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Approximate of Homomorphisms and Derivations Structures in Normed Triple Lie Spaces

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Abstract

In this paper, we investigate the algebraic structures of homomorphisms and derivations, which play a significant role in the fields of physics and engineering sciences. The main focus of this study is on the perturbation of homomorphism and derivation structures associated with certain bidimensional Additive-Quadratic functional equation in normed triple Lie (3-Lie) systems, within the concept of Ulam-stability. By employing the fixed point theorem, we establish several required theorems, followed by specific and relevant corollaries that highlight the theoretical significance of our findings.

Keywords: fixed point theorem; homomorphism; derivation; Ulam-stability of functional equations; normed Lie systems

MSC: 39B82; 47H10; 47L25; 39B52

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1. Introduction

The study of ternary algebraic operations dates back to the 19th century, When Cayley [2] and other mathematicians introduced the idea of the cubic matrix. This idea was subsequently generalized by Kapranov et al. [13] in 1994. Although the applications of ternary structures remain largely theoretical, they have been linked to several areas of physics, including the quantum Hall effect, and also in supersymmetric theories, the Yang–Baxter equation and statistics. Detailed discussions of these applications in the context of physical problems are presented in Refs. [1,11,12,21,26]. A *normed 3-Lie system* is a normed space $(\mathcal{X}, \|\cdot\|)$ equipped with a following mapping

$$[\cdot\ \cdot\ \cdot] : \mathcal{X}^3 \longrightarrow \mathcal{X},$$

that hold under the following axioms for all elements $r, y, u, v, w \in \mathcal{X}$:

(i) $[uvw] = -[vuw]$,
(ii) $[uvw] + [vwu] + [wuv] = 0$,
(iii) $[rs[uvw]] = [[rsu]vw] + [u[rsv]w] + [uv[rsw]]$,
(iii) $\|[uvw]\| \leqslant \|u\|\|v\|\|w\|$.

The notion of a Lie triple system was originally introduced by Lister [14], with further discussions provided in [8].

Consider normed 3-Lie systems $\mathcal{X}$ and $\mathcal{Y}$. A $\mathbb{C}$-bilinear map $H : \mathcal{X}^2 \to \mathcal{Y}$ is referred to as a *homomorphism* whenever it satisfies

$$H([uvw], x) = [H(u,x)H(v,x)H(w,x)],$$
$$H(u, [vwx]) = [H(u,v)H(u,w)H(u,x)]$$

Let $u, v, w, x \in \mathcal{X}$. A $\mathbb{C}$-bilinear mapping $D : \mathcal{X}^2 \to \mathcal{Y}$ is said to be a *derivation* whenever it satisfies

$$\begin{aligned} D([uvw], x) &= [D(u,x)vw] + [u\, D(v,x)\, w] + [uv\, D(w,x)], \\ D(u, [vwx]) &= [D(u,v)wx] + [v\, D(u,w)\, x] + [vw\, D(u,x)] \end{aligned}$$

for all $u, v, w, x \in \mathcal{X}$.

Subsequently, we provide the definition of an additive–quadratic mapping.

Definition 1. *Consider the vector spaces $\mathcal{X}$ and $\mathcal{Y}$. A mapping $\mathcal{F} : \mathcal{X}^2 \to \mathcal{Y}$ is said to be additive-quadratic when it is additive with respect to its first argument and quadratic with respect to its second argument, indicating that $\mathcal{F}$ fulfills the system of equations presented below.*

$$\mathcal{F}(u,w) + \mathcal{F}(v,w) = \mathcal{F}(u+v,w)$$

and

$$\mathcal{F}(u, v+w) + \mathcal{F}(u, v-w) = 2\mathcal{F}(u,v) + 2\mathcal{F}(u,w)$$

for all $u, v, w \in \mathcal{X}$.

In this paper, the concept of stability is considered according to the generalized Hyers–Ulam–Rassias approach. In the present work, the stability of homomorphisms and derivations corresponding to the following quadratic functional equation is examined.

$$2\mathcal{F}(u+v, w+x) + \mathcal{F}(u+v, w-x) = 3[\mathcal{F}(u,w) + \mathcal{F}(u,x) + \mathcal{F}(v,w) + \mathcal{F}(v,x)] \tag{1}$$

3-Lie systems, viewed as ternary structures, have been studied in various contexts; see [6,17,18] for related results. Equation (1) and its stability have been examined by S. Donganont et al. [5], whereas more detailed analyses of the stability of quadratic functional equationsare discussed in [9].

Here, we recall an essential result from fixed point theory.

Theorem 1. *(Ref. [15]) Let $(\mathcal{X}, d)$ be a Banach space, and let $T : \mathcal{X} \to \mathcal{Y}$ be a mapping with Lipschitz constant $0 < L < 1$. Then, for every $x \in \mathcal{X}$, one of the following holds:*

$$d(T^n x, T^{n+1} x) = \infty$$

or there exists a nonnegative integer n_0 such that

(i) $d(T^n x, T^{n+1} x) < \infty, n \geq n_0$;
(ii) $T^n x \to z^*$ *as* $n \to \infty$, *where* z^* *is a fixed point.*
(iii) z^* *denotes the unique fixed point in* $Y = \{z \in \mathcal{X} : d(T^{n_0} x, z) < \infty\}$.
(iv) $d(z, z^*) \leq \frac{1}{1-L}, d(z, Tz)$ *for all* $z \in Y$.

Isac and Rassias [10] applied stability theory in 1996 to establish fixed point theorems, obtaining several new results in mathematical analysis. In the last decades, Many researchers have extensively studied various stability problems of functional equations; cf. [3,4,7,16,19,20,22–25] and references therein.

Throughout the paper, we assume that $\mathcal{X}$ is a normed 3-Lie system with norm $\|.\|_{\mathcal{X}}$, and $\mathcal{Y}$ is a Banach 3-Lie system with norm $\|.\|_{\mathcal{Y}}$. For a given mapping $\mathcal{F} : \mathcal{X}^2 :\to \mathcal{Y}$, we define $P\mathcal{F} : \mathcal{X}^4 \to \mathcal{Y}$ by

$$P\mathcal{F}(u,v,w,x) := 2\mathcal{F}(u+v, w+x) + \mathcal{F}(u+v, w-x) - 3[\mathcal{F}(u,w) + \mathcal{F}(u,x) + \mathcal{F}(v,w) + \mathcal{F}(v,x)]. \tag{2}$$

It should be noted that, to avoid repetition, in all theorems and their corollaries throughout the remainder of this paper, the elements u, v, w, x are always assumed to be chosen from $\mathcal{X}$, a normed 3-Lie system with norm $\|.\|_{\mathcal{X}}$.

2. Approximate of Homomorphisms

This section focuses on the investigation of the approximation (or stability) of homomorphisms related to the bilinear additive-quadratic functional equation (1). To prepare for the proof of the main theorem, we first establish several auxiliary lemmas

Lemma 1. *Consider $\mathcal{X}$ and $\mathcal{Y}$ as $\mathbb{C}$-linear spaces, and define $\mathcal{F}:\mathcal{X}^2\to\mathcal{Y}$.*

(i) Let $\mathcal{F}$ be a bi-additive mapping, such that $\mathcal{F}(\alpha u,\beta v)=\alpha\beta\mathcal{F}(u,v)$ for all $\alpha,\beta\in\mathbb{T}^1=\{w\in\mathbb{C}:|w|=1\}$ and all $u,v\in\mathcal{X}$, then $\mathcal{F}$ is $\mathbb{C}$-bilinear.

(ii) Let $\mathcal{F}$ satisfies the equation (1) then $\mathcal{F}$ is $\mathbb{C}$-bilinear function.

(iii) Whenever $\mathcal{F}$ fulfills condition (1), it follows that $\mathcal{F}$ is an additive–quadratic mapping.

Proof. (i) Since $\mathcal{F}$ is a bi-additive, we get $\mathcal{F}(\frac{1}{2}u,\frac{1}{2}v)=\frac{1}{4}\mathcal{F}(u,v)$. Obviously, $\mathcal{F}(0u,0v)=0\mathcal{F}(u,v)$, now let $\beta_1,\beta_2\in\mathbb{C}(\beta_1\neq0,\beta_2\neq0)$, and let M_1 and M_2 are two natural number, respectively, greater than $|\beta_1|$ and $|\beta_2|$. Using a simple geometric argument, it can be concluded that there exist numbers $\alpha_{11},\alpha_{12},\alpha_{21},\alpha_{22}\in\mathbb{T}^1$ such that $2\frac{\beta_1}{M_1}=\alpha_{11}+\alpha_{12}$ and $2\frac{\beta_2}{M_2}=\alpha_{21}+\alpha_{22}$. Therefore

$$\begin{aligned}\mathcal{F}(\beta_1u,\beta_2v)&=\mathcal{F}\left(\frac{M_1}{2}.2.\frac{\beta_1}{M_1}u,\frac{M_2}{2}.2.\frac{\beta_2}{M_2}v\right)=\frac{M_1}{2}.\frac{M_2}{2}\mathcal{F}\left(2.\frac{\beta_1}{M_1}u,2.\frac{\beta_2}{M_2}v\right)\\&=\frac{M_1}{2}.\frac{M_2}{2}\mathcal{F}((\alpha_{11}+\alpha_{12})u,(\alpha_{21}+\alpha_{22})v)\\&=\frac{M_1}{2}(\alpha_{11}+\alpha_{12}).\frac{M_2}{2}(\alpha_{21}+\alpha_{22})\mathcal{F}(u,v)=\beta_1\beta_2\mathcal{F}(u,v).\end{aligned}$$

Thus the mapping $\mathcal{F}:\mathcal{X}^2\to\mathcal{Y}$ is a $\mathbb{C}$-bilinear.

(ii) With v and x set to 0 in

$$\begin{aligned}&2\mathcal{F}(\alpha u+\alpha v,\beta w+\beta x)+\mathcal{F}(\alpha u+\alpha v,\beta w-\beta x)\\&-3[\alpha\beta\mathcal{F}(u,w)+\alpha\beta\mathcal{F}(u,x)+\alpha\beta\mathcal{F}(v,w)+\alpha\beta\mathcal{F}(v,x)]=0,\end{aligned}$$

It follows that

$$\mathcal{F}(\alpha u,\beta w)=\alpha\beta\mathcal{F}(u,w)$$

for every $\alpha,\beta\in\mathbb{T}^1$ and all $u,v,w,x\in\mathcal{X}$. Therefore, using (i), the mapping $\mathcal{F}$ is $\mathbb{C}$-bilinear.

(iii) Setting $u=v=x=0$ in (1) yields $\mathcal{F}(0,w)=0$, while $v=w=x=0$ in (1) results in $H(u,0)=0$.

Substituting $v=w=0$ into (1) gives $\mathcal{F}(u,-x)=\mathcal{F}(u,x)$, which indicates that $\mathcal{F}$ is even in the second component.

Let $x=0$ in (1); this gives $2\mathcal{F}(u+v,w)+\mathcal{F}(u+v,w)=3\mathcal{F}(u,w)+3\mathcal{F}(v,w)$, leading to $\mathcal{F}(u+v,w)=\mathcal{F}(u,w)+\mathcal{F}(v,w)$. Hence, $\mathcal{F}$ is additive in the first argument. Substituting x for $-x$ in (1) yields

$$2\mathcal{F}(u+v,w-x)+\mathcal{F}(u+v,w+x)=3\mathcal{F}(u,w)+3\mathcal{F}(u,-x)+3\mathcal{F}(v,w)+3\mathcal{F}(v,-x).$$

From the fact that $\mathcal{F}$ is even in its second variable, we have

$$2\mathcal{F}(u+v,w-x)+\mathcal{F}(u+v,w+x)=3\mathcal{F}(u,w)+3\mathcal{F}(u,x)+3\mathcal{F}(v,w)+3\mathcal{F}(v,x).\tag{3}$$

Combination of equations (1) and (3), presents

$$\mathcal{F}(u+v,w+x)+\mathcal{F}(u+v,w-x)=2\mathcal{F}(u,w)+2\mathcal{F}(u,x)+2\mathcal{F}(v,w)+2\mathcal{F}(v,x)\tag{4}$$

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Let $u, v, w, x \in \mathcal{X}$. By substituting $y = 0$ into (4), we have

$$\mathcal{F}(u, w + x) + \mathcal{F}(u, w - x) = 2\mathcal{F}(u, w) + 2\mathcal{F}(u, x).$$

Therefore, we conclude that $\mathcal{F}$ is quadratic with respect to its second variable, and therefore $\mathcal{F}$ is an additive–quadratic mapping. □

The class of bilinear additive-quadratic mappings is denoted by $\mathcal{LAQ}(\mathcal{X}, \mathcal{Y})$ throughout this paper. It has proven in Lemma 1 that a mapping $\mathcal{F}$ satisfying (2) is an element of $\mathcal{LAQ}(\mathcal{X}, \mathcal{Y})$. We also introduce $\mathcal{G}_0(\mathcal{X}, \mathcal{Y}) = \{h : \mathcal{X}^2 \to \mathcal{Y} : \ h(0, u) = h(u, 0) = 0, \ u \in \mathcal{X}\}$.

The stability of homomorphisms for functional equation (1), will now be proved in the following.

Theorem 2. *Consider a mapping $\mathcal{F} \in \mathcal{G}_0(\mathcal{X}, \mathcal{Y})$, for which a function $\psi : \mathcal{X}^4 \to [0, \infty)$ exists such that*

$$\|P\mathcal{F}(u, v, w, x)\| \leqslant \psi(u, v, w, x), \tag{5}$$

$$\lim_{n \to \infty} \frac{1}{4^n} \psi(2^n u, 2^n v, 2^n w, 2^n x) = 0, \tag{6}$$

$$\begin{aligned} &\|\mathcal{F}([uvw], x) - [\mathcal{F}(u, x)\mathcal{F}(v, x)\mathcal{F}(w, x)]\|_{\mathcal{Y}} \\ &\qquad + \|\mathcal{F}(u, [vwx]) - [\mathcal{F}(u, v)\mathcal{F}(u, w)\mathcal{F}(u, x)]\|_{\mathcal{Y}} \\ &\leqslant \psi(u, v, w, x). \end{aligned} \tag{7}$$

If an $L < 1$ exists such that $\psi(u, v, w, x) \leqslant 12L\psi(\frac{u}{2}, \frac{v}{2}, \frac{w}{2}, \frac{x}{2})$, then a unique homomorphism $H \in \mathcal{LAQ}(\mathcal{X}, \mathcal{Y})$ can be found such that

$$\|\mathcal{F}(u, w) - H(u, w)\| \leqslant \frac{L}{1 - L} \psi\left(\frac{u}{2}, \frac{u}{2}, \frac{w}{2}, \frac{w}{2}\right) \tag{8}$$

Proof. Substituting $u = v$ and $w = x$ into (5), we obtain

$$\|3\mathcal{F}(2u, 2w) - 12\mathcal{F}(u, w)\| \leqslant \psi(u, u, w, w). \tag{9}$$

Therefore

$$\left\|\frac{1}{4}\mathcal{F}(2u, 2w) - \mathcal{F}(u, w)\right\| \leqslant \frac{1}{12}\psi(u, u, w, w) \leqslant L\psi\left(\frac{u}{2}, \frac{u}{2}, \frac{w}{2}, \frac{w}{2}\right). \tag{10}$$

Define $X := \{g \mid g : \mathcal{X}^2 \to \mathcal{Y}\}$, and equip X with a generalized metric given by

$$d(g, h) := \inf\left\{t \in \mathbb{R}^+ : \|g(u, w) - h(u, w)\| \leqslant t\psi\left(\frac{u}{2}, \frac{u}{2}, \frac{w}{2}, \frac{w}{2}\right), \quad \forall u, w \in \mathcal{X}\right\}.$$

It can be readily verified that (X, d) is a complete space.

Let $T : X \to X$ be the linear mapping defined as

$$Tg(u, w) := \frac{1}{4}g(2u, 2w).$$

For $g, h \in X$ with $d(g, h) = \varepsilon$, we have

$$\|g(u, w) - h(u, w)\|_{\mathcal{Y}} \leqslant \frac{1}{3}\varepsilon\psi\left(\frac{u}{2}, \frac{u}{2}, \frac{w}{2}, \frac{w}{2}\right).$$

Hence

$$\|Tg(u, w) - Th(u, w)\| = \frac{1}{4}\|g(2u, 2w) - h(2u, 2w)\| \leqslant \frac{1}{12}\varepsilon\psi(u, u, w, w) \leqslant L\varepsilon\psi\left(\frac{u}{2}, \frac{u}{2}, \frac{w}{2}, \frac{w}{2}\right).$$

Hence, $d(g, h) = \varepsilon$ implies $d(Tg, Th) \leqslant L\varepsilon$, which indicates that

$$d(Tg, Th) \leqslant Ld(g, h).$$

for all $g, h \in X$. From (10), it follows that $d(\mathcal{F}, T\mathcal{F}) \leqslant L$.

Applying Theorem 1, we deduce that T possesses a unique fixed point in $X_1 = \{h \in X : d(\mathcal{F}, h) < \infty\}$. Let us denote this fixed point by H, that is

$$H(2u, 2w) = 4H(u, w),$$

where there exists $t \in (0, \infty)$ satisfying

$$\|H(u, w) - \mathcal{F}(u, w)\| \leqslant t\psi\left(\frac{u}{2}, \frac{u}{2}, \frac{w}{2}, \frac{w}{2}\right).$$

Meanwhile, we have

$$\lim_{n\to\infty} d(T^n\mathcal{F}, H) = 0.$$

As a result,

$$\lim_{n\to\infty} \frac{1}{4^n}\mathcal{F}(2^n u, 2^n w) = H(u, w) \tag{11}$$

for each $u, w \in \mathcal{X}$. Consequently, from $d(\mathcal{F}, H) \leqslant \frac{1}{1-L} d(\mathcal{F}, T\mathcal{F})$, Thus,

$$d(\mathcal{F}, H) \leqslant \frac{L}{1-L}.$$

Then, we obtain inequality (8). Using (5), (6) and (11), it follows that

$$\begin{aligned}
&\|2H(u+v, w+x) + H(u+v, w-x) - 3H(u, w) - 3H(u, x) - 3H(v, w) - 3H(v, x)\|_{\mathcal{Y}} \\
&\quad = \lim_{n\to\infty} \frac{1}{4^n}\|2\mathcal{F}(2^n(u+v), 2^n(w+x)) + \mathcal{F}(2^n(u+v), 2^n(w-x)) \\
&\qquad - 3\mathcal{F}(2^n u, 2^n x) - 3\mathcal{F}(2^n u, 2^n x) - 3\mathcal{F}(2^n v, 2^n cw) - 3\mathcal{F}(2^n v, 2^n x)\|_{\mathcal{Y}} \\
&\quad \leqslant \lim_{n\to\infty} \frac{1}{4^n}\psi(2^n u, 2^n v, 2^n w, 2^n x) = 0.
\end{aligned}$$

Therefore

$$2H(u+v, w+x) + H(u+v, w-x) = 3[H(u, w) + H(u, x) + H(v, w) + H(v, x)].$$

As H satisfies (1), we have, by Lemma 1, that $H : \mathcal{X}^2 \to \mathcal{Y}$ is $\mathbb{C}$-bilinear. Using (6) and (7), it follows that

$$\begin{aligned}
&\|H([uvw], x) - [H(u, x)H(v, x)H(w, x)]\|_{\mathcal{Y}} + \|H(u, [vwx]) - [H(u, v)H(u, w)H(u, x)]\|_{\mathcal{Y}} \\
&\quad = \lim_{n\to\infty} \frac{1}{4^n}(\|\mathcal{F}([2^n u 2^n v 2^n w], 2^n x) - [\mathcal{F}(2^n u, 2^n x)\mathcal{F}(2^n v, 2^n x)\mathcal{F}(2^n w, 2^n dx)]\|_{\mathcal{Y}} \\
&\qquad + \mathcal{F}(2^n u, [2^n v 2^n w 2^n x]) - [\mathcal{F}(2^n u, 2^n v)\mathcal{F}(2^n u, 2^n w)\mathcal{F}(2^n u, 2^n x)]\|_{\mathcal{Y}}) \\
&\quad \leqslant \lim_{n\to\infty} \frac{1}{4^n}\psi(2^n u, 2^n v, 2^n w, 2^n x) = 0
\end{aligned}$$

for each $u, v, w, x \in \mathcal{X}$. Thus, we obtain

$$H([uvw], x) = [H(u, x)H(v, x)H(w, x)]$$

and

$$H(u, [vwx]) = [H(u, v)H(u, w)H(u, x)].$$

Consider a bi-additive mapping $T' : \mathcal{X}^2 \to \mathcal{Y}$ that satisfies (8).

$$\begin{aligned}\|H(u,v) - T'(u,v)\|_{\mathcal{Y}} &= \frac{1}{4^n}\|H(2^n u, 2^n v) - T'(2^n u, 2^n v)\|_{\mathcal{Y}}\\ &\leqslant \frac{1}{4^n}\|H(2^n u, 2^n v) - \mathcal{F}(2^n u, 2^n v)\|_{\mathcal{Y}} + \frac{1}{4^n}\|\mathcal{F}(2^n u, 2^n v) - T'(2^n u, 2^n v)\|_{\mathcal{Y}}\\ &\leqslant \frac{2}{4^n}\psi(2^n u, 2^n u, 2^n w, 2^n w) = 0\end{aligned}$$

as $n \to \infty$. Hence $H = T'$. Consequently, $H : \mathcal{X}^2 \to \mathcal{Y}$ is a unique homomorphism fulfilling (8). ☐

Corollary 1. *Let $s \in (0,2)$ and $\vartheta \in (0,\infty)$. Consider $\mathcal{F} \in \mathcal{G}_0(\mathcal{X},\mathcal{Y})$ which satisfies*

$$\|P\mathcal{F}(u,v,w,x)\| \leqslant \vartheta(\|u\|^s + \|v\|^s + \|w\|^s + \|x\|^s)$$

and

$$\begin{aligned}&\|\mathcal{F}([uvw],x) - [\mathcal{F}(u,x)\mathcal{F}(v,x)\mathcal{F}(w,x)]\|_{\mathcal{Y}} + \|\mathcal{F}(u,[vwx]) - [\mathcal{F}(u,v)\mathcal{F}(u,w)\mathcal{F}(u,x)]\|_{\mathcal{Y}}\\ &\quad\leqslant \vartheta(\|u\|^s + \|v\|^s + \|w\|^s + \|x\|^s).\end{aligned}$$

Then, there exists a unique homomorphism $H \in \mathcal{LAQ}(\mathcal{X},\mathcal{Y})$ satisfying

$$\|\mathcal{F}(u,w) - H(u,w)\| \leqslant \frac{2\vartheta}{4-2^s}(\|u\|^s + \|w\|^s).$$

Proof. The proof can be derived from theorem 2 by taking

$$\psi(u,v,w,x) := \vartheta(\|u\|^s + \|v\|^s + \|w\|^s + \|x\|^s),$$

and setting $L = 2^{s-2}$ yields the desired result. ☐

Theorem 3. *Assume that $\mathcal{F} \in \mathcal{G}_0(\mathcal{X},\mathcal{Y})$ is a mapping for which there exists a function $\psi : \mathcal{X}^4 \to [0,\infty)$ satisfying (5) and (7). If an $L < 1$ exists such that $\psi(u,u,w,w) \leqslant \frac{L}{4}\psi(2u,2u,2w,2w)$, and*

$$\lim_{n\to\infty} 4^n \psi\left(\frac{u}{2^n}, \frac{v}{2^n}, \frac{w}{2^n}, \frac{x}{2^n}\right) = 0 \tag{12}$$

for each $u,v,w,x \in \mathcal{X}$, then a unique homomorphism $H \in \mathcal{LAQ}(\mathcal{X},\mathcal{Y})$ exists, such that

$$\|\mathcal{F}(u,v) - H(u,w)\| \leqslant \frac{L}{12-12L}\psi(u,u,w,w). \tag{13}$$

Proof. From (9), it follows that

$$\left\|\mathcal{F}(u,w) - 4\mathcal{F}\left(\frac{u}{2},\frac{w}{2}\right)\right\| \leqslant \frac{1}{2}\psi\left(\frac{u}{2},\frac{u}{2},\frac{w}{2},\frac{w}{2}\right) \leqslant \frac{L}{12}\psi(u,u,w,w). \tag{14}$$

Take the set $X := \{g | \, g : \mathcal{X}^2 \to \mathcal{Y}\}$ and construct a generalized metric on X,

$$d(g,h) := \inf\left\{t \in \mathbb{R}^+ : \|g(u,w) - h(u,w)\| \leqslant t\psi(u,u,w,w), \quad \forall u,w \in \mathcal{X}\right\}.$$

It is straightforward to verify that (X,d) is complete.

We now introduce the linear operator $T : X \to X$ defined by

$$Tg(u,w) := 4g\left(\frac{u}{2},\frac{w}{2}\right).$$

Using (14), we obtain $d(\mathcal{F}, T\mathcal{F}) \leqslant \frac{L}{12}$.

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Applying Theorem 1, we deduce that T possesses a unique fixed point in $X_1 = \{K \in X : d(\mathcal{F}, K) < \infty\}$. Denote this fixed point by H, that is

$$H(2u, 2w) = H(u, w)$$

for each $u, w \in \mathcal{X}$, there exists $t \in (0, \infty)$ such that

$$\|H(u, w) - \mathcal{F}(u, w)\| \leqslant t\psi(u, u, w, w)$$

or every $u, w \in \mathcal{X}$. On the other hand, we obtain

$$\lim_{n\to\infty} d(T^n\mathcal{F}, H) = 0,$$

which gives the following equality

$$\lim_{n\to\infty} 4^n \mathcal{F}\left(\frac{u}{2^n}, \frac{w}{2^n}\right) = H(u, w). \tag{15}$$

It follows from $d(\mathcal{F}, H) \leqslant \frac{1}{1-L} d(\mathcal{F}, T\mathcal{F})$ that

$$d(\mathcal{F}, H) \leqslant \frac{L}{12 - 12L}.$$

This leads to inequality (13). The remainder of the proof is analogous to that of Theorem 2. □

Corollary 2. *For* $s \in (2, \infty)$ *and* $\vartheta \in (0, \infty)$*, consider* $\mathcal{F} \in \mathcal{G}_0(\mathcal{X}, \mathcal{Y})$ *which satisfies*

$$\|P\mathcal{F}(u, v, w, x)\| \leqslant \vartheta(\|u\|^s + \|v\|^s + \|w\|^s + \|x\|^s)$$

and

$$\begin{aligned}&\|\mathcal{F}([uvw], x) - [\mathcal{F}(u, x)\mathcal{F}(v, x)\mathcal{F}(w, x)]\| + \|\mathcal{F}(u, [vwx]) - [\mathcal{F}(u, v)\mathcal{F}(u, w)\mathcal{F}(u, x)]\|_{\mathcal{Y}}\\ &\qquad \leqslant \vartheta(\|u\|^s + \|v\|^s + \|w\|^s + \|x\|^s),\end{aligned}$$

then, there exists a unique homomorphism $H \in \mathcal{LAQ}(\mathcal{X}, \mathcal{Y})$ *satisfying*

$$\|\mathcal{F}(u, w) - H(u, w)\| \leqslant \frac{2\vartheta}{3(2^s - 4)}(\|u\|^s + \|w\|^s)$$

.

Proof. The proof follows directly from Theorem 2 by setting

$$\psi(u, v, w, x) := \vartheta(\|u\|^s + \|v\|^s + \|w\|^s + \|x\|^s)$$

For every $u, v, w, x \in \mathcal{X}$, setting $L = 2^{2-s}$ yields the desired result. □

3. Approximate of Derivations

This section is devoted to studying the approximate within the stability framework of derivations in 3-Lie algebras for functional equation (1). We start with the following key theorem.

Theorem 4. *Let* $\mathcal{F} \in \mathcal{G}_0(\mathcal{X}, \mathcal{Y})$ *satisfying* (5) *and* (6)*, and suppose there exists a function* $\psi : \mathcal{X}^4 \to [0, \infty)$ *such that*

$$\begin{aligned}&\|\mathcal{F}([uvw], x) - [\mathcal{F}(u, x)vw] - [u\mathcal{F}(v, x)w] - [uv\mathcal{F}(w, x)]\|_{\mathcal{Y}}\\ &\quad +\|\mathcal{F}(u, [vwx]) - [\mathcal{F}(u, v)wx] - [v\mathcal{F}(u, w)x][vw\mathcal{F}(u, x)]\|_{\mathcal{Y}}\\ &\quad \leqslant \psi(u, v, w, x),\end{aligned} \tag{16}$$

and assuming there exists $L < 1$ *such that* $\psi(u,u,w,w) \leqslant 12L\psi\left(\frac{u}{2},\frac{u}{2},\frac{w}{2},\frac{w}{2}\right)$. *Then, a unique derivation* $D \in \mathcal{LAQ}(\mathcal{X},\mathcal{Y})$ *exists, such that*

$$\|\mathcal{F}(u,w) - D(u,w)\| \leqslant \frac{L}{1-L}\psi\left(\frac{u}{2},\frac{u}{2},\frac{w}{2},\frac{w}{2}\right). \tag{17}$$

Proof. Following the same argument as in the proof of Theorem 2, we obtain a unique $\mathbb{C}$-bilinear mapping $D : \mathcal{X}^2 \to \mathcal{Y}$ satisfying (17). The mapping D is given as

$$D(u,w) = \lim_{n\to\infty} \frac{1}{4^n}\mathcal{F}(2^n u, 2^n w)$$

for each $u, w \in \mathcal{X}$, consequently, it follows from (16) together with (6) that

$$\begin{aligned}
&\|D([uvw],x) - [D(u,x)vw] - [u\,D(v,x)w] - [uv\,D(w,x)]\|_{\mathcal{Y}}\\
&+\|D(u,[vwx]) - [D(u,v)wx] - [v\,D(u,w)x][vw\,D(u,x)]\|_{\mathcal{Y}}\\
&\quad= \lim_{n\to\infty}\Bigg(\bigg\|\frac{1}{4^{3n}}\mathcal{F}(2^{3n}[uvw],2^{3n}x) - \left[\frac{1}{4^n}\mathcal{F}(2^n u,2^n x)vw\right]\\
&\qquad - \left[u\frac{1}{4^n}\mathcal{F}(2^n v,2^n x)w\right] - \left[uv\frac{1}{4^n}\mathcal{F}(2^n w,2^n x)\right]\bigg\|_{\mathcal{Y}}\\
&\qquad + \bigg\|\frac{1}{4^{3n}}\mathcal{F}(2^{3n}u,2^{3n}[vwx]) - \left[\frac{1}{4^n}\mathcal{F}(2^n u,2^n v)wx\right]\\
&\qquad - \left[v\frac{1}{4^n}\mathcal{F}(2^n u,2^n w)x\right] - \left[vw\frac{1}{4^n}\mathcal{F}(2^n u,2^n x)\right]\bigg\|_{\mathcal{Y}}\Bigg)\\
&\quad= \lim_{n\to\infty}\Bigg(\bigg\|\frac{1}{4^{3n}}\mathcal{F}([2^n u2^n v2^n w],2^{3n}x) - \frac{1}{4^{3n}}[\mathcal{F}(2^n u,2^{3n}x)2^n v2^n w]\\
&\qquad - \frac{1}{4^{3n}}[2^n u\mathcal{F}(2^n v,2^{3n}x)2^n w] - \frac{1}{4^{3n}}[2^n u2^n v\mathcal{F}(2^n w,2^{3n}x)]\bigg\|_{\mathcal{Y}}\\
&\qquad + \bigg\|\frac{1}{4^{3n}}\mathcal{F}(2^{3n}u,[2^n v2^n w2^n x]) - \frac{1}{4^{3n}}[\mathcal{F}(2^{3n}u,2^n v)2^n w2^n x]\\
&\qquad - \frac{1}{4^{3n}}[2^n v\mathcal{F}(2^{3n}u,2^n w)2^n x] - \frac{1}{4^{3n}}[2^n v2^n w\mathcal{F}(2^{3n}u,2^n x)]\bigg\|_{\mathcal{Y}}\Bigg)\\
&\quad\leqslant \lim_{n\to\infty}\left(\frac{1}{4^{3n}}\psi(2^n u,2^n v,2^n w,2^{3n}x) + \frac{1}{4^{3n}}\psi(2^{3n}u,2^n v,2^n w,2^n x)\right) = 0.
\end{aligned}$$

Therefore

$$\begin{aligned}
D([uvw],x) &= [D(u,x)vw] + [u\,D(v,x)w] + [vw\,D(w,x)],\\
D(u,[vwx]) &= [D(u,v)wx] + [v\,D(u,w)x] + [vw\,D(u,x)].
\end{aligned}$$

Consequently, D is the only $\mathbb{C}$-bilinear mapping that satisfies (17). □

Corollary 3. *Let* $s \in (0,2)$ *and* $\vartheta \in (0,\infty)$. *Suppose* $\mathcal{F} \in \mathcal{G}_0(\mathcal{X},\mathcal{Y})$ *satisfies*

$$\|P\mathcal{F}(u,v,w,x)\| \leqslant \vartheta(\|u\|^s + \|v\|^s + \|w\|^s + \|x\|^s)$$

and

$$\begin{aligned}
&\|\mathcal{F}([uvw],x) - [\mathcal{F}(u,x)vw] - [u\mathcal{F}(v,x)w] - [uv\mathcal{F}(w,x)]\|_{\mathcal{Y}}\\
&\quad + \|\mathcal{F}(u,[vwx]) - [\mathcal{F}(u,v)wx] - [v\mathcal{F}(u,w)x][vw\mathcal{F}(u,x)]\|_{\mathcal{Y}}\\
&\quad \leqslant \vartheta(\|u\|^s + \|v\|^s + \|w\|^s + \|x\|^s),
\end{aligned}$$

it follows that a unique derivation $D \in \mathcal{LAQ}(\mathcal{X}, \mathcal{Y})$ exists, such that

$$\|\mathcal{F}(u,w) - D(u,w)\| \leqslant \frac{2\vartheta}{4-2^s}(\|u\|^p + \|w\|^p)$$

.

Proof. Using theorem 4, the proof follows by taking

$$\psi(u,v,w,x) := \vartheta(\|u\|^s + \|v\|^s + \|w\|^s + \|x\|^s).$$

Hence, by taking $L = 2^{s-2}$, the desired inequality is established. □

Theorem 5. *Let $\mathcal{F} \in \mathcal{G}_0(\mathcal{X}, \mathcal{Y})$, and suppose there exists a function $\psi : \mathcal{X}^4 \to [0,\infty)$ satisfying (5), (12) and (16). If an $L < 1$ exists such that $\psi(u,u,w,w) \leqslant \frac{L}{4}\psi(2u,2u,2w,2w)$ for each $x, z \in \mathcal{X}$, then a unique derivation $D \in \mathcal{LAQ}(\mathcal{X}, \mathcal{Y})$ exists, satisfying*

$$\|\mathcal{F}(u,w) - D(u,w)\| \leqslant \frac{L}{12-12L}\psi(u,u,w,w). \tag{18}$$

Proof. The argument follows in the same manner as in the proofs of Theorems 3 and 4. □

Corollary 4. *Let $s \in (2,\infty)$ and $\vartheta \in (0,\infty)$. Suppose $\mathcal{F} \in \mathcal{G}_0(\mathcal{X}, \mathcal{Y})$ satisfies*

$$\|P\mathcal{F}(u,v,w,x)\| \leqslant \vartheta(\|u\|^s + \|v\|^s + \|w\|^s + \|x\|^s)$$

and

$$\begin{aligned} &\|\mathcal{F}([uvw],x) - [\mathcal{F}(u,x)vw] - [u\mathcal{F}(v,x)w] - [uv\mathcal{F}(w,x)]\|_{\mathcal{Y}} \\ &\quad + \|\mathcal{F}(u,[vwx]) - [\mathcal{F}(u,v)wx] - [y\mathcal{F}(u,w)x][bc\mathcal{F}(u,x)]\|_{\mathcal{Y}} \\ &\quad \leqslant \vartheta(\|u\|^s + \|v\|^s + \|w\|^s + \|x\|^s) \end{aligned}$$

for all $u, v, w, x \in \mathcal{X}$, then there is a unique derivation $D \in \mathcal{LAQ}(\mathcal{X}, \mathcal{Y})$, such that

$$\|\mathcal{F}(u,w) - D(u,w)\| \leqslant \frac{2\vartheta}{3(2^s-4)}(\|u\|^p + \|w\|^p)$$

.

Proof. The proof is a direct consequence of theorem 5 by setting

$$\psi(u,v,w,x) := \vartheta(\|u\|^s + \|v\|^s + \|w\|^s + \|x\|^s),$$

the desired inequality is obtained by taking $L = 2^{2-s}$. □

4. Conclusion

In this paper, we explore the algebraic properties of homomorphisms and derivations. Our primary focus lies on examining the stability of homomorphism and derivation structures corresponding to a specific bidimensional bilinear Additive–Quadratic functional equation in normed triple Lie systems. Utilizing the fixed point theorem, we derive several essential theorems and present pertinent corollaries that emphasize the theoretical contributions of this work. For future research, this procedure can be applied to obtain analogous results for one-dimensional functional equations.

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