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[Rômulo Damasclin Chaves dos Santos](#)^{*} and [Jorge Henrique de Oliveira Sales](#)

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Article

Voronovskaya Expansions and Spectral Convergence for Neural Operators on Complex Foliated Manifolds

Rômulo Damasclin Chaves dos Santos * and Jorge Henrique de Oliveira Sales

Santa Cruz State University
* Correspondence: rdcantos@uesc.br

Abstract

This paper introduces a rigorous geometric-analytic framework for symmetrized neural operators defined on holomorphic foliations in complex manifolds. By integrating tools from complex geometry, functional analysis, and neural operator theory, we establish a novel class of operators that respect the intrinsic symmetries imposed by the foliation’s holonomy pseudogroup. Our analysis proves universal approximation theorems in leafwise Sobolev and C^k spaces, demonstrating that symmetrized neural operators can approximate arbitrary smooth functions on both compact and non-compact leaves with explicit convergence rates. A key result is the derivation of Voronovskaya-type asymptotic expansions, which reveal a deep connection between these operators and the leafwise Laplace–Beltrami operator, providing precise leading-order behavior and remainder estimates. We further establish L^p -stability and spectral decomposition results, linking the dynamics of symmetrized neural operators to the underlying foliation geometry. The framework extends to singular foliations and non-compact leaves, introducing weighted Sobolev spaces and spectral convergence results near singularities. These operators are interpreted as discrete approximations to leafwise diffusion processes, bridging operator-theoretic neural networks with classical complex dynamical systems. The results open new directions in high-dimensional approximation theory, spectral analysis, and geometrically constrained machine learning, with applications to Calabi–Yau manifolds and logarithmic foliations.

Keywords: holomorphic foliations; neural operators; Voronovskaya expansions; singular foliations; non-compact leaves

1. Introduction

The interplay between **complex geometry** and **operator theory** has long been a fertile ground for breakthroughs in pure mathematics, particularly in the study of dynamical systems, invariant measures, and spectral properties. **Holomorphic foliations**, which decompose complex manifolds into families of analytic leaves, offer a powerful framework for exploring these phenomena, as they encode both local and global geometric structures through their holonomy pseudogroups and transverse dynamics [1–3]. Concurrently, the emergence of **neural operators** a generalization of classical neural networks designed to learn mappings between infinite-dimensional function spaces has revolutionized approximation theory and computational mathematics, enabling data-driven approaches to problems in partial differential equations, dynamical systems, and geometric analysis [4–6].

Despite the remarkable progress in both fields, their intersection, the study of neural operators on foliated complex manifolds, remains largely unexplored. This gap presents a unique opportunity: by integrating the geometric constraints imposed by holomorphic foliations with the expressive power of neural operators, we can not only generalize universal approximation theory to foliated settings but also uncover profound connections between **asymptotic operator expansions** and **holonomy-invariant dynamics**. This work initiates a rigorous investigation into *symmetrized neural operators* defined on the leaves of complex foliations, systematically incorporating the symmetries dictated by their holonomy groups and transverse structures.

Key Contributions

Our study introduces a **geometric-analytic framework** for symmetrized neural operators on holomorphic foliations, with the following principal contributions:

- **Rigorous Formulation of Foliated Neural Operators:** We define a class of neural operators on foliated complex manifolds that respect the intrinsic symmetries of the foliation. By leveraging **holonomy-invariant symmetrization**, we ensure compatibility with the transverse geometry and the pseudogroup structure of the foliation.
- **Universal Approximation Theorems:** We prove that symmetrized neural operators can approximate arbitrary smooth functions on both **compact and non-compact leaves**, with explicit convergence rates in **leafwise Sobolev and C^k spaces**. This extends classical approximation theory to geometrically constrained function spaces, providing a foundation for applications in complex dynamics and spectral geometry.
- **Voronovskaya-Type Asymptotic Expansions:** We derive asymptotic expansions for these operators, revealing a deep connection between their behavior and the **leafwise Laplace–Beltrami operator**. These expansions include **leading-order terms** and **precise remainder estimates**, offering insights into the spectral and dynamical properties of the operators.
- **Spectral Decomposition and L^p -Stability:** We establish that symmetrized neural operators admit a **spectral decomposition** in $L^2(L)$, with eigenvalues asymptotically linked to the spectrum of the leafwise Laplacian. Additionally, we prove **L^p -stability** for $1 \leq p \leq \infty$, ensuring robustness in both theoretical and applied settings.
- **Dynamical Interpretation and Ergodicity:** We interpret symmetrized neural operators as **discrete approximations to leafwise diffusion processes**, proving the existence of **invariant measures** and **ergodic properties** along the leaves. This bridges operator-theoretic neural networks with classical complex dynamical systems, enabling the analysis of long-term behavior and stability.
- **Extensions to Non-Compact Leaves and Singular Foliations:** We generalize our framework to **non-compact leaves** using weighted Sobolev spaces and to **singular foliations**, where leaf dimensions may vary. Our results include **spectral convergence** near singularities and the smooth extension of eigenfunctions across singular sets, broadening the applicability of the theory to more general geometric settings.

Broader Implications

This work not only advances the theoretical understanding of neural operators in geometric contexts but also opens new avenues for **geometrically structured machine learning**. By incorporating the symmetries and constraints of holomorphic foliations, our framework enables the development of **foliation-aware neural architectures** with guaranteed approximation properties and stability characteristics. Potential applications span **complex dynamics, spectral geometry, theoretical physics** (e.g., Calabi–Yau manifolds and string theory), and **numerical analysis** (e.g., structure-preserving discretizations for foliated PDEs).

The remainder of the paper is organized to systematically develop these ideas, beginning with foundational material on holomorphic foliations, leafwise function spaces, and neural operators, followed by detailed proofs of the main theorems and a discussion of future research directions.

2. Preliminaries and Mathematical Foundations

This section introduces the key mathematical structures, definitions, and results that form the foundation for the subsequent study of foliated neural operators, Voronovskaya-type expansions, and leafwise dynamical analysis. We focus on holomorphic foliations, leafwise function spaces, integral operators, and spectral theory.

2.1. Holomorphic Foliations

Definition 1 (Holomorphic Foliation). *Let M be a complex manifold of complex dimension m . A holomorphic foliation $\mathcal{F}$ of leaf dimension p is a decomposition of M into connected, immersed complex submanifolds $L \subset M$, called leaves, such that:*

1. *Locally, M admits charts (U, ϕ) with coordinates $(z_1, \dots, z_m)$ in which leaves are given by*

$$z_{p+1} = \text{const}, \dots, z_m = \text{const}. \quad (2.1)$$

2. *Transition functions between overlapping charts preserve the leaf structure holomorphically.*

The set of holonomy maps associated with loops in the leaves forms the *holonomy pseudogroup* G , which encodes the transverse geometry of the foliation [1,3].

2.2. Leafwise Function Spaces

Let L be a leaf of $\mathcal{F}$.

Definition 2 (Leafwise Smooth Functions). *The space of k -times continuously differentiable functions along L is denoted*

$$C^k(L) := \{f : L \rightarrow \mathbb{C} \mid f \text{ has continuous derivatives up to order } k\}. \quad (2.2)$$

Definition 3 (Leafwise Sobolev Spaces). *For $s \geq 0$, the leafwise Sobolev space $H^s(L)$ consists of functions $f \in L^2(L)$ whose weak derivatives up to order s are square-integrable along L . Its norm is*

$$\|f\|_{H^s(L)} := \left(\sum_{|\alpha| \leq s} \int_L |D^\alpha f|^2 d\text{vol}_L \right)^{1/2}. \quad (2.3)$$

These spaces provide quantitative control over function regularity and are fundamental for approximation and stability results.

2.3. Foliated Neural Operators

Definition 4 (Foliated Neural Operator). *Let L be a leaf of $\mathcal{F}$. A foliated neural operator is a map*

$$\mathcal{N}_{\mathcal{F}} : C^\infty(L) \rightarrow C^\infty(L) \quad (2.4)$$

that admits an integral representation

$$(\mathcal{N}_{\mathcal{F}} f)(x) = \int_L K(x, y) \sigma(f(y)) d\text{vol}_L(y), \quad x \in L, \quad (2.5)$$

where $K : L \times L \rightarrow \mathbb{C}$ is a holomorphic kernel and σ is a Lipschitz or holomorphic activation function.

2.4. Holonomy-Invariant Symmetrization

Definition 5 (Holonomy-Invariant Symmetrized Operator). *Given a foliated neural operator $\mathcal{N}_{\mathcal{F}}$ and the holonomy pseudogroup G of L , the holonomy-invariant symmetrization is*

$$\mathcal{S}[\mathcal{N}_{\mathcal{F}}](f) := \frac{1}{|G|} \sum_{g \in G} g^{-1} \circ \mathcal{N}_{\mathcal{F}} \circ g(f). \quad (2.6)$$

This operator commutes with the action of G :

$$\forall g \in G, \quad g \circ \mathcal{S}[\mathcal{N}_{\mathcal{F}}] = \mathcal{S}[\mathcal{N}_{\mathcal{F}}] \circ g. \quad (2.7)$$

2.5. Leafwise Laplace–Beltrami Operator

Definition 6 (Leafwise Laplacian). Let (L, g_L) be a Riemannian leaf induced from a Hermitian metric on M . The leafwise Laplace–Beltrami operator Δ_L acts on smooth functions $f \in C^\infty(L)$ by

$$\Delta_L f = \operatorname{div}_L(\nabla_L f), \quad (2.8)$$

where ∇_L and div_L denote the gradient and divergence along L , respectively.

2.6. Spectral Theory and Compact Operators

Compact, self-adjoint integral operators on $L^2(L)$ admit an orthonormal basis of eigenfunctions. For a symmetrized neural operator $\mathcal{S}[\mathcal{N}_{\mathcal{F}}]$ with smooth kernel, this yields

$$\mathcal{S}[\mathcal{N}_{\mathcal{F}}]\phi_j = \lambda_j \phi_j, \quad \{\phi_j\}_{j \in \mathbb{N}} \text{ orthonormal in } L^2(L), \quad (2.9)$$

with eigenvalues $\lambda_j \rightarrow 0$ as $j \rightarrow \infty$.

2.7. Ergodic Theory Preliminaries

Let L be compact and $T : L \rightarrow L$ a measure-preserving transformation. A measure μ_L is *invariant* if

$$\int_L \varphi d\mu_L = \int_L \varphi \circ T d\mu_L, \quad \forall \varphi \in C(L). \quad (2.10)$$

The Birkhoff ergodic theorem [10] ensures that for μ_L -almost every $x_0 \in L$,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} \varphi(T^k x_0) = \int_L \varphi d\mu_L. \quad (2.11)$$

Remark 1. These preliminaries establish the analytic, geometric, and dynamical framework for studying foliated neural operators, universal approximation, Voronovskaya-type expansions, L^p -stability, and ergodic properties along leaves.

3. Complex Foliations and Holonomy

3.1. Refined Definition and Structural Properties

Definition 7 (Holomorphic Foliation). Let M be a complex manifold of dimension n . A holomorphic foliation $\mathcal{F}$ of codimension q and leaf dimension $p = n - q$ satisfies the following equivalent conditions:

(1) **Partition by complex submanifolds:** There exists a partition of M into disjoint connected immersed complex submanifolds

$$\mathcal{F} = \{L_\alpha\}_{\alpha \in A}, \quad \dim_{\mathbb{C}} L_\alpha = p. \quad (3.1)$$

(2) **Holomorphic foliated atlas:** There exists an open cover $\{U_i\}$ of M and biholomorphisms

$$\phi_i : U_i \rightarrow \mathbb{D}^p \times \mathbb{D}^q \quad (3.2)$$

such that

- Each leaf component $L_\alpha \cap U_i$ maps to a plaque $\mathbb{D}^p \times \{y\}$.
- On overlaps $U_i \cap U_j$, the transition maps preserve the local product structure:

$$\phi_j \circ \phi_i^{-1}(z, w) = (\psi_{ij}^1(z, w), \psi_{ij}^2(w)), \quad (3.3)$$

with ψ_{ij}^1 holomorphic in (z, w) and ψ_{ij}^2 holomorphic in w only.

A foliation is *regular* if all leaves have the same dimension; otherwise, it is *singular*. In this work, we focus on regular holomorphic foliations.

Theorem 8 (Equivalent Characterizations of Holomorphic Foliations). *Let M be a complex manifold of dimension n . The following are equivalent:*

1. $\mathcal{F}$ is a holomorphic foliation of codimension q .
2. There exists a holomorphic involutive distribution $\mathcal{D} \subset T^{1,0}M$ of rank p .
3. There exists an open cover $\{U_i\}$ and holomorphic submersions $f_i : U_i \rightarrow \mathbb{C}^q$ with $f_i = \gamma_{ij} \circ f_j$ on overlaps for biholomorphisms γ_{ij} .
4. There exists a holomorphic vector bundle map $\pi : T^{1,0}M \rightarrow N$ of rank q whose kernel is involutive.

Proof. The detailed proof follows by constructing local distributions

$$\mathcal{D}_i := \ker(d\pi_2 \circ d\phi_i) \subset T^{1,0}U_i, \quad (3.4)$$

gluing via transition maps, and applying the holomorphic Frobenius theorem [1] to obtain local submersions defining leaves. Equivalence with the bundle map π follows from the integration of involutive distributions. $\square$

Proposition 1 (Local Structure Theorem). *Let $\mathcal{F}$ be a holomorphic foliation of codimension q on M . For any $x \in M$, there exist:*

- A neighborhood $U \cong \mathbb{D}^p \times \mathbb{D}^q$
- Holomorphic coordinates $(z_1, \dots, z_p, w_1, \dots, w_q)$
- Holomorphic vector fields $X_1, \dots, X_p$ spanning $\mathcal{D}|_U$, with $[X_i, X_j] = 0$

so that leaves in U are given by

$$w = \text{constant}, \quad X_i = \frac{\partial}{\partial z_i}. \quad (3.5)$$

Remark 2 (Singular Foliations). *The theory extends to singular foliations where leaf dimensions vary. The distribution $\mathcal{D}$ is defined outside an analytic subvariety $S \subset M$, and involutivity holds on $M \setminus S$.*

Remark 3 (Transverse Structure). *Transition maps ψ_{ij}^2 define a transverse holomorphic structure. The cocycle condition*

$$\psi_{ik}^2 = \psi_{ij}^2 \circ \psi_{jk}^2 \quad (3.6)$$

endows a holonomy pseudogroup acting on $\mathbb{D}^q$.

Definition 9 (Holonomy Group). *Let L be a leaf and $x \in L$. Fix a transverse section Σ through x . The holonomy group $\text{Hol}(L, x)$ is the image of*

$$\Phi : \pi_1(L, x) \rightarrow \text{Diff}_0(\Sigma, x), \quad (3.7)$$

assigning to a loop its germ of holonomy map obtained by sliding along leaves.

Proposition 2 (Holonomy and Leaf Geometry — Refined Version). *Let $\mathcal{F}$ be a holomorphic foliation on a complex manifold M , and let $L \in \mathcal{F}$ be a leaf with holonomy group $\text{Hol}(L, x)$ at $x \in L$. Then:*

1. **Finite holonomy:** *If $\text{Hol}(L, x)$ is a finite group, then L is properly embedded in the leaf space $M/\mathcal{F}$ and its closure $\bar{L}$ is compact in $M/\mathcal{F}$.*
2. **Trivial holonomy:** *If $\text{Hol}(L, x)$ is trivial (i.e., consists only of the identity), then L is locally closed in M . More precisely, there exists a neighborhood $U \subset M$ of x such that*

$$L \cap U \text{ is closed in } U,$$

and the foliation is locally diffeomorphic to a product $U \cong L \cap U \times T$, where T is a transverse section.

3. **Holonomy growth and transverse dynamics:** Let $g \in \text{Hol}(L, x)$, and denote by $\|g^n\|$ a suitable measure of the derivative growth along transverse directions. Then:
- Subexponential growth $\|g^n\| = O(e^{\epsilon n})$ implies quasi-periodic transverse dynamics.
 - Exponential growth $\|g^n\| \sim e^{\lambda n}$, $\lambda > 0$, indicates sensitive dependence on initial conditions and may yield transverse ergodicity of the foliation.

3.2. Leafwise Function Spaces and Transverse Measures

Let $\mathcal{F}$ be a holomorphic foliation on a complex manifold M , and let L denote a leaf. The complex structure induces a Hermitian metric h and a Kähler form ω_L along each leaf. A transverse measure μ can be defined on a transversal Σ to integrate over families of leaves, providing a global structure for leafwise integration.

Definition 10 (Leafwise Function Spaces). For $1 \leq p \leq \infty$ and $s \geq 0$, define:

1. L^p spaces on leaves:

$$L^p(L) := \left\{ f : L \rightarrow \mathbb{C} \text{ measurable} \mid \|f\|_{L^p(L)}^p := \int_L |f|^p \omega_L < \infty \right\}. \quad (3.8)$$

2. Sobolev spaces $W^{s,p}(L)$ for integer s :

$$W^{s,p}(L) := \left\{ f \in L^p(L) \mid \nabla_L^k f \in L^p(L), 0 \leq k \leq s \right\}, \quad \|f\|_{W^{s,p}(L)} := \sum_{k=0}^s \|\nabla_L^k f\|_{L^p(L)}. \quad (3.9)$$

3. Sobolev spaces $H^s(L)$ for real $s \geq 0$: via the leafwise Kodaira Laplacian Δ_L ,

$$H^s(L) := \{ f \in L^2(L) \mid (I + \Delta_L)^{s/2} f \in L^2(L) \}, \quad \|f\|_{H^s(L)} := \|(I + \Delta_L)^{s/2} f\|_{L^2(L)}. \quad (3.10)$$

Proposition 3 (Completeness and Sobolev Embeddings). For each leaf L :

1. $L^p(L)$, $W^{s,p}(L)$, and $H^s(L)$ are Banach spaces, and Hilbert spaces if $p = 2$.
2. For $s > \frac{\dim_{\mathbb{C}} L}{p}$, the Sobolev embedding holds locally:

$$H^s(L) \hookrightarrow C^0(L), \quad W^{s,p}(L) \hookrightarrow C^0(L), \quad (3.11)$$

with norms equivalent under leafwise charts and invariant under holonomy.

Proof. (Completeness) The L^p norms (3.8) define complete metric spaces since the Kähler form ω_L is smooth and positive-definite on the compact leaf charts. For Sobolev spaces $W^{s,p}(L)$, completeness follows from standard arguments using partitions of unity subordinate to leaf charts, and equivalence of local and global norms via transition maps. For $H^s(L)$, the Kodaira Laplacian is self-adjoint and elliptic, yielding a complete Hilbert space structure by spectral theory.

(Sobolev Embedding) Using local holomorphic charts $\phi : U \subset L \rightarrow \mathbb{D}^{\dim_{\mathbb{C}} L}$ and standard Sobolev inequalities in Euclidean space, we have for $s > \dim_{\mathbb{C}} L / p$:

$$\|f\|_{C^0(U)} \leq C \|f\|_{W^{s,p}(U)}.$$

Gluing via a partition of unity subordinate to the leaf atlas and using holonomy-invariance of norms yields the global embedding (3.11). $\square$

Remark 4 (Global Leafwise Spaces). A transverse measure μ allows defining global L^p spaces along the foliation:

$$L^p(\mathcal{F}, \mu) := \left\{ f : M \rightarrow \mathbb{C} \mid \int_{\Sigma} \|f|_L\|_{L^p(L)}^p d\mu(L) < \infty \right\}. \quad (3.12)$$

3.3. Symmetrized Neural Operators on Foliated Spaces

Definition 11 (Holonomy-Invariant Neural Operator). Let $\mathcal{F}$ be a holomorphic foliation and $\mathcal{H}$ its holonomy pseudogroup. A neural operator on $\mathcal{F}$ is a family

$$\{\mathcal{N}_L : \mathcal{X}(L) \rightarrow \mathcal{Y}(L)\}_{L \in M/\mathcal{F}}$$

satisfying:

- **Transversely continuous:** $L \mapsto \mathcal{N}_L$ is continuous.
- **Locally defined:** On each chart $U \cong \mathbb{D}^p \times \mathbb{D}^q$, $\mathcal{N}_L|_U$ is determined by a local operator $\mathcal{N}_U$.

Definition 12 (Symmetrized Neural Operator). Let G be a compact Lie group acting on M preserving $\mathcal{F}$, with normalized Haar measure μ_G . Define the symmetrized operator by

$$\mathcal{S}[\mathcal{N}](f)(x) := \int_G (g^* \circ \mathcal{N} \circ (g^{-1})^*)(f)(x) d\mu_G(g), \quad g^*f := f \circ g. \quad (3.13)$$

Proposition 4 (Properties of Symmetrization). The operator $\mathcal{S}$ defined in (3.13) satisfies:

1. **G-invariance:** $\mathcal{S}[\mathcal{N}] \circ g^* = g^* \circ \mathcal{S}[\mathcal{N}]$ for all $g \in G$.
2. **Idempotence:** $\mathcal{S} \circ \mathcal{S} = \mathcal{S}$.
3. **Continuity:** $\|\mathcal{S}[\mathcal{N}]\|_{L^p(L) \rightarrow L^p(L)} \leq \|\mathcal{N}\|_{L^p(L) \rightarrow L^p(L)}$.
4. **Holonomy compatibility:** If G contains the holonomy group, $\mathcal{S}[\mathcal{N}]$ descends naturally to the leaf space $M/\mathcal{F}$.

Proof. The G -invariance follows by invariance of the Haar measure under left translation:

$$\mathcal{S}[\mathcal{N}] \circ g^*(f) = \int_G h^* \mathcal{N}(h^{-1})^*(f \circ g) d\mu_G(h) = g^* \mathcal{S}[\mathcal{N}](f).$$

Idempotence holds because iterating the averaging operator over a compact group does not change the G -invariant subspace. Continuity is immediate from Minkowski's integral inequality. Holonomy compatibility is a consequence of the invariance of leafwise norms under holonomy transformations. $\square$

Remark 5 (Integral Kernel Representation). If $\mathcal{N}$ is given by a leafwise kernel K , then

$$(\mathcal{S}[\mathcal{N}]f)(x) = \int_G \int_L K(g(x), g(y)) \sigma(f(y)) \omega_L(y) d\mu_G(g), \quad (3.14)$$

with K holonomy-invariant and leafwise biholomorphic.

Remark 6 (Connection to Representation Theory). Symmetrization (3.13) projects operators onto the G -invariant subspace. The Peter–Weyl theorem guarantees a Fourier decomposition for compact G , enabling spectral analysis of symmetrized neural operators.

4. Neural Operators on Foliated Complex Manifolds

4.1. Foliated Neural Operators

Definition 13 (Foliated Neural Operator). Let L be a leaf of a holomorphic foliation $\mathcal{F}$. A foliated neural operator is a mapping

$$\mathcal{N}_{\mathcal{F}} : C^\infty(L) \rightarrow C^\infty(L) \quad (4.1)$$

that admits an integral representation of the form

$$(\mathcal{N}_{\mathcal{F}}f)(x) = \int_L K(x, y) \sigma(f(y)) d\text{vol}_L(y), \quad x \in L, \quad (4.2)$$

where $K : L \times L \rightarrow \mathbb{C}$ is holomorphic in each variable separately, satisfying the integrability condition

$$\int_L |K(x, y)| d\text{vol}_L(y) < \infty, \quad \forall x \in L, \quad (4.3)$$

ensuring that $\mathcal{N}_{\mathcal{F}} f$ is well-defined for $f \in L^p(L)$, $1 \leq p \leq \infty$. The function $\sigma : \mathbb{C} \rightarrow \mathbb{C}$ is a nonlinear activation, assumed globally Lipschitz or holomorphic, guaranteeing smoothness and stability of the output. This formulation allows $\mathcal{N}_{\mathcal{F}}$ to capture both local and nonlocal interactions along the leaf while respecting the complex geometric structure.

4.2. Continuity in Sobolev Spaces

Theorem 14 (Continuity of Foliated Neural Operators in Sobolev Spaces). *Let $\mathcal{N}_{\mathcal{F}}$ be a foliated neural operator as in (4.2), and assume $K \in H^s(L \times L)$ for some $s \geq 0$, with σ globally Lipschitz. Then*

$$\mathcal{N}_{\mathcal{F}} : H^s(L) \rightarrow H^s(L) \quad (4.4)$$

is a bounded operator. More precisely, there exists a constant $C_s > 0$ such that

$$\|\mathcal{N}_{\mathcal{F}} f\|_{H^s(L)} \leq C_s \|f\|_{H^s(L)}, \quad \forall f \in H^s(L). \quad (4.5)$$

Proof. For $s = 0$, consider the L^2 -norm:

$$\|\mathcal{N}_{\mathcal{F}} f\|_{L^2(L)}^2 = \int_L \left| \int_L K(x, y) \sigma(f(y)) d\text{vol}_L(y) \right|^2 d\text{vol}_L(x). \quad (4.6)$$

By applying the Cauchy–Schwarz inequality to the inner integral and Fubini’s theorem, we obtain

$$\|\mathcal{N}_{\mathcal{F}} f\|_{L^2(L)} \leq \sup_{x \in L} \left(\int_L |K(x, y)|^2 d\text{vol}_L(y) \right)^{1/2} \|\sigma(f)\|_{L^2(L)}. \quad (4.7)$$

Since σ is globally Lipschitz, there exists $C > 0$ such that $\|\sigma(f)\|_{L^2(L)} \leq C \|f\|_{L^2(L)}$, establishing boundedness in $L^2(L)$.

For $s > 0$, differentiating under the integral sign gives

$$\nabla_L^k (\mathcal{N}_{\mathcal{F}} f)(x) = \int_L \nabla_x^k K(x, y) \sigma(f(y)) d\text{vol}_L(y), \quad 1 \leq k \leq s. \quad (4.8)$$

Estimating each term as in (4.7) and summing over k leads to (4.5). $\square$

Remark 7. This result provides the foundation for defining symmetrized foliated neural operators acting holonomy-invariantly on $H^s(L)$, since continuity ensures that compositions and averaging over the holonomy group are well-defined in Sobolev spaces.

4.3. Holonomy-Invariant Symmetrization

Theorem 15 (Holonomy-Invariant Symmetrized Neural Operators). *Let $\mathcal{N}_{\mathcal{F}}$ be a foliated neural operator on a leaf L of a holomorphic foliation $\mathcal{F}$, and let G be the holonomy pseudogroup associated with L , assumed finite or compact with normalized Haar measure μ . Define the holonomy-invariant symmetrization by*

$$\mathcal{S}[\mathcal{N}_{\mathcal{F}}](f) := \begin{cases} \frac{1}{|G|} \sum_{g \in G} g^{-1} \circ \mathcal{N}_{\mathcal{F}} \circ g(f), & \text{if } G \text{ is finite,} \\ \int_G g^{-1} \circ \mathcal{N}_{\mathcal{F}} \circ g(f) d\mu(g), & \text{if } G \text{ is compact.} \end{cases} \quad (4.9)$$

Then the following properties hold:

1. **Holonomy invariance:**

$$g \circ \mathcal{S}[\mathcal{N}_{\mathcal{F}}] = \mathcal{S}[\mathcal{N}_{\mathcal{F}}] \circ g, \quad \forall g \in G. \quad (4.10)$$

2. **Boundedness:** $\mathcal{S}[\mathcal{N}_{\mathcal{F}}]$ is bounded in $H^s(L)$, with

$$\|\mathcal{S}[\mathcal{N}_{\mathcal{F}}]f\|_{H^s(L)} \leq \|\mathcal{N}_{\mathcal{F}}\|_{H^s(L)} \|f\|_{H^s(L)}. \quad (4.11)$$

Proof. For finite G , relabeling $k = hg^{-1} \in G$ in the sum gives (4.10). For compact G , the invariance follows from left-invariance of the Haar measure and the change of variable $k = hg^{-1}$. Boundedness (4.11) is immediate since each g acts as an isometry in $H^s(L)$ and the sum/integral preserves norms. $\square$

Remark 8. Holonomy-invariant symmetrization ensures that operator outputs are consistent across transverse directions and respect the geometric structure of the foliation. For integral operators with holomorphic kernels, symmetrization also preserves analyticity along leaves and commutes with leafwise differential operators, which is fundamental for spectral theory and Voronovskaya-type expansions.

5. Universal Approximation on Leaves

Theorem 16 (Universal Approximation on Foliated Leaves). *Let L be a compact leaf of a holomorphic foliation $\mathcal{F}$, and let $\mathcal{N}_{\mathcal{F}}$ be a symmetrized neural operator acting on $C^k(L)$ for $k \geq 0$. Then, for every $f \in C^k(L)$ and $\varepsilon > 0$, there exists a choice of $\mathcal{N}_{\mathcal{F}}$ such that*

$$\|\mathcal{N}_{\mathcal{F}}f - f\|_{C^k(L)} < \varepsilon. \quad (5.1)$$

Proof. Each leaf L is a compact complex manifold of dimension p . Classical results in complex analysis imply that for any local coordinate chart $U \subset L$, the restriction $f|_U$ can be approximated uniformly in the C^k norm by holomorphic polynomials P [7]:

$$\|f|_U - P\|_{C^k(U)} < \varepsilon. \quad (5.2)$$

Foliated neural operators of the integral form (4.2) can reproduce finite linear combinations of these polynomials by an appropriate choice of holomorphic kernel K and nonlinear activation σ . Therefore, for each local patch U , there exists a local operator $\mathcal{N}_{\mathcal{F}}^U$ such that

$$\|\mathcal{N}_{\mathcal{F}}^U f - f\|_{C^k(U)} < \varepsilon. \quad (5.3)$$

A smooth partition of unity $\{\chi_\alpha\}$ subordinate to a foliation atlas $\{U_\alpha\}$ allows extension of these local approximations to the entire leaf via

$$\mathcal{N}_{\mathcal{F}}f = \sum_{\alpha} \chi_{\alpha} \mathcal{N}_{\mathcal{F}}^{\alpha} f. \quad (5.4)$$

The holonomy-invariant symmetrization

$$\mathcal{S}[\mathcal{N}_{\mathcal{F}}] = \frac{1}{|G|} \sum_{g \in G} g^{-1} \circ \mathcal{N}_{\mathcal{F}} \circ g \quad (5.5)$$

preserves the approximation quality, since the sum of approximating operators remains an approximator and the action of $g \in G$ is linear. This establishes (5.1). $\square$

Theorem 17 (Approximation with Quantitative Rate in Sobolev Norms). *Let L be a compact leaf of dimension d and let $f \in H^{s+r}(L)$ with $r > 0$. There exists a sequence of symmetrized foliated neural operators $\{\mathcal{S}[\mathcal{N}_{\mathcal{F},n}]\}_{n \in \mathbb{N}}$ such that*

$$\|\mathcal{S}[\mathcal{N}_{\mathcal{F},n}]f - f\|_{H^s(L)} \leq Cn^{-r/d} \|f\|_{H^{s+r}(L)}, \quad (5.6)$$

where C depends only on the geometry of L and the choice of kernel, but not on f .

Proof. Let $\{U_\alpha\}_{\alpha \in A}$ be a finite foliation atlas covering the compact leaf L , and let $\{\chi_\alpha\}$ be a smooth partition of unity subordinate to this cover. On each local chart U_α , $f \in H^{s+r}(U_\alpha)$ can be approximated by holomorphic polynomials $P_{\alpha,n}$ such that

$$\|f - P_{\alpha,n}\|_{H^s(U_\alpha)} \leq C_\alpha n^{-r/d} \|f\|_{H^{s+r}(U_\alpha)}, \quad (5.7)$$

where C_α depends on the geometry of U_α but not on f .

For each $P_{\alpha,n}$, construct a local foliated neural operator $\mathcal{N}_{\mathcal{F},n}^{(\alpha)}$ reproducing $P_{\alpha,n}$ exactly. The holonomy-invariant symmetrization

$$\mathcal{S}[\mathcal{N}_{\mathcal{F},n}^{(\alpha)}] := \frac{1}{|G|} \sum_{g \in G} g^{-1} \circ \mathcal{N}_{\mathcal{F},n}^{(\alpha)} \circ g \quad (5.8)$$

ensures invariance under the holonomy pseudogroup G without altering the approximation property.

The global operator is then

$$\mathcal{S}[\mathcal{N}_{\mathcal{F},n}]f := \sum_{\alpha} \chi_\alpha \mathcal{S}[\mathcal{N}_{\mathcal{F},n}^{(\alpha)}]f. \quad (5.9)$$

Using the equivalence of Sobolev norms under partitions of unity and boundedness of χ_α in H^s , we obtain (5.6), with C depending on the atlas and leaf geometry but not on f . $\square$

Remark 9. Theorems 16 and 17 establish that symmetrized foliated neural operators provide both qualitative and quantitative universal approximation on compact leaves. Holonomy-invariant symmetrization ensures compatibility with the transverse foliation structure and underpins subsequent Voronovskaya-type asymptotic expansions and spectral analysis.

6. Voronovskaya-Type Asymptotic Expansion

Theorem 18 (Voronovskaya Expansion on Foliated Leaves). *Let $\{\mathcal{N}_{\mathcal{F},n}\}_{n \in \mathbb{N}}$ be a sequence of symmetrized foliated neural operators acting on a compact leaf L of a holomorphic foliation $\mathcal{F}$. Suppose $f \in C^{k+2}(L)$, $k \geq 0$. Then, for each $x \in L$, there exists an asymptotic expansion*

$$\mathcal{N}_{\mathcal{F},n}f(x) = f(x) + \frac{1}{2n} \Delta_L f(x) + R_n(f)(x), \quad (6.1)$$

where Δ_L is the Laplace–Beltrami operator along L , and the remainder satisfies

$$\|R_n(f)\|_{C^k(L)} = o(n^{-1}) \quad \text{as } n \rightarrow \infty. \quad (6.2)$$

Proof. Consider a local foliation chart (U, ϕ) with coordinates $(z_1, \dots, z_p)$, $p = \dim L$. For $f \in C^{k+2}(L)$, the second-order Taylor expansion around $x \in U$ is

$$f(y) = f(x) + \sum_{i=1}^p \frac{\partial f}{\partial z_i}(x)(y_i - x_i) + \frac{1}{2} \sum_{i,j=1}^p \frac{\partial^2 f}{\partial z_i \partial z_j}(x)(y_i - x_i)(y_j - x_j) + r_2(x, y), \quad (6.3)$$

with

$$\|r_2(x, \cdot)\|_{C^k(U)} \leq C \|y - x\|^3, \quad (6.4)$$

for some constant $C > 0$ depending on $\|f\|_{C^{k+2}(U)}$.

Substituting (6.3) into the integral representation of $\mathcal{N}_{\mathcal{F},n}$ and expanding the nonlinear activation σ around $f(x)$, the leading contributions are

$$\begin{aligned} \mathcal{N}_{\mathcal{F},n}f(x) &= f(x) + \sum_{i=1}^p \frac{\partial f}{\partial z_i}(x) \underbrace{\int_L K_n(x,y)(y_i - x_i) d\text{vol}_L(y)}_{=0 \text{ by symmetrization}} \\ &+ \frac{1}{2} \sum_{i,j=1}^p \frac{\partial^2 f}{\partial z_i \partial z_j}(x) \int_L K_n(x,y)(y_i - x_i)(y_j - x_j) d\text{vol}_L(y) + R_n(f)(x). \end{aligned} \quad (6.5)$$

The holonomy-invariant symmetrization

$$\mathcal{S}[\mathcal{N}_{\mathcal{F},n}](f) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \circ \mathcal{N}_{\mathcal{F},n} \circ g(f) \quad (6.6)$$

ensures cancellation of first-order terms (linear in $(y - x)$) under averaging over G , leaving the second-order term as dominant. This term is exactly the Laplace–Beltrami contribution along L :

$$\frac{1}{2n} \Delta_L f(x) = \frac{1}{2} \sum_{i=1}^p \frac{\partial^2 f}{\partial z_i^2}(x) \int_L K_n(x,y)(y_i - x_i)^2 d\text{vol}_L(y). \quad (6.7)$$

The remainder $R_n(f)$ arises from:

- Third-order derivatives in the Taylor expansion,
- Mixed terms in the expansion of σ ,
- Higher-order nonlinear contributions in f .

Using the Sobolev embedding $H^{k+2}(L) \hookrightarrow C^k(L)$ and standard kernel estimates, one obtains

$$\|R_n(f)\|_{C^k(L)} \leq Cn^{-1-\delta}, \quad \text{for some } \delta > 0, \quad (6.8)$$

establishing (6.2). Since the leaf is compact, the argument can be extended to all charts using a partition of unity, yielding the global expansion (8.11). $\square$

Remark 10. This Voronovskaya-type expansion rigorously connects the leafwise Laplace–Beltrami operator to the leading-order behavior of symmetrized neural operators. It provides a foundation for precise error estimates, L^p -stability, and spectral analysis on foliated complex manifolds [8,9].

Remark 11. Extensions to non-compact leaves, weighted Sobolev norms, or operators with holomorphic kernels on singular leaves are possible under suitable integrability and regularity conditions, preserving the asymptotic structure.

7. Stability and Spectral Properties

7.1. L^p -Stability

Theorem 19 (L^p -Stability). *Let L be a leaf of a holomorphic foliation $\mathcal{F}$ and let $\mathcal{S}[\mathcal{N}_{\mathcal{F}}]$ be a holonomy-invariant symmetrized neural operator with integral kernel $K \in L^\infty(L \times L)$. Then, for every $1 \leq p \leq \infty$, there exists a constant $C_p > 0$ such that*

$$\|\mathcal{S}[\mathcal{N}_{\mathcal{F}}]f\|_{L^p(L)} \leq C_p \|f\|_{L^p(L)}, \quad \forall f \in L^p(L), \quad (7.1)$$

where C_p depends only on $\|K\|_{L^\infty}$, the Lipschitz constant of σ , and the volume of L .

Proof. The operator admits the integral form

$$(\mathcal{S}[\mathcal{N}_{\mathcal{F}}]f)(x) = \int_L \tilde{K}(x,y) \sigma(f(y)) d\text{vol}_L(y), \quad (7.2)$$

with symmetrized kernel

$$\tilde{K}(x, y) = \frac{1}{|G|} \sum_{g \in G} K(g(x), g(y)) \in L^\infty(L \times L). \quad (7.3)$$

For $1 \leq p < \infty$, Minkowski's integral inequality yields

$$\|\mathcal{S}[\mathcal{N}_{\mathcal{F}}]f\|_{L^p(L)} \leq \|\tilde{K}\|_{L^\infty} \|\sigma(f)\|_{L^p(L)}. \quad (7.4)$$

Since σ is globally Lipschitz with constant L_σ ,

$$|\sigma(f(y))| \leq L_\sigma |f(y)| + |\sigma(0)|, \quad (7.5)$$

and therefore

$$\|\mathcal{S}[\mathcal{N}_{\mathcal{F}}]f\|_{L^p(L)} \leq \|\tilde{K}\|_{L^\infty} \left(L_\sigma \|f\|_{L^p(L)} + |\sigma(0)| (\text{vol}(L))^{1/p} \right), \quad (7.6)$$

which establishes (7.1). The case $p = \infty$ follows analogously by taking essential supremum. $\square$

7.2. Spectral Decomposition and Asymptotics

Proposition 5 (Spectral Decomposition of Symmetrized Neural Operators). *Let L be a compact leaf and $\mathcal{S}[\mathcal{N}_{\mathcal{F}}] : L^2(L) \rightarrow L^2(L)$ a self-adjoint symmetrized neural operator with smooth kernel. Then there exists an orthonormal basis $\{\phi_j\}_{j \in \mathbb{N}}$ of $L^2(L)$, consisting of eigenfunctions of the leafwise Laplacian Δ_L , such that*

$$\mathcal{S}[\mathcal{N}_{\mathcal{F}}]\phi_j = \lambda_j \phi_j, \quad j \in \mathbb{N}, \quad (7.7)$$

with eigenvalues satisfying the asymptotic expansion

$$\lambda_j = 1 - \frac{\mu_j}{2n} + o(n^{-1}) \quad \text{as } n \rightarrow \infty, \quad (7.8)$$

where $\Delta_L \phi_j = -\mu_j \phi_j$ and $\mu_j \geq 0$.

Proof. Self-adjointness and compactness of $\mathcal{S}[\mathcal{N}_{\mathcal{F}}]$ on $L^2(L)$ guarantee, via the spectral theorem, the existence of an orthonormal eigenbasis $\{\phi_j\}$.

Using the Voronovskaya-type expansion (Theorem 18) on ϕ_j ,

$$\mathcal{S}[\mathcal{N}_{\mathcal{F},n}]\phi_j = \phi_j + \frac{1}{2n} \Delta_L \phi_j + R_n(\phi_j) = \left(1 - \frac{\mu_j}{2n} + o(n^{-1}) \right) \phi_j, \quad (7.9)$$

where $\|R_n(\phi_j)\|_{L^2(L)} = o(n^{-1})$, which implies the asymptotic formula (7.8) for all j . $\square$

Remark 12. This spectral decomposition establishes a direct quantitative link between the geometry of the leaf (via Δ_L) and the dynamics of symmetrized neural operators. It provides explicit control over stability, convergence rates, and iterative behavior on each leaf. For non-compact leaves, analogous results hold in weighted L^2 spaces under decay conditions on K .

Remark 13. Combining Theorem 19 with the Voronovskaya asymptotics allows derivation of sharp operator norm bounds, spectral gaps, and error propagation estimates for iterative schemes based on foliated neural operators.

8. Complex Dynamical Interpretation

Symmetrized neural operators $\mathcal{S}[\mathcal{N}_{\mathcal{F}}]$ naturally induce discrete-time flows along the leaves of a holomorphic foliation $\mathcal{F}$. For a point $x_0 \in L$, where L is a compact leaf, the orbit is recursively defined by

$$x_{k+1} = \mathcal{S}[\mathcal{N}_{\mathcal{F}}](x_k), \quad k \in \mathbb{N}. \quad (8.1)$$

8.1. Holonomy-Equivariant Orbits

Holonomy invariance ensures that these orbits respect the transverse structure of the foliation. Specifically, for any $g \in G$ in the holonomy pseudogroup:

$$g(x_k) = \mathcal{S}[\mathcal{N}_{\mathcal{F}}](g(x_{k-1})), \quad (8.2)$$

showing that the transverse dynamics are equivariant under G . Invariant sets and measures for the leafwise dynamics are then automatically compatible with the foliation's transverse structure.

8.2. Invariant Measures and Ergodicity

On compact leaves, L^p -stability (Theorem 19) ensures boundedness of iterates, enabling the construction of invariant probability measures μ_L satisfying

$$\int_L \varphi d\mu_L = \int_L (\mathcal{S}[\mathcal{N}_{\mathcal{F}}]\varphi) d\mu_L, \quad \forall \varphi \in C(L). \quad (8.3)$$

Time averages along orbits converge μ_L -almost everywhere to spatial averages:

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} \varphi(x_k) = \int_L \varphi d\mu_L, \quad \text{for } \mu_L\text{-a.e. } x_0 \in L. \quad (8.4)$$

The measure-theoretic entropy

$$h_{\mu_L}(\mathcal{S}[\mathcal{N}_{\mathcal{F}}]) = \sup_{\mathcal{P}} \lim_{k \rightarrow \infty} \frac{1}{k} H_{\mu} \left(\bigvee_{i=0}^{k-1} \mathcal{S}[\mathcal{N}_{\mathcal{F}}]^{-i} \mathcal{P} \right), \quad (8.5)$$

quantifies the complexity of the orbit structure in terms of measurable partitions $\mathcal{P}$. Holonomy invariance restricts the set of invariant measures, producing rigidity effects analogous to classical results in complex foliations [1,3].

8.3. Leafwise Diffusion Interpretation

The Voronovskaya-type expansion (Theorem 18) shows that for large n , the leading-order behavior of $\mathcal{S}[\mathcal{N}_{\mathcal{F},n}]$ is controlled by the leafwise Laplacian:

$$x_{k+1} - x_k \approx \frac{1}{2n} \Delta_L x_k. \quad (8.6)$$

This provides a discrete approximation to a leafwise diffusion process, connecting neural operator dynamics to heat-like flows along the compact leaf.

Theorem 20 (Ergodic Diffusion on Foliated Leaves). *Let L be a compact leaf of a holomorphic foliation $\mathcal{F}$, and let $\{\mathcal{S}[\mathcal{N}_{\mathcal{F},n}]\}_{n \in \mathbb{N}}$ be a sequence of symmetrized neural operators with smooth kernels and Lipschitz activations. Then:*

1. **Convergence to Leafwise Diffusion:** Define the discrete flow

$$x_{k+1} = \mathcal{S}[\mathcal{N}_{\mathcal{F},n}](x_k), \quad k \in \mathbb{N}, \quad x_0 \in L. \quad (8.7)$$

As $n \rightarrow \infty$ and $k/n \rightarrow t \geq 0$, the flow converges in distribution to the solution of

$$\frac{\partial u}{\partial t}(t, x) = \frac{1}{2} \Delta_L u(t, x), \quad u(0, x) = x_0. \quad (8.8)$$

2. **Existence and Uniqueness of Invariant Measure:** There exists a unique probability measure μ_L on L , absolutely continuous w.r.t. vol_L , such that

$$\int_L \varphi d\mu_L = \int_L (S[\mathcal{N}_{\mathcal{F},n}]\varphi) d\mu_L, \quad \forall \varphi \in C(L). \quad (8.9)$$

3. **Ergodicity:** For μ_L -almost every $x_0 \in L$, time averages along the discrete orbit converge to spatial averages:

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} \varphi(x_k) = \int_L \varphi d\mu_L, \quad \forall \varphi \in C(L). \quad (8.10)$$

Proof. By Theorem 18, for sufficiently large n ,

$$S[\mathcal{N}_{\mathcal{F},n}]f = f + \frac{1}{2n} \Delta_L f + R_n(f), \quad \|R_n(f)\|_{C^1(L)} = o(n^{-1}), \quad (8.11)$$

for $f \in C^2(L)$.

Consider the discrete orbit $\{x_k\}$ (8.7). Subtracting x_k gives

$$x_{k+1} - x_k = \frac{1}{2n} \Delta_L x_k + R_n(x_k). \quad (8.12)$$

Rescaling discrete time via $t = k/n$ and letting $n \rightarrow \infty$, $R_n(x_k) \rightarrow 0$ in C^1 , yielding the continuous diffusion (8.8).

Compactness of L and ellipticity of Δ_L ensure existence and uniqueness of a smooth invariant measure μ_L satisfying (8.9), compatible with holonomy.

Finally, the classical ergodic theorem for measure-preserving transformations [10] gives (8.10), establishing convergence of time averages to spatial averages. $\square$

Remark 14. Theorem 20 formalizes symmetrized neural operators as discrete approximations of leafwise diffusion processes. It rigorously connects operator-theoretic neural dynamics to classical complex dynamical systems, enabling analysis of Lyapunov exponents, spectral gaps, and stochastic-like behavior along foliated leaves.

9. Extension to Non-Compact Leaves and Singular Foliations

9.1. Weighted Sobolev Spaces on Non-Compact Leaves

Let L be a non-compact leaf of a holomorphic foliation $\mathcal{F}$. To handle non-compactness, we introduce weighted function spaces controlling growth at infinity.

Definition 21 (Weighted Leafwise Sobolev Spaces). Let $\rho : L \rightarrow \mathbb{R}^+$ be a smooth weight function with $\rho(x) \rightarrow \infty$ as $x \rightarrow \infty$ in L . For $s \geq 0$ and $1 \leq p < \infty$, the weighted Sobolev space $W_\rho^{s,p}(L)$ consists of functions $f \in L_{\text{loc}}^p(L)$ satisfying

$$\|f\|_{W_\rho^{s,p}(L)}^p := \sum_{|\alpha| \leq s} \int_L |D^\alpha f(x)|^p \rho(x) d\text{vol}_L(x) < \infty. \quad (9.1)$$

For $p = 2$, we denote $H_\rho^s(L) := W_\rho^{s,2}(L)$.

9.2. Singular Holomorphic Foliations

We now consider singular holomorphic foliations, where leaf dimension may drop along an analytic subvariety.

Definition 22 (Singular Holomorphic Foliation). A singular holomorphic foliation $\mathcal{F}$ on a complex manifold M is defined by a holomorphic subbundle $\mathcal{D} \subset T^{1,0}M$ outside an analytic singular set $S \subset M$ with $\text{codim}(S) \geq 2$, such that $\mathcal{D}$ is involutive on $M \setminus S$. Leaves are maximal connected immersed complex submanifolds of $M \setminus S$ tangent to $\mathcal{D}$.

9.3. Universal Approximation on Non-Compact Leaves

Theorem 23 (Universal Approximation with Weighted Sobolev Control). Let L be a non-compact leaf of a holomorphic foliation $\mathcal{F}$ with weight function ρ . Suppose $f \in H_\rho^s(L)$ with $s > \dim L$. Then for every $\varepsilon > 0$, there exists a holonomy-invariant symmetrized neural operator $\mathcal{S}[\mathcal{N}_{\mathcal{F}}]$ such that

$$\|\mathcal{S}[\mathcal{N}_{\mathcal{F}}]f - f\|_{H_\rho^s(L)} < \varepsilon. \quad (9.2)$$

If f decays exponentially at infinity, the approximation can be made uniform with explicit geometric rate.

Proof. Let $\{K_j\}_{j \in \mathbb{N}}$ be a compact exhaustion of L with $K_j \subset \text{int}(K_{j+1})$ and $\bigcup_j K_j = L$. By Theorem 16, for each K_j there exists a local symmetrized neural operator $\mathcal{S}[\mathcal{N}_{\mathcal{F}}^{(j)}]$ such that

$$\|\mathcal{S}[\mathcal{N}_{\mathcal{F}}^{(j)}]f - f\|_{H^s(K_j)} < \varepsilon 2^{-j}. \quad (9.3)$$

Let $\{\chi_j\}$ be a partition of unity subordinate to $\{K_{j+1} \setminus K_{j-1}\}$ with $K_0 = \emptyset$, and define the global operator

$$\mathcal{S}[\mathcal{N}_{\mathcal{F}}]f := \sum_j \chi_j \mathcal{S}[\mathcal{N}_{\mathcal{F}}^{(j)}]f. \quad (9.4)$$

Then

$$\|\mathcal{S}[\mathcal{N}_{\mathcal{F}}]f - f\|_{H_\rho^s(L)} \leq \sum_j \|\chi_j\|_{C^s} \|\mathcal{S}[\mathcal{N}_{\mathcal{F}}^{(j)}]f - f\|_{H^s(K_j)} \sup_{x \in \text{supp}(\chi_j)} \rho(x)^{1/2}. \quad (9.5)$$

Choosing $\rho(x) \sim (1 + d(x, x_0))^{2s}$ ensures convergence.

Holonomy invariance is preserved by the gluing via the partition of unity.

If $|f(x)| \leq Ce^{-\lambda d(x, x_0)}$, choosing $\rho(x) = e^{\mu d(x, x_0)}$ with $\mu < \lambda$ gives exponential decay of the approximation error outside K_j , yielding a geometric convergence rate. $\square$

9.4. Spectral Convergence for Singular Foliations

Theorem 24 (Leafwise Spectral Convergence). Let $\mathcal{F}$ be a singular holomorphic foliation on a compact Kähler manifold M with singular set S , and L a regular leaf. Let $\{\mathcal{S}[\mathcal{N}_{\mathcal{F}, n}]\}$ be symmetrized neural operators with smooth kernels on $M \times M$. Then:

1. $\mathcal{S}[\mathcal{N}_{\mathcal{F}, n}]$ are compact on $L^2(M)$ and converge strongly to identity.
2. There exists a spectral gap $\gamma > 0$ such that

$$\sigma(\mathcal{S}[\mathcal{N}_{\mathcal{F}, n}]) \subset [1 - \gamma n^{-1}, 1] \cup \{\text{isolated eigenvalues}\}.$$

3. Eigenfunctions corresponding to eigenvalues near 1 concentrate on regular leaves and extend smoothly up to S .

Proof. We proceed in rigorous manner.

Since M is compact and the kernels of $\mathcal{S}[\mathcal{N}_{\mathcal{F},n}]$ are smooth on $M \times M$, each operator is Hilbert–Schmidt on $L^2(M)$. Therefore, by standard operator theory, $\mathcal{S}[\mathcal{N}_{\mathcal{F},n}]$ is compact:

$$\|\mathcal{S}[\mathcal{N}_{\mathcal{F},n}]f\|_{L^2(M)} \leq C\|f\|_{L^2(M)}, \quad \forall f \in L^2(M),$$

with C independent of n .

On the regular part $M \setminus S$, the Voronovskaya-type expansion (Theorem 18) yields

$$\mathcal{S}[\mathcal{N}_{\mathcal{F},n}]f = f + \frac{1}{2n}\Delta_{\mathcal{F}}f + R_n(f), \quad \|R_n(f)\|_{L^2(M \setminus S)} = o(n^{-1}), \quad (9.6)$$

for $f \in C^\infty(M \setminus S)$. Since S has measure zero in M , the $L^2(M)$ -norm is unaffected by S , ensuring

$$\lim_{n \rightarrow \infty} \|\mathcal{S}[\mathcal{N}_{\mathcal{F},n}]f - f\|_{L^2(M)} = 0,$$

i.e., strong convergence to the identity on $L^2(M)$.

Consider the leafwise Laplacian $\Delta_{\mathcal{F}}$ on a regular leaf $L \subset M \setminus S$. Being elliptic and defined on a compact leaf, it has discrete spectrum $0 = \mu_0 < \mu_1 \leq \mu_2 \leq \dots \rightarrow \infty$. Restricting to the orthogonal complement of constants, $\Delta_{\mathcal{F}} \leq -\gamma$ for some $\gamma > 0$. Combining with (9.6), the spectrum of $\mathcal{S}[\mathcal{N}_{\mathcal{F},n}]$ satisfies

$$\sigma(\mathcal{S}[\mathcal{N}_{\mathcal{F},n}]) \subset [1 - \gamma n^{-1}, 1] \cup \{\text{isolated eigenvalues}\},$$

for sufficiently large n .

Let $\phi_n \in L^2(M)$ be an eigenfunction of $\mathcal{S}[\mathcal{N}_{\mathcal{F},n}]$ with eigenvalue $\lambda_n \rightarrow 1$. Then

$$\frac{1 - \lambda_n}{n^{-1}} \rightarrow \mu \geq 0, \quad \text{and } \phi_n \rightharpoonup \phi \text{ weakly in } L^2(M),$$

where ϕ is an eigenfunction of $\Delta_{\mathcal{F}}$ on $M \setminus S$. Elliptic regularity implies $\phi \in C^\infty(L)$ on each regular leaf L .

Since S is an analytic set of codimension at least 2, bounded holomorphic functions on $M \setminus S$ extend uniquely across S (Riemann’s extension theorem). Consequently, ϕ extends smoothly to all of M , preserving leafwise smoothness up to the singular set.

Combining, we conclude that $\mathcal{S}[\mathcal{N}_{\mathcal{F},n}]$ are compact, converge strongly to identity, possess a spectral gap, and their eigenfunctions near the top of the spectrum concentrate on regular leaves while extending smoothly across S . $\square$

9.5. Concluding Remarks

Extensions to non-compact leaves and singular foliations reveal:

- Weight function ρ acts analogously to a potential in Schrödinger operators.
- Singularities impose effective boundary conditions on the leafwise Laplacian, reflected in the neural operator spectrum.
- Holonomy invariance ensures transverse regularity across singularities.

These results enable applications in mirror symmetry (non-compact special Lagrangian foliations) and complex dynamics (Riemann surface foliations with punctures).

10. Results

The theoretical framework developed in this work establishes a **unified geometric-analytic foundation** for the study of **symmetrized neural operators** on complex foliated manifolds. By integrating tools from **complex geometry**, **functional analysis**, and **neural operator theory**, we provide both **qualitative insights** and **quantitative estimates** into the behavior of these operators along the leaves of holomorphic foliations. Our results not only generalize classical approximation theory to geometrically

constrained settings but also reveal deep connections between operator dynamics, spectral theory, and leafwise diffusion processes.

10.1. Universal Approximation on Foliated Leaves

We rigorously prove that symmetrized neural operators, constructed via **holonomy-invariant averaging**, universally approximate smooth functions on compact leaves. Specifically, for any $f \in C^k(L)$ and $\varepsilon > 0$, there exists a symmetrized operator $\mathcal{S}[\mathcal{N}_{\mathcal{F}}]$ such that:

$$\|\mathcal{S}[\mathcal{N}_{\mathcal{F}}]f - f\|_{C^k(L)} < \varepsilon.$$

This result extends to **leafwise Sobolev spaces** with explicit convergence rates:

$$\|\mathcal{S}[\mathcal{N}_{\mathcal{F},n}]f - f\|_{H^s(L)} \leq Cn^{-r/d}\|f\|_{H^{s+r}(L)},$$

where $f \in H^{s+r}(L)$, $r > 0$, and $d = \dim L$. The constant C depends only on the geometry of L and the choice of kernel, ensuring robustness across different foliated structures.

10.2. Voronovskaya-Type Asymptotic Expansions

A central result of this work is the derivation of **Voronovskaya-type asymptotic expansions**, which establish a precise connection between the behavior of symmetrized neural operators and the **leafwise Laplace–Beltrami operator**:

$$\mathcal{N}_{\mathcal{F},n}f(x) = f(x) + \frac{1}{2n}\Delta_L f(x) + R_n(f)(x),$$

where the remainder term satisfies $\|R_n(f)\|_{C^k(L)} = o(n^{-1})$. This expansion provides a **leading-order characterization** of the operator dynamics, revealing how neural operators discretely approximate diffusion processes along the leaves. The result is derived using **holonomy-invariant symmetrization**, which ensures cancellation of first-order terms and preserves the geometric structure of the foliation.

10.3. Spectral Decomposition and L^p -Stability

We establish that symmetrized neural operators admit a **spectral decomposition** in $L^2(L)$, with eigenvalues asymptotically related to the spectrum of the leafwise Laplacian:

$$\lambda_j = 1 - \frac{\mu_j}{2n} + o(n^{-1}),$$

where $\Delta_L \phi_j = -\mu_j \phi_j$. This result links the **dynamics of neural operators** to the **underlying geometry** of the foliation, providing explicit control over stability and convergence rates. Additionally, we prove **L^p -stability** for $1 \leq p \leq \infty$:

$$\|\mathcal{S}[\mathcal{N}_{\mathcal{F}}]f\|_{L^p(L)} \leq C_p \|f\|_{L^p(L)},$$

where C_p depends on the kernel K , the Lipschitz constant of the activation function σ , and the volume of L . This stability result ensures the robustness of the operators in both theoretical and applied contexts.

10.4. Dynamical Interpretation and Ergodicity

Symmetrized neural operators induce **discrete-time flows** along the leaves, which converge to **leafwise diffusion processes** governed by the Laplace–Beltrami operator:

$$\frac{\partial u}{\partial t}(t, x) = \frac{1}{2}\Delta_L u(t, x).$$

We prove the existence of **unique invariant measures** μ_L on compact leaves and establish **ergodicity**:

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} \varphi(x_k) = \int_L \varphi d\mu_L, \quad \mu_L\text{-almost everywhere.}$$

This result bridges **neural operator theory** with **classical dynamical systems**, enabling the analysis of long-term behavior and stability in foliated spaces.

10.5. Extension to Non-Compact Leaves and Singular Foliations

To handle non-compact leaves, we introduce **weighted Sobolev spaces** $W_{\rho}^{s,p}(L)$ and prove universal approximation in these spaces:

$$\|S[\mathcal{N}_{\mathcal{F}}]f - f\|_{H_{\rho}^s(L)} < \varepsilon.$$

For singular foliations on compact Kähler manifolds, we establish **spectral convergence**:

$$\sigma(S[\mathcal{N}_{\mathcal{F},n}]) \subset [1 - \gamma n^{-1}, 1] \cup \{\text{isolated eigenvalues}\},$$

where eigenfunctions concentrate on regular leaves and extend smoothly across singularities. This extension significantly broadens the applicability of the framework to more general geometric settings, including **logarithmic foliations** and **Calabi–Yau manifolds**.

10.6. Holonomy-Invariant Symmetrization

The symmetrization procedure,

$$S[\mathcal{N}_{\mathcal{F}}](f) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \circ \mathcal{N}_{\mathcal{F}} \circ g(f),$$

ensures **compatibility with the transverse foliation structure** and preserves approximation properties across all geometric settings. This procedure is fundamental to the **holonomy-invariant** nature of the operators, guaranteeing that their outputs respect the symmetries imposed by the foliation's pseudogroup.

11. Conclusions

This work establishes a **comprehensive geometric-analytic framework** for symmetrized neural operators on complex foliated manifolds, bridging fundamental areas of **mathematics** with modern **operator learning theory**. Our results provide a rigorous foundation for the study of neural operators in geometrically constrained settings, with broad implications for approximation theory, spectral analysis, and dynamical systems.

11.1. Key Contributions

Our principal contributions include:

- A **rigorous formulation** of neural operators on foliated complex manifolds, incorporating a **holonomy-invariant symmetrization procedure** that respects the transverse geometry of the foliation.
- **Universal approximation theorems** in leafwise Sobolev and C^k spaces, ensuring broad functional expressivity for both compact and non-compact leaves.
- Derivation of **Voronovskaya-type asymptotic expansions**, linking operator behavior to the leafwise Laplace–Beltrami operator and providing precise remainder estimates.
- Establishment of **L^p -stability** and **spectral decomposition results**, explicitly connecting operator dynamics to the underlying foliation geometry.

- A **dynamical interpretation** of symmetrized neural operators as discrete approximations to leafwise diffusion processes, with proofs of invariant measures and ergodicity.
- Extension of the framework to **singular foliations** and **non-compact leaves**, including spectral convergence and regularity results near singularities.
- Introduction of **weighted Sobolev spaces** for non-compact leaves, enabling universal approximation under decay conditions.

11.2. Theoretical Implications

The framework reveals deep connections between:

- **Operator learning** and **geometric analysis** on foliated spaces, enabling the development of structured neural architectures that respect geometric constraints.
- **Neural network dynamics** and **classical diffusion processes**, providing a discrete approximation to leafwise Laplacian flows.
- **Spectral theory of neural operators** and **leafwise Laplacians**, with explicit links between eigenvalues and the geometry of the foliation.
- **Approximation theory** in geometrically constrained function spaces, extending classical results to foliated settings.

11.3. Applications and Future Directions

The results open several promising research directions:

- **Geometric Machine Learning:** Development of **foliation-aware neural architectures** for problems in complex geometry, theoretical physics, and high-dimensional data analysis.
- **Dynamical Systems:** Application to **invariant measure computation** and **stability analysis** in complex dynamical systems, including ergodic theory on foliated spaces.
- **Spectral Geometry:** Use of neural operators for **spectral approximation** on singular foliations and non-compact manifolds, with applications to quantum chaos and spectral gaps.
- **Mathematical Physics:** Exploration of foliations in **Calabi–Yau manifolds** and **string theory**, where holomorphic structures play a central role.
- **Numerical Analysis:** Development of **structure-preserving discretizations** for foliated PDEs, leveraging the stability and approximation properties of symmetrized neural operators.
- **Singularity Theory:** Extension to more general **singular foliations** and **stratified spaces**, with applications to algebraic geometry and topological data analysis.

11.4. Concluding Remarks

This work demonstrates that symmetrized neural operators provide a **powerful and versatile tool** for analyzing and approximating functions on geometrically complex spaces. The **holonomy-invariant approach** ensures compatibility with the underlying foliation structure, while the approximation and spectral results establish these operators as natural objects in geometric analysis. The extension to singular and non-compact settings significantly broadens the applicability of the theory, opening new avenues for research at the intersection of **geometry**, **analysis**, and **machine learning**.

The framework developed here not only advances theoretical understanding but also provides **practical foundations** for developing geometrically structured learning algorithms with **guaranteed approximation properties** and **stability characteristics**. Future work will explore applications to specific geometric problems, such as **logarithmic foliations in algebraic geometry** and **neural PDE solvers on foliated domains**, further bridging the gap between pure mathematics and computational science.

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Notation and Symbols

In this work, we adopt the following notation and symbols:

Sets, Spaces, and Manifolds

M	Complex manifold of (complex) dimension n
$\mathcal{F}$	Holomorphic foliation on M of codimension q and leaf dimension $p = n - q$
L	Leaf of the foliation $\mathcal{F}$
$M/\mathcal{F}$	Leaf space (quotient by the foliation)
$C^k(L)$	Space of k -times continuously differentiable functions on L
$L^p(L)$	Leafwise Lebesgue space with respect to the induced volume form
$H^s(L), W^{s,2}(L)$	Sobolev space of order s on the leaf L
$W_\rho^{s,p}(L)$	Weighted Sobolev space on non-compact leaves, with weight function ρ

Operators

$\mathcal{N}_{\mathcal{F}}$	Foliated neural operator acting along the leaf
$\mathcal{S}[\mathcal{N}_{\mathcal{F}}]$	Symmetrized neural operator (holonomy-invariant)
Δ_L	Leafwise Laplace–Beltrami operator
∇_L, div_L	Gradient and divergence along the leaf
T^k	k -fold iteration of an operator T

Groups and Symmetries

G	Holonomy pseudogroup or compact symmetry group
$g \in G$	Holonomy transformation or symmetry element
g^*	Pullback operator: $f \mapsto f \circ g$
$\text{Hol}(L, x)$	Holonomy group of leaf L at point x

Kernels and Activations

$K(x, y)$	Integral kernel defining a neural operator
σ	Nonlinear activation function (e.g., Lipschitz or holomorphic)
$\text{vol}_L, d\text{vol}_L$	Leafwise volume form
ω_L	Kähler form induced on leaf L

Dynamical Quantities

$x_{k+1} = \mathcal{S}[\mathcal{N}_{\mathcal{F}}](x_k)$	Discrete leafwise flow
μ_L	Invariant probability measure on a compact leaf
h_{μ_L}	Measure-theoretic entropy
ρ	Weight function for non-compact leaves

Spectral Notation

$\{\varphi_j\}_{j \in \mathbb{N}}$	Orthonormal eigenfunctions of Δ_L
$\Delta_L \varphi_j = -\mu_j \varphi_j$	Spectral decomposition of the leafwise Laplacian
λ_j	Eigenvalues of $\mathcal{S}[\mathcal{N}_{\mathcal{F}}]$
$\sigma(\mathcal{S}[\mathcal{N}_{\mathcal{F},n}])$	Spectrum of the symmetrized neural operator

Greek Letters and Indices	
α, β	Multi-indices for derivatives
μ_j	Eigenvalues of $-\Delta_L$
σ	Activation function or standard deviation (context-dependent)
ε	Approximation tolerance
γ	Spectral gap parameter
i, j, k	Coordinate directions along leaves (Latin indices)
α, β	Multi-indices or transverse directions (Greek indices)

Asymptotic and Norm Notation	
$\ \cdot\ _{C^k(L)}$	C^k supremum norm on L
$\ \cdot\ _{H^s(L)}$	Sobolev norm of order s on L
$\ \cdot\ _{W^{s,p}_\rho(L)}$	Weighted Sobolev norm on non-compact leaves
$o(n^{-1})$	Term vanishing faster than n^{-1} as $n \rightarrow \infty$
$A \lesssim B$	There exists $C > 0$ such that $A \leq CB$

Miscellaneous	
$\mathbb{N}, \mathbb{R}, \mathbb{C}$	Sets of natural, real, and complex numbers
Id	Identity operator
χ_α	Partition of unity subordinate to a foliation atlas
$\mathcal{D}$	Holomorphic distribution tangent to the foliation
S	Singular set of a singular foliation

References

1. Camacho, C., & Sad, P. (1982). Invariant varieties through singularities of holomorphic vector fields. *Annals of mathematics*, 115(3), 579-595. <https://doi.org/10.2307/2007013>.

2. Brunella, M. (2015). *Birational geometry of foliations*. Cham: Springer International Publishing.

3. Ghys, É. (1995). Topologie des feuilles génériques. *Annals of Mathematics*, 141(2), 387-422. <https://doi.org/10.2307/2118526>.

4. Lu, L., Jin, P., & Karniadakis, G. E. (2019). Deeponet: Learning nonlinear operators for identifying differential equations based on the universal approximation theorem of operators. *arXiv preprint arXiv:1910.03193*. <https://doi.org/10.48550/arXiv.1910.03193>.

5. Kovachki, N., Li, Z., Liu, B., Azizzadenesheli, K., Bhattacharya, K., Stuart, A., & Anandkumar, A. (2023). Neural operator: Learning maps between function spaces with applications to pdes. *Journal of Machine Learning Research*, 24(89), 1-97.

6. Lanthaler, S., Mishra, S., & Karniadakis, G. E. (2022). Error estimates for deeponets: A deep learning framework in infinite dimensions. *Transactions of Mathematics and Its Applications*, 6(1), tnac001. <https://doi.org/10.1093/imatrm/tnac001>.

7. Krantz, S. G. (2001). *Function theory of several complex variables* (Vol. 340). American Mathematical Soc..

8. ANASTASSIOU, G. A. (2025). Multivariate left side Canavati fractional Landau inequalities. *Journal of Applied and Pure Mathematics*, 7(1–2), 103-119.

9. Bahouri, H. (2011). *Fourier analysis and nonlinear partial differential equations*. Springer.

10. Walters, P. (2000). *An introduction to ergodic theory* (Vol. 79). Springer Science & Business Media.

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