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Posted Date: 10 June 2025

doi: 10.20944/preprints202506.0791.v1

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Pair of Associated η-Ricci–Bourguignon Almost Solitons with Generalized Conformal Killing Potential on Sasaki-like Almost Contact Complex Riemannian Manifolds

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Abstract: The subject of study is almost contact complex Riemannian manifolds, also known as almost contact B-metric manifolds. The considerations are restricted to a special class of these manifolds, namely those of the Sasaki-like type, because of their geometric construction and the explicit expression of their classification tensor by the pair of B-metrics. Here, each of the two B-metrics is considered as an η-Ricci–Bourguignon almost soliton, where η is the contact form. The soliton potential is chosen to be a conformal Killing vector field (in particular, concircular or concurrent) and then a generalization of the notion of conformality using contact conformal transformations of B-metrics. The resulting manifolds, equipped with the introduced almost solitons, are geometrically characterized. In the five-dimensional case, an explicit example on a Lie group depending on two real parameters is constructed, and the properties obtained in the theoretical part are confirmed.

Keywords: η-Ricci–Bourguignon almost soliton; almost contact B-metric manifold; almost contact complex Riemannian manifold; Sasaki-like manifold; conformal killing vector field

MSC: 53C25; 53D15; 53C50; 53C44; 53D35; 70G45

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1. Introduction

Let $\mathcal{M}$ be a smooth manifold equipped with a time-dependent family $g(t)$ of (pseudo-)Riemannian metrics. Suppose that the corresponding Ricci tensor $\rho(t)$ and scalar curvature $\tau(t)$ satisfy together with $g(t)$ the following flow equation

$$\frac{\partial}{\partial t}g = -2(\rho - \ell\tau g), \qquad g(0) = g_0,$$

where ℓ is a real constant. The evolution equation defined as described above is known as the *Ricci–Bourguignon flow* (or RB flow for short) and was introduced by J. P. Bourguignon in [1] to generalize other well-known geometric flows by choosing specific values of ℓ. Namely, these are: the Ricci flow for $\ell = 0$, the Einstein flow for $\ell = \frac{1}{2}$, the traceless Ricci flow for $\ell = \frac{1}{m}$, and the Schouten flow for $\ell = \frac{1}{2(m-1)}$, where m is the dimension of the manifold [2,3].

It is known that the solitons of an internal geometric flow on $\mathcal{M}$ are its fixed points or self-similar solutions. The corresponding soliton of the Ricci–Bourguignon flow is called the *Ricci–Bourguignon soliton* (abbreviated RB soliton) and is defined by [4]

$$\rho + \frac{1}{2}\mathcal{L}_\vartheta g + (\lambda + \ell\tau)g = 0, \tag{1}$$

where $\mathcal{L}_\vartheta g$ denotes the Lie derivative of g with respect to the vector field ϑ, called the soliton potential, and λ is called the soliton constant.

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An RB soliton is said to be *expanding*, *steady*, or *shrinking* if $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively.

In the case where λ is not a constant, but a differentiable function on $\mathcal{M}$, then (1) defines an *RB almost soliton* [5].

Other recent studies by other authors on RB (almost) solitons are published in [6–11].

An RB (almost) soliton is called *trivial* if the soliton potential is a *Killing vector field*. Such a vector field is an infinitesimal generator of isometry, which means that the flow of a Killing vector field ϑ preserves the given metric, i.e. $\mathcal{L}_\vartheta g = 0$ holds.

A slightly more general concept than the latter is that of conformal Killing vector fields, i.e. vector fields with a flow preserving a given conformal class of metrics. A *conformal Killing vector field* on a manifold with (pseudo-)Riemannian metric g is a vector field ϑ whose (locally defined) flow defines conformal transformations, i.e. preserve g up to scale and preserve the conformal structure. In other words, the Lie derivative of g with respect to ϑ is a multiple of g by a coefficient of some function on the manifold. This refers to the classical conformal transformation of the metric, which performs a change on the metric tensor by multiplying it by a scalar function known as the conformal factor. Conformal vector fields appear in various physical theories where angle-preserving symmetries are important, such as conformal field theory and its applications in condensed matter physics, statistical mechanics, quantum statistical mechanics, and string theory [12]. In particular, if the conformal factor is a constant, such vector fields are known as *homothetic* and they find application in the study of singularities in general relativity [13].

We focus our research on almost contact complex Riemannian manifolds (abbreviated accR manifolds), also known as almost contact B-metric manifolds. What is special about them is that they are equipped with a pair of pseudo-Riemannian metrics (known as B-metrics) g and $\tilde{g}$, which are interrelated by the almost contact structure. Some studies on such manifolds by other authors that have been published in recent years are [14–17].

One approach to using both B-metrics is given in [18]. There, a generalization of the RB almost soliton of (1) is studied, defined as follows

$$\rho + \frac{1}{2}\mathcal{L}_\vartheta g + \frac{1}{2}\mathcal{L}_\vartheta \tilde{g} + (\lambda + \ell\tau)g + (\tilde{\lambda} + \ell\tilde{\tau})\tilde{g} = 0, \tag{2}$$

where $\tilde{\lambda}$ is also a function on $\mathcal{M}$ and $\tilde{\tau}$ is the scalar curvature of $\tilde{g}$.

In [19] we launched another idea, different from that of (2), to include both B-metrics in the definition of RB almost solitons. Namely, to generate an η-RB almost soliton from each of the two B-metrics via the contact form η.

In this paper, we specialize accR manifolds into a type called *Sasaki-like accR manifolds*. Thus, we not only determine the Lie derivative of the metric, but we are also motivated by the rich geometric properties of manifolds of Sasaki type, their connections to physics, and their role as a generalization of Riemannian geometry. The Sasaki-like accR manifolds are defined geometrically by the condition that the complex cone of such a manifold is a holomorphic complex Riemannian manifold (also called a Kähler-Norden manifold).

The soliton potential is usually chosen to be at some special positions relative to the structures. A popular possibility is that the soliton potential is a vertical vector field, i.e. pointwise collinear with the Reeb vector field, as this case is studied in [19]. Another substantial case, which is the subject of study here, is when the soliton potential is a conformal Killing vector field, i.e. the Lie derivative of any B-metric with respect to the potential is the same metric multiplied by a function. We also study in this paper a more general case, when the Lie derivative of any B-metric with respect to the potential is obtained by a contact conformal transformation inherent to accR manifolds. In this case, we said that the potential is a generalized conformal Killing vector field.

We also consider the soliton potential to be conformal Killing of a special kind, namely a concircular vector field. Such type of vector fields on Riemannian manifolds have been introduced in [20]. A *concircular vector field* is defined by its covariant derivative being parallel to the identity tensor.

 doi:10.20944/preprints202506.0791.v1

It is well known that concircular vector fields play an important role in the theory of projective and conformal transformations and have applications in general relativity and mathematical physics [21].

An even narrower specialization is made for the soliton potential, with its covariant derivative coinciding with the identity tensor. Then the vector field is called *concurrent* [22]. Such vector fields can describe flows in several areas of theoretical physics. For example, in general relativity they are used to model the trajectories of galaxies and to characterize space-time structures.

2. accR Manifolds

Let $\mathcal{M}$ be a smooth $(2n+1)$-dimensional manifold equipped with an almost contact structure (φ, ξ, η) and a B-metric g. In more detail, φ is an endomorphism in the Lie algebra $\mathfrak{X}(\mathcal{M})$ of vector fields on $\mathcal{M}$, ξ is a Reeb vector field, η is its dual contact form, and g is a pseudo-Riemannian metric of signature $(n+1, n)$ for which the following identities hold:

$$\begin{gathered}\varphi\xi = 0, \qquad \varphi^2 = -\iota + \eta \otimes \xi, \qquad \eta \circ \varphi = 0, \qquad \eta(\xi) = 1, \\ g(\varphi x, \varphi y) = -g(x, y) + \eta(x)\eta(y),\end{gathered} \tag{3}$$

where ι is the identity in $\mathfrak{X}(\mathcal{M})$ [23]. In the last equality and further on, by x, y, z we denote arbitrary elements of $\mathfrak{X}(\mathcal{M})$ or vectors in the tangent space $T_p\mathcal{M}$ of $\mathcal{M}$ at an arbitrary point p of $\mathcal{M}$.

It is known that on $\mathcal{M}$ there exists an associated metric $\tilde{g}$ of g, which is also a B-metric and is defined by

$$\tilde{g}(x, y) = g(x, \varphi y) + \eta(x)\eta(y).$$

Such a manifold is called an *almost contact B-metric manifold* or an *almost contact complex Riemannian manifold* (abbreviated *accR manifold*) and we use $(\mathcal{M}, \varphi, \xi, \eta, g, \tilde{g})$ to denote it. The well-known Ganchev–Mihova–Gribachev classification of accR manifolds is introduced in [23]. It consists of 11 basic classes $\mathcal{F}_i$, $i \in \{1, 2, \dots, 11\}$, determined by conditions for the (0,3)-tensor F defined by

$$F(x, y, z) = g\big((\nabla_x \varphi)y, z\big),$$

where ∇ denotes the Levi-Civita connection of g. It has the following basic properties:

$$\begin{gathered}F(x, y, z) = F(x, z, y) = F(x, \varphi y, \varphi z) + \eta(y)F(x, \xi, z) + \eta(z)F(x, y, \xi), \\ F(x, \varphi y, \xi) = (\nabla_x \eta)(y) = g(\nabla_x \xi, y).\end{gathered}$$

The Lee forms of $(\mathcal{M}, \varphi, \xi, \eta, g, \tilde{g})$ are the following 1-forms associated with F:

$$\theta(z) = g^{ij}F(e_i, e_j, z), \quad \theta^*(z) = g^{ij}F(e_i, \varphi e_j, z), \quad \omega(z) = F(\xi, \xi, z),$$

where $\left(g^{ij}\right)$ is the inverse matrix of the matrix $\left(g_{ij}\right)$ of g with respect to a basis $\{e_i; \xi\}$ $(i = 1, 2, \dots, 2n)$ of T_pM.

2.1. Sasaki-Like accR Manifolds

A Sasaki-like accR manifold, according to the definition in [24], is such that its warped product manifold with the real line, called the complex cone of such a manifold, is a Kähler–Norden manifold. An accR manifold, which is Sasaki-like, is defined using F by the condition

$$F(x, y, z) = g(\varphi x, \varphi y)\eta(z) + g(\varphi x, \varphi z)\eta(y).$$

Therefore, Sasaki-like accR manifolds have the following Lee forms

$$\theta = -2n\,\eta, \qquad \theta^* = \omega = 0 \tag{4}$$

 doi:10.20944/preprints202506.0791.v1

and the following identities hold [24]

$$\nabla_x \xi = -\varphi x, \qquad (\nabla_x \eta)(y) = -g(x, \varphi y), \tag{5}$$
$$\rho(x, \xi) = 2n\,\eta(x), \qquad \rho(\xi, \xi) = 2n, \tag{6}$$

where ρ denotes the Ricci tensor for g.

Since the contact form η is closed on these manifolds, taking into account the second equality in (5), then θ in (4) is also closed.

Let $\tilde{\rho}$ denote the Ricci tensor defined by $\tilde{g}$. In [19] for every Sasaki-like accR manifold it is proved that ρ and $\tilde{\rho}$ coincide, i.e.

$$\tilde{\rho}(y, z) = \rho(y, z). \tag{7}$$

As a consequence of (6) and (7) we obtain the following for a Sasaki-like accR manifold:

$$\tilde{\rho}(x, \xi) = 2n\,\eta(x), \qquad \tilde{\rho}(\xi, \xi) = 2n. \tag{8}$$

Contracting (7), we obtain a dependence between the scalar curvature $\tilde{\tau}$ concerning $\tilde{g}$ and the associated quantity of τ regarding φ, defined by $\tau^* = g^{ij}\rho(e_i, \varphi e_j)$. This formula has the form $\tilde{\tau} = -\tau^* + \rho(\xi, \xi)$, which combining with the last result in (6) implies

$$\tilde{\tau} = -\tau^* + 2n. \tag{9}$$

2.2. *Almost Einstein-Like accR Manifolds*

For accR manifolds, the notion of an Einstein-like manifold was introduced in [25] and studied for Ricci-like solitons, but is applicable to any $(\mathcal{M}, \varphi, \xi, \eta, g, \tilde{g})$. Such a manifold is called *Einstein-like* if its Ricci tensor ρ satisfies the condition

$$\rho = a\,g + b\,\tilde{g} + c\,\eta \otimes \eta \tag{10}$$

for some triplet of constants (a, b, c). In particular, $(\mathcal{M}, \varphi, \xi, \eta, g, \tilde{g})$ is called an *η-Einstein manifold* if $b = 0$, or an *Einstein manifold* if $b = c = 0$.

In [26] another, more specialized notion of Einstein-like accR manifolds was introduced and studied, namely the so-called *v-Einstein* manifold (v is an abbreviation for vertically), if $a = b = 0$ and $c \neq 0$.

If a, b, c in (10) are functions on $\mathcal{M}$, then the manifold is called *almost Einstein-like*, *almost η-Einstein* and *almost Einstein*, respectively [27].

A corresponding definition for an Einstein-like accR manifold using Ricci tensor $\tilde{\rho}$ is appropriate and the corresponding condition of (10) is the following

$$\tilde{\rho} = \tilde{a}\,\tilde{g} + \tilde{b}\,g + \tilde{c}\,\eta \otimes \eta$$

for some triplet of constants $(\tilde{a}, \tilde{b}, \tilde{c})$. The notions arising from this idea, such as those for ρ above, are also relevant.

3. η-RB Almost Solitons

Many publications have studied more general solitons, for which an additional 1-form η is used in the definition, the so-called η-solitons. For example, some recent publications on η-RB solitons are [6,10]. This approach makes particular sense when the 1-form is part of the tensor structure of the manifold. In our case, this is the contact form η and we use this idea for the RB almost solitons defined by (1).

 doi:10.20944/preprints202506.0791.v1

This gives us an *η-Ricci–Bourguignon almost soliton* (in short, η-RB almost soliton), induced by the B-metric g in the following way:

$$\rho + \frac{1}{2}\mathcal{L}_{\vartheta} g + (\lambda + \ell\tau) g + \mu\,\eta \otimes \eta = 0, \tag{11}$$

where μ is also a function on $\mathcal{M}$ [6]. Hereafter, we denote this almost soliton by $(g;\vartheta;\lambda,\mu,\ell)$.

Recall that when λ and μ are constants on the manifold, $(g;\vartheta;\lambda,\mu,\ell)$ is called an *η-RB soliton*.

In this paper, we exploit the idea of including both B-metrics in the definition of η-RB almost solitons, but with exactly the same roles, unlike the so-called RB-like almost soliton defined by (2) in [18]. Recall that both B-metrics act on the vertical distribution span(ξ) as $\eta \otimes \eta$.

Analogously to (11), we can also consider an η-RB almost soliton induced by the other B-metric $\tilde{g}$, denoted by $(\tilde{g};\vartheta;\tilde{\lambda},\tilde{\mu},\tilde{\ell})$ and defined as follows

$$\tilde{\rho} + \frac{1}{2}\mathcal{L}_{\vartheta} \tilde{g} + (\tilde{\lambda} + \tilde{\ell}\tilde{\tau}) \tilde{g} + \tilde{\mu}\,\eta \otimes \eta = 0, \tag{12}$$

where $\tilde{\lambda}$ and $\tilde{\mu}$ are functions on $\mathcal{M}$, and $\tilde{\ell}$ is a constant.

Clearly, an η-RB almost soliton $(\tilde{g};\vartheta;\tilde{\lambda},\tilde{\mu},\tilde{\ell})$ with $\tilde{\mu} = 0$ is simply an RB almost soliton with respect to $\tilde{g}$.

Similarly, if $\tilde{\lambda}$ and $\tilde{\mu}$ are constants on $\mathcal{M}$, then the *solitons* of the corresponding type are said to be given.

Further in this paper we study manifolds for which we use the following

Definition 1 ([19])**.** *An accR manifold $(\mathcal{M},\varphi,\xi,\eta,g,\tilde{g})$ is said to be equipped with a pair of associated η-RB almost solitons with potential vector field ϑ if the corresponding Ricci tensors ρ, $\tilde{\rho}$ and scalar curvatures τ, $\tilde{\tau}$ satisfy* (11) *and* (12)*, respectively.*

3.1. The Potential Is a Conformal Killing Vector Field

A vector field on $\mathcal{M}$, e.g. the potential ϑ, is called *conformal Killing with respect to g* if there exists a function ψ on $\mathcal{M}$ such that [5]

$$\mathcal{L}_{\vartheta} g = 2\psi g, \tag{13}$$

where $\psi = \operatorname{div}\vartheta / \dim \mathcal{M}$. Particular cases of conformal vector fields are homothetic vector fields for which $\psi =$ const and isometric vector fields, also called Killing vector fields, for which $\psi = 0$. The conformal Killing vector field is *non-trivial* if $\psi \neq 0$, otherwise ϑ is called a *Killing vector field* with respect to g.

The counterpart notion regarding the other metric can also be considered. Namely, a vector field ϑ is called a *conformal with respect to $\tilde{g}$* if there exists a function $\tilde{\psi}$ on $\mathcal{M}$ such that

$$\mathcal{L}_{\vartheta} \tilde{g} = 2\tilde{\psi}\tilde{g}. \tag{14}$$

Similarly, it is defined by $\tilde{\psi}$ whether a conformal vector field is non-trivial or Killing with respect to $\tilde{g}$.

Theorem 1. *Let $(\mathcal{M},\varphi,\xi,\eta,g,\tilde{g})$ be a $(2n+1)$-dimensional Sasaki-like accR manifold that admits a pair of associated η-RB almost solitons $(g;\vartheta;\lambda,\mu,\ell)$ and $(\tilde{g};\vartheta;\tilde{\lambda},\tilde{\mu},\tilde{\ell})$, where the potential ϑ is a conformal Killing vector field with respect to g and $\tilde{g}$ with functions ψ and $\tilde{\psi}$, respectively.*

Then the manifold is v-Einstein regarding ρ and $\tilde{\rho}$ with one and the same triplet of constants $(a,b,c) = (\tilde{a},\tilde{b},\tilde{c}) = (0,0,2n)$, i.e. $\mathcal{M}$ has the following Ricci tensors with respect to the pair of B-metrics:

$$\rho = \tilde{\rho} = 2n\,\eta \otimes \eta, \tag{15}$$

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and equal scalar curvatures with respect to g and $\tilde{g}$ as follows:

$$\tau = \tilde{\tau} = 2n. \tag{16}$$

Furthermore, the following conditions for the used functions are valid:

$$\lambda + \psi = -2n\ell, \qquad \tilde{\lambda} + \tilde{\psi} = -2n\tilde{\ell}, \tag{17}$$

$$\mu = \tilde{\mu} = -2n. \tag{18}$$

Proof. Taking into account (11) and (13), we obtain the form of the Ricci tensor for g as follows

$$\rho = -(\lambda + \ell\tau + \psi)g - \mu\,\eta \otimes \eta. \tag{19}$$

Therefore, the manifold is almost η-Einstein. Similarly, (14) and (12) imply the following form of the Ricci tensor for $\tilde{g}$:

$$\tilde{\rho} = -(\tilde{\lambda} + \tilde{\ell}\tilde{\tau} + \tilde{\psi})\tilde{g} - \tilde{\mu}\,\eta \otimes \eta. \tag{20}$$

We use (6) and (8) with (19) and (20) to get the following conditions:

$$\lambda + \ell\tau + \psi = -(\mu + 2n), \tag{21}$$

$$\tilde{\lambda} + \tilde{\ell}\tilde{\tau} + \tilde{\psi} = -(\tilde{\mu} + 2n). \tag{22}$$

After that we substitute (21) and (22) into (19) and (20) respectively to obtain

$$\rho = (\mu + 2n)g - \mu\,\eta \otimes \eta, \tag{23}$$

$$\tilde{\rho} = (\tilde{\mu} + 2n)\tilde{g} - \tilde{\mu}\,\eta \otimes \eta. \tag{24}$$

Then, we take the appropriate traces of (23) and (24) to obtain τ and $\tilde{\tau}$ for g and $\tilde{g}$, respectively. The results are the following

$$\tau = 2n(\mu + 2n + 1), \tag{25}$$

$$\tilde{\tau} = 2n(\tilde{\mu} + 2n + 1). \tag{26}$$

Taking into account (7), then (23) and (24) imply

$$(\mu + 2n)g - (\tilde{\mu} + 2n)\tilde{g} - (\mu - \tilde{\mu})\eta \otimes \eta = 0,$$

which is equivalent to the following equality

$$(\mu + 2n)g(\varphi x, \varphi y) + (\tilde{\mu} + 2n)g(x, \varphi y) = 0. \tag{27}$$

Replacing y with φy in (27) gives the following equation:

$$(\tilde{\mu} + 2n)g(\varphi x, \varphi y) - (\mu + 2n)g(x, \varphi y) = 0. \tag{28}$$

Therefore, there exists a solution to the system of equations (27) and (28) for arbitrary x and y if and only if the conditions in (18) are satisfied. The latter result is taken into account in (25) and (26). Thus the corresponding scalar curvatures are obtained as in (16).

Similarly, (18), (23) and (24) imply the form of the two Ricci tensors given in (15).

Next we substitute (16) and (18) into (21) and (22) and get the equalities in (17). □

doi:10.20944/preprints202506.0791.v1

3.2. The Potential Is a Generalized Conformal Killing Vector Field

A vector field ϑ is called a *generalized conformal Killing vector field* with respect to g if the following condition is satisfied:

$$\frac{1}{2}(\mathcal{L}_\vartheta g) = e^{2u}\cos 2v\, g + e^{2u}\sin 2v\, \tilde{g} + \Big(e^{2w} - e^{2u}\cos 2v - e^{2u}\sin 2v\Big)\eta\otimes\eta,$$

where $u, v, w \in \mathcal{F}(\mathcal{M})$. For brevity we can denote $\psi = e^{2u}\cos 2v$, $\chi = e^{2u}\sin 2v$, $\omega = e^{2w}$ and the above definition condition takes the following form:

$$\frac{1}{2}(\mathcal{L}_\vartheta g) = \psi\, g + \chi\, \tilde{g} + (\omega - \psi - \chi)\eta\otimes\eta, \tag{29}$$

where $\psi, \chi, \omega \in \mathcal{F}(\mathcal{M})$.

If $\chi = 0$ and $\omega = \psi$ are valid, then (29) implies (13), the condition for a usual conformal Killing vector field regarding g.

Similarly, ϑ is called a generalized conformal Killing vector field with respect to $\tilde{g}$ if it satisfies:

$$\frac{1}{2}(\mathcal{L}_\vartheta \tilde{g}) = \tilde{\psi}\, \tilde{g} + \tilde{\chi}\, g + (\tilde{\omega} - \tilde{\psi} - \tilde{\chi})\eta\otimes\eta, \tag{30}$$

where $\tilde{\psi}, \tilde{\chi}, \tilde{\omega} \in \mathcal{F}(\mathcal{M})$.

In this case, the specialization $\tilde{\chi} = 0$ and $\tilde{\omega} = \tilde{\psi}$ gives us (14), the condition for a usual conformal Killing vector field regarding $\tilde{g}$.

Theorem 2. *Let $(\mathcal{M}, \varphi, \xi, \eta, g, \tilde{g})$ be a $(2n+1)$-dimensional Sasaki-like accR manifold that admits a pair of associated η-RB almost solitons $(g; \vartheta; \lambda, \mu, \ell)$ and $(\tilde{g}; \vartheta; \tilde{\lambda}, \tilde{\mu}, \tilde{\ell})$, where the potential ϑ is a generalized conformal Killing vector field with respect to g and $\tilde{g}$ with functions (ψ, χ, ω) and $(\tilde{\psi}, \tilde{\chi}, \tilde{\omega})$, respectively.*

Then the manifold is almost Einstein-like regarding ρ and $\tilde{\rho}$ with triplets of functions $(a, b, c) = (-\tilde{\chi}, -\chi, \chi + \tilde{\chi} + 2n)$ and $(\tilde{a}, \tilde{b}, \tilde{c}) = (-\chi, -\tilde{\chi}, \chi + \tilde{\chi} + 2n)$, i.e. $\mathcal{M}$ has the following Ricci tensors with respect to each of the B-metrics:

$$\rho = \tilde{\rho} = -\tilde{\chi}\, g - \chi\tilde{g} + (\chi + \tilde{\chi} + 2n)\eta\otimes\eta \tag{31}$$

and the scalar curvatures with respect to g and $\tilde{g}$ are as follows:

$$\tau = 2n(1 - \tilde{\chi}), \qquad \tilde{\tau} = 2n(1 - \chi). \tag{32}$$

Furthermore, the following conditions for the used functions are valid:

$$\omega - \psi + \mu + 2n = -\tilde{\chi}, \qquad \tilde{\omega} - \tilde{\psi} + \tilde{\mu} + 2n = -\chi, \tag{33}$$

$$\lambda - 2n\ell\psi + (1 + 2n\ell)(\omega + \mu) = -2n\{1 + (2n+1)\ell\}, \tag{34}$$

$$\tilde{\lambda} - 2n\tilde{\ell}\tilde{\psi} + (1 + 2n\tilde{\ell})(\tilde{\omega} + \tilde{\mu}) = -2n\{1 + (2n+1)\tilde{\ell}\}. \tag{35}$$

In particular, we get that the manifold is:

(i) almost η-Einstein regarding g if and only if χ is zero.
(ii) almost η-Einstein regarding $\tilde{g}$ if and only if $\tilde{\chi}$ is zero.
(iii) v-Einstein if and only if $\chi = \tilde{\chi} = 0$ holds.

Proof. Combining (11) and (29), we obtain the following

$$\rho = -(\lambda + \ell\tau + \psi)g - \chi\, \tilde{g} - (\omega - \psi - \chi + \mu)\eta\otimes\eta, \tag{36}$$

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We apply the last expression of ρ for (ξ, ξ) and due to the last equality in (6) we obtain

$$\ell\tau = -(\lambda + \omega + \mu + 2n). \tag{37}$$

Then, we put (37) into (36) and the form of the Ricci tensor for g is as follows

$$\rho = (\omega - \psi + \mu + 2n)g - \chi\tilde{g} - (\omega - \psi + \mu - \chi)\eta \otimes \eta. \tag{38}$$

Taking the trace of (38), we get an expression of the scalar curvature for g

$$\tau = 2n(\omega - \psi + \mu + 2n + 1). \tag{39}$$

Now we take the trace of ρ_{ij} from (38) with $\varphi_s^j g^{is}$ to obtain the associated quantity of τ as follows

$$\tau^* = 2n\chi. \tag{40}$$

We proceed similarly with the quantities for the other B-metric $\tilde{g}$. The equalities in (8), (12) and (30) imply the following expressions for the Ricci tensor and the scalar curvature regarding $\tilde{g}$:

$$\tilde{\rho} = (\tilde{\omega} - \tilde{\psi} + \tilde{\mu} + 2n)\tilde{g} - \tilde{\chi} g - (\tilde{\omega} - \tilde{\psi} + \tilde{\mu} - \tilde{\chi})\eta \otimes \eta, \tag{41}$$

$$\tilde{\tau} = 2n(\tilde{\omega} - \tilde{\psi} + \tilde{\mu} + 2n + 1). \tag{42}$$

Taking into account (9), (40) and (42), we obtain the relation in the second equality of (33), which is between some of the used functions. The equality in (38) simplifies the form of $\tilde{\rho}$ from (41) to the corresponding form in (31).

Similarly, (42) and the second equality in (33) give the expression of $\tilde{\tau}$ in (32).

The equalities in (7), (31) and (38), together with the last identity in (3), imply

$$(\omega - \psi + \mu + \tilde{\chi} + 2n)g(\varphi x, \varphi y) = 0,$$

which is true if and only if the first equality in (33) holds. Considering the same equality, the expressions for ρ and τ from (38) and (39) take the forms in (31) and (32), respectively.

Let us return to (37) and use (39). Then we get the relation in (34) between the functions λ, μ and the constant ℓ for the η-RB almost soliton concerning g and the functions ψ and ω for its potential ϑ.

In a similar way, the dependence in (35) is obtained for the corresponding functions of the η-RB almost soliton concerning $\tilde{g}$. □

3.3. The Potential Is Concircular or Concurrent

Recall that if for a vector field, e.g. ϑ, there exists a function β on the manifold such that $\nabla_x\vartheta = \beta x$ is valid for any vector field x, then ϑ is called a *concircular* vector field. In particular, if β is a constant, then ϑ is called a *concurrent* vector field.

Since it is known that $(\mathcal{L}_\vartheta g)(x, y) = g(\nabla_x\vartheta, y) + g(x, \nabla_y\vartheta)$ is true, then for a concircular vector field ϑ with respect to g it follows $\mathcal{L}_\vartheta g = 2\beta g$, i.e. every concircular vector field ϑ is conformal regarding g. A similar statement is also true regarding $\tilde{g}$, i.e. $\mathcal{L}_\vartheta \tilde{g} = 2\tilde{\beta}\tilde{g}$ is valid if ϑ is concircular with respect to $\tilde{g}$ with function $\tilde{\beta}$.

Corollary 1. *Let the conventions of Theorem 1 be given. A special case arises when the potential ϑ is concurrent with respect to each of the two B-metrics. Then $(g; \vartheta; \lambda, \mu, \ell)$ and $(\tilde{g}; \vartheta; \tilde{\lambda}, \tilde{\mu}, \tilde{\ell})$ are η-RB solitons.*

Proof. In the case under consideration, since $\psi = \beta$ and $\tilde{\psi} = \tilde{\beta}$ are constants and the results in (17) are known, then the functions λ and $\tilde{\lambda}$ become constants. Then, λ and $\tilde{\lambda}$, together with the constants μ and $\tilde{\mu}$ from (17), determine $(g; \vartheta; \lambda, \mu, \ell)$ and $(\tilde{g}; \vartheta; \tilde{\lambda}, \tilde{\mu}, \tilde{\ell})$ as η-RB solitons, respectively. □

Corollary 2. *Let $(\mathcal{M}, \varphi, \xi, \eta, g, \tilde{g})$ satisfy the conditions of Theorem 2. In particular, if the potential ϑ is:*

(i) concircular with respect to g and $\tilde{g}$ for functions β and $\tilde{\beta}$, respectively, then the manifold is v-Einstein with η-RB almost solitons $(g; \vartheta; \lambda, \mu, \ell)$ and $(\tilde{g}; \vartheta; \tilde{\lambda}, \tilde{\mu}, \tilde{\ell})$, so that

$$\rho = \tilde{\rho} = 2n\,\eta \otimes \eta, \qquad \tau = \tilde{\tau} = 2n, \tag{43}$$
$$\lambda + 2n\ell = -\beta, \qquad \tilde{\lambda} + 2n\tilde{\ell} = -\tilde{\beta}, \qquad \mu = \tilde{\mu} = -2n. \tag{44}$$

(ii) concurrent with respect to each of the two B-metrics, then $(g; \vartheta; \lambda, \mu, \ell)$ and $(\tilde{g}; \vartheta; \tilde{\lambda}, \tilde{\mu}, \tilde{\ell})$ are η-RB solitons.

Proof. Under the assumption about ϑ in (i) it follows $\psi = \omega = \beta$, $\tilde{\psi} = \tilde{\omega} = \tilde{\beta}$ and $\chi = \tilde{\chi} = 0$ hold. Then (31)–(35) is specialized in the corresponding form in (43) and (44).

The assumption in (ii) is a special case of that in (i), with β and $\tilde{\beta}$ being constants. Then the results in (44) show that the considered almost solitons become solitons of this type. □

3.4. Example of an η-RB Almost Soliton with a Generalized Conformal Killing Potential

We now consider an explicit example of a Sasaki-like accR manifold given as Example 2 in [24]. It is on a Lie group G of dimension 5, i.e. for $n = 2$, with a basis of left-invariant vector fields $\{e_0, \dots, e_4\}$. The corresponding Lie algebra is defined by the commutators

$$\begin{aligned} &[e_0, e_1] = pe_2 + e_3 + qe_4, && [e_0, e_2] = -pe_1 - qe_3 + e_4, \\ &[e_0, e_3] = -e_1 - qe_2 + pe_4, && [e_0, e_4] = qe_1 - e_2 - pe_3, \qquad p, q \in \mathbb{R}. \end{aligned} \tag{45}$$

An accR structure $(\varphi, \xi, \eta, g, \tilde{g})$ is defined as follows

$$\begin{aligned} &g(e_0, e_0) = g(e_1, e_1) = g(e_2, e_2) = -g(e_3, e_3) = -g(e_4, e_4) = 1, \\ &g(e_i, e_j) = 0, \quad i, j \in \{0, 1, \dots, 4\},\ i \neq j, \\ &\xi = e_0, \quad \varphi e_1 = e_3, \quad \varphi e_2 = e_4, \quad \varphi e_3 = -e_1, \quad \varphi e_4 = -e_2. \end{aligned}$$

After that, in [25], the components of the curvature tensor $R_{ijkl} = R(e_i, e_j, e_k, e_l)$ and those of the Ricci tensor $\rho_{ij} = \rho(e_i, e_j)$ are calculated. Thus, the form of its Ricci tensor is established as $\rho = 4\,\eta \otimes \eta$ and the manifold is v-Einstein. The scalar curvatures for g and $\tilde{g}$ are obtained as $\tau = \tilde{\tau} = 4$ and the constructed manifold is $*$-scalar flat, i.e. $\tau^* = 0$. These calculations confirm the results in (15) and (16) and the related conclusions of Theorem 1.

Using (11) and (12), we construct a pair of associated η-RB almost solitons $(g; \vartheta; \lambda, \mu, \ell)$ and $(\tilde{g}; \vartheta; \tilde{\lambda}, \tilde{\mu}, \tilde{\ell})$ on $(G, \varphi, \xi, \eta, g, \tilde{g})$ with the following functions

$$\lambda = 2(t - 2\ell), \quad \tilde{\lambda} = -2(t + 2\tilde{\ell}), \qquad \mu = \tilde{\mu} = -4, \tag{46}$$

assuming that $\eta = \mathrm{d}t$. We then choose the potential ϑ to be a generalized conformal Killing vector field with respect to each of the two B-metrics for a corresponding set of functions (ψ, χ, ω) and $(\tilde{\psi}, \tilde{\chi}, \tilde{\omega})$, i.e. (29) and (30) are valid. We choose the involved functions as follows

$$\psi = \omega = -\tilde{\psi} = -\tilde{\omega} = -2t, \qquad \chi = \tilde{\chi} = 0. \tag{47}$$

We verify that (46) and (47) satisfy (33), (34) and (35) according to Theorem 2 and (44) in Corollary 2.

The same choice of functions as in (46) also satisfies the requirement that ϑ be concircular with respect to g and $\tilde{g}$ for functions $\beta = -2t$ and $\tilde{\beta} = 2t$, respectively.

As another option, we choose β and $\tilde{\beta}$ to be constants, e.g. $\beta = -2$ and $\tilde{\beta} = 2$. Therefore, we obtain that ϑ is concurrent with respect to g and $\tilde{g}$, respectively, as well as $\lambda = 2 - 4\ell$ and $\tilde{\lambda} = -2 - 4\tilde{\ell}$

are constants. The last conclusion means that $(g;\vartheta;\lambda,\mu,\ell)$ and $(\tilde{g};\vartheta;\tilde{\lambda},\tilde{\mu},\tilde{\ell})$ are η-RB solitons in this case.

We establish that the constructed η-RB soliton $(g;\vartheta;\lambda,\mu,\ell)$ is expanding, steady, or shrinking if and only if we choose $\ell<\frac{1}{2}$, $\ell=\frac{1}{2}$, or $\ell>\frac{1}{2}$, respectively. Analogously, $(\tilde{g};\vartheta;\tilde{\lambda},\tilde{\mu},\tilde{\ell})$ is expanding, steady, or shrinking if and only if we set $\tilde{\ell}<-\frac{1}{2}$, $\tilde{\ell}=-\frac{1}{2}$, or $\tilde{\ell}>-\frac{1}{2}$, respectively.

We need to find whether such a vector field ϑ exists. Let it be $\vartheta=\vartheta^s e_s$ with respect to the considered basis for $s=0,1,2,3,4$. By virtue of the well-known Koszul formula and (45) we obtain the condition $p^4+q^4+2p^2q^2-2p^2+2q^2+1=0$ for parameters p and q, which has solutions $p_{1,2}=1\pm qi$, $p_{3,4}=-1\pm qi$. Since p and q are real, then we have $q=0$ and $p=1$ or $p=-1$. If we choose $q=0$ and $p=1$, we obtain for the potential $\vartheta=c_1e_1+c_2e_2-c_2e_3+c_1e_4$ for arbitrary constants c_1 and c_2. We verify that this ϑ is also concurrent with respect to $\tilde{g}$.

Funding: The research is partially supported by project FP25-FMI-010 “Innovative Interdisciplinary Research in Informatics, Mathematics, and Pedagogy of Education”, of the Scientific Fund of the Paisii Hilendarski University of Plovdiv, Bulgaria.

Institutional Review Board Statement: Not applicable.

Data Availability Statement: Data is contained within the article.

Conflicts of Interest: The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript; or in the decision to publish the results. The author declares no conflict of interest.

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