} - 1$, for the edges $e'_{2i,i} = u_{2i}v_i$ and $e_{2i,2i+1} = u_{2i}u_{2i+1}$, we have

$$n_{u_{2i}}(e'_{2i,i}|G_{E,O}) = \frac{3c+2}{2}, \;\; n_{v_i}(e'_{2i,i}|G_{E,O}) = \frac{3c-2}{2},$$

$$n_{u_{2i}}(e_{2i,2i+1}|G_{E,O}) = \frac{3c+2}{2}, \;\; n_{u_{2i+1}}(e_{2i,2i+1}|G_{E,O}) = \frac{3c-2}{2}.$$

Therefore,

$$\phi(e'_{2i,i}|G_{E,O}) = \phi(e_{2i,2i+1}|G_{E,O}) = \frac{3c+2}{2} - \frac{3c-2}{2} = 2.$$

Now, for the edge, $e'_{2i+2,i} = u_{2i+2}v_i$, we have

$$n_{u_{2i+2}}(e'_{2i+2,i}|G_{E,O}) = \frac{3c}{2}, \;\; n_{v_i}(e'_{2i+2,i}|G_{E,O}) = \frac{3c-2}{2},$$

and for the edge $e_{2i+1,2i+2} = u_{2i+1}u_{2i+2}$, we have

$$n_{u_{2i+2}}(e_{2i+1,2i+2}|G_{E,O}) = \frac{3c}{2}, \;\; n_{u_{2i+1}}(e_{2i+1,2i+2}|G_{E,O}) = \frac{3c-2}{2}.$$

Therefore,

$$\phi(e'_{2i+2,i}|G_{E,O}) = \phi(e_{2i+1,2i+2}|G_{E,O}) = \frac{3c}{2} - \frac{3c-2}{2} = 1.$$

Similarly, on the cycle C_i, where $i = \frac{c}{2} + 2, \ldots, c-1$, for the edge $e'_{2i+1,i} = u_{2i+1}v_i$, we have

$$n_{u_{2i+1}}(e'_{2i+1,i}|G_{E,O}) = \frac{3c}{2}, \;\; \text{and} \;\; n_{v_i}(e'_{2i+1,i}|G_{E,O}) = \frac{3c-2}{2},$$

and for the edge $e_{2i+1,2i+2} = u_{2i+1}u_{2i+2}$, we get

$$n_{u_{2i+1}}(e_{2i+1,2i+2}|G_{E,O}) = \frac{3c}{2}, \text{ and } n_{u_{2i+2}}(e_{2i+1,2i+2}|G_{E,O}) = \frac{3c-2}{2}.$$

Therefore, $\phi(e'_{2i+1,i}|G_{E,O}) = \phi(e_{2i+1,2i+2}|G_{E,O}) = 1$. For the other two edges, $e'_{2i+3,i} = u_{2i+3}v_i$, we have

$$n_{u_{2i+3}}(e'_{2i+3,i}|G_{E,O}) = \frac{3c+2}{2}, \text{ and } n_{v_i}(e'_{2i+3,i}|G_{E,O}) = \frac{3c-2}{2},$$

and for the edge $e_{2i+2,2i+3} = u_{2i+2}u_{2i+3}$, we get

$$n_{u_{2i+3}}(e_{2i+2,2i+3}|G_{E,O}) = \frac{3c+2}{2}, \text{ and } n_{u_{2i+2}}(e_{2i+2,2i+3}|G_{E,O}) = \frac{3c-2}{2}.$$

Therefore, $\phi(e'_{2i+3,i}|G_{E,O}) = \phi(e_{2i+2,2i+3}|G_{E,O}) = \dfrac{3c+2}{2} - \dfrac{3c-2}{2} = 2$. Now, for the edges, $e_{1,2c+1} = u_1u_{2c+1}$ and $e_{3,4} = u_3u_4$, the contribution is:

$$\phi(u_{1,2c+1}|G_{E,O}) = \phi(e_{3,4}|G_{E,O}) = \frac{3c}{2} - \frac{3c-2}{2} = 1,$$

and the remaining edge has the contribution $\phi(e_{c+2,c+3}|G_{E,O}) = \dfrac{3c-2}{2} - \dfrac{3c-2}{2} = 0$. Using the above results, from (1.2), we obtain:

$$Mo(G_{E,O}) = 6(c-4) + 12 + 2 + 3c - 1 = 9c - 11.$$

(b) In $G^k_{E,O}$ and $G^{k-1}_{E,O}$, the pendant edges contribute $\phi(e|G^k_{E,O}) = 3c + 2k - 1$ and $\phi(e|G^{k-1}_{E,O}) = 3c + 2k - 3$, respectively. Two new edges in $G^k_{E,O}$ each contribute $\phi(e|G^k_{E,O}) = 0$ to the Mostar index. For the remaining edges $e = uv$ of $G^k_{E,O}$, we have

$$\phi(e|G^k_{E,O}) = |n_u(e|G^{k-1}_{E,O}) + 1 - n_v(e|G^{k-1}_{E,O}) - 1| = |n_u(e|G^{k-1}_{E,O}) - n_v(e|G^{k-1}_{E,O})| = \phi(e|G^{k-1}_{E,O}).$$

Therefore, the remaining edges of $G^k_{E,O}$ and $G^{k-1}_{E,O}$ contribute the same amount. Thus, we conclude:

$$Mo(G^k_{E,O}) = Mo(G^{k-1}_{E,O}) + 2.$$

■

We can also provide several classes of constructions that yield the same value of the Mostar index as in Proposition 2.3. A general version of the construction is as follows.

General Construction II

Consider the base graph $G_{E,O}$ as described in Construction II. For $j = 6, 8, \ldots, c-2$, let u_j and u_{c+j+1} be vertices such that u_j lies between the 4-cycles $C_{\frac{j}{2}}$ and $C_{\frac{j}{2}-1}$, and u_{c+j+1} lies between the cycles $C_{\frac{c+j}{2}}$ and $C_{\frac{c+j}{2}-1}$. Define $G^{j,1}_{E,O}$ as the graph obtained by deleting the edges $u_jv_{\frac{j}{2}}$ and $u_{c+j+1}v_{\frac{c+j}{2}}$, subdividing the edges u_ju_{j+1} and $u_{c+j+1}u_{c+j+2}$, relabeling the vertices of C as $u_1, u_2, \ldots, u_{2c+3}$, and adding edges $u_{j+1}v_{\frac{j}{2}}$ and $u_{c+j+3}v_{\frac{c+j}{2}}$. This process effectively connects the cycles $C_{\frac{j}{2}}$ and $C_{\frac{j}{2}-1}$, as well as $C_{\frac{c+j}{2}}$ and $C_{\frac{c+j}{2}-1}$, through the newly added edges. Proceeding iteratively, let $G^{j,k}_{E,O}$ denote the graph obtained by subdividing the edges $u_{j+k-2}u_{j+k-1}$ and $u_{c+j+2k-2}u_{c+j+2k-1}$ of $G^{j,k-1}_{E,O}$ exactly once,

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or equivalently, inserting $k-1$ edges between the cycles $C_{\frac{j}{2}}$ and $C_{\frac{j}{2}-1}$, and between $C_{\frac{c+j}{2}}$ and $C_{\frac{c+j}{2}-1}$ in C of $G^{j,1}_{E,O}$, like in Figure 2. After each step, relabel the vertices of the cycle C in $G^{j,k}_{E,O}$. Clearly, $G^{j,k}_{E,O}$ is a c-cyclic graph with $3c+2k+1$ vertices and $4c+2k$ edges, where $k=1,2,\ldots,$ and $j=6,8,\ldots,c-2$.

Proposition 2.4. *For every fixed even integer $c \geq 4$, $Mo(G^{j,k}_{E,O}) = Mo(G^{j,k-1}_{E,O}) + 2$, where $j=6,8,\ldots,c-2$ and $k=1,2,\ldots$.*

Proof. In $G^{j,k}_{E,O}$ and $G^{j,k-1}_{E,O}$, the pendant edges contribute $\phi(e|G^{j,k}_{E,O}) = 3c+2k-1$ and $\phi(e|G^{j,k-1}_{E,O}) = 3c+2k-3$, respectively. Two new edges in $G^{j,k}_{E,O}$ each contribute $\phi(e|G^{j,k}_{E,O}) = 0$ to the Mostar index. The remaining edges of $G^{j,k}_{E,O}$ and $G^{j,k-1}_{E,O}$ contribute the same amount. Therefore, $Mo(G^{j,k}_{E,O}) = Mo(G^{j,k-1}_{E,O}) + 2$. ■

Next, we consider the case when c is odd.

Construction III

Consider a cycle $C = u_1u_2\ldots u_{2c-1}u_1$ of length $2c-1$. Add a vertex v_1 and connect it to u_1 and u_2 to form a 3-cycle $u_1u_2v_1u_1$. For each path $u_iu_{i+1}u_{i+2}$ in C where $i=3,5,\ldots,2c-3$, introduce a vertex $v_{\frac{i+1}{2}}$ and connect it to u_i and u_{i+2}, forming a 4-cycle $u_iu_{i+1}u_{i+2}v_{\frac{i+1}{2}}u_i$. Let the resultant graph be denoted as $G_{O,E}$ (see, Figure 4). Define C_1 as the 3-cycle $u_1u_2v_1u_1$, and for $i=2,3,\ldots,c-1$, let C_i denote the 4-cycle containing the vertex v_i. Now, construct $G^1_{O,E}$ (see, Figure 4) from $G_{O,E}$ by deleting the edge $u_cv_{\frac{c-1}{2}}$, subdividing the edges u_1u_{2c-1} and $u_{c-1}u_c$, relabeling the vertices of C as $u_1,u_2,\ldots,u_{2c+1}$, and adding the edge $v_{\frac{c-1}{2}}u_c$. Alternatively, this can be described as inserting a new edge between the cycles C_1 and C_{c-1}, as well as between $C_{\frac{c-1}{2}}$ and $C_{\frac{c+1}{2}}$ in C. Proceeding recursively, $G^k_{O,E}$ is obtained from $G^{k-1}_{O,E}$ by inserting new edges between C_1 and C_{c-1}, and between $C_{\frac{c-1}{2}}$ and $C_{\frac{c+1}{2}}$ in C, or equivalently by subdividing the edges $u_1u_{2c-1+2(k-1)}$ and u_cu_{c+1} exactly once, or by subdividing the edges u_1u_{2c+1} and u_cu_{c+1} in $G^1_{O,E}$ exactly $k-1$ times. Clearly, $G_{O,E}$ is a c-cyclic graph of order $3c-2$ and size $4c-3$, while $G^k_{O,E}$ is a c-cyclic graph of order $3c-2+2k$ and size $4c-3+2k$ for $k=1,2,\ldots$. In all these graphs, let $e_{i,j}$ denote the edge connecting vertices u_i and u_j (where $i \leq j$), and let $e'_{i,j}$ denote the edge connecting u_i and v_j (where $i \geq j$).

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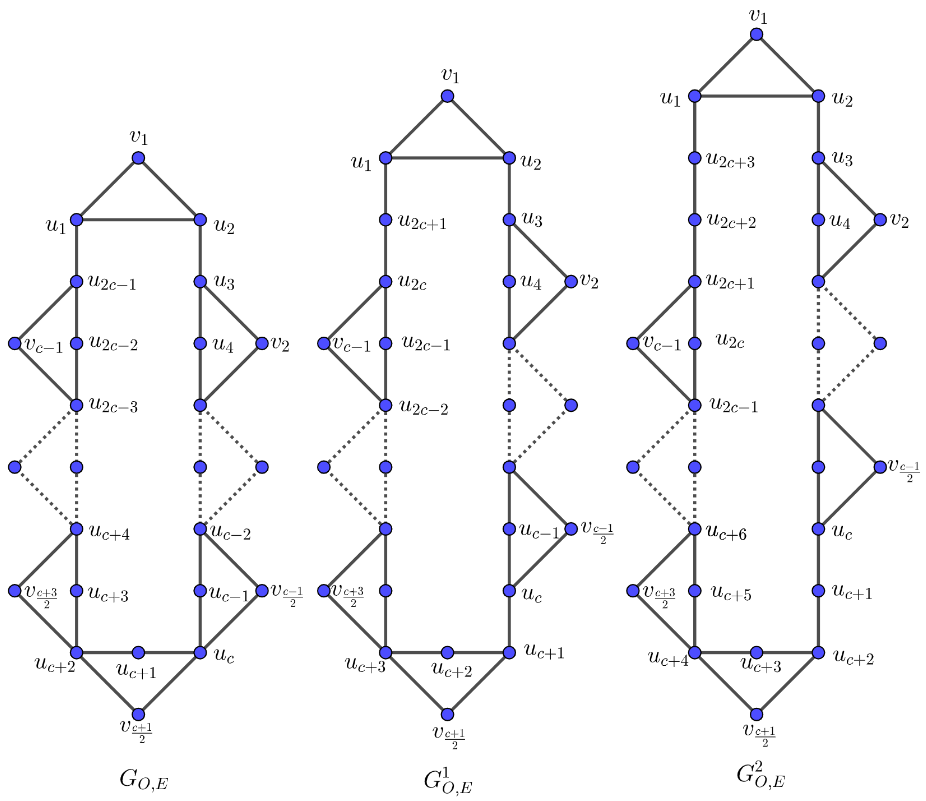

Figure 4. The graphs mentioned in Construction III.

Proposition 2.5. *For odd $c \geq 3$, (a) $Mo(G_{O,E}) = 9c - 13$, (b) $Mo(G^1_{O,E}) = Mo(G_{O,E}) + 2$ and $Mo(G^k_{O,E}) = Mo(G^{k-1}_{O,E}) + 2$, $j = 2, 3, \ldots$.*

Proof. (a) In $G_{O,E}$, two edges $e'_{1,1} = u_1v_1$ and $e'_{2,1} = u_2v_1$ contribute as follows:

$$\phi(e'_{1,1}|G_{O,E}) = \phi(e'_{2,1}|G_{O,E}) = \frac{n-1}{2} = \frac{3c-3}{2}.$$

These contribute to the Mostar index, and the edge $e_{1,2} = u_1u_2$ contributes:

$$\phi(e_{1,2}|G_{O,E}) = |n_{u_1}(e_{1,2}|G_{O,E}) - n_{u_2}(e_{1,2}|G_{O,E})| = \frac{3c-5}{2} - \frac{3c-5}{2} = 0.$$

In each of the cycles C_i for $i = 2, 3, \ldots c-1$, where $i \neq \frac{c+1}{2}$, two edges $e'_{2i-1,i} = u_{2i-1}v_i$ and $e_{2i-1,2i} = u_{2i-1}u_{2i}$ contribute as follows:

$$n_{u_{2i-1}}(e'_{2i-1,i}|G_{O,E}) = \frac{3c-3}{2} \quad n_{v_i}(e'_{2i-1,i}|G_{O,E}) = \frac{3c-5}{2},$$

$$n_{u_{2i-1}}(e_{2i-1,2i}|G_{O,E}) = \frac{3c-3}{2} \quad n_{u_{2i}}(e_{2i-1,2i}|G_{O,E}) = \frac{3c-5}{2}.$$

Therefore, the contribution of these two edges are:

$$\phi(e'_{2i-1,i}|G_{O,E}) = \phi(e_{2i-1,2i}|G_{O,E}) = \frac{3c-3}{2} - \frac{3c-5}{2} = 1.$$

For the edges, $e_{2i,2i+1} = u_{2i}u_{2i+1}$ and $e'_{2i+1,i} = u_{2i+1}v_i$, we have:

$$n_{u_{2i+1}}(e_{2i,2i+1}|G_{O,E}) = \frac{3c-1}{2} \quad n_{u_{2i}}(e_{2i,2i+1}|G_{O,E}) = \frac{3c-5}{2},$$

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$$n_{u_{2i+1}}(e'_{2i+1,i}|G_{O,E}) = \frac{3c-1}{2} \quad n_{v_i}(e'_{2i+1,i}|G_{O,E}) = \frac{3c-5}{2}.$$

Thus, the contributions of these two edges are: $\phi(e'_{2i+1,i}|G_{O,E}) = \phi(e_{2i,2i+1}|G_{O,E}) = \frac{3c-1}{2} - \frac{3c-5}{2} = 2$. In $C_{\frac{c+1}{2}}$, each of the four edges $e = uv$ contributes:

$$n_u(e|G_{O,E}) = \frac{3c-1}{2} \text{ when } u = u_c, u_{c+2},$$

$$n_v(e|G_{O,E}) = \frac{3c-5}{2} \text{ when } v = u_{c+1}, v_{\frac{c+1}{2}}.$$

Therefore, each of these four edges contributes:

$$\phi(e|G_{O,E}) = \frac{3c-1}{2} - \frac{3c-5}{2} = 2.$$

The remaining edges in $G_{O,E}$ have zero contribution. Using the above results, from (1.2), we obtain

$$Mo(G_{O,E}) = 2\left(\frac{3c-3}{2}\right) + 6(c-3) + 8 = 9c - 13.$$

(b) In $G^k_{O,E}$ and $G^{k-1}_{O,E}$, two edges $e'_{1,1} = u_1v_1$ and $e'_{2,1} = u_2v_1$ contribute

$$\phi(e'_{1,1}|G^k_{O,E}) = \phi(e'_{2,1}|G^k_{O,E}) = \frac{3c-3+2k}{2},$$

$$\phi(e'_{1,1}|G^{k-1}_{O,E}) = \phi(e'_{2,1}|G^{k-1}_{O,E}) = \frac{3c-3+2(k-1)}{2}.$$

The edge $e_{1,2} = u_1u_2$ contribute $\phi(e_{1,2}|G^k_{O,E}) = \phi(e_{1,2}|G^{k-1}_{O,E}) = 0$. Two new edges $e \in E(G^k_{O,E})$ have contribution $\phi(e|G^k_{O,E}) = 0$. For all remaining edges $e = uv$ in $G^k_{O,E}$, we have

$$n_u(e|G^k_{O,E}) = n_u(e|G^{k-1}_{O,E}) + 1 \text{ and } n_v(e|G^k_{O,E}) = n_v(e|G^{k-1}_{O,E}) + 1.$$

Therefore,

$$\phi(e|G^k_{O,E}) = |n_u(e|G^{k-1}_{O,E}) + 1 - n_v(e|G^{k-1}_{O,E}) - 1| = |n_u(e|G^{k-1}_{O,E}) - n_v(e|G^{k-1}_{O,E})| = \phi(e|G^{k-1}_{O,E}).$$

Thus, all the remaining edges of $G^k_{O,E}$ have the same contribution as in $G^{k-1}_{O,E}$. Consequently,

$$Mo(G^k_{O,E}) = Mo(G^{k-1}_{O,E}) + 2.$$

■

We can also provide several classes of constructions to obtain the same value of the Mostar index as in Proposition 2.5. A general version of the construction is as follows.

General Construction III

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Consider the base graph $G_{O,E}$ as in Construction III. For $j = 3,5,\ldots,c-2$, consider the vertices u_j and u_{c+j-1}. The vertex u_j lies between the 4-cycles $C_{\frac{j+1}{2}}$ and $C_{\frac{j+1}{2}-1}$, while u_{c+j-1} lies between the cycles $C_{\frac{c+j}{2}}$ and $C_{\frac{c+j}{2}-1}$. For each $j = 3,5,\ldots,c-2$, let $G^{j,1}_{O,E}$ be the graph obtained by deleting the edges $u_j v_{\frac{j+1}{2}}$ and $u_{c+j-1} v_{\frac{c+j}{2}}$, subdividing the edges $u_j u_{j+1}$ and $u_{c+j-1} u_{c+j}$, relabeling the vertices $u_1, u_2, \ldots, u_{2c-1}$ of C as $u_1, u_2, \ldots, u_{2c+1}$, and attaching the edges $u_{j+1} v_{\frac{j+1}{2}}$ and $u_{c+j+3} v_{\frac{c+j}{2}}$ (or equivalently, adding an edge between the cycles $C_{\frac{j+1}{2}}$ and $C_{\frac{j+1}{2}-1}$, as well as between the cycles $C_{\frac{c+j}{2}}$ and $C_{\frac{c+j}{2}-1}$). Proceeding in this manner, let $G^{j,k}_{O,E}$ be the graph obtained by subdividing the edges $u_{j+k-2}u_{j+k-1}$ and $u_{c+j+2k-2}u_{c+j+2k-1}$ of $G^{j,k-1}_{O,E}$ exactly once (or equivalently, inserting $k-1$ edges between the cycles $C_{\frac{j+1}{2}}$ and $C_{\frac{j+1}{2}-1}$, as well as between the cycles $C_{\frac{c+j}{2}}$ and $C_{\frac{c+j}{2}-1}$ in $G^{j,1}_{O,E}$). At each step, relabel the vertices of the cycle C in $G^{j,k}_{O,E}$. Clearly, $G^{j,k}_{O,E}$ is a c-cyclic graph with $3c-2+2k$ vertices and $4c-3+2k$ edges, where $k = 2,3,\ldots$ and $j = 3,5,\ldots,c-2$.

Proposition 2.6. *For every fixed odd integer $c \geq 3$, $Mo(G^{j,k}_{O,E}) = Mo(G^{j,k-1}_{O,E}) + 2$, where $j = 3,5,\ldots,c-2$ and $k = 2,3,\ldots$.*

Proof. In $G^{j,k}_{O,E}$ and $G^{j,k-1}_{O,E}$, the two edges $e'_{1,1} = u_1v_1$ and $e'_{2,1} = u_2v_1$ contribute:

$$\phi(e'_{1,1}|G^{j,k}_{O,E}) = \phi(e'_{2,1}|G^{j,k}_{O,E}) = \frac{3c-3+2k}{2}$$

and

$$\phi(e'_{1,1}|G^{j,k-1}_{O,E}) = \phi(e'_{2,1}|G^{j,k-1}_{O,E}) = \frac{3c-3+2(k-1)}{2}.$$

The edge $e_{1,2} = u_1u_2$ contribute $\phi(e_{1,2}|G^{j,k}_{O,E}) = \phi(e_{1,2}|G^{j,k-1}_{O,E}) = 0$. Two new edges $e \in E(G^{j,k}_{O,E})$ have contribution $\phi(e|G^{j,k}_{O,E}) = 0$. All remaining edges in $G^{j,k}_{O,E}$ have the same contribution as in $G^{j,k-1}_{O,E}$. Therefore, $Mo(G^{j,k}_{O,E}) = Mo(G^{j,k-1}_{O,E}) + 2$. ■

Construction IV

Consider a cycle $C = u_1u_2\ldots u_{2c}u_1$ of length $2c$, where c is odd. Introduce a vertex v_1 and connect u_1v_1 and u_2v_1 to form a 3-cycle $u_1u_2v_1u_1$. For each path $u_iu_{i+1}u_{i+2}$ in C starting from $i = 2$ and incrementing by 2, add a vertex $v_{\frac{i+2}{2}}$, and connect $u_iv_{\frac{i+2}{2}}$ and $u_{i+2}v_{\frac{i+2}{2}}$ to form a 4-cycle $u_iu_{i+1}u_{i+2}v_{\frac{i+2}{2}}u_i$ for $i = 2,4,\ldots,c-1$. Similarly, for $i = c+2, c+4, \ldots, 2c-3$, add a vertex $v_{\frac{i+1}{2}}$, and connect $u_iv_{\frac{i+1}{2}}$ and $u_{i+2}v_{\frac{i+1}{2}}$ to form a 4-cycle $u_iu_{i+1}u_{i+2}v_{\frac{i+1}{2}}u_i$. Denote the resulting graph by $G_{O,O}$. Let C_1 denote the 3-cycle $u_1u_2v_1u_1$, and let C_i denote the 4-cycle containing v_i for $i = 2,3,\ldots,c-1$. Define $G^1_{O,O}$ as the graph obtained from $G_{O,O}$ by deleting the edge $u_{c-1}v_{\frac{c-1}{2}}$ and subdividing the edges $u_{c-2}u_{c-1}$ and u_1u_{2c} exactly once. Relabel the vertices of C as $u_1u_2\ldots u_{2c+2}$, and reattach the edge $u_{c-1}v_{\frac{c-1}{2}}$ (or insert a new edge between the cycles C_1 and C_{c-1} in C, and another new edge between $C_{\frac{c-1}{2}}$ and $C_{\frac{c+1}{2}}$ in C). Iteratively, define $G^k_{O,O}$ as the graph obtained from $G^{k-1}_{O,O}$ by inserting new edges between C_1 and C_{c-1} and between $C_{\frac{c-1}{2}}$ and $C_{\frac{c+1}{2}}$, or equivalently, by subdividing the edges $u_1u_{2c+2(k-1)}$ and $u_{c-1}u_c$ of $G^{k-1}_{O,O}$ exactly once. Thus, $G_{O,O}$ is a c-cyclic graph of order $3c-1$ and size $4c-2$, and $G^k_{O,O}$ is a c-cyclic graph of order $3c-1+2k$ and size $4c-2+2k$ for $k = 1,2,\ldots$. In these graphs, let $e_{i,j}$ denote the edge connecting u_i and u_j (where $i \leq j$), and $e'_{i,j}$ denote the edge connecting u_i and v_j (where $i \geq j$).

Proposition 2.7. *For odd $c \geq 3$, (a) $Mo(G_{O,O}) = 11c - 20$, (b) $Mo(G^1_{O,O}) = Mo(G_{O,O}) + 2$ and $Mo(G^k_{O,O}) = Mo(G^{k-1}_{O,O}) + 2$, $j = 2,3,\ldots$.*

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Proof. (a) In the cycle C_1, two edges incident on v_1 have contribution

$$\phi(e'_{1,1}|G_{O,O}) = |n_{u_1}(e'_{1,1}|G_{O,O}) - n_{v_1}(e'_{1,1}|G_{O,O})| = \frac{3c-3}{2} - 1 = \frac{3c-5}{2},$$

$$\phi(e'_{2,1}|G_{O,O}) = |n_{u_2}(e'_{2,1}|G_{O,O}) - n_{v_1}(e'_{2,1}|G_{O,O})| = \frac{3c-1}{2} - 1 = \frac{3c-3}{2}.$$

The remaining edge in the cycle, $e_{1,2} = u_1u_2$ has contribution:

$$\phi(e_{1,2}|G_{O,O}) = |n_{u_2}(e_{1,2}|G_{O,O}) - n_{u_1}(e_{1,2}|G_{O,O})| = \frac{3c-1}{2} - \frac{3c-3}{2} = 1.$$

Now, except for the cycle $C_{\frac{c+1}{2}}$, in all other cycles C_i (for $i = 2, 3, \ldots \frac{c-1}{2}$), we have the following contributions for edges $e_{2i-2,2i-1} = u_{2i-2}u_{2i-1}$ and $e'_{2i-2,i} = u_{2i-2}v_i$:

$$n_{u_{2i-2}}(e_{2i-2,2i-1}|G_{O,O}) = \frac{3c+1}{2}, \ n_{u_{2i-1}}(e_{2i-2,2i-1}|G_{O,O}) = \frac{3c-3}{2},$$

$$n_{u_{2i-2}}(e'_{2i-2,i}|G_{O,O}) = \frac{3c+1}{2}, \ n_{v_i}(e'_{2i-2,i}|G_{O,O}) = \frac{3c-3}{2}.$$

For edges $e_{2i-1,2i} = u_{2i-1}u_{2i}$ and $e'_{2i,i} = u_{2i}v_i$, we have

$$n_{u_{2i}}(e_{2i-1,2i}|G_{O,O}) = \frac{3c+1}{2}, \ n_{u_{2i-1}}(e_{2i-1,2i}|G_{O,O}) = \frac{3c-3}{2},$$

$$n_{u_{2i}}(e'_{2i,i}|G_{O,O}) = \frac{3c+1}{2}, \ n_{v_i}(e'_{2i,i}|G_{O,O}) = \frac{3c-3}{2}.$$

Therefore, all four edges have a contribution $\phi(e_{2i-1,2i}|G_{O,O}) = \phi(e'_{2i,i}|G_{O,O}) = \frac{3c+1}{2} - \frac{3c-3}{2} = 2$ each. For $i = \frac{c+3}{2}, \ldots, c-1$, the edges $e_{2i-1,2i} = u_{2i-1}u_{2i}$ and $e'_{2i-1,i} = u_{2i-1}v_i$, we have

$$n_{u_{2i-1}}(e_{2i-1,2i}|G_{O,O}) = \frac{3c+1}{2}, \ n_{u_{2i}}(e_{2i-1,2i}|G_{O,O}) = \frac{3c-3}{2},$$

$$n_{u_{2i-1}}(e'_{2i-1,i}|G_{O,O}) = \frac{3c+1}{2}, \ n_{v_i}(e'_{2i-1,i}|G_{O,O}) = \frac{3c-3}{2}.$$

For edges $e_{2i,2i+1} = u_{2i}u_{2i+1}$ and $e'_{2i+1,i} = u_{2i+1}v_i$, we have

$$n_{u_{2i+1}}(e_{2i,2i+1}|G_{O,O}) = \frac{3c+1}{2} \text{ and } n_{u_{2i}}(e_{2i,2i+1}|G_{O,O}) = \frac{3c-3}{2},$$

$$n_{u_{2i+1}}(e'_{2i+1,i}|G_{O,O}) = \frac{3c+1}{2} \text{ and } n_{v_i}(e'_{2i+1,i}|G_{O,O}) = \frac{3c-3}{2}.$$

Therefore, all four edges have a contribution $\phi(e|G_{O,O}) = \frac{3c+1}{2} - \frac{3c-3}{2} = 2$ each. In $C_{\frac{c+1}{2}}$, two edges $e'_{c-1,\frac{c+1}{2}} = u_{c-1}v_{\frac{c+1}{2}}$ and $e_{c-1,c} = u_{c-1}u_c$ have contributions

$$\phi(e_{c-1,c}|G_{O,O}) = \phi(e'_{c-1,\frac{c+1}{2}}|G_{O,O}) = \frac{3c+1}{2} - \frac{3c-3}{2} = 2.$$

The other two edges $e'_{c+1,\frac{c+1}{2}} = u_{c+1}v_{\frac{c+1}{2}}$ and $e_{c,c+1} = u_cu_{c+1}$, have contributions

$$\phi(e_{c,c+1}|G_{O,O}) = \phi(e'_{c+1,\frac{c+1}{2}}|G_{O,O}) = \frac{3c-1}{2} - \frac{3c-1}{2} = 0.$$

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Among the remaining edges,

$$\phi(e_{1,2c}|G_{O,O}) = 2,\ \phi(e_{2c-1,2c}|G_{O,O}) = 0 \text{ and } \phi(e_{c+1,c+2}|G_{O,O}) = 1.$$

Using the above results, from (1.2), we obtain

$$Mo(G_{O,O}) = 8(c-3) + 8 + \frac{3c-3}{2} + \frac{3c-5}{2} = 11c - 20.$$

(b) In $G^k_{O,O}$ and $G^{k-1}_{O,O}$, the edges incident on v_1 contribute

$$\phi(e|G^k_{O,O}) = \frac{3c-1+2k}{2} - 1 = \frac{3c-3+2k}{2},$$

$$\phi(e|G^k_{O,O}) = \frac{3c-3+2k}{2} - 1 = \frac{3c-5+2k}{2},$$

$$\phi(e|G^{k-1}_{O,O}) = \frac{3c-1+2(k-1)}{2} - 1 = \frac{3c-3+2(k-1)}{2},$$

and

$$\phi(e|G^{k-1}_{O,O}) = \frac{3c-3+2(k-1)}{2} - 1 = \frac{3c-5+2(k-1)}{2},$$

respectively. Two new edges in $G^k_{O,O}$ each contribute $\phi(e|G^k_{O,O}) = 0$ to the Mostar index. For the remaining edges of $G^k_{O,O}$ and $G^{k-1}_{O,O}$, we have

$$\phi(e|G^k_{O,O}) = |n_u(e|G^{k-1}_{O,O}) + 1 - n_v(e|G^{k-1}_{O,O}) - 1| = |n_u(e|G^{k-1}_{O,O}) - n_v(e|G^{k-1}_{O,O})| = \phi(e|G^{k-1}_{O,O}).$$

Therefore, the remaining edges of $G^k_{O,O}$ and $G^{k-1}_{O,O}$ have the same contribution. Thus, $Mo(G^k_{O,O}) = Mo(G^{k-1}_{O,O}) + 2$. ■

We can also provide several class of constructions for obtaining the same value of Mostar index as in Proposition 2.7. A general version of the construction is as follows.

General Construction IV

Consider the base graph $G_{O,O}$ as in Construction IV. For $j = 2, 4, \ldots, c-3$, consider the vertices u_j and u_{c+j}. Then, u_j is in between the 4-cycles $C_{\frac{j+2}{2}}$ and $C_{\frac{j+2}{2}-1}$, and u_{c+j} is in between the cycles $C_{\frac{c+j+1}{2}}$ and $C_{\frac{c+j+1}{2}-1}$. Now, for each $j = 2, 4, \ldots, c-3$, let $G^{j,1}_{O,O}$ be the graph obtained by deleting the edges $u_j v_{\frac{j+2}{2}}$, $u_{c+j} v_{\frac{c+j-1}{2}}$ and subdividing the edges $u_j u_{j+1}$, $u_{c+j} u_{c+j+1}$, and relabeling the vertices of C as $u_1, u_2, \ldots, u_{2c+2}$. Also, attach the edges $u_{j+1} v_{\frac{j+2}{2}}$ and $u_{c+j+3} v_{\frac{c+j+1}{2}}$ (or add an edge between the cycles $C_{\frac{j+2}{2}}$, $C_{\frac{j+2}{2}-1}$ and $C_{\frac{c+j+1}{2}}$, $C_{\frac{c+j+1}{2}-1}$ in C). Proceeding in this manner, let $G^{j,k}_{O,O}$ be the graph obtained by subdividing the edges $u_{j+k-2}u_{j+k-1}$, $u_{c+j+2k-2}u_{c+j+2k-1}$ of $G^{j,k-1}_{O,O}$ exactly once (or inserting $k-1$ edges between the cycles $C_{\frac{j+2}{2}}$, $C_{\frac{j+2}{2}-1}$ and $C_{\frac{c+j+1}{2}}$, $C_{\frac{c+j+1}{2}-1}$ in C of $G^{j,1}_{O,O}$). As shown in Figure 2, in each of the steps, relabel the vertices of the cycle C in $G^{j,k}_{E,O}$. Clearly, $G^{j,k}_{O,O}$ is a c-cyclic graph with $3c-1+2k$ vertices and $4c-2+2k$ edges, where $k = 2, 3, \ldots$ and $j = 2, 4, \ldots, c-3$.

Proposition 2.8. *For every fixed odd integer $c \geq 3$, $Mo(G^{j,k}_{O,O}) = Mo(G^{j,k-1}_{O,O}) + 2$, where $j = 2, 4, \ldots, c-3$ and $k = 2, 3, \ldots$.*

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Proof. In $G_{O,O}^{j,k}$ and $G_{O,O}^{j,k-1}$, the edges incident on v_1 contribute

$$\phi(e|G_{O,O}^{j,k}) = \frac{3c-3+2k}{2},\ \phi(e|G_{O,O}^{j,k}) = \frac{3c-5+2k}{2}$$

and

$$\phi(e|G_{O,O}^{j,k-1}) = \frac{3c-3+2(k-1)}{2},\ \phi(e|G_{O,O}^{j,k-1}) = \frac{3c-5+2(k-1)}{2},$$

respectively. Two new edges in $G_{O,O}^{j,k}$ each contribute $\phi(e|G_{O,O}^{j,k}) = 0$ to the Mostar index. The remaining edges have the same contribution in $G_{O,O}^{j,k}$ as in $G_{O,O}^{j,k-1}$. Therefore, $Mo(G_{O,O}^{j,k}) = Mo(G_{O,O}^{j,k-1}) + 2$. ■

Using these constructions, we are ready to resolve Problem 1 by proving Conjecture 1. The proof of Conjecture 1 is as follows. Here, we denote $\mathbb{Z}_{\geq 0}$ as the set of non-negative integers, and the set of integers is represented by the letter $\mathbb{Z}$.

Theorem 2.9. *For any fixed $c \geq 3$, there exists a c-cyclic graph with Mostar index $Mo(G) = p$, where $p \in \mathbb{Z}_{\geq 0} \backslash A$, and A is given by $A = A_1 \cup A_2$ when c is even, and $A = A_3 \cup A_4$ when c is odd, with*

$$\begin{aligned}
A_1 &= \Big\{2x \in \mathbb{Z} \,|\, 0 \leq x \leq 4c-5,\ \mathbf{c} \textit{ is even}\Big\},\\
A_2 &= \Big\{2x+1 \in \mathbb{Z} \,|\, 1 \leq x \leq \frac{9c}{2} - 7,\ \mathbf{c} \textit{ is even}\Big\},\\
A_3 &= \Big\{2x \in \mathbb{Z} \,|\, 0 \leq x \leq \frac{11c-15}{2},\ \mathbf{c} \textit{ is odd}\Big\},\\
A_4 &= \Big\{2x+1 \in \mathbb{Z} \,|\, 1 \leq x \leq \frac{11c-23}{2},\ \mathbf{c} \textit{ is odd}\Big\}.
\end{aligned}$$

Proof. We consider the following two cases:

Case 1: c is even. The graph $G_{E,E}$ (see, Figure 1) is a c-cyclic graph (even $c \geq 4$) described in Construction I. By Proposition 2.1, we have

$$Mo(G_{E,E}) = 8c-8,\ Mo(G_{E,E}^1) = Mo(G_{E,E}) + 2, \text{ and } Mo(G_{E,E}^k) = Mo(G_{E,E}^{k-1}) + 2 \text{ for } k = 2,3\ldots$$

From the above, we conclude that every even number greater than or equal to $8c-8$ can be the Mostar index of some c-cyclic graph for every fixed c (even $c \geq 4$). Thus we have $Mo(G_{E,E}) \geq 2k$, where k is a positive integer with $k \geq 4c-4$ (even $c \geq 4$).

The graph $G_{E,O}$ (see, Figure 3) is a c-cyclic graph (even $c \geq 4$) described in Construction II. By Proposition 2.3, we have

$$Mo(G_{E,O}) = 9c-11,\ Mo(G_{E,O}^1) = Mo(G_{E,O}) + 2, \text{ and } Mo(G_{E,O}^k) = Mo(G_{E,O}^{k-1}) + 2 \text{ for } k = 2,3\ldots$$

From the above, we conclude that every odd number greater than or equal to $9c-11$ can be the Mostar index of some c-cyclic graph for every fixed c (even $c \geq 4$). Thus we have $Mo(G_{E,E}) \geq 2k+1$, where k is a positive integer with $k \geq \frac{9c}{2} - 6$ (even $c \geq 4$).

For c-cyclic graph for every fixed c (even $c \geq 4$), we obtain $Mo(G) = p$, where $p \in \mathbb{Z}_{\geq 0} \backslash A$, and A is given by

$$A = A_1 \cup A_2,$$

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with

$$A_1 = \{2x \in \mathbb{Z} \,|\, 0 \le x \le 4c - 5, \ \mathbf{c} \text{ is even}\},$$
$$A_2 = \{2x + 1 \in \mathbb{Z} \,|\, 1 \le x \le \frac{9c}{2} - 7, \ \mathbf{c} \text{ is even}\}.$$

Case 2: c is odd. The graph $G^k_{O,E}$ (see, Figure 4) is a c-cyclic graph (odd $c \ge 3$) described in Construction III. By Proposition 2.5, we have

$$Mo(G_{O,E}) = 11c - 13, \ Mo(G^1_{O,E}) = Mo(G_{O,E}) + 2 \text{ and } Mo(G^k_{O,E}) = Mo(G^{k-1}_{O,E}) + 2 \text{ for } k = 2, 3 \ldots$$

From the above, we conclude that every even number greater than or equal to $11c - 13$ can be the Mostar index of some c-cyclic graph for every fixed c, when c is odd. Thus we have $Mo(G_{E,E}) \ge 2k$, where k is a positive integer with $k \ge \frac{11c-13}{2}$ (c is odd).

The graph $G_{O,O}$ (see, Figure 5) is a c-cyclic graph (odd $c \ge 3$) described in Construction IV. By Proposition 2.7, we have

$$Mo(G_{O,O}) = 11c - 20, \ Mo(G^1_{O,O}) = Mo(G_{O,O}) + 2 \text{ and } Mo(G^k_{O,O}) = Mo(G^{k-1}_{O,O}) + 2 \text{ for } k = 2, 3 \ldots$$

From the above, we conclude that every odd number greater than or equal to $11c - 20$ can be the Mostar index of some c-cyclic graph for every fixed c, when c is odd. Thus we have $Mo(G_{O,O}) \ge 2k + 1$, where k is a positive integer with $k \ge \frac{11c-21}{2}$ (odd $c \ge 3$).

For c-cyclic graph for every fixed c (odd $c \ge 3$), we obtain $Mo(G) = p$, where $p \in \mathbb{Z}_{\ge 0} \backslash A$, and A is given by

$$A = A_3 \cup A_4,$$

with

$$A_3 = \{2x \in \mathbb{Z} \,|\, 0 \le x \le \frac{11c - 15}{2}, \ \mathbf{c} \text{ is odd}\},$$
$$A_4 = \{2x + 1 \in \mathbb{Z} \,|\, 1 \le x \le \frac{11c - 23}{2}, \ \mathbf{c} \text{ is odd}\}.$$

This completes the proof of the theorem. ■

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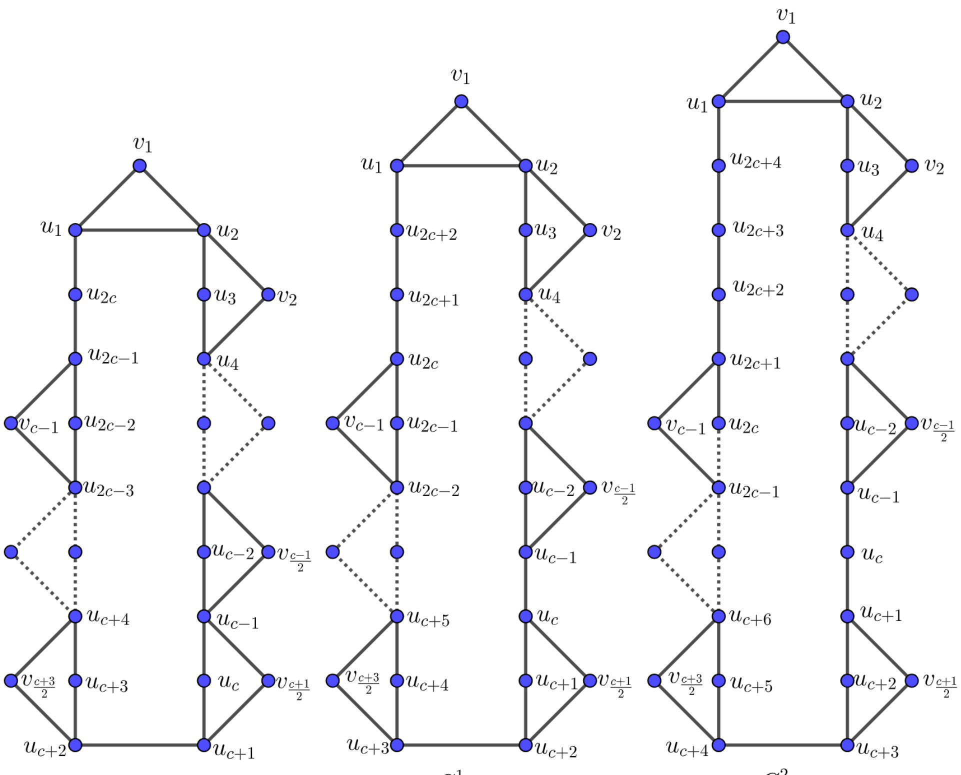

Figure 5. The Graphs Mentioned in Construction IV.

Remark 2.10. *Using Theorem 2.9, we provide a solution to Problem 1.*

Using Theorem 2.9, we can also solve the inverse Mostar index problem for connected graphs.

Corollary 2.11. *For every positive integer $p \geq 19$, there exist a graph G with Mostar index $Mo(G) = p$.*

Proof. For $c = 4$, from Theorem 2.9, we have

$$A_1 = \{2x \in \mathbb{Z} \,|\, 0 \leq x \leq 11\},\ A_2 = \{2x + 1 \in \mathbb{Z} \,|\, 1 \leq x \leq 11\},$$

that is,

$$A_1 = \{0, 2, 4, 6, \ldots, 20, 22\},\ \ A_2 = \{1, 3, 5, \ldots, 21, 23\}.$$

Similarly, for $c = 3$, from Theorem 2.9, we have

$$A_3 = \{0, 2, 4, \ldots, 16, 18\},\ \ A_4 = \{1, 3, 5, \ldots, 9, 11\}.$$

This completes the proof of the result. ■

Remark 2.12. *We can also prove Theorem 2.9, using the general constructions I,II, III and IV and Proposition 2.2, Proposition 2.4, Proposition 2.6 and Proposition 2.8.*

3. Conclusions

Several variants of the Mostar index have been proposed recently, including the edge Mostar index $Mo_e(G)$, the total Mostar index $Mo^*(G)$, and the total edge Mostar index $Mo_e^*(G)$ [4]. For a graph G, these indices are defined as follows:

$$Mo^*(G) = \sum_{\{u,v\} \subseteq V} \left| n_u(e|G) - n_v(e|G) \right|,$$

$$Mo_e(G) = \sum_{uv \in E(G)} \left| m_u(e|G) - m_v(e|G) \right|,$$

$$Mo_e^*(G) = \sum_{\{u,v\} \subseteq V} \left| m_u(e|G) - m_v(e|G) \right|,$$

where $m_u(e|G)$ is the number of edges closer to u than to v, and similarly for $m_v(e|G)$. In this paper, we have resolved an open problem and conjecture related to the inverse Mostar index problem for c-cyclic graphs with $c \geq 3$. Despite this significant progress, several open problems remain in this area. For instance, the inverse edge Mostar index problem for c-cyclic graphs with $c \geq 4$ has yet to be explored. Additionally, the infinite realizability of odd integers for both the Mostar index and the edge Mostar index remains an open question. Future research opportunities include addressing analogous problems for other bond-additive indices, such as the total Mostar index and the total edge Mostar index. These extensions represent promising directions for further studies in this field.

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