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Two Boundary-Certificate Proofs of the Crouzeix Theorem: Fiberwise Fourier Recurrence and Defect–Gram Telescoping

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Abstract

We give two proofs of the finite-dimensional scalar Crouzeix theorem: the numerical range of every complex matrix is a 2-spectral set for polynomials and for rational functions without poles on the numerical range. Both proofs start from the positive double-layer calculus and reduce the sharp local estimate to a top singular pair, but their certificates are different. The first disintegrates the boundary density over the fibres of a finite Blaschke product, uses conjugate Cauchy companions and Fourier cancellation, and forces an impossible unbounded scalar recurrence. The second keeps the boundary variable, constructs a positive defect-Gram family, and obtains a weighted telescoping identity whose strictly positive first defect excludes norm at or above 2. We identify the common scalar moment sequence underlying the two certificates. A shared finite confluent Schur completion, real-analytic convex outer exhaustion, and intrinsic rational calculus give the global theorem. Two pinned Lean 4 developments separately kernel-check the corresponding polynomial and reduced-rational endpoints.

Keywords: numerical range; Crouzeix theorem; spectral set; double-layer potential; boundary fibres; Fourier recurrence; defect-Gram matrix; Schur algorithm; rational functional calculus; Lean 4

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1. Introduction

For a square matrix A, the numerical range $W(A)$ is a compact convex set that contains the spectrum. Crouzeix proved in 2004 that the holomorphic functional calculus over $W(A)$ admits a dimension-independent bound and conjectured that the optimal universal constant is 2 [3]. His subsequent general estimate gave the constant 11.08 [4]; further structural and numerical aspects were developed in [5]. Crouzeix and Palencia later established the published universal bound $1+\sqrt{2}$ [6]. A broad account of the problem, its extremal formulation, and the classes known at that time is given by Bickel et al. [2].

The integral machinery behind many of these estimates begins with the operator-valued boundary representation of Delyon and Delyon [8]. Its convex-domain and numerical-range forms were developed in [1,4], and the Crouzeix–Palencia symmetrized argument was clarified in [19]. Recent treatments of the double-layer potential and configuration constants include [14,22]. These works explain both the power of the positive boundary calculus and why an estimate of a symmetrized functional calculus does not, by itself, yield the sharp constant for the unsymmetrized value $f(A)$.

Two contemporaneous 2026 preprints also announce the constant-2 theorem: Jin [10] uses an auxiliary-basis positive-real completion and ordered weighted Gramians, whereas Lorist and Schwenninger [11] use a dilation perturbation lemma together with all iterates f^n. Neither announced theorem is used in the manuscript proof below. In particular, the present local argument is not deduced from the sign condition discussed in Remark 2 of [11]; its strictness is supplied by a positive first boundary defect. The proof given here is finite-dimensional and scalar-valued. It makes no claim about the complete matrix-valued version, whose relation to scalar spectral sets is discussed in [7].

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The paper contains two boundary-certificate proofs. Preliminary versions were released separately by the present author as the Boundary–Fourier route [12] and the Defect–Gram route [13]. The present manuscript consolidates the two proofs, removes duplicated global reductions, and proves their precise relationship. It is therefore submitted as a substantial follow-up version of the Boundary–Fourier preprint rather than as an unrelated new manuscript. For a finite Blaschke product on a conformal carrier, it disintegrates the positive double-layer density over boundary preimages. Conjugate Cauchy companions for all powers then give exact Fourier coefficients. A sum-of-squares remainder and a top singular pair produce an affine recurrence whose orbit would be unbounded if the norm exceeded 2.

The second proof uses the same positive boundary calculus in a different way. It keeps the boundary point as the integration variable and forms a defect-Gram family from the resolvent defects $(f(\sigma)^j I - f(A)^j)(\sigma I - A)^{-1}$. Diagonal and adjacent Gram identities collapse at a top singular pair to a scalar recurrence. A geometrically weighted telescope has only nonnegative terms and a strictly positive first defect, excluding norm at or above 2.

The two arguments are genuinely different certificates, not unrelated computations. Their scalar sequences coincide after commuting holomorphic functions of A, and the positive fibre kernel in the first proof is the disintegration of one half of the boundary density in the second. We make this relationship explicit after proving both local estimates. Neither certificate is deduced from the other, and either one can be combined with the shared global reduction: a finite confluent Schur completion replaces a normalized polynomial without loss at every Jordan jet, a real-analytic strictly convex outer exhaustion passes to $W(A)$, and direct polynomial approximation of pole terms yields the rational formulation.

Section 2 establishes the required numerical-range geometry. Sections 3 and 4 give the two local proofs, and Section 5 identifies their common objects and their distinct contradiction mechanisms. Sections 6 and 7 carry out the shared lossless inner replacement and outer-domain limit. Section 8 proves the rational equivalence. The final sections record sharpness, the scope of the two Lean 4 developments, availability, authorship, and disclosure information.

Throughout, vectors in $\mathbb{C}^n$ are columns, $\langle u, v\rangle = u^* v$, and $\|\cdot\|$ denotes both the Euclidean vector norm and its induced operator norm. For $A \in \mathbb{C}^{n\times n}$, its numerical range is

$$W(A) = \{x^* A x : x \in \mathbb{C}^n,\ \|x\| = 1\}.$$

Theorem 1. *For every $A \in \mathbb{C}^{n\times n}$ and every polynomial p,*

$$\|p(A)\| \le 2 \max_{z \in W(A)} |p(z)|. \tag{1}$$

Moreover, $W(A)$ is a compact convex set containing $\operatorname{spec}(A)$.

For a compact set $K \subset \mathbb{C}$ containing $\operatorname{spec}(A)$, *call K a C-*spectral set for rational functions *if*

$$\|r(A)\| \le C \max_{z \in K} |r(z)| \tag{2}$$

for every rational function r having no pole on K, where $r(A)$ is specified intrinsically in Section 8. For $K = W(A)$ and any fixed $C \ge 0$, the rational estimate (2) is equivalent to its restriction to polynomials. Consequently, (1) is equivalent to the assertion that $W(A)$ is a 2-spectral set for rational functions.

We prove every problem-specific ingredient. We use as foundational only the Riemann mapping theorem, the maximum-modulus principle, Schwarz's lemma, Cauchy's integral and derivative formulas, Jordan normal form, the singular value decomposition, the Jordan separation theorem, the conformal area formula, and elementary separation and support facts for planar compact convex sets. Standard background on conformal boundary behavior and convex support functions may be found in [18,20].

2. Geometry of the Numerical Range

Convexity is the classical Toeplitz–Hausdorff theorem [9,23]; the two-dimensional compression has the elliptical-range description of Murnaghan [17]. We include a short self-contained proof tailored to the finite-dimensional setting.

Lemma 2. *The set $W(A)$ is compact and convex, and* $\operatorname{spec}(A) \subset W(A)$.

Proof. The unit sphere is compact and $x \mapsto x^*Ax$ is continuous, so $W(A)$ is compact. If $Av = \lambda v$ and $\|v\| = 1$, then $v^*Av = \lambda$, proving $\operatorname{spec}(A) \subset W(A)$.

For convexity, take $z_1 = x^*Ax$ and $z_2 = y^*Ay$, and compress A to $S = \operatorname{span}\{x, y\}$. If $\dim S = 1$, then $z_1 = z_2$. Suppose $\dim S = 2$ and, in an orthonormal basis of S, write the compression as C. Put

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

A direct multiplication shows that the rank-one orthogonal projections on S are exactly

$$P(r) = \frac{1}{2}(I + r_1\sigma_1 + r_2\sigma_2 + r_3\sigma_3), \qquad r = (r_1, r_2, r_3) \in \mathbb{R}^3, \quad |r| = 1.$$

Thus the numerical range of C is

$$\frac{\operatorname{tr} C}{2} + L(\mathbb{S}^2), \qquad L(r) = \frac{1}{2}\sum_{k=1}^{3} r_k \operatorname{tr}(C\sigma_k),$$

where $L : \mathbb{R}^3 \to \mathbb{C} \simeq \mathbb{R}^2$ is real-linear. Since $\ker L \neq \{0\}$,

$$L(\mathbb{S}^2) = L(\overline{\mathbb{B}^3}). \tag{3}$$

Indeed, if $|r| \leq 1$, choose $0 \neq k \in \ker L$. The quadratic function $t \mapsto |r + tk|^2$ tends to infinity and has a minimum no larger than $|r|^2 \leq 1$, so it equals 1 for some real t; then $L(r + tk) = L(r)$. The reverse inclusion in (3) is immediate. The image of the ball is convex. It contains z_1, z_2, its entire segment lies in the compression numerical range, and that compression numerical range lies in $W(A)$. Hence $W(A)$ is convex. □

3. First Local Proof: Boundary Fibres and Fourier Recurrence

Let Ω be a bounded convex domain with real-analytic strictly convex boundary and $W(A) \Subset \Omega$. Let $\phi : \Omega \to \mathbb{D}$ be a conformal map. Schwarz reflection across the analytic boundary implies that ϕ extends conformally across $\partial\Omega$; see [18, Ch. 3]. Let

$$f = B \circ \phi, \tag{4}$$

where B is a nonconstant finite Blaschke product. Thus f is analytic on a neighborhood of $\overline{\Omega}$ and $|f| = 1$ on $\partial\Omega$.

Proposition 3 (Boundary–Fourier local estimate)**.** *Every f of the form* (4) *satisfies*

$$\|f(A)\| \leq 2.$$

The assertion also holds when B is a unimodular constant.

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Proof. Give $\partial\Omega$ its counterclockwise orientation. At $\zeta \in \partial\Omega$, let $n(\zeta)$ be the outward unit normal, regarded as a complex number, and let ds denote arclength. Thus

$$d\zeta = in(\zeta)\, ds. \tag{5}$$

If $\zeta = \zeta(s)$ is an arclength parametrization and $f(\zeta(s)) = e^{it(s)}$, then

$$\kappa(\zeta) := t'(s) > 0, \qquad f'(\zeta)n(\zeta) = f(\zeta)\kappa(\zeta). \tag{6}$$

Indeed, $B \circ \phi$ restricts to an orientation-preserving finite covering of the unit circle. In particular, all boundary solutions of $f(\zeta) = \alpha$, $\alpha \in \mathbb{T}$, are simple.

For $\alpha \in \mathbb{T}$, put

$$K_\alpha(z) = \sum_{\substack{\zeta\in\partial\Omega \\ f(\zeta)=\alpha}} \frac{n(\zeta)}{\kappa(\zeta)(\zeta - z)}. \tag{7}$$

This kernel has positive real part at A. Convexity and strict containment give

$$\mathrm{Re}\big(\overline{n(\zeta)}(\zeta I - A)\big) \succ 0.$$

If T is strictly accretive, then

$$\mathrm{Re}(T^{-1}) = T^{-*}(\mathrm{Re}\, T)T^{-1} \succeq 0. \tag{8}$$

Since

$$n(\zeta)(\zeta I - A)^{-1} = \big(\overline{n(\zeta)}(\zeta I - A)\big)^{-1},$$

we obtain

$$H_\alpha := \mathrm{Re}\, K_\alpha(A) \succeq 0. \tag{9}$$

Let $dm(\alpha) = dt/(2\pi)$ be normalized arclength measure on $\mathbb{T}$. Changing variables by $\alpha = f(\zeta)$ on every sheet of the boundary covering and using (6), for $m \geq 0$ we obtain

$$\begin{aligned}\int_{\mathbb{T}} \alpha^m K_\alpha(z)\, dm(\alpha) &= \frac{1}{2\pi}\int_{\partial\Omega} \frac{f(\zeta)^m n(\zeta)}{\zeta - z}\, ds \\ &= \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(\zeta)^m}{\zeta - z}\, d\zeta = f(z)^m,\end{aligned} \tag{10}$$

and, for $m \geq 1$,

$$\int_{\mathbb{T}} \bar{\alpha}^m K_\alpha(z)\, dm(\alpha) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{\overline{f(\zeta)^m}}{\zeta - z}\, d\zeta =: g_m(z). \tag{11}$$

These identities may be evaluated at A. With

$$F = f(A), \qquad G_m = g_m(A),$$

they give

$$\int_{\mathbb{T}} H_\alpha\, dm(\alpha) = I, \qquad \int_{\mathbb{T}} \alpha^m H_\alpha\, dm(\alpha) = \frac{F^m + G_m^*}{2} \quad (m \geq 1). \tag{12}$$

The positivity and normalization in (9)–(12) imply

$$\|F^m + G_m^*\| \leq 2 \qquad (m \geq 1). \tag{13}$$

Indeed, for unit vectors u, v, positivity first gives $|v^* H_\alpha u|^2 \leq (v^* H_\alpha v)(u^* H_\alpha u)$, and Cauchy–Schwarz in $L^2(dm)$ then gives

$$\left| v^*\Big(\int_{\mathbb{T}} \alpha^m H_\alpha\, dm\Big)u \right| \leq \Big(\int_{\mathbb{T}} v^* H_\alpha v\, dm\Big)^{1/2}\Big(\int_{\mathbb{T}} u^* H_\alpha u\, dm\Big)^{1/2} = 1.$$

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The same scalar argument, with A replaced by $z \in \Omega$, gives

$$|f(z)^m + \overline{g_m(z)}| \le 2, \qquad \|g_m\|_{\infty,\Omega} \le 3. \tag{14}$$

For $\alpha \in \mathbb{T}$, define

$$P_\alpha(z) = \frac{1}{1-\overline{\alpha} f(z)}, \qquad R_\alpha(z) = P_\alpha(z) - K_\alpha(z), \qquad C_\alpha = \alpha I - F. \tag{15}$$

At a boundary solution $f(\zeta) = \alpha$, the principal part of P_α is

$$\frac{n(\zeta)}{\kappa(\zeta)(\zeta - z)}$$

by (6); hence all apparent boundary poles cancel in R_α. Since $P_\alpha(z) = \sum_{m\ge 0} \overline{\alpha}^m f(z)^m$ in Ω, comparison with (10) and (11) yields

$$\int_{\mathbb{T}} \alpha^m R_\alpha(z)\, dm(\alpha) = 0 \qquad (m \ge 0), \tag{16}$$

$$\int_{\mathbb{T}} \overline{\alpha}^m R_\alpha(z)\, dm(\alpha) = -g_m(z) \qquad (m \ge 1). \tag{17}$$

Only these Fourier-coefficient identities are used; no pointwise convergence of a Fourier series is needed.

Because $I - \overline{\alpha} F = \overline{\alpha} C_\alpha$, we have $P_\alpha(A) = \alpha C_\alpha^{-1}$. The matrix C_α is invertible: the maximum-modulus principle and spectral mapping give $\operatorname{spec}(F) = f(\operatorname{spec}(A)) \subset \mathbb{D}$. Direct multiplication gives

$$C_\alpha^* \operatorname{Re}(P_\alpha(A)) C_\alpha = I - \operatorname{Re}(\overline{\alpha} F). \tag{18}$$

Using $P_\alpha = K_\alpha + R_\alpha$, we therefore have the exact decomposition

$$\mathcal{E}_\alpha := I - \operatorname{Re}(\overline{\alpha} F) = Q_\alpha + C_\alpha^* \operatorname{Re}(R_\alpha(A)) C_\alpha, \qquad Q_\alpha := C_\alpha^* H_\alpha C_\alpha \succeq 0. \tag{19}$$

Assume for contradiction that

$$\lambda := \|F\| > 2. \tag{20}$$

Choose unit singular vectors x, y such that

$$Fx = \lambda y, \qquad F^* y = \lambda x. \tag{21}$$

For every function h holomorphic on Ω, set $\tau(h) = x^* h(A) x$. Because $h(A)$ commutes with F, (21) gives

$$y^* h(A) y = \tau(h), \qquad x^* h(A) y = \frac{1}{\lambda} \tau(fh). \tag{22}$$

For $m \ge 1$, define

$$\beta_m = \operatorname{Re} \tau(f^m g_m). \tag{23}$$

These numbers are uniformly bounded. Indeed, choose a positively oriented contour $\gamma \Subset \Omega$ of winding number one around $\operatorname{spec}(A)$. The holomorphic functional calculus gives

$$\|h(A)\| \le M \|h\|_{\infty,\Omega}, \qquad M := \frac{\operatorname{length}(\gamma)}{2\pi} \max_{z\in\gamma} \|(zI - A)^{-1}\| < \infty. \tag{24}$$

Thus (14) implies

$$|\beta_m| \le 3M \qquad (m \ge 1). \tag{25}$$

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The case $m = 1$ of (13), together with (22), gives

$$2 \geq \mathrm{Re}\big(y^*(F + G_1^*)x\big) = \lambda + \frac{\beta_1}{\lambda}.$$

Hence

$$\beta_1 \leq -\lambda(\lambda - 2) < 1. \tag{26}$$

Fix $N \geq 1$ and set

$$v_N(\alpha) = x + \sum_{j=1}^{N} \bar{\alpha}^j F^{j-1} y. \tag{27}$$

Multiplication by C_α telescopes:

$$w_N(\alpha) := C_\alpha v_N(\alpha) = \alpha x - (\lambda - 1)y - \bar{\alpha}^N F^N y. \tag{28}$$

Fourier orthogonality gives the exact identity

$$\int_{\mathbb{T}} v_N(\alpha)^* \mathcal{E}_\alpha v_N(\alpha)\, dm(\alpha) = 2 - \lambda. \tag{29}$$

For completeness, the mean of $\|v_N\|^2$ is $1 + \sum_{k=0}^{N-1} \|F^k y\|^2$, whereas the mean of $\bar{\alpha} v_N^* F v_N$ is $\lambda + \sum_{k=1}^{N-1} \|F^k y\|^2$.

Apply the positive decomposition (19) to v_N and use (29). We get

$$2 - \lambda \geq \int_{\mathbb{T}} w_N(\alpha)^* \,\mathrm{Re}(R_\alpha(A)) w_N(\alpha)\, dm(\alpha). \tag{30}$$

Put $d = \lambda - 1$. The three Fourier exponents in w_N are $1, 0, -N$. Expanding and using (16)–(17), the only nonzero pairings give

$$\int_{\mathbb{T}} w_N^* R_\alpha(A) w_N\, dm = d\, x^* G_1 y + x^* G_{N+1} F^N y - d\, y^* G_N F^N y. \tag{31}$$

By (22), the real part of the right side is

$$\frac{\lambda - 1}{\lambda}\beta_1 + \frac{1}{\lambda}\beta_{N+1} - (\lambda - 1)\beta_N. \tag{32}$$

It follows that

$$\beta_{N+1} \leq -\lambda(\lambda - 2) - (\lambda - 1)\beta_1 + \lambda(\lambda - 1)\beta_N. \tag{33}$$

This contradicts (25). To see it explicitly, let

$$q = \lambda(\lambda - 1) > 1, \qquad c = -\lambda(\lambda - 2) - (\lambda - 1)\beta_1,$$

and define $U_1 = \beta_1$, $U_{N+1} = c + qU_N$. Induction in (33) gives $\beta_N \leq U_N$. If $s = -c/(q-1)$ is the fixed point, then

$$(q - 1)(\beta_1 - s) = \lambda(\lambda - 2)(\beta_1 - 1) < 0 \tag{34}$$

by (20) and (26). Therefore $U_N = s + q^{N-1}(\beta_1 - s) \to -\infty$, contradicting $|\beta_N| \leq 3M$. Hence $\|F\| \leq 2$. The constant unimodular case is immediate. $\square$

4. Second Local Proof: Defect-Gram Recurrence and Telescope

Let $\Omega \subset \mathbb{C}$ be a bounded convex domain with C^1 Jordan boundary, and assume

$$W(A) \Subset \Omega. \tag{35}$$

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Set $\mathcal{A}(\Omega) = \mathrm{Hol}(\Omega) \cap C(\overline{\Omega})$. We shall prove the following local estimate.

Proposition 4 (Defect–Gram local estimate). *If $f \in \mathcal{A}(\Omega)$ and $|f(\sigma)| = 1$ for every $\sigma \in \partial\Omega$, then*

$$\|f(A)\| < 2.$$

Give $\partial\Omega$ its counterclockwise orientation. At $\sigma \in \partial\Omega$, let n_σ be the outward unit normal and put

$$R_\sigma = (\sigma I - A)^{-1}, \qquad Q_\sigma = \mathrm{Re}(\overline{n_\sigma}(\sigma I - A)), \qquad \mu_\sigma = \frac{1}{\pi} R_\sigma^* Q_\sigma R_\sigma. \tag{36}$$

The resolvent exists by Lemma 2 and (35). The supporting-line property of a convex C^1 domain says

$$\mathrm{Re}(\overline{n_\sigma}(\sigma - z)) > 0 \quad (z \in W(A)).$$

The inequality is strict uniformly over the compact set $W(A)$. Therefore, for every unit vector v,

$$v^* Q_\sigma v = \mathrm{Re}(\overline{n_\sigma}(\sigma - v^* A v)) > 0,$$

so

$$Q_\sigma \succ 0. \tag{37}$$

For a continuous scalar function u on the boundary, define

$$\mathcal{L}(u) = \int_{\partial\Omega} u(\sigma) \mu_\sigma \, ds.$$

For $h \in \mathcal{A}(\Omega)$, define

$$g_h(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\overline{h(\sigma)}}{\sigma - z} \, d\sigma, \qquad z \in \Omega. \tag{38}$$

Differentiation on compact subsets shows that g_h is holomorphic. Jordan form and Cauchy's derivative formula give

$$g_h(A) = \frac{1}{2\pi i} \int_{\partial\Omega} \overline{h(\sigma)} R_\sigma \, d\sigma. \tag{39}$$

Indeed, on a Jordan block $J = \lambda I + N$ of size m the right side is

$$\sum_{k=0}^{m-1} N^k \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\overline{h(\sigma)}}{(\sigma - \lambda)^{k+1}} \, d\sigma = \sum_{k=0}^{m-1} \frac{g_h^{(k)}(\lambda)}{k!} N^k = g_h(J).$$

We also need the analytic half of Cauchy's formula at exactly the regularity $\mathcal{A}(\Omega)$. Fix $a \in \Omega$ and let $h_t(z) = h(a + t(z - a))$, $0 < t < 1$. Convexity gives $a + t(\overline{\Omega} - a) \Subset \Omega$, so h_t is holomorphic on a neighborhood of $\overline{\Omega}$. Uniform continuity of h gives $h_t \to h$ uniformly on $\overline{\Omega}$ as $t \uparrow 1$. The matrix Cauchy formula for h_t, checked on each Jordan block, is

$$\frac{1}{2\pi i} \int_{\partial\Omega} h_t(\sigma) R_\sigma \, d\sigma = h_t(A).$$

On the left one may pass to the limit uniformly. On an interior contour surrounding $\mathrm{spec}(A)$, uniform convergence gives $h_t(A) \to h(A)$. Hence

$$\frac{1}{2\pi i} \int_{\partial\Omega} h(\sigma) R_\sigma \, d\sigma = h(A). \tag{40}$$

From (36),

$$\mu_\sigma = \frac{1}{2\pi} (n_\sigma R_\sigma + \overline{n_\sigma} R_\sigma^*). \tag{41}$$

Since $d\sigma = in_\sigma ds$, equations (39)–(41) imply

$$\mathcal{L}(h) = h(A) + g_h(A)^*. \tag{42}$$

For $h = 1$, $g_1 = 1$ by Cauchy's formula, and therefore

$$\int_{\partial\Omega} \mu_\sigma \, ds = 2I. \tag{43}$$

All these are ordinary norm-convergent integrals of continuous matrix-valued functions.

Put $F = f(A)$ and, for $j \geq 0$, set

$$D_j(\sigma) = (f(\sigma)^j I - F^j)R_\sigma, \quad G_j = g_{f^j}(A), \quad H_j = \mathcal{L}(f^j) = F^j + G_j^*, \tag{44}$$

and

$$E_{rs} = \int_{\partial\Omega} D_r(\sigma)^* Q_\sigma D_s(\sigma) \, \frac{ds}{\pi} \qquad (r, s \geq 0). \tag{45}$$

Functions of A commute with R_σ, and functions of A^* commute with R_σ^*. Expanding without moving a factor across Q_σ, and using (42) and (43), gives

$$E_{rs} = 2F^{*r}F^s - H_r^* F^s - F^{*r} H_s + \mathcal{L}(\overline{f}^{\,r} f^s). \tag{46}$$

This is a Gram matrix: for arbitrary vectors $v_0, \ldots, v_N$,

$$\sum_{r,s=0}^{N} v_r^* E_{rs} v_s = \int_{\partial\Omega} \left\| Q_\sigma^{1/2} \sum_{s=0}^{N} D_s(\sigma) v_s \right\|^2 \frac{ds}{\pi} \geq 0. \tag{47}$$

On the boundary, $\overline{f}^{\,r} f^s = f^{s-r}$ for $s \geq r$, and $\mathcal{L}(\overline{h}) = \mathcal{L}(h)^*$. Consequently,

$$E_{jj} = 2I - G_j F^j - F^{*j} G_j^*, \tag{48}$$

$$E_{j+1,j} = -G_{j+1} F^j - F^{*(j+1)} G_j^* + F^* + G_1. \tag{49}$$

Proof of Proposition 4. If $c = \|F\| \leq 1$, there is nothing to prove. Suppose $c > 1$. Choose unit top right and left singular vectors x, y; thus

$$Fx = cy, \qquad F^* y = cx. \tag{50}$$

No orthogonality between x and y is asserted or needed. If $q(A)$ is any holomorphic function of A, it commutes with F, and

$$x^* q(A) x = c^{-1} y^* F q(A) x = c^{-1} y^* q(A) F x = y^* q(A) y. \tag{51}$$

For $j \geq 1$, define

$$a_j = \mathrm{Re}(y^* G_j F^j y) = \mathrm{Re}(x^* G_j F^j x), \tag{52}$$

where the equality follows from (51), and put

$$U_j(\sigma) = D_{j+1}(\sigma)x - D_j(\sigma)y, \qquad S_j = \int_{\partial\Omega} \|Q_\sigma^{1/2} U_j(\sigma)\|^2 \frac{ds}{\pi} \geq 0.$$

Equations (48) and (52) yield

$$x^* E_{j+1,j+1} x = 2 - 2a_{j+1}, \qquad y^* E_{jj} y = 2 - 2a_j.$$

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For the adjacent term, (50) gives

$$\begin{aligned} x^*G_{j+1}F^jy &= c^{-1}y^*G_{j+1}F^{j+1}y, \\ \mathrm{Re}(x^*F^{*(j+1)}G_j^*y) &= ca_j, \\ x^*F^*y &= c, \\ x^*G_1y &= c^{-1}y^*G_1Fy. \end{aligned}$$

Substitution into (49) gives

$$\mathrm{Re}(x^*E_{j+1,j}y) = -\frac{a_{j+1}}{c} - ca_j + c + \frac{a_1}{c}.$$

Expanding the square defining S_j now proves the exact recurrence

$$\frac{S_j}{2} = 2 - c - \frac{a_1}{c} + (c-1)a_j - \left(1 - \frac{1}{c}\right)a_{j+1}. \tag{53}$$

The companion integral (39) and $|f| = 1$ give the uniform bound

$$\|G_j\| \le \frac{\mathrm{length}(\partial\Omega)}{2\pi} \max_{\sigma\in\partial\Omega} \|R_\sigma\| =: C_{\Omega,A}. \tag{54}$$

Because $c > 1$, f cannot be constant. The maximum-modulus principle gives $|f(z)| < 1$ in Ω. By Jordan form,

$$\mathrm{spec}(F) = \{f(\lambda) : \lambda \in \mathrm{spec}(A)\};$$

indeed, $f(\lambda I + N)$ is triangular with diagonal $f(\lambda)$. Hence the spectral radius of F is less than 1. Jordan form then gives $\|F^j\| \to 0$, and (54) implies

$$|a_j| \le C_{\Omega,A}\|F^j\| \longrightarrow 0. \tag{55}$$

Multiply (53) by c^{-j} and sum from $j = 1$ to N. A finite telescoping computation gives

$$\sum_{j=1}^{N} \frac{S_j}{2c^j} = \left(2 - c - \frac{a_1}{c}\right)\frac{1-c^{-N}}{c-1} + \frac{c-1}{c}a_1 - \frac{c-1}{c^{N+1}}a_{N+1}. \tag{56}$$

Using (55) and simplifying,

$$\sum_{j=1}^{\infty} \frac{S_j}{2c^j} = \frac{(c-2)(a_1-1)}{c-1}. \tag{57}$$

Let

$$e = x^*E_{11}x = \int_{\partial\Omega} \|Q_\sigma^{1/2}D_1(\sigma)x\|^2 \frac{ds}{\pi}. \tag{58}$$

Equation (48) says $e = 2(1 - a_1)$, so

$$\sum_{j=1}^{\infty} \frac{S_j}{2c^j} + \frac{c-2}{2(c-1)}e = 0. \tag{59}$$

For each boundary point, every eigenvalue of F has modulus less than 1, whereas $|f(\sigma)| = 1$. Thus

$$D_1(\sigma) = (f(\sigma)I - F)R_\sigma$$

is invertible. Along with (37), this proves $e > 0$. Furthermore,

$$D_{j+1}(\sigma) = f(\sigma)D_j(\sigma) + D_1(\sigma)F^j,$$

so (50) gives

$$U_1(\sigma) = D_1(\sigma)\big(f(\sigma)x + (c-1)y\big). \tag{60}$$

The vector in parentheses cannot vanish. If it did, the norms of the two unit vectors and $|f(\sigma)| = 1$ would force $c = 2$ and $y = -f(\sigma)x$; then $Fx = cy$ would make x an eigenvector of F with an eigenvalue of modulus 2, contrary to the spectral-radius conclusion. Thus (37), invertibility of $D_1(\sigma)$, and (60) give $S_1 > 0$.

If $c > 2$, both terms on the left of (59) are nonnegative and the second is positive, a contradiction. If $c = 2$, the first term is positive because $S_1 > 0$, again a contradiction. Therefore $c < 2$. □

5. Relationship Between the Two Certificates

The two local proofs organize the same positive boundary calculus in different ways. We record the exact relationship, both to clarify the shared structure and to separate the two contradiction mechanisms.

Use the hypotheses and notation of Section 3. For a boundary point ζ, write

$$R_\zeta = (\zeta I - A)^{-1}, \qquad \Delta_\zeta = \mathrm{Re}\Big(\overline{n(\zeta)}(\zeta I - A)\Big), \qquad \mu_\zeta = \frac{1}{\pi} R_\zeta^* \Delta_\zeta R_\zeta.$$

This is precisely the density used in Section 4. The inverse-accretive identity (8) gives

$$\mu_\zeta = \frac{1}{\pi}\,\mathrm{Re}\Big(n(\zeta)(\zeta I - A)^{-1}\Big). \tag{61}$$

Lemma 5 (Fibre disintegration). *For every continuous scalar function u on $\mathbb{T}$,*

$$\int_{\mathbb{T}} u(\alpha) H_\alpha \, dm(\alpha) = \frac{1}{2}\int_{\partial\Omega} u(f(\zeta))\mu_\zeta \, ds. \tag{62}$$

Proof. Equations (7) and (61) give

$$H_\alpha = \pi \sum_{f(\zeta)=\alpha} \frac{\mu_\zeta}{\kappa(\zeta)}.$$

On every sheet of the covering $f : \partial\Omega \to \mathbb{T}$, $dm = dt/(2\pi) = \kappa(\zeta)ds/(2\pi)$ by (6). Substitution cancels κ and proves (62). Taking $u = 1$ also explains the normalizations $\int_{\mathbb{T}} H_\alpha \, dm = I$ and $\int_{\partial\Omega} \mu_\zeta \, ds = 2I$. □

There is also one common scalar companion sequence. Choose the same top right singular vector x for $F = f(A)$ in both proofs and use the same conjugate Cauchy companion $G_j = g_{f^j}(A)$. The first proof calls its moment

$$\beta_j = \mathrm{Re}\big(x^* F^j G_j x\big),$$

whereas the second calls it

$$a_j = \mathrm{Re}\big(x^* G_j F^j x\big).$$

Both F^j and G_j are holomorphic functions of A, so they commute. Consequently,

$$\beta_j = a_j \qquad (j \geq 1). \tag{63}$$

The equality of the input data does not identify the certificates. The first proof pushes the positive density to $\mathbb{T}$, subtracts the fibrewise Cayley kernel, and uses exact Fourier cancellation to derive an unstable affine orbit. It excludes $\|F\| > 2$. The second proof polarizes the defects before integrating, retains their Gram positivity term by term, and sums an exact recurrence with geometric weights. Its positive first defect also excludes the equality case $\|F\| = 2$ for nonconstant boundary-inner f.

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Feature	Boundary–Fourier certificate	Defect–Gram certificate
Positive object	Fibre kernel $H_\alpha \succeq 0$ on $\mathbb{T}$	Boundary block Gram family (E_{rs})
Organization	Fibres of $f : \partial\Omega \to \mathbb{T}$	Powers f^r, f^s at each boundary point
Scalar data	β_j	The same sequence a_j
Main algebra	Cayley remainder and Fourier coefficients	Diagonal and adjacent Gram identities
Final contradiction	Unbounded affine orbit	Nonnegative weighted telescope
Excluded case	$\|F\| > 2$	$\|F\| \geq 2$ for nonconstant f
Boundary hypothesis	Real-analytic carrier and finite Blaschke pullback	C^1 convex carrier and any boundary-inner f

Thus the routes share their geometric and functional-calculus foundation but provide distinct sharp local certificates. Either local proposition is sufficient for the common globalization below.

6. Finite Confluent Inner Replacement

We next replace a normalized polynomial by one boundary-inner scalar function without changing its value at the matrix. The recursion used below is the finite multipoint form of the classical Schur algorithm [21]; all details needed here, including repeated nodes, are proved directly. For $a, c \in \mathbb{D}$, write

$$b_a(z) = \frac{z-a}{1-\bar{a}z}, \qquad \tau_c(w) = \frac{w-c}{1-\bar{c}w}.$$

A Schur function is a holomorphic map from $\mathbb{D}$ into $\overline{\mathbb{D}}$.

Lemma 6 (Finite confluent Schur completion)**.** *Let s be a Schur function and let $a_1, \ldots, a_N \in \mathbb{D}$, with repetitions allowed. There is a finite Blaschke product B of degree at most N such that*

$$s - B \quad \text{is divisible in } \mathrm{Hol}(\mathbb{D}) \text{ by } \prod_{k=1}^{N} b_{a_k}. \tag{64}$$

A degree-zero finite Blaschke product is a unimodular constant. In particular, if a occurs m times, then $B^{(\ell)}(a) = s^{(\ell)}(a)$ for $0 \leq \ell < m$.

Proof. If q is Schur, $a \in \mathbb{D}$, and $c = q(a)$, then either $|c| = 1$, in which case $q \equiv c$ by the maximum-modulus principle, or $|c| < 1$ and

$$(Sq)(z) = \frac{\tau_c(q(z))}{b_a(z)} \tag{65}$$

is Schur. The singularity is removable, and this assertion follows by applying Schwarz's lemma to $\tau_c \circ q \circ b_a^{-1}$, which fixes zero. Conversely, if r is Schur, then

$$q(z) = \tau_c^{-1}(b_a(z)r(z)) = \frac{c + b_a(z)r(z)}{1 + \bar{c}\, b_a(z)r(z)} \tag{66}$$

is Schur and $q(a) = c$.

Starting with $s_0 = s$, put $c_k = s_{k-1}(a_k)$ and apply (65) successively while $|c_k| < 1$. If $|c_k| = 1$ at some stage, s_{k-1} is constant. Reconstructing by (66) for the preceding stages shows that s itself is a finite Blaschke product of degree at most $k-1$; take $B = s$.

If all N steps occur, choose an arbitrary unimodular constant B_N and reconstruct backwards:

$$B_{k-1}(z) = \tau_{c_k}^{-1}(b_{a_k}(z)B_k(z)), \qquad k = N, N-1, \ldots, 1. \tag{67}$$

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Each B_{k-1} is rational, has no pole on $\overline{\mathbb{D}}$, is unimodular on $\partial\mathbb{D}$, and has degree one more than B_k; hence $B = B_0$ is a finite Blaschke product of degree at most N. The identity

$$\frac{c+u}{1+\bar{c}u} - \frac{c+v}{1+\bar{c}v} = \frac{(1-|c|^2)(u-v)}{(1+\bar{c}u)(1+\bar{c}v)} \tag{68}$$

shows at each backward step that $s_{k-1} - B_{k-1}$ gains the factor b_{a_k} and retains every factor of $s_k - B_k$. The denominators do not vanish in $\mathbb{D}$. Backward induction proves (64). Since b_a has a simple zero at a, the jet assertion follows. □

The Riemann map used below needs only continuous boundary values. This is a special case of the classical boundary-extension theory for conformal maps [18]; we include a proof at the required regularity.

Lemma 7 (Boundary extension for smooth convex domains)**.** *If U, V are bounded convex domains with C^1 Jordan boundaries, every conformal bijection $F : U \to V$ extends to a homeomorphism $\overline{U} \to \overline{V}$.*

Proof. Fix $\xi \in \partial U$ and $u_0 \in U$. For every sufficiently small regular $r > 0$, the set $C_r = U \cap \partial B(\xi, r)$ is one open arc separating the cap $U_r = U \cap B(\xi, r)$ from u_0. To check this locally, translate and rotate so that the boundary is the graph $y = g(x)$, where g is convex, $g(0) = g'(0) = 0$, and U is above the graph. On $x > 0$ the function $x^2 + g(x)^2$ is strictly increasing, and on $x < 0$ it is strictly decreasing. Thus a sufficiently small centered circle has exactly two boundary intersections and its portion in U is one arc.

The conformal area formula gives

$$\int_U |F'(z)|^2 \, dA(z) = \mathrm{area}(V) < \infty.$$

If $\ell(r)$ is the length of $F(C_r)$, Cauchy–Schwarz and length$(C_r) \le 2\pi r$ give

$$\int_0^R \frac{\ell(r)^2}{r} \, dr \le 2\pi \int_{U \cap B(\xi, R)} |F'(z)|^2 \, dA(z) < \infty.$$

Hence there are decreasing regular radii $r_j \downarrow 0$ with $\ell(r_j) \to 0$.

For such a radius, the finite-length arc $F(C_r)$ has endpoint limits in $\overline{V}$. They lie on ∂V: an interior endpoint limit would, by continuity of F^{-1} there, force points tending to ∂U to converge to an interior point. The closure Γ_r of $F(C_r)$ is either a crosscut of V or a Jordan loop meeting ∂V only at its common endpoint, and $\operatorname{diam} \Gamma_r \le \ell(r)$.

We use the following elementary small-crosscut observation. If such an arc Γ separates a bounded convex Jordan domain V, then, as $\operatorname{diam} \Gamma \to 0$, the component not containing a fixed $v_0 \in V$ has diameter tending to zero. For distinct endpoints a, b, a circle parametrization of ∂V and uniform continuity of it and its inverse show that one of the two boundary arcs β from a to b has diameter tending to zero with $|a - b|$. The Jordan curve $\Gamma \cup \beta$ bounds one component, whose closure is contained in $\operatorname{conv}(\Gamma \cup \beta)$ because V is convex; its diameter tends to zero. Once it is smaller than $\operatorname{dist}(v_0, \partial V)$, it is the component not containing v_0. If the endpoints agree, use the Jordan loop Γ and the same convex-hull argument.

The homeomorphism $F : U \to V$ maps the two components cut off by C_{r_j} to those cut off by $F(C_{r_j})$. Therefore

$$\operatorname{diam} F(U_{r_j}) \longrightarrow 0.$$

The nested caps form a neighborhood basis at ξ; their image closures are nested nonempty compact sets with diameters tending to zero. Their intersection is one point, which is the limit of $F(z)$ as $z \in U$ tends to ξ.

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These boundary limits define a continuous extension. Indeed, if $\xi_k \in \partial U$ tends to ξ, choose $z_k \in U$ with $|z_k - \xi_k| < 1/k$ and $|F(z_k) - F(\xi_k)| < 1/k$. Then $z_k \to \xi$, so $F(\xi_k) \to F(\xi)$. Mixed interior-boundary sequences are handled the same way. Apply the argument also to F^{-1}; the two extensions are inverse by approximation with interior points. Thus the extension is a homeomorphism. □

Lemma 8 (Lossless boundary-inner replacement)**.** *Let Ω be a bounded real-analytic strictly convex domain with $W(A) \Subset \Omega$, and let p be a nonzero polynomial. Set*

$$M_\Omega = \max_{z \in \overline{\Omega}} |p(z)|.$$

There is a finite Blaschke product B and a conformal map $\phi : \Omega \to \mathbb{D}$, continuous on the closures, such that $f = B \circ \phi \in \mathcal{A}(\Omega)$, $|f| = 1$ on $\partial\Omega$, and

$$f(A) = \frac{p(A)}{M_\Omega}. \tag{69}$$

The degree of B is at most the degree of the minimal polynomial of A.

Proof. Factor the minimal polynomial as

$$\mu_A(z) = \prod_{j=1}^{s} (z - \lambda_j)^{m_j}, \qquad N = \sum_{j=1}^{s} m_j.$$

The Riemann mapping theorem gives a conformal bijection $\phi : \Omega \to \mathbb{D}$. Lemma 7, applied also to $\psi = \phi^{-1}$, gives homeomorphic extensions to the closures and maps $\partial\Omega$ onto $\partial\mathbb{D}$.

Since $M_\Omega > 0$, the function

$$s(w) = \frac{p(\psi(w))}{M_\Omega}$$

is a Schur function continuous on $\overline{\mathbb{D}}$. Put $\alpha_j = \phi(\lambda_j)$ and apply Lemma 6 to the list in which α_j occurs m_j times. It gives a finite Blaschke product B of degree at most N. Let $f = B \circ \phi$. It lies in $\mathcal{A}(\Omega)$ and is unimodular on the boundary.

Near λ_j,

$$b_{\alpha_j}(\phi(z)) = (z - \lambda_j) v_j(z), \qquad v_j(\lambda_j) = \frac{\phi'(\lambda_j)}{1 - |\alpha_j|^2} \neq 0.$$

Thus $f - p/M_\Omega$ has a zero of order at least m_j at each λ_j. Holomorphic functions with these same jets have equal matrix values: on the generalized eigenspace for λ_j, write $A = \lambda_j I + N_j$, with $N_j^{m_j} = 0$, and use

$$h(A) = \sum_{k=0}^{m_j - 1} \frac{h^{(k)}(\lambda_j)}{k!} N_j^k.$$

This proves (69), including nonsemisimple A. □

7. Real-Analytic Strictly Convex Outer Exhaustion

Lemma 9. *Every compact convex $K \subset \mathbb{C}$, including a segment or a point, has compact real-analytic strictly convex supersets K_m such that $K \subset \operatorname{int} K_m$ and $K_m \to K$ in Hausdorff distance.*

Proof. Let

$$h(\theta) = \max_{z \in K} \operatorname{Re}(e^{-i\theta} z)$$

be the support function. In the distributional sense,

$$h + h'' \geq 0. \tag{70}$$

For completeness, approximate K in Hausdorff distance by convex polygons. Their support functions converge uniformly to h. A polygonal support function satisfies $h + h'' = 0$ between switching angles and has a nonnegative jump in its derivative at every switch; hence its $h + h''$ is a nonnegative atomic measure. Passing against nonnegative smooth test functions proves (70).

Let η_m be periodic Poisson kernels with parameters tending to 1. They are positive real-analytic approximate identities. Set

$$h_m^0 = h * \eta_m, \qquad \varepsilon_m = \|h_m^0 - h\|_\infty + \frac{1}{m}, \qquad H_m = h_m^0 + \varepsilon_m.$$

Then

$$H_m + H_m'' = (h + h'') * \eta_m + \varepsilon_m > 0, \qquad H_m \geq h + \frac{1}{m}. \tag{71}$$

Define

$$\gamma_m(\theta) = e^{i\theta}(H_m(\theta) + iH_m'(\theta)). \tag{72}$$

Then $\gamma_m'(\theta) = ie^{i\theta}(H_m + H_m'')$, so the curve is regular. For every α,

$$\frac{d}{d\theta}\operatorname{Re}(e^{-i\alpha}\gamma_m(\theta)) = -(H_m + H_m'')\sin(\theta - \alpha). \tag{73}$$

Thus the indicated functional has its unique maximum at $\theta = \alpha$, with value $H_m(\alpha)$.

The curve is injective. If $0 < \beta - \alpha \leq \pi$, then

$$\gamma_m(\beta) - \gamma_m(\alpha) = \int_\alpha^\beta ie^{i\theta}(H_m + H_m'')\,d\theta$$

has positive scalar projection onto $ie^{i(\alpha+\beta)/2}$, because the relevant cosine is nonnegative and positive in the interior. For $\pi < \beta - \alpha < 2\pi$, use the complementary interval. Hence γ_m is a real-analytic Jordan curve. Its convex hull K_m has support function H_m by (73); the unique support point in every direction is $\gamma_m(\alpha)$. Thus ∂K_m is exactly this curve and contains no nontrivial segment, so K_m is strictly convex.

The support function of $K + m^{-1}\overline{\mathbb{D}}$ is $h + m^{-1}$. Equation (71) and the support-half-plane description of compact convex sets give

$$K + m^{-1}\overline{\mathbb{D}} \subset K_m,$$

so $K \subset \operatorname{int} K_m$. Finally,

$$\|H_m - h\|_\infty \leq 2\|h * \eta_m - h\|_\infty + \frac{1}{m} \longrightarrow 0.$$

For compact convex sets C, D, $d_H(C, D) = \|h_C - h_D\|_\infty$: the inclusion $C \subset D + \delta\overline{\mathbb{D}}$ is equivalent, by the support-half-plane description, to $h_C \leq h_D + \delta$, and one applies this in both directions. Therefore $K_m \to K$ in Hausdorff distance. □

Proof of the polynomial inequality in Theorem 1. Put $K = W(A)$. It is compact and convex by Lemma 2. Let K_m be supplied by Lemma 9, and write $\Omega_m = \operatorname{int} K_m$. If p is nonzero, then

$$M_m = \max_{z \in K_m} |p(z)| > 0.$$

Lemma 8 supplies $f_m = B_m \circ \phi_m$ with $f_m(A) = p(A)/M_m$ and $|f_m| = 1$ on $\partial\Omega_m$. Because each Ω_m has real-analytic boundary, the local estimate may be obtained from either Proposition 3 or Proposition 4. The first gives $\|p(A)\| \leq 2M_m$ directly. The second gives the strict inequality when B_m is nonconstant, while a constant B_m gives $\|p(A)\| = M_m$. Thus, by either proof route,

$$\|p(A)\| \leq 2M_m. \tag{74}$$

Hausdorff convergence and uniform continuity of p on a fixed compact neighborhood imply

$$M_m \longrightarrow M_K := \max_{z\in K} |p(z)|.$$

Indeed, $K \subset K_m$ gives $M_m \geq M_K$, while every point of K_m is within $d_H(K_m, K)$ of a point of K, giving the reverse limsup. Letting $m \to \infty$ in (74) proves (1). The zero polynomial is immediate. This also covers the case of a nonzero polynomial vanishing on a segment or singleton K: every M_m is positive and its limit is zero. $\square$

8. Rational Spectral Sets

We first make the rational functional calculus explicit. If $r = a/b$, where a, b are polynomials and b has no zero on a compact set $K \supset \mathrm{spec}(A)$, define

$$r(A) = a(A)b(A)^{-1}. \tag{75}$$

The matrix $b(A)$ is invertible: in Jordan form, each block $b(\lambda I + N)$ is upper triangular with diagonal entry $b(\lambda) \neq 0$. The definition is representation-independent. If $a/b = c/d$ as rational functions, then the polynomial identity $ad = bc$ gives $a(A)d(A) = b(A)c(A)$. All polynomial functions of A commute; multiplying by the commuting inverses of $b(A)$ and $d(A)$ gives $a(A)b(A)^{-1} = c(A)d(A)^{-1}$.

On a Jordan block $J = \lambda I + N$ of size m, this definition agrees with the Taylor formula

$$r(J) = \sum_{j=0}^{m-1} \frac{r^{(j)}(\lambda)}{j!} N^j. \tag{76}$$

To see this without assuming the conclusion, Taylor's formula modulo $(z-\lambda)^m$ gives polynomials A_0, B_0 of degree below m such that $a(z) \equiv A_0(z)$ and $b(z) \equiv B_0(z)$ modulo $(z-\lambda)^m$. Because $B_0(\lambda) \neq 0$, the truncated Taylor polynomial of $1/B_0$ multiplies B_0 to 1 modulo $(z-\lambda)^m$. Substitution $z = \lambda I + N$ therefore shows that $b(J)^{-1}$ is the corresponding truncated Taylor polynomial, and multiplying by $a(J)$ yields precisely the truncated Taylor expansion of a/b, namely (76). This proves, in particular, that repeated Jordan blocks and repeated poles in a partial-fraction expansion cause no ambiguity.

Proposition 10. *Let $A \in \mathbb{C}^{n\times n}$ and let $K \subset \mathbb{C}$ be compact and convex with* $\mathrm{spec}(A) \subset K$*. For a fixed $C \geq 0$, the following are equivalent:*

$$\|p(A)\| \leq C \max_K |p| \qquad \textit{for every polynomial } p; \tag{P}$$

$$\|r(A)\| \leq C \max_K |r| \qquad \textit{for every rational } r \textit{ with no pole on } K. \tag{R}$$

Proof. Clearly (R) implies (P), because polynomials are rational functions with no finite poles. Assume (P), and let r have no pole on K. If r has no finite pole, then it is a polynomial and there is nothing to prove. We may therefore list its distinct finite poles as $w_1, \ldots, w_s$, all outside K. Choose $\varepsilon > 0$ smaller than $\min_j \mathrm{dist}(w_j, K)$ and put

$$L = K + \{z : |z| < \varepsilon\}.$$

Then L is an open bounded convex neighborhood of K and $\overline{L}$ misses every pole.

Fix one pole $w \notin \overline{L}$. Choose $z_0 \in \overline{L}$ minimizing $|w - z|$ and put $u = (w - z_0)/|w - z_0|$. For $z \in \overline{L}$, convexity puts $z_0 + t(z - z_0) \in \overline{L}$ for $0 \leq t \leq 1$. The right derivative at $t = 0$ of the squared distance to w is nonnegative, hence

$$\mathrm{Re}\big(\overline{(w - z_0)}(z - z_0)\big) \leq 0.$$

Writing $d = |w - z_0| > 0$ and $\xi = \overline{u}(z - w)$, we obtain $\operatorname{Re}\xi \le -d$ on $\overline{L}$. Let $M = \max_{z\in\overline{L}} |\xi|$ and choose a real $R > M^2/(2d)$. Then

$$|\xi + R|^2 = R^2 + 2R\operatorname{Re}\xi + |\xi|^2 \le R^2 - 2Rd + M^2 < R^2.$$

Consequently

$$q := \sup_{z\in\overline{L}} |1 + \xi/R| < 1,$$

and the geometric expansion

$$\frac{1}{z-w} = -\frac{\overline{u}}{R}\sum_{k=0}^{\infty}\left(1 + \frac{\overline{u}(z-w)}{R}\right)^k \tag{77}$$

converges uniformly on $\overline{L}$. Its partial sums are polynomials in z. Products of uniformly convergent sequences are uniformly convergent on this compact set, so powers of these partial sums approximate $(z-w)^{-j}$ uniformly for every positive integer j. The partial-fraction expansion of r, with every multiplicity retained, therefore yields polynomials p_m satisfying

$$\max_{z\in\overline{L}} |p_m(z) - r(z)| \longrightarrow 0. \tag{78}$$

It remains to justify convergence after applying the matrix calculus when A is not semisimple. Around every distinct eigenvalue λ choose a positively oriented circle $|z - \lambda| = \delta_\lambda$ whose closed disk lies in L and on which r is holomorphic. Cauchy's derivative formula and (78) give, for every $j \ge 0$,

$$|p_m^{(j)}(\lambda) - r^{(j)}(\lambda)| \le \frac{j!}{\delta_\lambda^j}\max_{|z-\lambda|=\delta_\lambda} |p_m(z) - r(z)| \longrightarrow 0. \tag{79}$$

For each Jordan block $J = \lambda I + N$, the polynomial Taylor formula and (76) show

$$p_m(J) - r(J) = \sum_{j=0}^{\dim J-1} \frac{p_m^{(j)}(\lambda) - r^{(j)}(\lambda)}{j!} N^j \longrightarrow 0$$

in operator norm. There are finitely many blocks; conjugating back by one fixed Jordan similarity proves

$$\|p_m(A) - r(A)\| \longrightarrow 0. \tag{80}$$

Meanwhile (78) on K gives

$$\left|\max_K |p_m| - \max_K |r|\right| \le \max_K |p_m - r| \longrightarrow 0.$$

Apply (P) to p_m and pass to the limit, using (80), to obtain (R) with exactly the same constant C. $\square$

Apply Proposition 10 with $K = W(A)$, whose compactness, convexity, and spectral inclusion were proved in Lemma 2. The polynomial estimate already proved, with $C = 2$, is therefore equivalent to the rational spectral-set statement in Theorem 1. This completes the proof.

9. Sharpness

The constant cannot be reduced. Let

$$J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad p(z) = z.$$

The numerical range of J is the closed disk centered at the origin with radius $1/2$, while $\|p(J)\| = \|J\| = 1$. Hence

$$\frac{\|p(J)\|}{\max_{z\in W(J)} |p(z)|} = 2.$$

10. Formal Verification and Scope

Two Lean 4 developments [16], built on Mathlib [15], separately check the finite-dimensional scalar polynomial endpoint and a reduced-rational endpoint under immutable dependency revisions. The Boundary–Fourier closure ends at the declaration

CrouzeixFormalization.luoCrouzeixEndpoint,

and the Defect–Gram closure ends at

CrouzeixFormalization.DefectGramProof.defectGramCrouzeixEndpoint.

Both inhabit the same explicitly stated proposition type. Both import a pinned revision of the Lean package accompanying Jin [10] for lower-level numerical-range, functional-calculus, contour, and compactness infrastructure, and both separately use the vendored RMT4 development for the Riemann-mapping theorem. Neither proof closure invokes the supporting package's separately stated constant-2 endpoint.

The Lean kernel reports exactly `propext`, `Classical.choice`, and `Quot.sound` for both route endpoints. The verification scripts reject `sorry`, `admit`, custom axioms, `unsafe`, and `native_decide`. The Boundary–Fourier release documents its transitive use of the pinned package's lower-level infrastructure and its separation from that package's endpoint. The Defect–Gram release additionally performs a recursive dependency-closure audit that rejects packaged constant-2 endpoints, positive-real completion, ordered-Gramian machinery, and the alternative route.

The formal developments and this manuscript reach the same polynomial and rational conclusions, but they are not claimed to be line-by-line translations. In particular, the manuscript uses the finite confluent completion in Lemma 6, while Lean closes the corresponding fixed-domain stage through checked finite-Blaschke limits, Schur recursion and its termination dichotomy, contour convergence, and conformal normalization. The two releases document their source-to-declaration maps and this fidelity boundary. No infinite-dimensional, matrix-valued, or completely bounded endpoint is asserted here.

Author Contributions: Qinyu Luo: conceptualization, methodology, formal analysis, writing—original draft, writing—review and editing, validation, supervision, and project administration. The author approved the final manuscript and accepts responsibility for its contents.

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AI Research Contribution Statement: GPT-5.6 Sol (OpenAI) made substantive contributions to the discovery, development, and verification of both proof routes. Its contributions included developing and auditing the boundary-fibre kernel, Cayley-remainder Fourier cancellation, and scalar recurrence in the first route; formulating the boundary defect-Gram approach, deriving its adjacent recurrence and weighted telescoping certificate, and establishing its strict local estimate in the second; organizing the common finite-Blaschke, Schur, outer-limit, and rational reductions; and producing substantial parts of both Lean 4 formalizations and their verification infrastructure. Qinyu Luo directed the research process, selected and consolidated the proof strategy, reviewed the mathematical arguments and formal statements, supervised the verification and release process, and accepts responsibility for the published work.

Data and Code Availability: The Boundary–Fourier manuscript is available as https://doi.org/10.20944/preprints202608.1661.v1. Its source, formalization, verification scripts, dependency lockfile, and source map are archived in the immutable research release https://doi.org/10.5281/zenodo.22052523 and corresponding repository tag

https://github.com/LOGIC-10/CrouzeixConjecture/releases/tag/v0.1.0.

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The corresponding Defect–Gram materials are archived under the immutable Zenodo version DOI

https://doi.org/10.5281/zenodo.22105428

and the matching repository release

https://github.com/LOGIC-10/Crouzeix-Defect-Gram-Proof/releases/tag/v1.2.0.

The two repositories record exact source hashes, releases, dependency revisions, and third-party notices sufficient to reproduce both kernel checks. The present integrated manuscript is the publication-level synthesis of those two disclosed routes. It uses no research dataset; its arXiv source package contains the complete LaTeX source for the main file and both included route files.

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References

1. C. Badea, M. Crouzeix, and B. Delyon, Convex domains and K-spectral sets, *Math. Z.* **252** (2006), no. 2, 345–365. https://doi.org/10.1007/s00209-005-0857-y
2. K. Bickel, P. Gorkin, A. Greenbaum, T. Ransford, F. L. Schwenninger, and E. Wegert, Crouzeix's conjecture and related problems, *Comput. Methods Funct. Theory* **20** (2020), 701–728. https://doi.org/10.1007/s40315-020-00350-9
3. M. Crouzeix, Bounds for analytical functions of matrices, *Integral Equations Operator Theory* **48** (2004), 461–477. https://doi.org/10.1007/s00020-002-1188-6
4. M. Crouzeix, Numerical range and functional calculus in Hilbert space, *J. Funct. Anal.* **244** (2007), no. 2, 668–690. https://doi.org/10.1016/j.jfa.2006.10.013
5. M. Crouzeix, Some constants related to numerical ranges, *SIAM J. Matrix Anal. Appl.* **37** (2016), no. 1, 420–442. https://doi.org/10.1137/15M1020411
6. M. Crouzeix and C. Palencia, The numerical range is a $(1+\sqrt{2})$-spectral set, *SIAM J. Matrix Anal. Appl.* **38** (2017), no. 2, 649–655. https://doi.org/10.1137/17M1116672
7. K. R. Davidson, V. I. Paulsen, and H. J. Woerdeman, Complete spectral sets and numerical range, *Proc. Amer. Math. Soc.* **146** (2018), no. 3, 1189–1195. https://doi.org/10.1090/proc/13801
8. B. Delyon and F. Delyon, Generalization of von Neumann's spectral sets and integral representation of operators, *Bull. Soc. Math. France* **127** (1999), no. 1, 25–41. https://doi.org/10.24033/bsmf.2340
9. F. Hausdorff, Der Wertvorrat einer Bilinearform, *Math. Z.* **3** (1919), 314–316. https://doi.org/10.1007/BF01292610
10. S. Jin, The numerical range is a 2-spectral set, *Preprints* 2026, 202607.1919, version 4. https://doi.org/10.20944/preprints202607.1919.v4
11. E. Lorist and F. L. Schwenninger, A solution to Crouzeix's conjecture, arXiv:2608.03841v2 (2026). https://arxiv.org/abs/2608.03841v2
12. Q. Luo, The sharp numerical-range spectral-set inequality, *Preprints* 2026, 202608.1661, version 1. https://doi.org/10.20944/preprints202608.1661.v1
13. Q. Luo, *A Defect-Gram Proof of the Crouzeix Theorem*, Zenodo, version 1.2.0 (2026), https://doi.org/10.5281/zenodo.22105428; matching GitHub research release, https://github.com/LOGIC-10/Crouzeix-Defect-Gram-Proof/releases/tag/v1.2.0
14. B. Malman, J. Mashreghi, R. O'Loughlin, and T. Ransford, Double-layer potentials, configuration constants, and applications to numerical ranges, *Int. Math. Res. Not. IMRN* **2025** (2025), no. 8, Article rnaf084. https://doi.org/10.1093/imrn/rnaf084
15. The Mathlib Community, The Lean mathematical library, in *Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020)*, ACM, 2020, pp. 367–381. https://doi.org/10.1145/3372885.3373824
16. L. de Moura and S. Ullrich, The Lean 4 theorem prover and programming language, in *Automated Deduction—CADE 28*, Lecture Notes in Computer Science 12699, Springer, Cham, 2021, pp. 625–635. https://doi.org/10.1007/978-3-030-79876-5_37

17. F. D. Murnaghan, On the field of values of a square matrix, *Proc. Natl. Acad. Sci. USA* **18** (1932), no. 3, 246–248. https://doi.org/10.1073/pnas.18.3.246
18. C. Pommerenke, *Boundary Behaviour of Conformal Maps*, Grundlehren der mathematischen Wissenschaften 299, Springer, Berlin, 1992. https://doi.org/10.1007/978-3-662-02770-7
19. T. Ransford and F. L. Schwenninger, Remarks on the Crouzeix–Palencia proof that the numerical range is a $(1+\sqrt{2})$-spectral set, *SIAM J. Matrix Anal. Appl.* **39** (2018), no. 1, 342–345. https://doi.org/10.1137/17M1143757
20. R. Schneider, *Convex Bodies: The Brunn–Minkowski Theory*, 2nd expanded ed., Encyclopedia of Mathematics and its Applications 151, Cambridge University Press, Cambridge, 2014. https://doi.org/10.1017/CBO9781139003858
21. I. Schur, Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind. I, *J. Reine Angew. Math.* **147** (1917), 205–232. https://doi.org/10.1515/crll.1917.147.205
22. F. L. Schwenninger and J. de Vries, The double-layer potential for spectral constants revisited, *Integral Equations Operator Theory* **97** (2025), Article 13, 22 pp. https://doi.org/10.1007/s00020-025-02800-2
23. O. Toeplitz, Das algebraische Analogon zu einem Satze von Fejér, *Math. Z.* **2** (1918), 187–197. https://doi.org/10.1007/BF01212904