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The Straight Proof to Riemann Hypothesis

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Abstract: This paper provides the straight proof for the Riemann Hypothesis.The proof has been formulated with the modified version of Riemann Xi function $\xi(s)$, that shows the relationship with upper incomplete gamma function.

Keywords: Riemann Xi function; upper incomplete gamma function; Riemann Hypothesis

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1. Introduction

Riemann Zeta function [1] is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

$$Re(s) > 1$$

where s is a complex variable.

Through analytic continuation $\zeta(s)$ is extended to entire complex plane,with one simple pole at s=1 with residue 1. $\zeta(s)$ satisfies following functional equation [1] which relates $\zeta(s)$ with $\zeta(1-s)$,

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{(1-s)}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) \tag{2}$$

or equally re-written as,

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s) \tag{3}$$

where $\Gamma(s)$ is the Gamma function. This functional equation established the fact that $\zeta(-2s) = 0$ i.e. all trivial zeros of $\zeta(s)$ are located at negative even integers.As there are no zeros on the right-half plane[$Re(s) > 1$], the non-trivial zeros of $\zeta(s)$ lies in a critical strip [$0 \le Re(s) \le 1$].Jacques Hadamard [2] and De La Vallée Poussin [3] proved that $\zeta(s) \ne 0$ at s=1 and s=0.

2. Proof of Riemann Hypothesis

The proof has been formulated step by step in the following manner;

- Deriving new expression for $\xi(s)$.
- Analyzing $\xi(s)$ at $s = \alpha + it$.
- Final Proof
- Conclusion

2.1. Deriving New Expression of $\xi(s)$

$\xi(s)$ is an entire function defined as,[1]

$$\xi(s) = \frac{s(s-1)}{2}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) \tag{4}$$

and satisfies the functional equation $\xi(s) = \xi(1-s)$.

It is also known,[1]

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \frac{1}{s(s-1)} + \int_1^{\infty} \psi(x)\left(x^{\frac{s}{2}-1} + x^{\frac{1-s}{2}-1}\right)dx \tag{5}$$

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where

$$\psi(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$$

So that,[1]

$$\xi(s) = \frac{1}{2} - \frac{s(1-s)}{2} \int_1^{\infty} \psi(x)\left(x^{\frac{s}{2}-1} + x^{\frac{1-s}{2}-1}\right)dx \tag{6}$$

Next,the following new expression for $\xi(s)$ will be proved rigorously,

$$\xi(s) = \frac{1}{2} - \frac{s(1-s)}{2}\left[\pi^{-\frac{s}{2}}\zeta(s)\Gamma\left(\frac{s}{2},\pi n^2\right) + \pi^{-\frac{(1-s)}{2}}\zeta(1-s)\Gamma\left(\frac{1-s}{2},\pi n^2\right)\right] \tag{7}$$

where $\Gamma(a,x)$ [4] is the upper incomplete gamma function defined as,

$$\Gamma(a,x) = \int_x^{\infty} e^{-x}x^{a-1}dx \tag{8}$$

2.2. Proof for New Expression of $\xi(s)$:

Equation 6 can be further written as,

$$\xi(s) = \frac{1}{2} - \frac{s(1-s)}{2}\left[\int_1^{\infty} \psi(x)\left(x^{\frac{s}{2}-1}\right)dx + \int_1^{\infty} \psi(x)\left(x^{\frac{1-s}{2}-1}\right)dx\right] \tag{9}$$

Now,let's solve for the terms in square bracket first, Let

$$A = \int_1^{\infty} \psi(x)\left(x^{\frac{s}{2}-1}\right)dx$$

and

$$B = \int_1^{\infty} \psi(x)\left(x^{\frac{1-s}{2}-1}\right)dx$$

Substituting original definition for $\psi(x)$ into 'A' and re-writing,gives us,

$$A = \int_1^{\infty} \sum_{n=1}^{\infty} e^{-\pi n^2 x}\left(x^{\frac{s}{2}-1}\right)dx$$

Let $u = \pi n^2 x$, so $dx = \frac{du}{\pi n^2}$,when $x = 1, u = \pi n^2$ and $x = \infty, u = \infty$

$$A = \int_{\pi n^2}^{\infty} \sum_{n=1}^{\infty} e^{-u}\left(\frac{u}{\pi n^2}\right)^{\frac{s}{2}-1}\frac{du}{\pi n^2}$$

$$= \sum_{n=1}^{\infty} \int_{\pi n^2}^{\infty} e^{-u}\left(\frac{u}{\pi n^2}\right)^{\frac{s}{2}-1}\frac{du}{\pi n^2}$$

Taking the constant term outside the integral,and keeping in order,we have

$$= \pi^{-\frac{s}{2}} \sum_{n=1}^{\infty} \frac{1}{n^s} \int_{\pi n^2}^{\infty} e^{-u}\left(u^{\frac{s}{2}-1}\right)du$$

So,it can be written as,

$$= \pi^{-\frac{s}{2}} \sum_{n=1}^{\infty} \frac{1}{n^s}\Gamma\left(\frac{s}{2},\pi n^2\right)$$

or,

$$= \pi^{-\frac{s}{2}}\zeta(s)\Gamma\left(\frac{s}{2},\pi n^2\right)$$

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Same procedure for 'B',

$$B = \int_{\pi n^2}^{\infty} \sum_{n=1}^{\infty} e^{-u} \left(\frac{u}{\pi n^2}\right)^{\frac{1-s}{2}-1} \frac{du}{\pi n^2}$$

$$= \sum_{n=1}^{\infty} \int_{\pi n^2}^{\infty} e^{-u} \left(\frac{u}{\pi n^2}\right)^{\frac{1-s}{2}-1} \frac{du}{\pi n^2}$$

Taking the constant term outside the integral,and keeping in order,we have

$$= \pi^{-\frac{(1-s)}{2}} \sum_{n=1}^{\infty} \frac{1}{n^{1-s}} \int_{\pi n^2}^{\infty} e^{-u} \left(u^{\frac{1-s}{2}-1}\right) du$$

So,it can be written as,

$$= \pi^{-\frac{(1-s)}{2}} \sum_{n=1}^{\infty} \frac{1}{n^{1-s}} \Gamma\left(\frac{1-s}{2}, \pi n^2\right)$$

or

$$= \pi^{-\frac{(1-s)}{2}} \zeta(1-s) \Gamma\left(\frac{1-s}{2}, \pi n^2\right)$$

Putting back the value of 'A' and 'B' in Equation 9,the desired equation is successfully derived,

$$\xi(s) = \frac{1}{2} - \frac{s(1-s)}{2} \left[\pi^{-\frac{s}{2}} \zeta(s) \Gamma\left(\frac{s}{2}, \pi n^2\right) + \pi^{-\frac{(1-s)}{2}} \zeta(1-s) \Gamma\left(\frac{1-s}{2}, \pi n^2\right)\right]$$

2.3. Analyzing $\xi(s)$ at $(s = \alpha + it)$

$$\xi(s) = \frac{1}{2} - \frac{s(1-s)}{2} \left[\pi^{-\frac{s}{2}} \zeta(s) \Gamma\left(\frac{s}{2}, \pi n^2\right) + \pi^{-\frac{(1-s)}{2}} \zeta(1-s) \Gamma\left(\frac{1-s}{2}, \pi n^2\right)\right]$$

writing $\zeta(s)$ in summation form and $\Gamma(s)$ in integral form,we have,

$$\xi(s) = \frac{1}{2} - \frac{s(1-s)}{2} \left[\pi^{-\frac{s}{2}} \sum_{n=1}^{\infty} \frac{1}{n^s} \int_{\pi n^2}^{\infty} e^{-u} \left(u^{\frac{s}{2}-1}\right) du + \pi^{-\frac{(1-s)}{2}} \sum_{n=1}^{\infty} \frac{1}{n^{1-s}} \int_{\pi n^2}^{\infty} e^{-u} \left(u^{\frac{1-s}{2}-1}\right) du\right]$$

now, let's evaluate them in isolation, letting,

$$W = \frac{s(1-s)}{2}$$

$$A = \pi^{-\frac{s}{2}} \sum_{n=1}^{\infty} \frac{1}{n^s} \int_{\pi n^2}^{\infty} e^{-u} \left(u^{\frac{s}{2}-1}\right) du$$

$$B = \pi^{-\frac{(1-s)}{2}} \sum_{n=1}^{\infty} \frac{1}{n^{1-s}} \int_{\pi n^2}^{\infty} e^{-u} \left(u^{\frac{1-s}{2}-1}\right) du$$

so that,it is in the form,

$$\xi(s) = \frac{1}{2} - W[A+B] \tag{10}$$

for W at $s = \alpha + it$,

$$W = \frac{(\alpha+it)(1-\alpha-it)}{2}$$

$$= \frac{t^2 - \alpha^2 + \alpha + i(t - 2\alpha t)}{2} \tag{11}$$

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in **Mod-Arg form**,

$$= \frac{\sqrt{(t^2 - \alpha^2 + \alpha)^2 + (t - 2\alpha t)^2}}{2} . e^{i \arctan\left(\frac{t-2\alpha t}{t^2-\alpha^2+\alpha}\right)} \tag{12}$$

for A at $s = \alpha + it$,

$$A = \pi^{\frac{-\alpha-it}{2}} \sum_{n=1}^{\infty} \frac{1}{n^{\alpha+it}} \int_{\pi n^2}^{\infty} e^{-u} u^{\frac{\alpha+it}{2}-1} du$$

$$= \pi^{-\frac{\alpha}{2}} . \pi^{-\frac{it}{2}} \sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} . n^{-it} \int_{\pi n^2}^{\infty} e^{-u} u^{\frac{\alpha}{2}-1} . u^{\frac{it}{2}} du$$

$$= \pi^{-\frac{\alpha}{2}} \cdot e^{i(-\frac{t}{2} ln(\pi))} \sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} \cdot e^{i(-t ln(n))} \int_{\pi n^2}^{\infty} e^{-u} u^{\frac{\alpha}{2}-1} \cdot e^{i(\frac{t}{2} ln(u))} du$$

$$A = \pi^{-\frac{\alpha}{2}} \sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} \cdot e^{i(-t ln(n) - \frac{t}{2} ln(\pi))} \int_{\pi n^2}^{\infty} e^{-u} u^{\frac{\alpha}{2}-1} . e^{i(\frac{t}{2} ln(u))} du \tag{13}$$

in **Mod-Arg form**,

$$= \pi^{-\frac{\alpha}{2}} \sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} \int_{\pi n^2}^{\infty} e^{-u} u^{\frac{\alpha}{2}-1} du \cdot e^{i(-t ln(n) - \frac{t}{2} ln(\pi) + \frac{t}{2} ln(u))} \tag{14}$$

for B at $s = \alpha + it$,

$$B = \pi^{-\frac{(1-\alpha-it)}{2}} \sum_{n=1}^{\infty} \frac{1}{n^{(1-\alpha-it)}} \int_{\pi n^2}^{\infty} e^{-u} u^{\frac{1-\alpha-it}{2}-1} du$$

$$= \pi^{-\frac{(1-\alpha)}{2}} . \pi^{\frac{it}{2}} \sum_{n=1}^{\infty} \frac{1}{n^{(1-\alpha)}} . n^{it} \int_{\pi n^2}^{\infty} e^{-u} u^{\frac{1-\alpha}{2}-1} . u^{\frac{-it}{2}} du$$

$$= \pi^{-\frac{(1-\alpha)}{2}} . e^{i(\frac{t}{2} ln(\pi))} \sum_{n=1}^{\infty} \frac{1}{n^{(1-\alpha)}} . e^{i(t ln(n))} \int_{\pi n^2}^{\infty} e^{-u} u^{\frac{1-\alpha}{2}-1} . e^{-i(\frac{t}{2} ln(u))} du$$

$$B = \pi^{-\frac{(1-\alpha)}{2}} \sum_{n=1}^{\infty} \frac{1}{n^{(1-\alpha)}} . e^{i(t ln(n) + \frac{t}{2} ln(\pi))} \int_{\pi n^2}^{\infty} e^{-u} u^{\frac{1-\alpha}{2}-1} . e^{-i(\frac{t}{2} ln(u))} du \tag{15}$$

in **Mod-Arg form**,

$$= \pi^{-\frac{(1-\alpha)}{2}} \sum_{n=1}^{\infty} \frac{1}{n^{(1-\alpha)}} \int_{\pi n^2}^{\infty} e^{-u} u^{\frac{1-\alpha}{2}-1} du \cdot e^{i(t ln(n) + \frac{t}{2} ln(\pi) - \frac{t}{2} ln(u))} \tag{16}$$

so,in general it can be written as,

$$\xi(s) = \underbrace{\frac{1}{2}}_{\text{First term}} - \underbrace{|W| e^{i \arctan\left(\frac{t-2\alpha t}{t^2-\alpha^2+\alpha}\right)}}_{\text{Second term}} \left[\underbrace{|A| \cdot e^{i(-t ln(n) - \frac{t}{2} ln(\pi) + \frac{t}{2} ln(u))}}_{\text{Third term}} + \underbrace{|B| \cdot e^{i(t ln(n) + \frac{t}{2} ln(\pi) - \frac{t}{2} ln(u))}}_{\text{Fourth term}} \right] \tag{17}$$

For $\xi(s) = 0$, it is evident that the product of the **Second term** with the **Sum of Third and Fourth terms** must be a real number. In the following section, we will demonstrate why this condition holds exclusively for $s = \frac{1}{2} + it$ and not for any $\alpha > \frac{1}{2}$ or $\alpha < \frac{1}{2}$.

Now let's analyze the equation term by term, as **First term** is constant, we'll start from **Second term**,

2.3.1. Analyzing Second Term

$$|W| \cdot e^{i \arctan\left(\frac{t-2\alpha t}{t^2-\alpha^2+\alpha}\right)} \tag{18}$$

where

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$$|W| = \frac{\sqrt{(t^2 - a^2 + a)^2 + (t - 2at)^2}}{2}$$

and let

$$\theta = \arctan\left(\frac{t - 2at}{t^2 - a^2 + a}\right)$$

In the region, $0 \leq \alpha \leq 1$,as 't' gets finitely larger and larger , θ approaches nearer and nearer to zero.

$At\ s = \frac{1}{2} + it$

$$\theta = \arctan\left(\frac{t - 2 \cdot \frac{1}{2} \cdot t}{t^2 - (\frac{1}{2})^2 + \frac{1}{2}}\right)$$
$$= \arctan(0)$$
$$= 0$$

similarly,

$$|W| = \frac{\sqrt{(t^2 - (\frac{1}{2})^2 + \frac{1}{2})^2 + (t - 2 \cdot \frac{1}{2} \cdot t)^2}}{2}$$
$$= \frac{t^2 + \frac{1}{4}}{2}$$

therefore,

$$|W| \cdot e^{i \arctan\left(\frac{t-2at}{t^2-a^2+a}\right)}$$
$$= \frac{t^2 + \frac{1}{4}}{2} \cdot e^{i(0)}$$
$$= \frac{t^2 + \frac{1}{4}}{2}$$

on the other hand,

- for $\alpha > \frac{1}{2}$, θ is negative.
- for $\alpha < \frac{1}{2}$, θ is positive.
- the value of θ exhibits reflective symmetry about the vertical line $s = \frac{1}{2} + it$ with values on either side of this line being equal in magnitude but opposite in sign.

Imaginary part gets smaller with 't' but do not vanish for any finite value of 't'.So only at $s = \frac{1}{2} + it$,the expression is real-valued.

2.3.2. Analyzing the Sum of Third and Fourth term

$$|A| \cdot e^{i(-tln(n) - \frac{t}{2}ln(\pi) + \frac{t}{2}ln(u))} + |B| \cdot e^{i(tln(n) + \frac{t}{2}ln(\pi) - \frac{t}{2}ln(u))} \tag{19}$$

where

$$|A| = \pi^{-\frac{\alpha}{2}} \sum_{n=1}^{\infty} \frac{1}{n^\alpha} \int_{\pi n^2}^{\infty} e^{-u} u^{\frac{\alpha}{2}-1} du$$

and

$$|B| = \pi^{-\frac{(1-\alpha)}{2}} \sum_{n=1}^{\infty} \frac{1}{n^{(1-\alpha)}} \int_{\pi n^2}^{\infty} e^{-u} u^{\frac{1-\alpha}{2}-1} du$$

The argument of two terms are complex conjugate of each other.

$$e^{i(-tln(n)-\frac{t}{2}ln(\pi)+\frac{t}{2}ln(u))} = cos\left(-tln(n) - \frac{t}{2}ln(\pi) + \frac{t}{2}ln(u)\right) + isin\left(-tln(n) - \frac{t}{2}ln(\pi) + \frac{t}{2}ln(u)\right)$$

$$= cos\left(tln(n) + \frac{t}{2}ln(\pi) - \frac{t}{2}ln(u)\right) - isin\left(tln(n) + \frac{t}{2}ln(\pi) - \frac{t}{2}ln(u)\right)$$

$$= e^{-i(tln(n)+\frac{t}{2}ln(\pi)-\frac{t}{2}ln(u))}$$

similarly,

$$e^{i(tln(n)+\frac{t}{2}ln(\pi)-\frac{t}{2}ln(u))} = cos\left(tln(n) + \frac{t}{2}ln(\pi) - \frac{t}{2}ln(u)\right) + isin\left(tln(n) + \frac{t}{2}ln(\pi) - \frac{t}{2}ln(u)\right)$$

$$= e^{i(tln(n)+\frac{t}{2}ln(\pi)-\frac{t}{2}ln(u))}$$

This shows that they always have opposite argument. Let,

$$\beta = \left(tln(n) + \frac{t}{2}ln(\pi) - \frac{t}{2}ln(u)\right)$$

$At\ s = \frac{1}{2} + it$

$$|A| = \pi^{-\frac{1}{4}} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}} \int_{\pi n^2}^{\infty} e^{-u} u^{-\frac{3}{4}} du$$

$$|B| = \pi^{-\frac{1}{4}} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}} \int_{\pi n^2}^{\infty} e^{-u} u^{-\frac{3}{4}} du$$

$$|A| = |B|$$

therefore, the sum at $s = \frac{1}{2} + it$ becomes,

$$= \pi^{-\frac{1}{4}} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}} \int_{\pi n^2}^{\infty} e^{-u} u^{-\frac{3}{4}} du \left(e^{i(-tln(n)-\frac{t}{2}ln(\pi)+\frac{t}{2}ln(u))} + e^{i(tln(n)+\frac{t}{2}ln(\pi)-\frac{t}{2}ln(u))}\right)$$

Due to opposite argument,while summing,imaginary parts cancels out and remaining terms are,

$$= \pi^{-\frac{1}{4}} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}} \int_{\pi n^2}^{\infty} e^{-u} u^{-\frac{3}{4}} du \cdot 2cos\left(tln(n) + \frac{t}{2}ln(\pi) - \frac{t}{2}ln(u)\right)$$

Applying identity $cos(A - B)$,

$$= 2\pi^{-\frac{1}{4}} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}} \int_{\pi n^2}^{\infty} e^{-u} u^{-\frac{3}{4}} du \Big[\cos\left(tln(n) + \frac{t}{2}ln(\pi)\right) \cos\left(\frac{t}{2}ln(u)\right) + \sin\left(tln(n) + \frac{t}{2}ln(\pi)\right) \sin\left(\frac{t}{2}ln(u)\right)\Big]$$

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Arranging them,we have,

$$=2\pi^{-\frac{1}{4}}\sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{2}}}\cdot cos\left(tln(n)+\frac{t}{2}ln(\pi)\right)\int_{\pi n^2}^{\infty}e^{-u}u^{-\frac{3}{4}}cos\left(\frac{t}{2}ln(u)\right)du$$
$$+2\pi^{-\frac{1}{4}}\sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{2}}}\cdot sin\left(tln(n)+\frac{t}{2}ln(\pi)\right)\int_{\pi n^2}^{\infty}e^{-u}u^{-\frac{3}{4}}sin\left(\frac{t}{2}ln(u)\right)du \quad (20)$$

on the other hand,

- for for $\alpha<\frac{1}{2}$ and $\alpha>\frac{1}{2}$,As 't' increases, the argument of the sum fluctuates between negative and positive values. Although the imaginary part diminishes as t grows, it remains non-zero for any finite value of 't'.
- The magnitude of **Third** and **Fourth** term are equal only at $\alpha=\frac{1}{2}$ as $\Gamma(s)$ and $\Gamma(1-s)$ are equal specifically at $Re(s)=\frac{1}{2}$ and not at other points.Thus,it is real-valued at $s=\frac{1}{2}+it$.
- The values of the sum exhibit symmetry around $s=\frac{1}{2}+it$.

2.4. Final Proof

Given that for $\xi(s)=0$, the imaginary part must vanish, an analysis of the **Second term** and the **Sum of the Third and Fourth terms** reveals that this condition is only satisfied when $s=\frac{1}{2}+it$.
As 't' gets larger,for $\alpha<\frac{1}{2}$ and $\alpha>\frac{1}{2}$,we cannot disregard the possibility of obtaining a real value when performing multiplication on complex terms.But following proof provides insight on why imaginary parts cannot vanish on those regions.

Proof; Why Imaginary Part Do Not Vanish at $\alpha<\frac{1}{2}$ and $\alpha>\frac{1}{2}$?

This equally means why there are no zeros for $\alpha<\frac{1}{2}$ and $\alpha>\frac{1}{2}$.
Since the value of $\xi(s)$ is symmetric about the line $s=\frac{1}{2}+it$ within the region $0\le\alpha\le1$, it suffices to examine the interval $0\le\alpha<\frac{1}{2}$. Let's evaluate the product of **Second term** with **Sum of Third and Fourth term**,

Also acknowledge that θ is positive in this region and β fluctuates from positive to negative.We can consider following form of equation in general,

$$R_1\cdot e^{i\theta}\left(R_2\cdot e^{i\beta}+R_3\cdot e^{-i\beta}\right) \quad (21)$$

where

$$R_1=|W|$$

,

$$R_2=|A|$$

$$R_3=|B|$$

$$\theta=arctan\left(\frac{t-2\alpha t}{t^2-\alpha^2+\alpha}\right)$$

$$\beta=\left(tln(n)+\frac{t}{2}ln(\pi)-\frac{t}{2}ln(u)\right)$$

solving equation further,we have,

$$(R_1cos\theta+iR_1sin\theta)(R_2cos\beta+iR_2sin\beta+R_3cos\beta-iR_3sin\beta)$$

$$(R_1cos\theta+iR_1sin\theta)[(R_2+R_3)cos\beta+i(R_2-R_3)sin\beta]$$

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Separating real and imaginary parts,

$$Re = R_1(R_2 + R_3)cos\theta cos\beta - R_1(R_2 - R_3)sin\theta sin\beta$$

$$Im = R_1(R_2 - R_3)cos\theta sin\beta + R_1(R_2 + R_3)sin\theta cos\beta$$

For this expression to become real, imaginary part must vanish,

$$R_1(R_2 - R_3)cos\theta sin\beta + R_1(R_2 + R_3)sin\theta cos\beta = 0$$

Case 1:

put $\theta = n\pi$,

$$R_1(R_2 - R_3)cos\theta sin\beta + R_1(R_2 + R_3)sin\theta cos\beta = 0$$

$$R_1(R_2 - R_3)cos(n\pi)sin\beta + R_1(R_2 + R_3)sin(n\pi)cos\beta = 0$$

$$R_1(R_2 - R_3)sin\beta = 0$$

$$\beta = n\pi$$

Case 2:

similarly put $\theta = \frac{\pi}{2} + n\pi$

$$R_1(R_2 - R_3)cos\theta sin\beta + R_1(R_2 + R_3)sin\theta cos\beta = 0$$

$$R_1(R_2 - R_3)cos\left(\frac{\pi}{2} + n\pi\right)sin\beta + R_1(R_2 + R_3)sin\left(\frac{\pi}{2} + n\pi\right)cos\beta = 0$$

$$R_1(R_2 + R_3)cos\beta = 0$$

$$\beta = \frac{\pi}{2} + n\pi$$

Case 1 and **Case 2** are the only possible combination of θ and β which can vanish imaginary number.However such combination is impossible for equation 21.Because, in the region $0 \le \alpha \le 1$,where,

$$\theta = arctan\left(\frac{t - 2\alpha t}{t^2 - \alpha^2 + \alpha}\right)$$

as 't' gets finitely larger and larger θ approaches nearer and nearer to 0.Due to the range of arctan(x) confined between $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$,it cannot equal $n\pi$ or $\frac{\pi}{2} + n\pi$ for any integer n.

Let's suppose

$$\beta = \left(tln(n) + \frac{t}{2}ln(\pi) - \frac{t}{2}ln(u)\right) = n\pi$$

Case 1 states that if $\beta = n\pi$, θ must equal to $n\pi$,but it is contradiction as θ never equals $n\pi$.

Let's suppose,

$$\beta = \left(tln(n) + \frac{t}{2}ln(\pi) - \frac{t}{2}ln(u)\right) = \frac{\pi}{2} + n\pi$$

Case 2 states that if $\beta = \frac{\pi}{2} + n\pi$, θ must equal to $\frac{\pi}{2} + n\pi$,but it is contradiction as θ never equals $\frac{\pi}{2} + n\pi$.

The conclusion is that no such combination of θ and β can satisfy the condition necessary for the vanishing of the imaginary parts. Any other combination will retain the imaginary part, which demonstrates that there are no zeros for $\xi(s)$ when $\alpha < \frac{1}{2}$ or $\alpha > \frac{1}{2}$.

At $s = \frac{1}{2} + it$

At $s = \frac{1}{2} + it$,we have observed that both the **Second term** and the **Sum of the Third and Fourth terms** are real-valued.Combining them,we have,

$$\xi\left(\frac{1}{2}+it\right) = \frac{1}{2} - \frac{\left(t^2+\frac{1}{4}\right)}{2}\Bigg[2\pi^{-\frac{1}{4}}\sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{2}}}\cdot cos\left(tln(n)+\frac{t}{2}ln(\pi)\right)\int_{\pi n^2}^{\infty}e^{-u}u^{-\frac{3}{4}}cos\left(\frac{t}{2}ln(u)\right)du$$
$$+2\pi^{-\frac{1}{4}}\sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{2}}}\cdot sin\left(tln(n)+\frac{t}{2}ln(\pi)\right)\int_{\pi n^2}^{\infty}e^{-u}u^{-\frac{3}{4}}sin\left(\frac{t}{2}ln(u)\right)du\Bigg] \quad (22)$$

As no imaginary parts are involved and it becomes function of 't'.

$$\xi(t) = \frac{1}{2} - \left(t^2+\frac{1}{4}\right)\Bigg[\pi^{-\frac{1}{4}}\sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{2}}}\cdot cos\left(tln(n)+\frac{t}{2}ln(\pi)\right)\int_{\pi n^2}^{\infty}e^{-u}u^{-\frac{3}{4}}cos\left(\frac{t}{2}ln(u)\right)du$$
$$+\pi^{-\frac{1}{4}}\sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{2}}}\cdot sin\left(tln(n)+\frac{t}{2}ln(\pi)\right)\int_{\pi n^2}^{\infty}e^{-u}u^{-\frac{3}{4}}sin\left(\frac{t}{2}ln(u)\right)du\Bigg] \quad (23)$$

The above equation is equal to Reimann definition of $\xi(t)$ on his paper,[5]

$$\xi(t) = \frac{1}{2} - \left(tt+\frac{1}{4}\right)\int_{1}^{\infty}\psi(x)x^{-\frac{3}{4}}cos\left(\frac{t}{2}ln(x)\right)dx$$

Thus $\xi(s)$ is real valued only at $s = \frac{1}{2} + it$.

2.5. Conclusion

The $\xi(s)$ is defined in term of $\zeta(s)$ as in equation 4 shows that zeros of $\xi(s)$ are precisely the non-trivial zeros of $\zeta(s)$.We have proved that it can be only possible at $s = \frac{1}{2} + it$.

Hence Proved.

"*All non-trivial zeros of the Riemann zeta function have a real part one-half.*" [6]

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