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Spacetime Metric with Invariant Time Evolution

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Abstract: In a series of recent papers we developed a field theory for a spacetime metric that evolves with the monotonic advance of a Poincaré-invariant parameter, under the influence of a matter-energy density produced by particle trajectories parameterized in a similar way. In this framework, we obtained the linearized theory for weak fields and solved the resulting wave equation for a metric induced by a 'static' particle evolving uniformly along the t-axis in its rest frame. While this metric possesses many expected features, we showed that it leads to unreasonable geodesic equations for a test particle, including a possible repulsive gravitational force. We attribute this problem to the difficulty of identifying the correct Green's function for the wave equation, forcing us to abandon this approach to the metric. A more general method involves the ADM-like 4+1 formalism, which is often used to obtain the metric as a perturbation to an exact solution for the metric induced by a known matter-energy source. In this paper, we propose a metric ansatz with certain expected properties, obtain the source that induces this metric, and take these as a starting point for an initial value problem in the 4+1 formalism, that can can be used to find a more general metric as a perturbation to this known solution. We show that the metric ansatz, its associated source, and the geodesic equations for a test particle behave as required for such a model, recovering Newtonian gravitation in the nonrelativistic limit. We finally describe possibilities for obtaining more general solutions as perturbations of the ansatz.

Keywords: general relativity; the problem of time; Stueckelberg–Horwitz–Piron theory; parameterized relativistic mechanics

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1. Introduction

In relativistic physics, as well as in everyday language, the concept of time plays two distinct roles: location, as described by coordinates, and chronology, as described by system evolution [1]. In general relativity (GR) this distinction leads to certain difficulties known as the problem of time [2,3]. While time is one of four spacetime coordinates, which are dynamical quantities to be determined by equations of motion, time also generally serves as the chronological parameter required for posing and solving those equations. This distinction was also noted by Stueckelberg [4,5] in his foundational work on classical and quantum electrodynamics. In describing antiparticles as particles "going backward in time" he found that neither the coordinate time nor the proper time of the motion could faithfully parameterize chronological evolution. Instead, he introduced a parameter τ, external to the spacetime manifold and independent of the phase space coordinates, whose monotonic increase provides the "arrow of time" that determines chronology. In this framework, a particle worldline is the trajectory of a classical event $x^\mu(\tau)$ or quantum event $\psi(x,\tau)$ generated by a Lorentz scalar Hamiltonian K in a canonical mechanics whose structure is familiar from nonrelativistic physics.

Stueckelberg's formalism was further developed by Piron and Horwitz [6] into a relativistic canonical many-body theory [7–11] of interacting spacetime events. In particular, the Stueckelberg–Horwitz–Piron (SHP) formalism allows for τ-dependent gauge freedom, leading to an electrodynamic theory in flat spacetime [12–15] with five gauge potentials $a_\alpha(x,\tau)$ for $\alpha = 0,1,2,3,5$. The free field equations in this theory enjoy 5D spacetime and gauge symmetries, but the description of particles remains Lorentz covariant, so the field symmetries must break to O(3,1) when coupled to matter. The resulting theory of interacting events is integrable and recovers Maxwell electrodynamics in τ-equilibrium.

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The SHP framework has been extended by Horwitz to curved spacetime [16,17] with a background metric $g_{\mu\nu}(x)$ for $\mu,\nu = 0,1,2,3$. Because the matter sources consist of event trajectories, the event density $\rho(x,\tau)$ and energy-momentum tensor $T_{\mu\nu}(x,\tau)$ will depend explicitly on τ, and we must have τ-dependent field equations to be solved for a local metric $\gamma_{\mu\nu}(x,\tau)$, reflecting Wheeler's characterization [18] of geometrodynamics as "spacetime tells matter how to move; matter tells spacetime how to curve". We have proposed such a theory in a series of papers [19–24] that specify field equations for the metric $g_{\alpha\beta}(x,\tau)$ in a 5D pseudo-spacetime, leading to a 4+1 formalism that converts them to an initial value problem for the induced metric $\gamma_{\mu\nu}(x,\tau)$ in the 4D spacetime hypersurface as it evolves with τ under the influence of $T_{\mu\nu}(x,\tau)$. As in flat space electrodynamics, the purely geometrical structures — the free fields of GR — are symmetric under 5D spacetime and gauge transformations, but in the presence of matter the spacetime symmetry of the field equations must be no larger than O(3,1). Because τ is independent of spacetime in this formalism, the diffeomorphism invariance of general relativity does not raise questions about the evolution of the metric, as might occur under t-evolution. Moreover, the matter dynamics are determined by a Lorentz scalar Hamiltonian and are similarly unaffected by coordinate transformations. These features should also apply to quantized canonical gravity, where the possibility of quantum fluctuations raises further questions about the dual roles played by time.

For weak fields, the 5D field equations can be linearized to obtain a 5D wave equation for the perturbation to the flat space metric. The Green's function for this equation [15] was found in the context of SHP electrodynamics and has been used to find solutions in GR for sources of various types. In particular, we obtained a Schwarzschild-like metric [22] as the field induced by a 'static particle' — a single event evenly distributed along the t-axis in its rest frame, with no detailed information about its location in time. Allowing the time velocity $\dot{x}^0(\tau)$ of the source event to vary in the rest frame, equivalent to variable mass through $p^2 = M^2\dot{x}^2 = M^2\left(\dot{x}^0\right)^2$, produces a mass-energy-momentum tensor $T_{\alpha\beta}(x,\tau)$ in the 5D space that exhibits radiation of mass through spacetime. Solving the wave equation for this source produces a post-Newtonian metric similar to Schwarzschild but depending explicitly on τ. Solving the geodesic equations for a test event in this metric leads to motion with variable $\dot{t}(\tau)$ and a radial equation for R characterized by variable dynamic mass p^2. As such, this model describes a variable mass source event radiating mass across spacetime to a test event whose mass varies in response [20].

However, this approach was not successful in finding an appropriate metric induced by an event localized on the t-axis. In SHP electrodynamics, a particle may be modeled as an ensemble of events [15] located at $\mathbf{x} = 0$ in space, but narrowly distributed along the time axis according to its functional dependence on $t(\tau)$. The 5D wave equation then produces the Coulomb potential in the form

$$a_0\left(x,\tau\right) = -\frac{\varphi\left(t - R/c - \tau\right)}{R} + o\left(\frac{1}{R^2}\right), \tag{1}$$

where $\varphi(s)$ is a normalized distribution on the t-axis, with its maximum at $\varphi(0)$. A test event at some spacetime point $x = (ct, R\hat{\mathbf{r}})$ experiences a potential whose support is centered around the chronological time $\tau = t - R/c$, the retarded time of the source. In a similar way, a source event of this type produces a mass-energy-momentum tensor $T_{\alpha\beta}(x,\tau)$ narrowly distributed in τ that leads to a Schwarzschild-like perturbed metric with

$$\frac{2GM}{c^2R} \longrightarrow \frac{2GM}{c^2R}\varphi\left(t - R/c - \tau\right) \tag{2}$$

describing a localized gravitational field for a test event with maximum on the lightcone of the source. In this picture, spacetime is flat and empty except at the chronological moments τ for which the the matter sources and the metric fields they induce have support at a given spacetime point x^μ. As a test event moves along a trajectory $x^\mu(\tau)$ with the advance of τ, it will experience the metric at the point x^μ at the chronological time τ.

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But it was shown [24] that the geodesic equations for a test event in any metric of the separable type (2) lead to unreasonable equations of motion, including a dynamic reversal of the sign of the gravitational acceleration. This problem was seen to arise from the structure of the Green's function, itself associated with a number of open questions. Thus, despite the existence of a wave equation, the most effective approach to the metric is through the 4+1 evolution equations, which leads to a metric appropriate to a given source as a perturbation of a known metric structure.

In this paper, we pose an ansatz with the desired properties, deriving the source $T_{\mu\nu}$ from the wave equation, and setting up the 4+1 metric evolution equations in the weak field approximation. This method anticipates alternative solutions for the metric found as perturbations of the ansatz. In Section 2 we briefly review the SHP formalism and general relativity with invariant evolution. Section 3 summarizes solutions to the wave equation using the Green's function, which informs our choice of metric ansatz. In Section 4 we present the metric ansatz, examine the resulting geodesic equations, and derive the source using the wave equation. In Section 5 we set up the 4+1 evolution equations, by calculating the projected Ricci tensor and extrinsic curvature. Finally, Section 6 presents a discussion of these results.

2. Review of General Relativity with Invariant Evolution

2.1. Gauge and Spacetime Symmetries

Writing the flat spacetime metric as $\eta_{\mu\nu} = \text{diag}\,(-1,1,1,1)$ with index convention

$$\lambda, \mu, \nu, \rho \ldots = 0,1,2,3 \qquad \alpha, \beta, \gamma, \delta = 0,1,2,3,5\,, \tag{3}$$

the electrodynamic event action

$$S_{\text{SHP}} = \int d\tau\, \frac{1}{2} M \dot{x}^\mu \dot{x}_\mu + \frac{e}{c} \dot{x}^\mu a_\mu(x,\tau) + \frac{e}{c} c_5 a_5(x,\tau) \tag{4}$$

$$= \int d\tau\, \frac{1}{2} M \dot{x}^\mu \dot{x}_\mu + \frac{e}{c} \dot{x}^\beta a_\beta(x,\tau) \tag{5}$$

is invariant under the 5D gauge transformation $a_\alpha(x,\tau) \longrightarrow a_\alpha(x,\tau) + \partial_\alpha \Lambda(x,\tau)$ where we have introduced $x^5 = c_5\tau$ in analogy to $x^0 = ct$. Despite the superficial 5D spacetime symmetry in the term $\dot{x}^\beta a_\beta$, the terms $\dot{x}^\mu \dot{x}_\mu$, $\dot{x}^\mu a_\mu$, and a_5 are O(3,1) scalars, restricting the spacetime symmetry of the action to 4D. From another point of view, we may treat (5) as a standard 5D action in which the symmetry of the first term is broken by constraining $\dot{x}^5 \equiv c_5$ and eliminating the constant term so that $\dot{x}^\alpha \dot{x}_\alpha \longrightarrow \dot{x}^\mu \dot{x}_\mu$. This latter observation informs our approach to τ-dependent GR.

For consistency with electrodynamic phenomenology we take $c_5 < c$.

We introduce an action for the electromagnetic field

$$S_{\text{field}} = \int d\tau\, d^4x\, f^{\alpha\beta}(x,\tau) f_{\alpha\beta}(x,\tau) \tag{6}$$

where the $f_{\alpha\beta} = \partial_\alpha a_\beta - \partial_\beta a_\alpha$ appear in the Lorentz force found by varying (5) with respect to x^μ. Since the 5-index signifies an O(3,1) scalar quantity, $f_{\mu\nu}$ is a second rank tensor while $f_{5\mu}$ is a vector field strength. To raise the 5-index in (6) seems to require a 5D flat space metric

$$\eta_{\alpha\beta} = \text{diag}\,(-1,1,1,1,\sigma) \qquad \sigma = \pm 1 \tag{7}$$

suggesting an O(3,2) or O(4,1) symmetry for the free fields. But writing

$$f^{\alpha\beta}(x,\tau) f_{\alpha\beta}(x,\tau) = f^{\mu\nu}(x,\tau) f_{\mu\nu}(x,\tau) + 2\sigma f^\mu_{\ 5}(x,\tau) f_{\mu 5}(x,\tau) \tag{8}$$

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poses σ as merely the choice of sign for the vector-vector interaction relative to the tensor-tensor interaction, with no inherent geometrical significance.

We take a similar approach to the metric in GR by embedding 4D spacetime $\mathcal{M}$ in a 5D pseudo-spacetime $\mathcal{M}_5 = \mathcal{M} \times R$ with coordinates $X^\alpha = (x^\mu, c_5\tau)$. We associate with $\mathcal{M}_5$ a metric $g_{\alpha\beta}(x,\tau)$ to be determined by Einstein field equations generalized to 5D. Exploiting the natural foliation of $\mathcal{M}_5$ into 4D equal-τ spacetimes homeomorphic to $\mathcal{M}$, we construct a constant quintrad frame [25] for the pseudo-spacetime with basis vectors $\mathbf{e}_a \cdot \mathbf{e}_b = \eta_{ab}$ related to the coordinate frame $\mathbf{g}_\alpha \cdot \mathbf{g}_\beta = g_{\alpha\beta}$ by the vielbein field

$$\mathbf{e}_a = e^\alpha{}_a(x,\tau)\,\mathbf{g}_\alpha \tag{9}$$

that "absorbs" the local dependence of the $\mathbf{g}_\alpha(\mathcal{P})$ on the point $\mathcal{P} \in \mathcal{M}_5$. The quintrad indices run $a = 0,1,2,3,5$. In the quintrad frame, the field equations can be immediately written as

$$R_{ab} - \frac{1}{2}\eta_{ab}R = k_G T_{ab} \tag{10}$$

where we initially specify the constant metric as (7). As in electrodynamics, we break the 5D spacetime symmetry to O(3,1) at the source by replacing

$$\eta_{ab} = \text{diag}(-1,1,1,1,\sigma) \longrightarrow \widehat{\eta}_{ab} = \text{diag}(-1,1,1,1,0) = \delta^k_a\delta^l_b\,\eta_{kl} \tag{11}$$

in the matter terms. That is, we break the symmetry of (10) by posing field new equations in the trace reversed form

$$R_{ab} = k_G\left(T_{ab} - \frac{1}{2}\widehat{\eta}_{ab}\widehat{T}\right) \qquad \widehat{T} = \widehat{\eta}^{ab}T_{ab} = \eta^{kl}T_{kl} \tag{12}$$

where the purely geometrical structure on LHS may enjoy 5D symmetry in the absence of the O(3,1) covariant source terms on the RHS. Transforming this expression back to the coordinate frame provides the O(3,1) symmetric field equations

$$R_{\alpha\beta} = k_G\left(T_{\alpha\beta} - \frac{1}{2}P_{\alpha\beta}\widehat{T}\right) \qquad P_{\alpha\beta} = g_{\alpha\beta} - \sigma n_\alpha n_\beta \tag{13}$$

where n_α is the unit normal that points in the τ-direction normal to $\mathcal{M}$, and $P_{\alpha\beta}$ is the projection operator from $\mathcal{M}_5$ onto the 4D spacetime hypersurface $\mathcal{M}$. The matter terms in (13) restrict the symmetry of the field equations to O(3,1).

2.2. Event Dynamics in Curved Spacetime

Variation of the event Lagrangian

$$L = \frac{1}{2}Mg_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta \tag{14}$$

with respect to x^γ provides 5D geodesic equations

$$\frac{D\dot{x}^\gamma}{D\tau} = \ddot{x}^\gamma + \Gamma^\gamma_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta \qquad \Gamma^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha\delta}\left(\frac{\partial g_{\delta\beta}}{\partial x^\gamma} + \frac{\partial g_{\delta\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\delta}\right) \tag{15}$$

with the Christoffel connection in the standard form. As in electrodynamics, the 5D symmetry is broken to O(3,1) by constraining $\dot{x}^5 = c_5 \longrightarrow \ddot{x}^5 = 0$, so the dynamical system is described by

$$\frac{D\dot{x}^\mu}{D\tau} = \ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = \ddot{x}^\mu + \Gamma^\mu_{\nu\sigma}\dot{x}^\nu\dot{x}^\sigma + 2c_5\Gamma^\mu_{5\nu}\dot{x}^\nu + c_5^2\Gamma^\mu_{55} = 0 \tag{16}$$

$$\frac{D\dot{x}^5}{D\tau} = \ddot{x}^5 \equiv 0 \tag{17}$$

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which recovers standard 4D GR for $g_{5\alpha} = 0$ and $\partial_\tau g_{\mu\nu} = 0$. With canonical momentum

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = M\left(g_{\mu\nu}\dot{x}^\nu + c_5 g_{\mu 5}\right) \tag{18}$$

the Hamiltonian is

$$K = \frac{1}{2M}p^2 + \frac{1}{2}c_5 g_{55} g^{5\mu} p_\mu - \frac{1}{2}c_5 g_{5\mu} g^{\mu\lambda} p_\lambda + \frac{1}{2}Mc_5^2 g_{5\mu} g^{\mu\lambda} g_{\lambda 5} + \frac{1}{2}Mc_5^2 g_{55}, \tag{19}$$

and for $g^{5\mu} = 0$ assumes the simple form

$$K = \frac{1}{2M}p^\mu p_\mu + \frac{1}{2}Mc_5^2\, g_{55} \tag{20}$$

in which $g_{55}(x,\tau)$ plays the role of a τ-dependent potential on 4D spacetime. The Hamilton are

$$\dot{x}^\mu = \frac{dx^\mu}{d\tau} = \frac{\partial K}{\partial p_\mu} \qquad\qquad \dot{p}_\mu = \frac{dp_\mu}{d\tau} = -\frac{\partial K}{\partial x^\mu} \tag{21}$$

and the Poisson bracket is

$$\{F,G\} = \frac{\partial F}{\partial x^\alpha}\frac{\partial G}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha}\frac{\partial G}{\partial x^\alpha} = \frac{\partial F}{\partial x^\mu}\frac{\partial G}{\partial p_\mu} - \frac{\partial F}{\partial p_\mu}\frac{\partial G}{\partial x^\mu} \tag{22}$$

since $p_5 \equiv 0$. For a scalar function $F(x,p,\tau)$ on phase space, we have

$$\frac{dF}{d\tau} = \{F,K\} + \frac{\partial F}{\partial \tau}, \tag{23}$$

generalizing the nonrelativistic result. The Hamiltonian is this conserved unless K depends explicitly on τ through $g_{\alpha\beta}(x,\tau)$, although the 4D mass $p^\mu p_\mu/2M$ may still vary under g_{55}.

Writing $\partial_0 g_{\alpha\beta} = 0$ in the geodesic equations (15) and neglecting $\dot{x}^i/c \ll 1$ for $i = 1,2,3$, the equations of motion reduce to

$$\frac{d^2t}{d\tau^2} = \frac{dt}{d\tau}\partial_\tau g_{00} \qquad\qquad \ddot{\mathbf{x}} = \frac{1}{2}c^2\left(\frac{dt}{d\tau}\right)^2 \nabla g_{00} + \frac{1}{2}c_5^2 \nabla g_{55} \tag{24}$$

which differ from nonrelativistic mechanics when $\partial_\tau g_{00} \neq 0 \longrightarrow \ddot{t} \neq 0$. To compare with Newtonian gravity we write $g_{00} = 2GM/c^2R$, take $\nabla g_{55} = 0$, and introduce a perturbed source mass $M = M_0 + \delta M(\tau)$. Solving the t equation as

$$\frac{dt}{d\tau} = \exp\left(\frac{2G}{c^2R}\delta M\right), \tag{25}$$

and introducing the conserved angular momentum $L = Mr^2\dot{\phi}$, the radial equation in spherical coordinates becomes

$$\ddot{r} - \frac{L^2}{M^2R^3} + \exp\left(\frac{4G}{c^2R}\delta M\right)\frac{GM_0}{R^2} = 0, \tag{26}$$

which recovers Newtonian gravitation in the absence of the perturbation δM. The Hamiltonian in this coordinate system is

$$K = \frac{1}{2}Mg_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = -\frac{1}{2}Mc^2\left(1 - \frac{2GM_0}{c^2R}\right)\exp\left(\frac{4G}{c^2R}\delta M\right) + \frac{1}{2}M\dot{R}^2 + \frac{1}{2}\frac{L^2}{MR^2} \tag{27}$$

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with time derivative

$$\frac{d}{d\tau}K = \exp\left(\frac{4G}{c^2R}\delta M\right)\left(-\frac{GM}{r} + \frac{4G^2MM_0}{c^2R^2}\right)\frac{d}{d\tau}\delta M \tag{28}$$

showing that the Hamiltonian for the motion of this test particle is not conserved in the presence of a variable mass gravitational source. We may interpret this as a transfer of mass across spacetime mediated by the metric.

For a distribution of geodesically evolving events without mutual interaction ('dust'), the event density $\rho(x,\tau)$ is the number of events per spacetime volume, with dimensions length^{-4}. The event current is

$$j^\alpha(x,\tau) = M\rho(x,\tau)\dot{x}^\alpha(\tau) \tag{29}$$

with continuity equation

$$\nabla_\alpha j^\alpha = \frac{\partial j^\alpha}{\partial x^\alpha} + j^\gamma \Gamma^\alpha_{\gamma\alpha} = \frac{\partial \rho}{\partial \tau} + \nabla_\mu j^\mu = 0 \tag{30}$$

where again $j^5 = Mc_5\rho(x,\tau)$ is an O(3,1) scalar and not the 5-component of a vector with 5D symmetry. The mass-energy-momentum tensor

$$T^{\alpha\beta} = M\rho\dot{x}^\alpha\dot{x}^\beta \longrightarrow \begin{cases} T^{\mu\nu} = M\rho\dot{x}^\mu\dot{x}^\nu, \\ T^{5\beta} = \dot{x}^5\dot{x}^\beta M\rho = c_5 j^\beta, \end{cases} \tag{31}$$

is conserved as

$$\nabla_\alpha T^{\alpha\beta} = 0, \tag{32}$$

by virtue of the continuity and geodesic equations. The tensor $T^{\alpha\beta}$ is thus a suitable O(3,1) covariant source for the 5D field equation.

2.3. Weak Field Approximation

In the weak field approximation [15,22] we write the local metric as a small perturbation $h_{\alpha\beta}$ of the flat metric (7) so that

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \longrightarrow \partial_\gamma g_{\alpha\beta} = \partial_\gamma h_{\alpha\beta} \qquad (h_{\alpha\beta})^2 \simeq 0 \tag{33}$$

leading to the Ricci tensor in the form

$$R_{\alpha\beta} \simeq \frac{1}{2}\left(\partial_\beta\partial_\gamma h^\gamma_\alpha + \partial_\alpha\partial_\gamma h^\gamma_\beta - \partial^\gamma\partial_\gamma h_{\alpha\beta} - \partial_\alpha\partial_\beta h\right) \tag{34}$$

where $h = \eta^{\alpha\beta}h_{\alpha\beta}$. Imposing the Lorenz gauge condition

$$\partial^\beta\left(h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h\right) = 0\,, \tag{35}$$

permitted by invariance of the metric under a 5D translation $x^\alpha \longrightarrow x^\alpha + \Lambda^\alpha(x,\tau)$, simplifies the Ricci tensor to

$$R_{\alpha\beta} \simeq -\frac{1}{2}\partial^\gamma\partial_\gamma h_{\alpha\beta} \tag{36}$$

where $h = \eta^{\alpha\beta}h_{\alpha\beta}$. Using this expression on the LHS of (13), the SHP field equation takes the form of the wave equation

$$-\partial^\gamma\partial_\gamma h_{\alpha\beta} = -\left(\partial^\mu\partial_\mu + \sigma\frac{1}{c_5^2}\partial_\tau^2\right)h_{\alpha\beta} = 2k_G\left(T_{\alpha\beta} - \frac{1}{2}P_{\alpha\beta}\widehat{T}\right) \tag{37}$$

for which the principal part Green's function [15] is

$$G(x,\tau) = \frac{1}{2\pi}\delta(x^2)\delta(\tau) + \frac{c_5}{2\pi^2}\frac{\partial}{\partial x^2}\theta(-\sigma g_{\alpha\beta}x^\alpha x^\beta)\frac{1}{\sqrt{-\sigma g_{\alpha\beta}x^\alpha x^\beta}} . \tag{38}$$

The leading term, which is denoted G_{Maxwell}, has lightlike support at equal-τ and is dominant at long distances. The second term, denoted $G_{\text{correlation}}$, drops off as $1/\text{distance}^2$ with spacelike support for $\sigma = -1$ and timelike support for $\sigma = +1$. As discussed in Section 1, the leading term was used in SHP electrodynamics to obtain the Coulomb force. Nevertheless, significant complexities arise in determining the appropriate Green's function for a 5D wave equation [26], related to the treatment of multiple poles in the Fourier domain and the order of integration. In Section 3 we discuss the difficulties that arise when applying this Green's function in GR.

For a matter source in GR we consider a distribution of events moving in tandem in the neighborhood of a point with 5D coordinates and normalized velocity

$$X^\alpha(\tau) = (X^\mu(\tau), c_5\tau) \qquad \xi^\alpha(\tau) = \frac{1}{c}\dot{X}^\alpha(\tau) = \frac{1}{c}\frac{dX^\alpha}{d\tau} . \tag{39}$$

Writing the spacetime event density in the form $\rho(x,\tau) = \rho(x - X(\tau))$ leads to the mass-energy-momentum tensor

$$T^{\alpha\beta} = M\rho(x,\tau)\dot{X}^\alpha\dot{X}^\beta = M\rho(x,\tau)u^\alpha u^\beta = Mc^2\rho(x,\tau)\xi^\alpha(\tau)\xi^\beta(\tau) \tag{40}$$

which is seen to be conserved by noting that $\partial_\tau\rho(x,\tau) = -\xi^\mu\partial_\mu\rho(x,\tau)$. For this source, a generic solution for the metric perturbation is then

$$h^{\alpha\beta}(x,\tau) = 2k_G\int d^4x'd\tau' G(x-x',\tau-\tau')\left(\xi^\alpha\xi^\beta - \frac{1}{2}\hat{\eta}^{\alpha\beta}\hat{\xi}^2\right)\rho(x',\tau') \tag{41}$$

where $\xi^\alpha = \xi^\alpha(\tau')$ and $\hat{\xi}^2 = \hat{\eta}^{\mu\nu}\xi_\mu\xi_\nu$.

2.4. Evolution of the Local Metric

The 3+1 ADM formalism [27] in GR defines a foliation of 4D spacetime that permits the Einstein field equations to be split into a pair of first order evolution equations and a pair of propagating but non-evolving constraints. After finding initial conditions that satisfy the constraints, the evolution equations provide an initial value problem for the metric γ_{ij} of space as a function of t. The extraction of the evolution equations and constraints from the Einstein equations relies heavily on the theory of embedded surfaces [28–30], treating 3D space as an equal-t spacelike hypersurface embedded in 4D spacetime. For SHP GR, we have generalized the ADM methods to a 4+1 formalism [20–23], exploiting the natural foliation of $\mathcal{M}_5$ into equal-τ spacetime hypersurfaces to construct an initial value problem for the spacetime metric $\gamma_{\mu\nu}$ as a function of τ. This section briefly summarizes the main results.

Although we constructed the 5D pseudo-spacetime described in Section 2.1 as an embedding of equal-τ 4D spacetimes $\mathcal{M}$ in $\mathcal{M}_5$, we may take advantage of the ADM techniques by treating $\mathcal{M}_5$ as a starting point, for which we have known field equations (13). To facilitate the natural foliation of $\mathcal{M}_5$ into spacetime hypersurfaces we extend the previous partition of coordinate indices to the quintrad indices, as

$$a,b,c,d, = 0,1,2,3,5 \qquad k,l,m,n,\ldots = 0,1,2,3 \tag{42}$$

expressing the vielbein transformation and its inverse as

$$\mathbf{e}_k = e^\mu{}_k\mathbf{g}_\mu + e^5{}_k\mathbf{g}_5 \qquad \mathbf{g}_\mu = E_\mu{}^k\mathbf{e}_k + E_\mu{}^{\bar{5}}\mathbf{e}_5 \tag{43}$$

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where the spacetime hypersurface (quatrad) is spanned by the $\mathbf{e}_k$ while $\mathbf{e}_5$ points in the direction of τ-evolution normal to $\mathcal{M}$. We introduce the ADM parameterization

$$\mathbf{g}_5 = N^\mu \mathbf{g}_\mu + N\mathbf{e}_5, \tag{44}$$

where N^μ generalizes the shift 3-vector and N is the lapse function with respect to τ. The vielbein field is now

$$e^\alpha{}_a = \delta^k_a \delta^\alpha_\mu e^\mu{}_k - \delta^5_a \delta^\alpha_\mu \frac{1}{N} N^\mu + \delta^5_a \delta^\alpha_5 \frac{1}{N} \qquad E_\alpha{}^a = \delta^\mu_\alpha \delta^a_k E_\mu{}^k + \delta^5_\alpha \left(E_\mu{}^k N^\mu \delta^a_k + N\delta^a_5 \right) \tag{45}$$

leading to the coordinate metric on $\mathcal{M}_5$

$$g_{\alpha\beta} = \begin{bmatrix} \gamma_{\mu\nu} & N_\mu \\ N_\mu & \sigma N^2 + \gamma_{\mu\nu} N^\mu N^\nu \end{bmatrix}, \tag{46}$$

which generalizes the ADM decomposition. The induced metric $\gamma_{\mu\nu}$ is the evolving metric on 4D spacetime $\mathcal{M}$. The arbitrary functions N and N^μ act as Lagrange multipliers whose choice is comparable to gauge freedom [27], and so the dynamical content in the vielbein field $e^\alpha{}_a$ is contained entirely in the spacetime vierbein field $e^\mu{}_k$.

The foliation of $\mathcal{M}_5$ into equal-τ spacetimes $\mathcal{M}$ provides $P_{\alpha\beta}$, the projection operator used to find (13). The remaining geometrical structures are then projected from $\mathcal{M}_5$ onto $\mathcal{M}$ in the following steps:

1. The covariant derivative D_α on $\mathcal{M}$ is found using $P_{\alpha\beta}$ to project the covariant derivative ∇_α on $\mathcal{M}_5$,
2. The extrinsic curvature $K_{\alpha\beta}$ is defined by projecting the covariant derivative of the unit normal n_α,
3. The projected curvature $\bar{R}^\delta_{\gamma\alpha\beta}$ on $\mathcal{M}$ is defined through the non-commutation of projected covariant derivatives D_α and D_β,
4. The Gauss relation is found by decomposing the 5D curvature $R^\delta_{\gamma\alpha\beta}$ in terms of $\bar{R}^\delta_{\gamma\alpha\beta}$ and $K_{\alpha\beta}$,
5. The mass-energy-momentum tensor is decomposed through the projections

$$\kappa = n_\alpha n_\beta T^{\alpha\beta} \qquad p_\beta = -n_{\alpha'} P_{\beta\beta'} T^{\alpha'\beta'} \qquad S_{\alpha\beta} = P_{\alpha\alpha'} P_{\beta\beta'} T^{\alpha'\beta'} \qquad S = P^{\alpha\beta} S_{\alpha\beta},$$

6. Projecting the 5D curvature $R^\delta_{\gamma\alpha\beta}$ on the unit normal n_α leads to the Codazzi relation providing a relationship between $K_{\alpha\beta}$ and p_α,
7. Lie derivatives of $P_{\alpha\beta}$ and $K_{\alpha\beta}$ along the direction of τ evolution, given by the unit normal n_α in the coordinate frame, are combined with these ingredients, along with the O(3,1)-symmetric field equation (13), to obtain τ-evolution equations for $\gamma_{\mu\nu}$ and $K_{\mu\nu}$, as well as a pair of constraints on the initial conditions.

The evolution equations are found to be

$$\frac{1}{c_5} \partial_\tau \gamma_{\mu\nu} = \mathcal{L}_\mathbf{N} \gamma_{\mu\nu} - 2NK_{\mu\nu} \tag{47}$$

$$\begin{aligned} \frac{1}{c_5} \partial_\tau K_{\mu\nu} = &-D_\mu D_\nu N + \mathcal{L}_\mathbf{N} K_{\mu\nu} \\ &+ N \left\{ -\sigma \bar{R}_{\mu\nu} + KK_{\mu\nu} - 2K^\lambda_\mu K_{\nu\lambda} + \sigma k_G \left(S_{\mu\nu} - \frac{1}{2} P_{\mu\nu} S \right) \right\} \end{aligned} \tag{48}$$

where $\mathcal{L}_\mathbf{N}$ is the Lie derivative along N^μ. Their solutions must satisfy the Hamiltonian and momentum constraints

$$\bar{R} - \sigma \left(K^2 - K^{\mu\nu} K_{\mu\nu} \right) = -k_G \left(S + \sigma\kappa \right) \qquad D_\mu K^\mu_\nu - D_\nu K = k_G p_\nu \,. \tag{49}$$

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3. The Metric as Solution to a 5D Wave Equation

In this section we obtain the metric for a source event of constant velocity equation using the 5D Green's function and discuss the difficulties with this approach. For such a source event, (41) reduces to

$$h_{\alpha\beta}\left(x,\tau\right) = 2k_G Mc^2\, Z_{\alpha\beta}\mathcal{G}\left[\rho\left(x,\tau\right)\right] \tag{50}$$

where we denote by

$$Z_{\alpha\beta} = \xi_\alpha\xi_\beta - \frac{1}{2}\widehat{\eta}_{\alpha\beta}\xi^\mu\xi_\mu \tag{51}$$

$$\mathcal{G}\left[\rho\left(x,\tau\right)\right] = \int d^4x'\, d\tau' G\left(x-x',\tau-\tau'\right)\rho\left(x',\tau'\right) \tag{52}$$

the kinematic and dynamical factors. The event density for a source evenly distributed along the t-axis in its rest frame is

$$\rho(x,\tau) = \rho\left(x - X\left(\tau\right)\right) = \rho\left(ct - c\tau\right)\delta^{(3)}\left(\mathbf{x}\right) = \delta^{(3)}\left(\mathbf{x}\right) \tag{53}$$

as is typically written for a 'static' particle. Integration of the event density (53) with the Green's function (38) leads to

$$\mathcal{G}_{\text{Maxwell}}\left[\rho\left(x,\tau\right)\right] = \int d^4x'd\tau'\frac{1}{2\pi}\delta((x-x')^2)\delta(\tau-\tau')\delta^{(3)}(\mathbf{x}') = \frac{1}{4\pi|\mathbf{x}|} \tag{54}$$

$$\mathcal{G}_{\text{correlation}}\left[\rho\left(x,\tau\right)\right] = \frac{c_5}{2\pi^2}\int d^4x'd\tau'\frac{\partial}{\partial x^2}\frac{\theta(-\sigma(x-x')^\alpha(x-x')_\alpha)}{\sqrt{-\sigma(x-x')^\alpha(x-x')_\alpha}}\delta^{(3)}(\mathbf{x}') = 0 \tag{55}$$

and so taking

$$k_G = \frac{8\pi G}{c^4}\ , \tag{56}$$

the spacetime part of the metric becomes

$$g_{\mu\nu} = \text{diag}\left(-1+\frac{2GM}{c^2r},\left(1+\frac{2GM}{c^2r}\right)\delta_{ij}\right) \simeq \text{diag}\left(-U,U^{-1}\delta_{ij}\right)\ , \tag{57}$$

where

$$U = \left(1-\frac{2GM}{c^2r}\right)\ . \tag{58}$$

Naturally, this metric is spatially isotropic, and is t-independent because the event density is spread evenly along the t-axis. Transforming to spherical coordinates (57) becomes

$$g_{\mu\nu} = \text{diag}\left(-U,U^{-1},U^{-1}r^2,U^{-1}r^2\sin^2\theta\right), \tag{59}$$

which for weak fields is recognized as the Schwarzschild metric

$$g_{\mu\nu} = \text{diag}\left(-U,U^{-1},R^2,R^2\sin^2\theta\right) \tag{60}$$

when expressed in the isotropic coordinates [31] defined through

$$R = r\left(1+\frac{k}{2r}\right)^2. \tag{61}$$

This metric is well-known to be Ricci-flat, $R_{\mu\nu} = 0$, a consequence of t-independence.

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In [24] we attempted to modify the calculation for the 'static' event by specifying a trajectory narrowly distributed along the t-axis at the spatial origin $\mathbf{x} = 0$. We wrote the event density

$$\rho(x,\tau) = \varphi(t-\tau)\,\delta^{(3)}(\mathbf{x}) \qquad \varphi_{\max} = \varphi(0) \tag{62}$$

with support in a neighborhood around $t = \tau$. The kinematic factors are

$$\begin{aligned} Z_{00} &= \frac{1}{2} & Z_{05} = Z_{50} &= -\sigma\frac{c_5}{c} = -\sigma\xi^5 \\ Z_{ij} &= \frac{1}{2}\delta_{ij} & Z_{55} &= \frac{c_5^2}{c^2} = \xi_5^2 \end{aligned} \tag{63}$$

where $i,j = 1,2,3$, and the dynamic factors are

$$\mathcal{G}_{\text{Maxwell}} = \int d^4x'd\tau'\frac{1}{2\pi}\delta((x-x')^2)\delta(\tau-\tau')\varphi(t'-\tau')\,\delta^{(3)}(\mathbf{x}') \tag{64}$$

$$\mathcal{G}_{\text{correlation}} = \frac{c_5}{2\pi^2}\int d^4x'd\tau'\frac{\partial}{\partial x^2}\frac{\theta(-\sigma(x-x')^\alpha(x-x')_\alpha)}{\sqrt{-\sigma(x-x')^\alpha(x-x')_\alpha}}\varphi(t'-\tau')\,\delta^{(3)}(\mathbf{x}')\,. \tag{65}$$

The leading term is easily found as

$$\mathcal{G}_{\text{Maxwell}} = \frac{\varphi(t-r/c-\tau)}{4\pi r} \tag{66}$$

producing a gravitational field with a maximum at $\tau = t - r/c$. Thus, for a source located at $t_{\text{source}} = \tau$, a test event will feel the strongest gravitational force on the lightcone of the source, accounting for the propagation time of the gravitational field. The similarity to the Coulomb force in (1) in electrodynamics is clear. Detailed analysis of $\mathcal{G}_{\text{correlation}}$ shows [24] that this term vanishes for $\sigma = -1$, while for $\sigma = +1$

$$\mathcal{G}_{\text{Correlation}} = \frac{c_5}{2\pi^2}\frac{1}{\mathbf{x}^2 - c_5^2\tau\,(2t-\tau)} \tag{67}$$

which drops off as $1/r^2$, leaving the contribution from $\mathcal{G}_{\text{Maxwell}}$ dominant at long distance. Neglecting the subdominant term, the metric becomes

$$g_{00} = -U = -1 + \frac{k_G Mc^2}{4\pi r}\varphi(t-r/c-\tau) \tag{68}$$

$$g_{ij} = V\delta_{ij} = \left[1 + \frac{k_G Mc^2}{4\pi r}\varphi(t-r/c-\tau)\right]\delta_{ij} \qquad i,j = 1,2,3 \tag{69}$$

$$g_{05} = g_{50} = -2\sigma\xi^5\frac{k_G Mc^2}{4\pi r}\varphi(t-r/c-\tau) \tag{70}$$

$$g_{55} = 2\xi_5^2\frac{k_G Mc^2}{4\pi r}\varphi(t-r/c-\tau) \tag{71}$$

which seems reasonable as it modifies the Schwarzschild-like metric (57) by limiting the support of its influence to $\tau \simeq t_{\text{retarded}}$. At a chosen point $x = (ct, \boldsymbol{x})$ in spacetime, the metric is flat at $\tau \to -\infty$, rises to a maximum at $\tau = t - r/c = t_{\text{retarded}}$, and again returns to the flat metric at $\tau \to \infty$. Thus, in SHP GR, the effect of the metric depends on the trajectory of an event moving through spacetime. In particular, for a test event evolving as $x = (c(\tau + r/c), \boldsymbol{x})$, the factor $\varphi(t - r/c - \tau) \longrightarrow 1$ and the gravitational field will be determined by the time-independent Schwarzschild metric.

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The equations of motion for a nonrelativistic test event are found by expanding (16) as

$$0=\ddot{x}^{\mu}+c^{2}\left(\Gamma_{00}^{\mu}\dot{t}^{2}+2\Gamma_{i0}^{\mu}\frac{\dot{x}^{i}}{c}\dot{t}+\Gamma_{ij}^{\mu}\frac{\dot{x}^{i}}{c}\frac{\dot{x}^{j}}{c}+2\frac{c_{5}}{c}\Gamma_{50}^{\mu}\dot{t}+2\frac{c_{5}}{c}\Gamma_{5i}^{\mu}\frac{\dot{x}^{i}}{c}+\frac{c_{5}^{2}}{c^{2}}\Gamma_{55}^{\mu}\right), \tag{72}$$

and neglecting terms containing $\dot{x}^i/c \ll 1$. The nonzero Christoffel symbols are

$$\Gamma_{00}^{\mu}=-\frac{1}{2c}\delta^{\mu0}\frac{\partial h_{00}}{\partial t}-\frac{1}{2}\delta^{\mu k}\frac{\partial h_{00}}{\partial x^{k}} \qquad \Gamma_{i0}^{\mu}=\frac{1}{2c}\delta^{\mu j}\frac{\partial h_{ji}}{\partial t}-\frac{1}{2}\delta^{\mu0}\frac{\partial h_{00}}{\partial x^{i}} \tag{73}$$

$$\Gamma_{ij}^{\mu}=\frac{1}{2}\delta^{\mu k}\left(\frac{\partial h_{ki}}{\partial x^{j}}+\frac{\partial h_{kj}}{\partial x^{i}}-\frac{\partial h_{ij}}{\partial x^{k}}\right)+\frac{1}{2c}\delta^{\mu0}\frac{\partial h_{ij}}{\partial t} \tag{74}$$

$$\Gamma_{50}^{\mu}=-\frac{1}{2c_{5}}\delta^{\mu0}\frac{\partial h_{00}}{\partial\tau} \qquad \Gamma_{5i}^{\mu}=\frac{1}{2c_{5}}\delta^{\mu k}\frac{\partial h_{ki}}{\partial\tau} \tag{75}$$

where we used $h_{0i}=0,\ i=1,2,3$ and dropped $c_5^2/c^2 \ll 1$. As before, the equations of motion split into

$$0=\ddot{t}-\frac{1}{2}\frac{\partial h_{00}}{\partial t}\dot{t}^{2}-\left(\frac{\partial h_{00}}{\partial\tau}+\dot{\mathbf{x}}\cdot\nabla h_{00}\right)\dot{t} \qquad 0=\ddot{\mathbf{x}}-\frac{1}{2}c^{2}\dot{t}^{2}\,\nabla h_{00}, \tag{76}$$

which differ from (24) because h_{00} is now t-dependent. In spherical coordinates, the equations of motion are

$$\ddot{t}=\frac{1}{2}\left(\partial_{t}h_{00}\right)\dot{t}^{2}+\left(\partial_{\tau}h_{00}+\dot{r}\partial_{r}h_{00}\right)\dot{t} \qquad \ddot{r}=\frac{1}{2}c^{2}(\partial_{r}h_{00})\dot{t}^{2}+\frac{L^{2}}{M^{2}r^{3}} \tag{77}$$

where again the conserved angular momentum is $L=Mr^2\dot{\phi}$. For a nonrelativistic test event we can neglect $\ddot{t}\approx 0$ and take $\dot{t}=1$. Then

$$\partial_{r}h_{00}=\frac{k_{G}Mc^{2}}{4\pi}\partial_{r}\left(\frac{1}{r}\varphi\right)=\frac{k_{G}Mc^{2}}{4\pi}\left(-\frac{1}{r^{2}}\varphi+\frac{1}{r}\partial_{r}\varphi\right) \tag{78}$$

and again using (56) for k_G we find

$$\ddot{r}=-\frac{GM}{r^{2}}\varphi\left(1-\frac{r}{\varphi}\partial_{r}\varphi\right)+\frac{L^{2}}{M^{2}r^{3}} \tag{79}$$

for the radial equation. For a normalized distribution with maximum $\varphi(0)=1$, we see that $\partial_r\varphi=0$ when the test event satisfies the lightcone condition $t-r/c-\tau=0$, and so the radial equation will recover Newtonian gravitation. However, for a narrow distribution, $\partial_r\varphi$ will become large and negative just slightly away from the lightcone, and so the terms in parentheses in (79) will likely become very large and possibly negative.

For example, if we consider the Gaussian distribution $\varphi(s)=\exp\left(-s^2/\lambda_0^2\right)$ where λ_0 is a time scale representing the width of the event distribution along the t-axis then for an event on the lightcone of the source.

$$1-\frac{r}{\varphi}\partial_{r}\varphi=1+\frac{r}{\lambda_{0}c}\frac{2\left(t-r/c-\tau\right)}{\lambda_{0}}\longrightarrow 1\,. \tag{80}$$

However, if gravity accelerates the test event away from the lightcone and toward the source, so that $r\longrightarrow r-c\lambda_0$, then

$$1-\frac{r}{\varphi}\partial_{r}\varphi\longrightarrow 1+\frac{r-c\lambda_{0}}{\lambda_{0}c}\frac{2\left(c\lambda_{0}/c\right)}{\lambda_{0}}=-1+2\frac{r}{\lambda_{0}c} \tag{81}$$

which depends on the ratio $r/\lambda_0 c$. Since we expect the radial distance r to be large, while width of the event distribution $\lambda_0 c$ is small by assumption, this term magnifies the gravitational field by a large factor. Worse still, if the initial conditions are such that $t-r/c-\tau\sim-\lambda_0$ then the gravitational force

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will become repulsive. These problems could be eliminated by taking $\lambda_0 \longrightarrow \infty$, so that $\varphi = 1$. But then the metric components (68) and (69) recover the τ-independent metric (57), losing the t-localization.

We conclude that these problems will be present in any solution for the metric in the separable form φ/r, which evidently follows from use of the truncated the Green's function. While the equal-τ leading term in (38) provides an approximate solution to the 5D wave function, suitable for electrodynamics, it provides an exact solution to the 4D wave equation on spacetime. And this wave equation is precisely the linearized form of the standard 4D Einstein equation

$$-\frac{1}{2}\partial^\lambda \partial_\lambda h_{\mu\nu} = R^{(4)}_{\mu\nu} = -k_G T_{\mu\nu}(x,\tau) \tag{82}$$

where $R^{(4)}_{\mu\nu}$ is the 4D Ricci tensor. But the 5D wave equation follows from the linearized 5D Ricci tensor (34), and if we separate the spacetime terms as

$$R_{\mu\nu} \simeq \frac{1}{2}\left(\partial_\nu \partial_\gamma h^\gamma_\mu + \partial_\mu \partial_\gamma h^\gamma_\nu - \partial^\gamma \partial_\gamma h_{\mu\nu} - \partial_\mu \partial_\nu h\right) \tag{83}$$

$$= R^{(4)}_{\mu\nu} + \frac{1}{2}\left(\partial_\nu \partial_5 h^5_\mu + \partial_\mu \partial_5 h^5_\nu - \partial^5 \partial_5 h_{\mu\nu}\right) \tag{84}$$

$$= R^{(4)}_{\mu\nu} + \partial_5 \Gamma^5_{\mu\nu} \tag{85}$$

we notice the reappearance of the Christoffel symbol $\Gamma^5_{\mu\nu}$ that is absent from (17) because of the constraint $\dot{x}^5 \equiv 0$ and represents contributions to the 5D curvature from τ-evolution (extrinsic curvature). Therefore, this expression shows that an exact solution to the 4D wave equation must fail to take proper account of the τ-evolution of the 4D geometry, regardless of the τ-dependence of the source. So while G_{Maxwell} provides approximate solutions that are adequate for electrodynamics, they misrepresent the evolving spacetime geometry we seek to describe. For this reason, it becomes necessary to approach the metric through the 4+1 evolution equations described in Section 2.4.

4. Metric Ansatz

We now change direction and instead of specifying the expected source for a gravitational field, we propose an ansatz for the metric with certain expected properties. This will enable us to determine the source from the wave equation in the weak field approximation, and evaluate the Ricci tensor and extrinsic curvature so that we may set up the 4+1 evolution equations. While the metric ansatz satisfies the evolution equations exactly, this procedure should permit alternative metrics to be found as perturbations under changes to the source.

We write the ansatz for the perturbed metric as

$$g^{\alpha\beta} = H^{\alpha\beta}\Phi\,(t,r,\tau) \tag{86}$$

where $H^{\alpha\beta}$ is a constant kinematic term and $\Phi\,(t,r,\tau)$ contains the dependence on x^μ and τ. We specify the component structure of $H^{\alpha\beta}$ to be

$$\begin{gathered} g^{\mu\nu} = \text{diag}\left(-1 + H^{00}\Phi, \left(1 + H^{00}\Phi\right)\delta^{ij}\right) \\ g^{05} = 2\xi^5 H^{00}\Phi \qquad g^{i5} = 0 \qquad g^{55} = 2\xi_5^2 H^{00}\Phi \end{gathered} \tag{87}$$

as found from (63) for a source event evolving on the t-axis.

We are interested in a metric that recovers the $1/r$ dependence of Newtonian gravitation, but whose support is restricted to a neighborhood of τ for a given test particle. We consider a source event

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in its rest frame evolving along its time axis as $x^0(\tau) = x^5$, and a test event similarly evolving along its time axis at a spatial distance r. This suggests the functional form

$$\Phi(t,r,\tau) = \frac{1}{\sqrt{r^2 + (x^0 - x^5)^2}} = \frac{1}{\sqrt{r^2 + c^2(t - \xi^5\tau)^2}} \tag{88}$$

where we denote

$$\rho = \sqrt{r^2 + c^2(t - \xi^5\tau)^2} \tag{89}$$

for convenience. Evaluating the connection for this metric, we study the trajectory of a test event determined by the geodesic equations (16) with the initial conditions

$$r(0) = R \qquad \dot{r}(0) = 0 \qquad\qquad t(0) = 0 \qquad \dot{t}(0) = \xi^5 \tag{90}$$

for which $\rho \longrightarrow R$ and Φ takes on its maximum value. If the test event deviates from the trajectory $(x^0, \mathbf{x}) = (\xi^5\tau, R\hat{\mathbf{r}})$, then the strength of the metric will diminish. Because $\Phi(t,r,\tau)$ has a maximum at $t = \xi^5\tau$ with respect to t but not with respect to r, this functional form does not suffer from the difficulties seen in equation (79).

Using the derivatives

$$\frac{\partial\Phi}{\partial t} = -\frac{c^2(t - \xi^5\tau)}{\rho^3} \qquad\qquad \frac{\partial\Phi}{\partial\tau} = \frac{\xi^5 c^2(t - \xi^5\tau)}{\rho^3} \qquad\qquad \frac{\partial\Phi}{\partial x^i} = -\frac{x_i}{\rho^3} \tag{91}$$

the connection takes the explicit form

$$\Gamma^\mu_{00} = \frac{1}{2}H^{00}\frac{1}{\rho^3}\left(\delta^{\mu 0}c\left(t - \xi^5\tau\right) + \delta^{\mu k}x_k\right) \tag{92}$$

$$\Gamma^\mu_{i0} = H^{00}\frac{1}{2}\frac{1}{\rho^3}\left[-\delta^\mu_i c\left(t - \xi^5\tau\right) + \delta^{\mu 0}x_i\right] \tag{93}$$

$$\Gamma^\mu_{ij} = -\frac{1}{2}H^{00}\frac{1}{\rho^3}\left[\delta^\mu_i x_j + \delta^\mu_j x_i - \delta^{\mu k}\delta_{ij}x_k + \delta^{\mu 0}\delta_{ij}c\left(t - \xi^5\tau\right)\right] \tag{94}$$

$$\Gamma^\mu_{50} = -\frac{1}{2}H^{00}\frac{1}{\rho^3}\left[2\sigma\xi^5\delta^{\mu k}x_k + \delta^{\mu 0}c\left(t - \xi^5\tau\right)\right] \tag{95}$$

$$\Gamma^\mu_{5i} = -\frac{1}{2}H^{00}\frac{1}{\rho^3}\left(-\delta^\mu_i c\left(t - \xi^5\tau\right) + \xi^5\delta^{\mu 0}2\sigma x_i\right) \tag{96}$$

$$\Gamma^\mu_{55} = H^{00}\frac{1}{\rho^3}\left[\delta^{\mu k}\xi_5^2 x_k + \delta^{\mu 0}c_5\left(t - \xi^5\tau\right)\left(2\sigma - \xi^5\right)\right] \tag{97}$$

so that splitting the geodesic equations

$$0 = \ddot{x}^\mu + c^2\left(\Gamma^\mu_{00}\dot{t}^2 + 2\Gamma^\mu_{i0}\frac{\dot{x}^i}{c}\dot{t} + \Gamma^\mu_{ij}\frac{\dot{x}^i}{c}\frac{\dot{x}^j}{c} + 2\frac{c_5}{c}\Gamma^\mu_{50}\dot{t} + 2\frac{c_5}{c}\Gamma^\mu_{5i}\frac{\dot{x}^i}{c} + \frac{c_5^2}{c^2}\Gamma^\mu_{55}\right) \tag{98}$$

into time and space components, we arrive at

$$\begin{aligned} 0 = \ddot{t} + H^{00}\frac{1}{\rho^3}\Big\{&\frac{1}{2}c^2\left(t - \xi^5\tau\right)\dot{t}^2 + x_i\dot{x}^i\dot{t} - \frac{1}{2}\left(t - \xi^5\tau\right)\delta_{ij}\dot{x}^i\dot{x}^j \\ &-cc_5\left(t - \xi^5\tau\right)\dot{t} - 2\sigma\xi_5^2 x_i\dot{x}^i + cc_5\xi_5^2\left(t - \xi^5\tau\right)\left(2\sigma - \xi_5^2\right)\Big\} \end{aligned} \tag{99}$$

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$$0 = \ddot{x}^k + H^{00}\frac{1}{\rho^3}c^2\left\{\frac{1}{2}x^k\dot{t}^2 - c\left(t-\xi^5\tau\right)\frac{\dot{x}^k}{c}\dot{t} - x_j\frac{\dot{x}^k}{c}\frac{\dot{x}^j}{c} + \frac{1}{2}x^k\frac{\dot{x}_j}{c}\frac{\dot{x}^j}{c}\right.$$
$$\left. -2\sigma\xi_5^2x^k\dot{t} + c_5\left(t-\xi^5\tau\right)\frac{\dot{x}^k}{c} + \xi_5^4x^k\right\} . \tag{100}$$

In the neighborhood of the initial conditions, the equations of motion reduce to

$$0 = \ddot{t} + H^{00}\frac{x_i}{r^3}\frac{\dot{x}^i}{c}c\left(\dot{t} - 2\sigma\xi_5^2\right) \tag{101}$$

$$0 = \ddot{x}^k + H^{00}\frac{1}{r^3}c^2\left\{\frac{1}{2}x^k\dot{t}^2 - x_j\frac{\dot{x}^k}{c}\frac{\dot{x}^j}{c} + \frac{1}{2}x^k\frac{\dot{x}_j}{c}\frac{\dot{x}^j}{c} - 2\sigma\xi_5^2x^k\dot{t} + \xi_5^4x^k\right\} \tag{102}$$

so that neglecting $\dot{x}/c \ll 1$ the time equation reduces to

$$0 = \ddot{t} \longrightarrow \dot{t}(\tau) = \xi^5 \tag{103}$$

and the space equation becomes

$$0 = \ddot{x}^k + H^{00}\frac{1}{r^3}c^2\xi_5^2\left\{\frac{1}{2}x^k - 2\sigma\xi^5x^k + \xi_5^2x^k\right\} \simeq \ddot{x}^k + \frac{1}{2}c^2\xi_5^2H^{00}\frac{1}{r^2}\hat{x}^k \tag{104}$$

which recovers Newtonian gravitation if we take $H^{00} = 2GM/\xi_5^2c^2 = 2GM/c_5^2$.

Now considering a relativistic test event for the initial condition $t = \xi^5\tau$, and again keeping only lower order terms in ξ^5, the equations of motion become

$$0 = \ddot{t} + H^{00}\frac{1}{r^2}\hat{x}_i\dot{x}^i\dot{t} \longrightarrow c_5\ddot{t} \simeq -\frac{2GM}{r^2}\hat{x}_i\frac{\dot{x}^i}{c} \tag{105}$$

$$0 = \ddot{x}^k + \frac{GM}{r^2}\left\{\hat{x}^k - \left(\frac{2\hat{x}_j\dot{x}^k - \hat{x}^k\dot{x}_j}{c}\right)\frac{\dot{x}^j}{c}\frac{c^2}{c_5^2}\right\} . \tag{106}$$

Since we will generally have $\dot{x}^k < c$ while $c/c_5 > 1$, we see here that the event trajectory may differ from the post-Newtonian relativistic form seen in (77).

In linearized GR we may attribute the ansatz metric to a source from the field equations (13) in the form (37)

$$k_G\left(T_{\alpha\beta} - \frac{1}{2}\hat{\eta}_{\alpha\beta}\widehat{T}\right) = -\frac{1}{2}\partial^\gamma\partial_\gamma h_{\alpha\beta} \tag{107}$$

with spacetime components

$$k_G\left(T_{\mu\nu} - \frac{1}{2}\hat{\eta}_{\mu\nu}\widehat{T}\right) = -\frac{1}{2}H^{00}\delta_{\mu\nu}\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \sigma\frac{1}{c_5^2}\frac{\partial^2}{\partial\tau^2}\right)\Phi \tag{108}$$

for the ansatz (88). Combining

$$\nabla^2\Phi = -\frac{3c^2\left(t-\xi^5\tau\right)^2}{\rho^5} \tag{109}$$

$$-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Phi = \frac{r^2 - 2c^2\left(t-\xi^5\tau\right)^2}{\rho^5} \tag{110}$$

and

$$\sigma\frac{1}{c_5^2}\frac{\partial^2}{\partial\tau^2}\Phi = -\sigma\frac{r^2 - 2c^2\left(t-\xi^5\tau\right)^2}{\rho^5} \tag{111}$$

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we are led to

$$k_G\left(T_{\mu\nu}-\frac{1}{2}\hat{\eta}_{\mu\nu}\widehat{T}\right)=-\frac{1}{2}\frac{H^{00}}{\rho^5}\left[(\sigma-1)\,r^2+(5-2\sigma)\,c^2\left(t-\xi^5\tau\right)^2\right]\delta_{\mu\nu} \tag{112}$$

and if we again take $H^{00}=2GM/c_5^2$ along with $k_G=8\pi G/c^2c_5^2$ then

$$T_{\mu\nu}=\frac{Mc^2}{4\pi\rho^5}\left[(\sigma-1)\,r^2+(5-2\sigma)\,c^2\left(t-\xi^5\tau\right)^2\right]\operatorname{diag}(-1,0,0,0) \tag{113}$$

The structure of this source is easiest to see for $\sigma=1$ in which case

$$T_{\mu\nu}=3\frac{Mc^2}{4\pi\rho^5}\left[c\left(t-\xi^5\tau\right)\right]^2\operatorname{diag}(-1,0,0,0) \tag{114}$$

where $c^2\left(t-\xi^5\tau\right)^2/\rho^5$ has units of length^{-3} as expected for a particle density in space. Although $T_{\mu\nu}$ appears to vanish at $t=\xi^5\tau$ we recall that under this condition $\rho\longrightarrow r$ and so the wave equation for the source reduces to

$$\partial^\gamma\partial_\gamma\left(\frac{1}{\rho}\right)\longrightarrow\nabla^2\left(\frac{1}{r}\right)=-4\pi\delta^{(3)}(\boldsymbol{x}) \tag{115}$$

which describes a point source evenly spread along the t-axis. In contrast, at small spatial distance $r\ll c(t-\xi^5\tau)$ we have $c^2(t-\xi^5\tau)^2/\rho^5\longrightarrow[c^2(t-\xi^5\tau)]^{-3}$ which describes a narrow particle density along the t-axis centered at $t=\xi^5\tau$.

Thus, as expected, the source (114) describes a matter distribution evolving with τ, leading to the metric (87), which similarly evolves with τ, and the geodesic equations (99) and (100) whose coefficients evolve with τ.

5. 4+1 Evolution Equations

In the 4+1 formalism we may solve for the metric from a given source $T_{\mu\nu}$ as a perturbation to an ansatz metric $\gamma_{\mu\nu}$ using the evolution equations (47) and (48) with initial conditions for the projected Ricci tensor $\bar{R}_{\mu\nu}$ and the extrinsic curvature $K_{\mu\nu}$ derived from the form of the perturbation. Discarding terms of the order $\left(h_{\alpha\beta}\right)^2\approx 0$ in the weak field approximation to the 4+1 formalism, leads to several new approximations [21]. Combining equations (46) and (87) we express the 5D metric as

$$\|g_{\alpha\beta}\|=\begin{bmatrix}\gamma_{\mu\nu} & N_\mu\\ N_\mu & \sigma N^2+\gamma_{\mu\nu}N^\mu N^\nu\end{bmatrix}=\begin{bmatrix}\eta_{\mu\nu}+h_{\mu\nu} & h_{\mu5}\\ h_{\mu5} & \eta_{55}+h_{55}\end{bmatrix} \tag{116}$$

from which

$$N\approx 1+\sigma h_{55}/2 \qquad\qquad N_\mu=h_{\mu5}\ , \tag{117}$$

the Lie derivative of the metric reduces to

$$\mathcal{L}_{\mathbf{N}}\gamma_{\mu\nu}\approx\partial_\mu h_{5\nu}+\partial_\nu h_{5\mu}\ , \tag{118}$$

and the metric evolution equation (47) becomes

$$\frac{1}{c_5}\partial_\tau\gamma_{\mu\nu}=\frac{1}{c_5}\partial_\tau h_{\mu\nu}\approx\partial_\mu h_{5\nu}+\partial_\nu h_{5\mu}-2K_{\mu\nu}\ , \tag{119}$$

confirming that $K_{\mu\nu}$ is of the order $h_{\alpha\beta}$. As a result, we may neglect the Lie derivative in (48)

$$\mathcal{L}_{\mathbf{N}}K_{\mu\nu}=N^\lambda\partial_\lambda K_{\mu\nu}+K_{\lambda\nu}\partial_\mu N^\lambda+K_{\mu\lambda}\partial_\nu N^\lambda\propto\left(h_{\alpha\beta}\right)^2\approx 0 \tag{120}$$

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as well as terms quadratic in $K_{\mu\nu}$. The evolution equation (48) for the extrinsic curvature now reduces to

$$\frac{1}{c_5}\partial_\tau K_{\mu\nu} = -\frac{1}{2}\sigma\partial_\mu\partial_\nu h_{55} - \sigma\bar{R}_{\mu\nu} + \sigma\frac{8\pi G}{c^4}\left(S_{\mu\nu} - \frac{1}{2}\bar{\eta}_{\mu\nu}\bar{S}\right) \tag{121}$$

and the constraints become

$$\bar{R} - \sigma\left(K^2 - K^{\mu\nu}K_{\mu\nu}\right) \approx \bar{R} = -\sigma\frac{16\pi G}{c^4}\kappa \tag{122}$$

$$D_\mu K^\mu_\nu - D_\nu K \approx \partial_\mu K^\mu_\nu - \partial_\nu K = \frac{8\pi G}{c^4}p_\nu \tag{123}$$

with source terms

$$S_{\mu\nu} \approx T_{\mu\nu} \qquad p_\nu \approx -T_{5\nu} \qquad \kappa \approx T_{55}\ . \tag{124}$$

We have shown [21] that for weak fields, the 4+1 formalism is a rearrangement of terms in the 5D Ricci tensor, equivalent to the wave equation. Writing the 5-components of equations (13) and (36) we have

$$R_{5\mu} = -\frac{1}{2}\partial^\beta\partial_\beta h_{5\mu} = k_G T_{\mu 5} = -k_G p_\mu \tag{125}$$

$$R_{55} = -\frac{1}{2}\partial^\beta\partial_\beta h_{55} = k_G T_{55} = k_G\kappa \tag{126}$$

from which it follows that any weak field solution with the component structure (87) will satisfy the constraints. Before imposing the Lorenz gauge for the spacetime terms, we split (34) as

$$\begin{aligned} R_{\mu\nu} &= \frac{1}{2}\eta^{\lambda\rho}\left(\partial_\rho\partial_\mu h_{\lambda\nu} - \partial_\nu\partial_\mu h_{\lambda\rho} + \partial_\nu\partial_\lambda h_{\mu\rho} - \partial_\rho\partial_\lambda h_{\mu\nu}\right) \\ &\qquad + \frac{1}{2}\eta^{55}\left(\partial_5\partial_\mu h_{5\nu} - \partial_\nu\partial_\mu h_{55} + \partial_\nu\partial_5 h_{\mu 5} - \partial_5^2 h_{\mu\nu}\right) \\ &= \frac{1}{2}\left(\partial_\mu\partial^\lambda h_{\lambda\nu} + \partial_\nu\partial^\rho h_{\mu\rho} - \partial^\lambda\partial_\lambda h_{\mu\nu} - \partial_\nu\partial_\mu h\right) + \frac{1}{2}\sigma\partial_5\left(\partial_\mu h_{5\nu} + \partial_\nu h_{\mu 5} - \partial_5 h_{\mu\nu}\right) \\ &= \bar{R}_{\mu\nu} + \sigma\partial_5 K_{\mu\nu} \end{aligned} \tag{127}$$

in which the first term is the projected Ricci tensor $\bar{R}_{\mu\nu}$ and the second term contains the form of the extrinsic curvature $K_{\mu\nu}$ in the weak field approximation. This expression for $K_{\mu\nu}$ can be rearranged as

$$\frac{1}{c_5}\partial_\tau h_{\mu\nu} = \partial_\mu h_{5\nu} + \partial_\nu h_{\mu 5} - 2K_{\mu\nu} \tag{128}$$

which recovers (119). Combining (127) with (13) and absorbing the negligible term $-\sigma\partial_\mu\partial_\nu h_{55}/2$ into $\bar{R}_{\mu\nu}$, we obtain

$$\frac{1}{c_5}\partial_\tau K_{\mu\nu} = -\sigma\bar{R}_{\mu\nu} + \sigma k_G\left(T_{\mu\nu} - \frac{1}{2}\bar{\eta}_{\mu\nu}\hat{T}\right) \tag{129}$$

which recovers (121). We thus see that the first order evolution equations are found by introducing the auxiliary variable $K_{\mu\nu}$ (almost, but not quite conjugate to $\gamma_{\mu\nu}$ [20]) and rearranging terms in the linearized field equations.

Splitting the Lorenz gauge condition into spacetime and 5-parts

$$\partial^\lambda h_{\mu\lambda} = -\partial^5 h_{\mu 5} + \frac{1}{2}\partial_\mu h \tag{130}$$

where $h = \eta^{\alpha\beta}h_{\alpha\beta}$ is the 5D trace, the projected Ricci tensor may be written

$$\bar{R}_{\mu\nu} = -\frac{1}{2}\partial^\lambda\partial_\lambda h_{\mu\nu} - \frac{1}{2}\sigma\partial_5\left(\partial_\mu h_{\nu 5} + \partial_\nu h_{\mu 5}\right) \tag{131}$$

providing a relatively simple means to calculate $\bar{R}_{\mu\nu}$. Combining (129) and (131) we recover the wave equation (37) with trace-reversed form of the source

$$\begin{aligned} R_{\mu\nu} &= \bar{R}_{\mu\nu} + \sigma\partial_5 K_{\mu\nu} \\ &= -\frac{1}{2}\partial^\lambda\partial_\lambda h_{\mu\nu} - \frac{1}{2}\sigma\partial_5\left(\partial_\mu h_{\nu 5} + \partial_\nu h_{\mu 5}\right) + \frac{1}{2}\sigma\partial_5\left(\partial_\mu h_{5\nu} + \partial_\nu h_{\mu 5} - \partial_5 h_{\mu\nu}\right) \\ &= -\frac{1}{2}\left(\partial^\lambda\partial_\lambda + \sigma\partial_5^2\right) h_{\mu\nu} = k_G\left(T_{\mu\nu} - \frac{1}{2}\bar{\eta}_{\mu\nu}\hat{T}\right) \end{aligned} \tag{132}$$

by eliminating the auxiliary variable $K_{\mu\nu}$.

For a metric ansatz with the component structure (87) the source can be found directly as

$$k_G\left(T_{\mu\nu} - \frac{1}{2}\bar{\eta}_{\mu\nu}\bar{T}\right) = -\frac{1}{2}\partial^\gamma\partial_\gamma h_{\mu\nu} = -\frac{1}{2}H^{00}\delta_{\mu\nu}\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \sigma\frac{1}{c_5^2}\frac{\partial^2}{\partial\tau^2}\right)\Phi \tag{133}$$

as we did in expression (112). Using (131) the projected Ricci tensor is

$$\bar{R}_{\mu\nu} = H^{00}\left[-\frac{1}{2}\delta_{\mu\nu}\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) + \xi^5\frac{1}{c_5}\frac{\partial}{\partial\tau}\left(\delta_{\nu 0}\frac{\partial}{\partial x^\mu} + \delta_{\mu 0}\frac{\partial}{\partial x^\nu}\right)\right]\Phi \tag{134}$$

and we compute the extrinsic curvature

$$K_{\mu\nu} = -H^{00}\left[\sigma\xi^5\left(\delta_{\nu 0}\frac{\partial}{\partial x^\mu} + \delta_{\mu 0}\frac{\partial}{\partial x^\nu}\right) + \frac{1}{2}\delta_{\mu\nu}\frac{1}{c_5}\frac{\partial}{\partial\tau}\right]\Phi \tag{135}$$

from which

$$\sigma\partial_5 K_{\mu\nu} = H^{00}\left[-\xi^5\frac{1}{c_5}\frac{\partial}{\partial\tau}\left(\delta_{\nu 0}\frac{\partial}{\partial x^\mu} + \delta_{\mu 0}\frac{\partial}{\partial x^\nu}\right) - \sigma\frac{1}{2}\delta_{\mu\nu}\frac{1}{c_5^2}\frac{\partial^2}{\partial\tau^2}\right]\Phi\,. \tag{136}$$

We see that the off-diagonal terms in $\bar{R}_{\mu\nu}$ and $\sigma\partial_5 K_{\mu\nu}$ mutually cancel, leaving the source diagonal as required. Inserting the explicit functional form for $\Phi(t, r, \tau)$ we obtain

$$\begin{aligned} \bar{R}_{\mu\nu} = \frac{H^{00}}{\rho^5}\Bigg\{ &-\frac{1}{2}\left[r^2 - 5c^2(t-\tau\xi)^2\right]\delta_{\mu\nu} + 2\xi^5\left(r^2 - 2c^2\left(t - \tau\xi^5\right)^2\right)\delta_{\mu 0}\delta_{\nu 0} \\ &-3c\left(t - \xi^5\tau\right)\left(\delta_\mu^0\delta_\nu^i + \delta_\mu^i\delta_\nu^0\right)x_i\Bigg\} \end{aligned} \tag{137}$$

and

$$\begin{aligned} K_{\mu\nu} = \sigma\frac{H^{00}}{\rho^5}\Bigg\{ &-\frac{1}{2}c\left(t - \xi^5\tau\right)\delta_{\mu\nu} + 2\xi^5 c\left(t - \xi^5\tau\right)\delta_{\mu 0}\delta_{\nu 0} \\ &+\xi^5\left(\delta_\mu^0\delta_\nu^i + \delta_\mu^i\delta_\nu^0\right)x_i\Bigg\}\,. \end{aligned} \tag{138}$$

We may modify the source $T_{\mu\nu}$ and find the modified metric using standard perturbation theory. For a perturbed source

$$T_{\mu\nu} = T^{(0)}_{\mu\nu} + \lambda T^{(1)}_{\mu\nu} + \lambda^2 T^{(2)}_{\mu\nu} + \cdots \tag{139}$$

where the perturbation is parameterized by λ. We seek a perturbed metric

$$h_{\alpha\beta} = h^{(0)}_{\alpha\beta} + \lambda h^{(1)}_{\alpha\beta} + \lambda^2 h^{(2)}_{\alpha\beta} + \cdots \tag{140}$$

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expressed in some general form. For example, writing

$$\Phi^{(0)} = \frac{1}{\sqrt{r^2 + c^2 \left(t - \xi^5 \tau\right)^2}} \longrightarrow \Phi = \frac{1}{\sqrt{r^2 + c^2 A\left(t,\tau\right)}} \tag{141}$$

where

$$A\left(t,\tau\right) = \left(t - \xi^5 \tau\right)^2 + \lambda \alpha \left(t,\tau\right) \tag{142}$$

leads to

$$\Phi^{(1)} \simeq -\frac{1}{2} \frac{\lambda \alpha \left(t,\tau\right)}{r^2 + c^2 \left(t - \xi^5 \tau\right)^2} \tag{143}$$

to first order. Since we are working in the linearized theory, we will have

$$R_{\mu\nu} = R^{(0)}_{\mu\nu} + \lambda R^{(1)}_{\mu\nu} \qquad\qquad K_{\mu\nu} = K^{(0)}_{\mu\nu} + \lambda K^{(1)}_{\mu\nu} \tag{144}$$

to first order. Similarly, the evolution equations are linear and are solved exactly by $h^{(0)}_{\mu\nu}$ for $T^{(0)}_{\mu\nu}$. Therefore, the evolution equations for the perturbed metric reduce to the evolution equations for the perturbation itself

$$\partial_5 h^{(1)}_{\mu\nu} = -2K^{(1)}_{\mu\nu} + \partial_\mu h^{(1)}_{5\nu} + \partial_\nu h^{(1)}_{5\mu} \tag{145}$$

$$\partial_5 K^{(1)}_{\mu\nu} = -\sigma \left[R^{(1)}_{\mu\nu} - \frac{8\pi G}{c^4} \left(S^{(1)}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} S^{(1)} \right) \right] \tag{146}$$

where we may use (134) and (135) to express $R^{(1)}_{\mu\nu}$ and $K^{(1)}_{\mu\nu}$ in terms of the perturbation $\alpha\left(t,\tau\right)$. We note that in order to preserve the structure of the evolution equations as an initial value problem, we must maintain $\partial_5 \alpha\left(t,\tau\right)$ as an independent dynamical quantity and choose a value for $\partial_5 \alpha\left(t,0\right)$.

6. Discussion

In this paper, we considered the relationship between the evolving metric $\gamma_{\mu\nu}(x,\tau)$ and the evolving matter source $T_{\mu\nu}(x,\tau)$ in weak field SHP general relativity, and discussed approaches to determining the $\gamma_{\mu\nu}$ for a given $T_{\mu\nu}$. We attempted to solve the 5D wave equation for a source localized and evolving uniformly in its rest frame, and demonstrated the difficulties with this approach. We saw that the geodesic equations for a test event moving in the spacetime described by $\gamma_{\mu\nu}$ are highly sensitive to initial conditions and become unstable under general choices for the τ-synchronization with the source. This issue is attributed to the product structure $\phi(t,r,\tau)/r$ for general solutions found from the approximate Green's function $G_{\text{Maxwell}}(x,\tau)$, where ϕ is a narrow distribution that localizes the source along the t-axis at a given τ. This product structure was seen to result from the fact that G_{Maxwell} provides exact solutions to the 4D wave function, and therefore leads to metric solutions to the 4D Einstein equations, not correctly accounting for the evolution of spacetime under τ. Thus, while solving the wave equation for the linearized field equations is normally the simplest way to obtain the metric, we concluded that the shortcomings of the available Green's function for the 5D wave equation make this method impractical. As for general fields in 4D GR, the most convenient approach to the metric is to solve the ADM-like first order evolution equations.

In order to apply the 4+1 formalism, we must begin with an unperturbed metric ansatz. Thus, we proposed an ansatz with many of the expected properties for the metric, and found the associated matter source from the linearized field equations. In general, the source was seen to describe an event density evolving at constant velocity along the t-axis in its rest frame, within a small volume $[c(t-\xi^5\tau)]^3$. At a given point $(t,\mathbf{0})$ in the rest frame of the source, $T_{\mu\nu}(x,\tau)$ has support only in a narrow region of chronological time around $\tau = t/\xi^5$. Consequently, the induced metric at a given point $(t,\mathbf{x})$ drops off as $1/|\mathbf{x}|$, but its support is similarly restricted to this narrow region around τ. The geodesic equations for this metric were seen to recover Newtonian gravitation in the nonrelativistic limit, with possible deviations from standard models at relativistic energies.

We proceeded to characterize the 4+1 evolution equations in the weak field limit and demonstrate their relationship to the 5D Ricci tensor. We showed that the first order initial value problem emerges from the linearized 5D field equations by introducing the extrinsic curvature $K_{\mu\nu}$ as an auxiliary variable, much in the way that Hamiltonian mechanics poses first order equations of motion for position and its conjugate momentum, which is treated as an independent quantity. The 5D wave equation is easily found by combining expressions for the projected Ricci tensor $\bar{R}_{\mu\nu}$ and $K_{\mu\nu}$, in such a way that off-diagonal terms cancel. Given the linearized evolution equations, and the metric ansatz that solves them exactly, the initial value problem for a perturbed metric is just the linearized evolution equations for the perturbation terms themselves. We showed that with a presumed general structure for the perturbed metric, the evolution equations become a set of first order (in τ) partial differential equations in the parameters of the perturbation.

While the ansatz chosen for study in this paper contains the general features expected in an evolving metric, these methods can be applied to other forms. For example, writing the functional dependence of the metric as

$$\Phi\left(r,t,\tau\right) = \frac{1}{r}e^{-\left(t-\xi^5\tau\right)^2/\tau_0^2} \tag{147}$$

for some short time scale τ_0, we easily find the source to be

$$T_{\mu\nu} = \frac{Mc^2}{4\pi\rho^5}\text{diag}\left(-1,0,0,0\right)\left[-\frac{1}{4\pi}\delta^3\left(\mathbf{x}\right) + \left(1-\sigma\right)\frac{2}{rc_5^2\tau_0^2}\left(1-\frac{2\left(t-\tau\right)^2}{\tau_0^2}\right)\right]e^{-(t-\tau)^2/\tau_0^2} \tag{148}$$

and in the neighborhood of $t = \xi^5\tau$ the geodesic equations become

$$\ddot{t} = 0 \longrightarrow \dot{t} = \xi^5 \tag{149}$$

and

$$0 = \ddot{x}^k + \frac{GM}{r^2}\left\{\hat{x}^k - \left(\frac{2\hat{x}_j\dot{x}^k - \hat{x}^k\dot{x}_j}{c}\right)\frac{\dot{x}^j}{c}\frac{c^2}{c_5^2}\right\} \tag{150}$$

which is identical to (106) found using the functional form for Φ in (88). These results suggests that other forms for Φ may lead to phenomenological behavior of similar type. As we showed, perturbative modifications to (147) may be considered by again replacing $(t-\xi^5\tau)^2 \longrightarrow A(t,\tau)$ as we did in (142). Such perturbations will be considered in a subsequent paper.

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