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Proof of the Riemann Hypothesis

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Abstract

This paper presents an operator-theoretic proof of the Riemann Hypothesis. The guiding principle is not to postulate a Hilbert–Polya type operator whose spectrum is identified with the zeros of the completed zeta function. Instead, the proof first constructs the analytic, arithmetic, and boundary components that make such an operator inevitable. In this sense, Sections 2–5 form the structural core of the paper. Section 2 builds the weighted Hilbert-space operator setting and its compact spectral theory. Section 3 constructs, on a coefficient Hilbert space, the arithmetic trace that evaluates the Euler-product prime-power contribution. Section 4 constructs the singular-boundary data and transport structure internally within the analytic Hilbert-space framework. Section 5 then places these pieces in the orthogonal ambient decomposition $X = \mathcal{K}_R \oplus J_{\mathrm{arith}}\mathcal{H}_{\mathrm{arith}} \oplus \operatorname{Ran}\Pi_{\mathrm{res}}$. In this decomposition the prime-power term is accounted for on the arithmetic summand, the residual term is removed by passage to the canonical representative modulo $\operatorname{Ran}\Pi_{\mathrm{res}}$, and the remaining effective object is a $\mathcal{K}_R$-component obtained before any determinant identity or zero-localization argument is invoked. The main novelty of the construction is therefore a conversion of the explicit formula into an orthogonal-decomposition problem. The completed zeta function enters the comparison only through the canonical ξ-finite-part functional extracted from $\xi_c(w) = \xi(\frac{1}{2} + w)$ by the classical finite-window contour finite-part rule. This is not an external datum and is not defined by the desired equality $F_K \equiv \xi$. It is realized through a fixed comparison-independent finite-part mechanism, while the analytic Hilbert space, the arithmetic trace, the singular-boundary carrier, and the residual quotient are constructed prior to the spectral determinant. Thus the proof is arranged so that the provenance of every object is visible: neither the location of the zeros of ξ, nor a positivity, Herglotz, or spectral-localization statement equivalent to the Riemann Hypothesis, is assumed. Section 6 closes the construction. The functional equation of ξ induces a boundary reflection which descends to a self-adjoint involution on $\mathcal{K}_R$. The associated signed boundary-distribution comparison kernel is then realized, by quantitative Sobolev-reference Schatten estimates, as a self-adjoint Hilbert–Schmidt operator $K = K^* \in \mathfrak{S}_2$. This operator defines the regularized Fredholm determinant $F_K(s) = e^{a_K + b_K(s-\frac{1}{2})} \det_2\big(I + i(s - \frac{1}{2})K\big)$, with the constants fixed only by the central value and first logarithmic derivative. The finite-window residual-free equality is then transported to a central Cauchy–Laplace identity through a regularization scheme, a Cauchy–Laplace comparison subspace, and a projected finite-window topology fixed before the relevant pairings are evaluated. On the K-side, the finite-part realization is connected to the determinant trace by finite-window and finite-rank cyclic contraction identities before any comparison with ξ is made. The two resulting central transforms identify the logarithmic derivatives of F_K and ξ; the central normalization and the identity theorem give $F_K(s) \equiv \xi(s)$ on the whole complex plane. Finally, since K is self-adjoint, the zeros of F_K produced by its nonzero eigenvalues have the form $s = \frac{1}{2} + \frac{i}{\lambda_j}, \qquad \lambda_j \in \mathbb{R} \setminus \{0\}$. The global identity $F_K \equiv \xi$ therefore places every nontrivial zero of ξ, and hence of ζ, on the critical line.

Keywords: Riemann hypothesis; millennium problems; Fredholm determinant; operator theory; analytic number theory

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1. Introduction

1.1. *Problem Setting and Outline of the Proof*

The purpose of this paper is to prove operator-theoretically that all nontrivial zeros of the completed zeta function

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$$

lie on the critical line

$$\operatorname{Re} s = \frac{1}{2}.$$

The proof does not identify zeros with eigenvalues from the outset. Instead, it separates the prime-power contribution on the arithmetic side from the $\mathcal{K}_R$-component on a common Hilbert space. After passage to the canonical representative modulo $\operatorname{Ran}\Pi_{\mathrm{res}}$, the residual component is removed, and the remaining $\mathcal{K}_R$-projected component is represented by its Π_R-projection in $\mathcal{K}_R$.

The structure of the proof consists of five stages:

$$\text{analytic operator setup} \longrightarrow \text{coefficient-space arithmetic construction}$$

$$\longrightarrow \text{singular boundary-data construction} \longrightarrow \text{orthogonal-decomposition comparison framework}$$

$$\longrightarrow \text{spectral determinant closure.}$$

This separation specifies which objects are constructed analytically, where the arithmetic trace evaluates the prime-power contribution, where the residual quotient is taken, where the self-adjoint Hilbert–Schmidt operator appears, and where the zero locations are finally determined. The analytic and arithmetic preparations in Sections 2–5 are an essential part of the argument; Section 6 uses their output rather than replacing them by an independent determinant ansatz.

At the same time, the proof keeps four construction-level dependency points explicit. The finite readout reconstruction map $\mathcal{V}_{R\xi,M}$ is constructed by finite-dimensional linear algebra as $G_M^{\dagger}$, not assumed as an auxiliary right inverse. The local principal-part ledger is computed from a universal local normal form and a common finite-jet counterterm, rather than from side-dependent symbolic coefficients. The full-test-space unsmoothed transport-generator kernel of Section 4 remains optional and is not used as an unconditional input in the determinant comparison. Finally, the only ξ-specific object used before the comparison is the canonical ξ-finite-part functional extracted from $\xi_c(w) = \xi(1/2+w)$ by the finite-window contour finite-part rule; the proof does not assert that K_R reconstructs ξ without reference to this functional.

The central chain of the proof is as follows. From the residual-free comparison interface obtained in Section 5, Section 6 constructs the continuous comparison map

$$\mathcal{D}_R \xrightarrow{\operatorname{Tr}^{\mathrm{cmp}}_{\partial,R}} \mathcal{D}'_{R,\mathrm{adm}} \xrightarrow{\mathrm{LCI}_R} X \xrightarrow{\Pi_R} \mathcal{K}_R.$$

The functional equation induces a boundary reflection

$$\Theta_R : \mathcal{D}'_R \to \mathcal{D}'_R,$$

which is defined before any use of zero-location information. Its compatibility with admissible boundary distributions and with the residual-free comparison interface allows it to descend to a bounded self-adjoint involution

$$\mathcal{S}_R : \mathcal{K}_R \to \mathcal{K}_R.$$

This gives the signed boundary-distribution comparison kernel

$$\mathfrak{k}_R(f,g) = \langle \mathcal{S}_R \Pi_R \mathcal{J}_R f,\, \Pi_R \mathcal{J}_R g \rangle_X.$$

The self-adjointness $\mathcal{S}_R^* = \mathcal{S}_R$ is obtained by transporting the reflected boundary pairing to the Hilbert inner product on the residual-free comparison range; consequently, the Hermitian symmetry of this kernel uses only $\mathcal{S}_R^* = \mathcal{S}_R$ and the Hilbert-space inner product. The Sobolev eigenvalue growth, boundary-trace smoothing, and boundedness of the comparison map then realize the kernel as a self-adjoint Hilbert–Schmidt operator

$$K = K^*, \qquad K \in \mathfrak{S}_2.$$

No positivity, Herglotz property, or zero-localization statement equivalent to the Riemann Hypothesis is assumed in this construction.

The regularized Fredholm determinant

$$F_K(s) = e^{a_K + b_K(s-\frac{1}{2})} \det{}_2\big(I + i(s - \tfrac{1}{2})K\big)$$

is then formed from this K. The constants a_K, b_K fix only the central value and the first logarithmic derivative; they do not prescribe the zeros of F_K and do not encode the locations of the zeros of ξ.

The comparison $F_K \equiv \xi$ is obtained through a separate central Cauchy–Laplace argument. Section 6 first fixes the central kernel

$$h_w(u) = \frac{e^{wu} - 1}{u}, \qquad h_w(0) = w,$$

the raw finite-window cutoffs, the central finite-jet map J_M^{cen}, the universal principal-part map P_M^{cen}, the principal-part embedding ι_M^{cen}, and the central comparison topology on $\mathcal{C}_{\mathrm{cen}}$. The regularized finite-window input is defined before either pairing is evaluated by the algebraic subtraction

$$\Psi_{w,M}^{\mathrm{fw}} = \widetilde{\Psi}_{w,M}^{\mathrm{fw}} - \iota_M^{\mathrm{cen}} P_M^{\mathrm{cen}} J_M^{\mathrm{cen}}(h_{w,M}^{\mathrm{fw}}) \quad \text{in } \mathcal{C}_{\mathrm{cen}}^0.$$

The common local-principal-part lemma shows that the same finite-jet counterterm removes the local singular principal part of the Archimedean, arithmetic-trace, and singular-boundary contributions. Its definition is independent of the values of μ_L and μ_ξ. Section 6 then proves the finite-window convergence

$$\Psi_{w,M}^{\mathrm{fw}} \longrightarrow \Psi_w^{\mathrm{cen}} \quad \text{in the central Cauchy–Laplace comparison subspace } \mathcal{C}_{\mathrm{CL}}(r),$$

locally uniformly in w, and proves first, on the finite-part side and before the determinant trace is used, that the μ_L^{fp}-pairing is continuous on this Cauchy–Laplace subspace. The same subspace is the only part of the ambient central space used in the proof. The open-band passage from finite-window logarithmic tests is justified through the projected finite-window comparison topology, so no compactly supported test is treated as strictly band-limited. On the determinant side, the finite-part/trace bridge is not introduced as a definition: the finite-window restricted functional is first represented by the cyclic trace of the finite-window boundary kernel K_M, and only then reduced to finite-window and finite-rank cyclic contraction identities,

$$(M,N) \longmapsto \operatorname{tr}_{E_N}\Big(w(I+iwK_{M,N})^{-1}K_{M,N}^2\Big),$$

and only after the M- and N-limits does it prove

$$\mu_L^{\mathrm{fp}} = \mu_L^{\mathrm{det}} \quad \text{on } \{\Psi_w^{\mathrm{cen}}\}_{|w|<r}.$$

Only after this theorem do we write μ_L without a superscript on the central Cauchy–Laplace family. Combining the finite-window residual-free equality with these analytic facts gives

$$\langle \mu_L, \Psi_w^{\mathrm{cen}} \rangle = \langle \mu_\xi, \Psi_w^{\mathrm{cen}} \rangle.$$

Separate transform lemmas identify the left-hand side with the central logarithmic derivative of F_K, and the right-hand side with that of ξ. Therefore

$$\frac{d}{dw}\log F_K\Big(\frac{1}{2}+w\Big) = \frac{d}{dw}\log \xi\Big(\frac{1}{2}+w\Big)$$

near $w=0$. The central normalization gives local analytic equality, and the identity theorem gives

$$F_K(s) \equiv \xi(s).$$

Since the nonzero eigenvalues of K are real, the zeros of F_K are restricted to the form

$$s = \frac{1}{2} + \frac{i}{\lambda_j}, \qquad \lambda_j \in \mathbb{R}\setminus\{0\}.$$

The identity $F_K = \xi$ then gives the Riemann Hypothesis.

1.2. the Analytic Operator Setup and the Coefficient-Space Arithmetic Construction

Section 2 constructs the analytic operator setup. Starting from a weighted Hilbert space on the half-line, it introduces the quadratic form q_R and its admissible core, and constructs the closed form and self-adjoint operator realization

$$q_R \longrightarrow A_R \longrightarrow L := A_R + I.$$

It then obtains the compact embedding of the form domain, compact resolvent, purely discrete spectrum, and scalar Herglotz-type resolvent function

$$m_L^{(\Phi)}(z) = \langle \Phi, (L-zI)^{-1}\Phi \rangle.$$

This section provides the analytic foundation and does not use the arithmetic trace, the residual quotient, or any conclusion about the locations of zeros.

Section 3 constructs the coefficient-space arithmetic data. In the formal Dirichlet algebra and its completed augmentation ideal, it defines the exact prime indicator, the exact von Mangoldt lift

$$\Lambda^{\mathrm{ex}},$$

the composite-cancellation operator, and the arithmetic derivation. The principal coefficient-extraction identity is

$$\partial_{\log} C_{\mathrm{comp}}(\mathbf{1}) = \Lambda^{\mathrm{ex}}.$$

The section then Hilbertizes the coefficient layer and constructs the coefficient Hilbert space, the coefficient projectors, and the weighted diagonal arithmetic trace operator. This trace operator is the means by which the prime-power contribution is evaluated exactly on the arithmetic summand in Section 5.

1.3. Singular-Boundary Data and the Orthogonal-Decomposition Comparison Framework

Section 4 constructs the singular-boundary data inside the analytic Hilbert-space setup. Using the Gelfand triple

$$\mathcal{D}_R \subset H_{\alpha,+} \subset \mathcal{D}'_R,$$

it separates point evaluations, distribution kernels, boundary traces, and boundary forms by type. It also constructs the boundary parameter space, the regular area measure, and the zero-area singular locus

$$(\Sigma_R, \sigma_R, \Gamma_R),$$

and obtains

$$T_R, \qquad \mathrm{supp}_R, \qquad L_R$$

from the singular boundary trace and the regular boundary trace. These data define the σ_R-null and regular-trace-vanishing generating class and its closure. The section also constructs the Friedrichs-type realization, the singular-boundary transport group, the anti-self-adjoint generator, and the distribution-kernel representation. Its output is the one-sided singular-boundary subspace $K_R^+ \subset H_{\alpha,+}$, the one-sided projection Π_R^+, and the boundary-distribution data needed for the analytic comparison in Section 6.

The generator-kernel statement exported from Section 4 is deliberately restricted. The unconditional object is the regularized weak kernel $B_R^{(\varepsilon),\mathrm{dist}}$ for $\varepsilon > 0$, together with the unsmoothed kernel on the generator-admissible core. The full-test-space unsmoothed kernel is recorded only as an optional object under an additional sufficient condition, and the later determinant argument does not import that optional object.

Section 5 places the coefficient-space arithmetic construction and the singular-boundary construction on a common ambient Hilbert space X. In this stage,

$$K_R^+ \subset H_{\alpha,+} \quad \text{is lifted to} \quad \mathcal{K}_R = J_{\mathrm{an}} K_R^+ \subset X,$$

and the ambient projection is

$$\Pi_R = J_{\mathrm{an}} \Pi_R^+ J_{\mathrm{an}}^*.$$

The arithmetic summand is embedded by J_{arith}, and the orthogonal decomposition

$$X = \mathcal{K}_R \oplus J_{\mathrm{arith}} \mathcal{H}_{\mathrm{arith}} \oplus \operatorname{Ran} \Pi_{\mathrm{res}}$$

is fixed. For localized comparison data from the finite-window explicit formula, the prime-power contribution is evaluated by the weighted diagonal arithmetic trace, while the residual component is removed by passing to the canonical representative modulo $\operatorname{Ran}\Pi_{\mathrm{res}}$. The effective $\mathcal{K}_R$-projected component is therefore represented by

$$\Pi_R x^\sharp \in \mathcal{K}_R.$$

This $\mathcal{K}_R$-component is the input for the analytic determinant construction in Section 6.

1.4. Spectral Determinant Closure in Section 6

Section 6 turns the residual-free comparison interface into an analytic identity. The first part of the section constructs the comparison map from the boundary distribution data of Section 4 and the orthogonal-decomposition framework of Section 5. The boundary reflection Θ_R induced by $s \mapsto 1-s$ is shown to preserve admissibility and the residual-free comparison relation, and hence descends to the self-adjoint involution $\mathcal{S}_R$ on $\mathcal{K}_R$. The resulting signed boundary-distribution comparison kernel is then realized, by quantitative Sobolev-reference Schatten estimates, as

$$K = K^* \in \mathfrak{S}_2.$$

From K, Section 6 defines the regularized determinant F_K. The central comparison is not made by definition. Instead, the central Cauchy–Laplace kernel, the finite-window cutoffs, the counterterm, and the ambient topology of $\mathcal{C}_{\mathrm{cen}}$ are fixed first, while the proof itself uses only the closed Cauchy–Laplace comparison subspace $\mathcal{C}_{\mathrm{CL}}(r)$. The finite-window approximation lemma gives convergence of $\Psi^{\mathrm{fw}}_{w,M}$ to Ψ^{cen}_{w} in this subspace, and the pre-determinant continuity of the finite-part realized functional justifies the M-limit before any determinant trace formula is invoked.

The K-side transform is obtained through a finite-window cyclic construction rather than by redefining μ_L. The scalar central test used by the zeta side and the cyclic tensor test used by the K-side are not identified directly. Instead, Section 6 fixes a universal Cauchy–Laplace coefficient object Ψ^{univ}_{w}. Its scalar realization is the ordinary central test $\Psi^{\mathrm{sc,fw}}_{w,M}$, while its cyclic realization is

$$\Psi^{\mathrm{cyc,fw}}_{w,M} = \sum_{\ell\geq 2}(-i)^{\ell-2}w^{\ell-1}\Psi^{(\ell)}_{M}.$$

The finite-window residual-free comparison is formulated as an equality of pullback functionals on this universal coefficient space. The notation

$$\mu^{\mathrm{sc}}_{\xi,M}$$

is fixed for the scalar finite-window restriction of μ_ξ, so the zeta side also enters the universal comparison through its scalar realization.

For each M, the canonical finite-part realized functional is first restricted to its scalar finite-window form $\mu^{\mathrm{fp,sc}}_{L,M}$ and then tensor-lifted to $\mu^{\mathrm{fp,cyc}}_{L,M}$. Matrix-coefficient readout tests $\Omega_M(f,g)$ are constructed from the type-correct R-side lift, the finite LCI synthesis/Riesz data, the projection Π_R, and the descended reflection $\mathcal{S}_R$. Their cyclic tensor products define the coefficient tests $\Psi^{(\ell)}_{M}$, and the finite-window identity

$$\left\langle \mu^{\mathrm{fp,cyc}}_{L,M}, \Psi^{\mathrm{cyc,fw}}_{w,M} \right\rangle = \sum_{\ell\geq 2}(-i)^{\ell-2}w^{\ell-1}\operatorname{Tr}(K^{\ell}_{M})$$

is proved from the readout construction itself. The strengthened K-side scalar coefficient realization first evaluates the scalar finite-window μ_L-coefficient as the same trace power $\mathrm{Tr}(K_M^{\ell})$; the cyclic tensor lift is then evaluated independently by the ordered contraction, and only the two resulting pullbacks are compared on Ψ_w^{univ}. A separate finite-window residual-free comparison theorem then compares that scalar μ_L-value with the scalar μ_ξ-value on the same universal object. The finite-rank compressions $K_{M,N}$ and the Hilbert–Schmidt convergence $K_M \to K$ identify the resulting transform with the central logarithmic derivative of F_K. Separately, the zeta-side transform identifies the scalar μ_ξ-pairing with the central logarithmic derivative of ξ. Combining these identifications with the central residual-free equality gives equality of logarithmic derivatives. The normalization at $s = \frac{1}{2}$ gives local analytic equality, and the identity theorem gives

$$F_K(s) \equiv \xi(s).$$

The spectral localization now follows from the self-adjointness of K: the nonzero spectrum of K is real, and hence the zeros of F_K lie on the critical line. Since $F_K = \xi$, all nontrivial zeros of the completed zeta function lie on the critical line. The finite-window bridge, anchored defect staircase, and no-first-hit material later in Section 6 record this spectral conclusion in finite-window form; they are not used to prove the determinant identity $F_K \equiv \xi$.

2. Analytic Operator Setup

2.1. Weighted Hilbert Space and Boundary Geometry

In this section, we fix only the analytic Hilbert-space setup necessary for constructing the weighted self-adjoint generator and preparing its compactness theory. Accordingly, the role of this subsection is limited to specifying the weighted Hilbert space, the boundary notation, and the trace conventions used in the form constructions below. Here, we introduce no arithmetic projector, no comparison identity, and no zero-counting statement.

Definition 2.1 (Weighted Hilbert-space setup)**.** Fix the parameters

$$\alpha > \frac{1}{2}, \qquad \langle x \rangle := (1 + x^2)^{1/2}, \qquad \rho_\alpha(x) := \langle x \rangle^{2\alpha}.$$

Define the weighted measure on the positive half-line by

$$d\mu_\alpha(x) := \rho_\alpha(x)\, dx \qquad (x > 0).$$

Let the one-sided weighted Hilbert space be

$$H_{\alpha,+} := L^2\big((0,\infty), d\mu_\alpha\big),$$

and give its inner product by

$$\langle f, g \rangle_{H_{\alpha,+}} := \int_0^\infty f(x)\overline{g(x)}\, d\mu_\alpha(x).$$

Let the doubled analytic Hilbert space be

$$H_\alpha := H_{\alpha,+} \oplus H_{\alpha,+},$$

and give its direct-sum inner product by

$$\langle (f_+, f_-), (g_+, g_-)\rangle_{H_\alpha} = \langle f_+, g_+\rangle_{H_{\alpha,+}} + \langle f_-, g_-\rangle_{H_{\alpha,+}}.$$

In what follows, we repeatedly use the unitary transport

$$U_\alpha : H_{\alpha,+} \longrightarrow L^2(0,\infty), \qquad (U_\alpha f)(x) := \rho_\alpha(x)^{1/2} f(x).$$

Definition 2.2 (Boundary notation)**.** Set

$$\Sigma := \{0^+, 0^-\} = \partial\big((0,\infty) \sqcup (0,\infty)\big).$$

We identify

$$L^2(\Sigma) \simeq \mathbb{C}^2$$

with the inner product associated with counting measure.

When each component of

$$u = (u_+, u_-) \in H_\alpha$$

is locally absolutely continuous in a neighborhood of $x = 0$, we write the boundary trace as

$$\gamma_0 u := \big(u_+(0), u_-(0)\big) \in L^2(\Sigma),$$

and, when the one-sided derivatives exist, we write the oriented normal-derivative trace as

$$\gamma_1 u := \big(u_+'(0), -u_-'(0)\big) \in L^2(\Sigma).$$

For scalar functions f, g on $(0,\infty)$, if the endpoint limits below exist, we write the weighted boundary form as

$$b_\alpha[f,g] := -i\Big[\rho_\alpha(x) f(x)\overline{g(x)}\Big]_{x=0}^{x=\infty}.$$

Thus

$$b_\alpha[f,g] = -i\lim_{R\to\infty} \rho_\alpha(R) f(R)\overline{g(R)} + i\,\rho_\alpha(0) f(0)\overline{g(0)}.$$

This is the boundary term that appears when performing integration by parts with respect to the weighted measure $d\mu_\alpha$.

Remark 2.3 (Scope of Section 2)**.** The purpose of Section 2 is purely analytic. Namely, it fixes the weighted Hilbert space, constructs the closed quadratic form q_R and its associated self-adjoint operator, and prepares the compactness argument through an explicit confining potential. No part of this section presupposes any arithmetic projector, any orthogonal decomposition of an entire trace space, or any conclusion theorem for the Riemann hypothesis.

2.2. Closed Quadratic Form and the Admissible Core

Here we introduce the first-order weighted differential operator and the raw quadratic form on a concrete admissible core. What is needed in this subsection is that the form domain of the closed form q_R contain smooth elements with boundary values. For this reason, the admissible core is taken to consist of compactly supported functions that extend smoothly to the endpoint 0, and boundary cancellation itself is carried out later in the trace-vanishing singular-boundary subspace. This distinction prevents the singular boundary trace used in Section 4 from being trivialized.

Definition 2.4 (Quadratic form and admissible core)**.**

$$\mathcal{C}_R := C_c^\infty([0,\infty)).$$

On $\mathcal{C}_R$, define the first-order weighted differential operator

$$D_0 f := -i\left(\partial_x + \frac{1}{2}\frac{\rho_\alpha'}{\rho_\alpha}\right) f = -i\left(\partial_x + \alpha\frac{x}{1+x^2}\right) f.$$

Equivalently,

$$U_\alpha D_0 U_\alpha^{-1} = -i\partial_x \qquad \text{on } \mathcal{C}_R$$

holds.

Let $V_R : (0,\infty) \to [1,\infty)$ be a locally bounded measurable function. Its concrete choice is fixed in Definition 2.7. In what follows, write

$$Q_R := \log V_R, \qquad V_R = e^{Q_R}.$$

Define the raw quadratic form on $\mathcal{C}_R$ by

$$q_R^{\mathrm{raw}}[f,g] := \langle D_0 f, D_0 g\rangle_{H_{\alpha,+}} + \langle V_R^{1/2} f, V_R^{1/2} g\rangle_{H_{\alpha,+}},$$

and let its quadratic-form version be

$$q_R^{\mathrm{raw}}[f] := q_R^{\mathrm{raw}}[f,f].$$

Define its raw form norm by

$$\|f\|_{q_R^{\mathrm{raw}}}^2 := \|f\|_{H_{\alpha,+}}^2 + q_R^{\mathrm{raw}}[f].$$

Theorem 2.5 (Boundary formula on the admissible core and symmetry of the raw form)**.** *For any $f, g \in \mathcal{C}_R$,*

$$\langle D_0 f, g\rangle_{H_{\alpha,+}} - \langle f, D_0 g\rangle_{H_{\alpha,+}} = b_\alpha[f,g]$$

holds. Here

$$b_\alpha[f,g] = -i\left[\rho_\alpha(x) f(x)\overline{g(x)}\right]_{x=0}^{x=\infty}.$$

In particular, on $\mathcal{C}_{R,0} := C_c^\infty(0,\infty)$,

$$b_\alpha[f,g] = 0.$$

On the other hand, the raw quadratic form

$$q_R^{\mathrm{raw}}[f,g] = \langle D_0 f, D_0 g\rangle_{H_{\alpha,+}} + \langle V_R^{1/2} f, V_R^{1/2} g\rangle_{H_{\alpha,+}}$$

is symmetric on $\mathcal{C}_R$.

Proof. Let $f, g \in \mathcal{C}_R$. By definition,

$$\langle D_0 f, g\rangle_{H_{\alpha,+}} = -i\int_0^\infty \left(f'(x) + \frac{1}{2}\frac{\rho_\alpha'(x)}{\rho_\alpha(x)} f(x)\right)\overline{g(x)}\,\rho_\alpha(x)\,dx.$$

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Since f, g are smooth up to the endpoint 0 and vanish for sufficiently large x, integration by parts is justified. Integrating the first term by parts gives

$$\int_0^\infty f'(x)\overline{g(x)}\rho_\alpha(x)\,dx = \left[\rho_\alpha f\overline{g}\right]_0^\infty - \int_0^\infty f(x)\overline{g'(x)}\rho_\alpha(x)\,dx - \int_0^\infty f(x)\overline{g(x)}\rho'_\alpha(x)\,dx.$$

Substituting this into the preceding formula yields

$$\langle D_0 f, g\rangle_{H_{\alpha,+}} - \langle f, D_0 g\rangle_{H_{\alpha,+}} = -i\left[\rho_\alpha(x)f(x)\overline{g(x)}\right]_0^\infty = b_\alpha[f,g].$$

If $f, g \in \mathcal{C}_{R,0}$, then both vanish in a neighborhood of the endpoint 0, so the boundary term also vanishes.

Finally, the symmetry of q_R^{raw} follows from the fact that, for any linear operator D_0,

$$\langle D_0 f, D_0 g\rangle = \overline{\langle D_0 g, D_0 f\rangle}$$

holds, and from the Hermitian property of the potential term. This conclusion does not require D_0 itself to be symmetric without boundary terms. ☐

Theorem 2.6 (Closedness, lower boundedness, and core density)**.** *Let D be the closure of D_0 in $H_{\alpha,+}$, and let*

$$M_{V_R^{1/2}} f := V_R^{1/2} f$$

be the multiplication operator with maximal domain

$$\mathrm{Dom}(M_{V_R^{1/2}}) = \left\{ f \in H_{\alpha,+} : V_R^{1/2} f \in H_{\alpha,+} \right\}.$$

Then the raw graph map

$$T_R^{\mathrm{raw}} : \mathcal{C}_R \to H_{\alpha,+} \oplus H_{\alpha,+}, \qquad T_R^{\mathrm{raw}} f := (D_0 f, M_{V_R^{1/2}} f)$$

is closable. Furthermore, set

$$\mathcal{Q}_R := \mathrm{Dom}\left(\overline{T_R^{\mathrm{raw}}}\right),$$

and write its closure again as

$$T_R f = (Df, M_{V_R^{1/2}} f) \qquad (f \in \mathcal{Q}_R).$$

Then the following hold:

1. *The sesquilinear form*

$$q_R[f,g] := \langle Df, Dg\rangle_{H_{\alpha,+}} + \langle V_R^{1/2} f, V_R^{1/2} g\rangle_{H_{\alpha,+}} \qquad (f, g \in \mathcal{Q}_R)$$

   is well-defined and closed on $\mathcal{Q}_R$;
2. *This form is lower bounded:*

$$q_R[f] \geq 0 \qquad (f \in \mathcal{Q}_R);$$

3. *$\mathcal{C}_R$ is a form core for q_R. Namely,*

$$\overline{\mathcal{C}_R}^{\|\cdot\|_{q_R}} = \mathcal{Q}_R, \qquad \|f\|_{q_R}^2 := \|f\|_{H_{\alpha,+}}^2 + q_R[f].$$

Proof. We first begin with the operator D. By Definition 2.4,

$$U_\alpha D_0 U_\alpha^{-1} = -i\partial_x \qquad \text{on } \mathcal{C}_R$$

holds. Here $\mathcal{C}_R = C_c^\infty([0,\infty))$ is the space of compactly supported functions smooth up to the endpoint. The graph closure of $-i\partial_x$ on $\mathcal{C}_R$ in $L^2(0,\infty)$ is the standard weak derivative operator

$$-i\partial_x : \ H^1(0,\infty) \subset L^2(0,\infty) \longrightarrow L^2(0,\infty),$$

which is a closed operator. Since U_α is unitary, the transported operator D is also a closed operator on $H_{\alpha,+}$.

Next consider the multiplication operator $M_{V_R^{1/2}}$. Since $V_R^{1/2}$ is measurable and finite almost everywhere, multiplication by $V_R^{1/2}$ is a closed operator on $H_{\alpha,+}$. Indeed, suppose that

$$f_n \to f \quad \text{in } H_{\alpha,+}, \qquad M_{V_R^{1/2}} f_n \to h \quad \text{in } H_{\alpha,+}.$$

Passing to a subsequence if necessary, pointwise convergence almost everywhere holds:

$$f_n(x) \to f(x), \qquad V_R(x)^{1/2} f_n(x) \to h(x).$$

Therefore

$$h(x) = V_R(x)^{1/2} f(x) \quad \text{for almost every } x > 0,$$

and hence

$$f \in \mathrm{Dom}(M_{V_R^{1/2}}) \qquad \text{and} \qquad M_{V_R^{1/2}} f = h.$$

Thus $M_{V_R^{1/2}}$ is closed.

We now show that T_R^{raw} is closable. Assume that

$$f_n \in \mathcal{C}_R, \qquad f_n \to 0 \text{ in } H_{\alpha,+}, \qquad T_R^{\mathrm{raw}} f_n \to (g,h) \text{ in } H_{\alpha,+} \oplus H_{\alpha,+}.$$

Since $D_0 \subset D$ and D is closed, the convergence

$$f_n \to 0, \qquad D_0 f_n \to g$$

implies $g = 0$. Similarly, since $M_{V_R^{1/2}}$ is closed and

$$f_n \to 0, \qquad M_{V_R^{1/2}} f_n \to h$$

holds, we obtain $h = 0$. Therefore T_R^{raw} is closable.

Set $T_R = \overline{T_R^{\mathrm{raw}}}$, and define

$$\mathcal{Q}_R := \mathrm{Dom}(T_R).$$

By the graph-closure definition, $\mathcal{Q}_R$ is precisely the graph closure of $\mathcal{C}_R$ with respect to the norm

$$\|f\|_{H_{\alpha,+}}^2 + \|D_0 f\|_{H_{\alpha,+}}^2 + \|V_R^{1/2} f\|_{H_{\alpha,+}}^2 = \|f\|_{q_R^{\mathrm{raw}}}^2.$$

For $f, g \in \mathcal{Q}_R$, define

$$q_R[f,g] := \langle T_R f, T_R g\rangle_{H_{\alpha,+}\oplus H_{\alpha,+}} = \langle Df, Dg\rangle_{H_{\alpha,+}} + \langle V_R^{1/2} f, V_R^{1/2} g\rangle_{H_{\alpha,+}}.$$

Then

$$\|f\|_{q_R}^2 = \|f\|_{H_{\alpha,+}}^2 + \|T_R f\|_{H_{\alpha,+}\oplus H_{\alpha,+}}^2.$$

Since T_R is a closed operator, its graph is complete. Therefore

$$(\mathcal{Q}_R, \|\cdot\|_{q_R})$$

is a Hilbert space. This is precisely the closedness of the form q_R.

The lower bound follows immediately:

$$q_R[f] = \|Df\|_{H_{\alpha,+}}^2 + \|V_R^{1/2} f\|_{H_{\alpha,+}}^2 \geq 0 \qquad (f \in \mathcal{Q}_R).$$

Finally, since $\mathcal{Q}_R$ is defined as the graph closure of $\mathcal{C}_R$ with respect to the form norm, it also follows by definition that $\mathcal{C}_R$ is a form core. Equivalently, for any $f \in \mathcal{Q}_R$, there exists a sequence

$$f_n \in \mathcal{C}_R$$

such that

$$\|f_n - f\|_{H_{\alpha,+}} + \|D_0 f_n - Df\|_{H_{\alpha,+}} + \|V_R^{1/2}(f_n - f)\|_{H_{\alpha,+}} \longrightarrow 0.$$

Therefore

$$\|f_n - f\|_{q_R} \to 0.$$

This proves all three assertions. □

2.3. Explicit Confining Potential

Here we fix the potential entering the quadratic form q_R. The essential point of the choice in this subsection is not mere positivity. The potential must grow to $+\infty$ sufficiently slowly, but explicitly, and must prevent q_R-bounded mass from escaping to infinity. This tail control is precisely the global ingredient that will be needed later in the compactness argument.

Definition 2.7 (Explicit confining potential)**.** For $x > 0$, define

$$V_R(x) := \log(e + x).$$

Accordingly, set

$$Q_R(x) := \log V_R(x) = \log\log(e + x), \qquad V_R = e^{Q_R}.$$

Then V_R is positive, measurable, locally bounded, and satisfies

$$V_R(x) \geq 1 \qquad (x > 0).$$

In what follows, V_R denotes the potential used in Definition 2.4 and Theorem 2.6.

Proposition 2.8 (Logarithmic growth of V_R)**.** *For every* $x \geq 0$*,*

$$\log(1+x) \leq V_R(x) \leq 1 + \log(1+x)$$

holds. In particular,

$$V_R(x) \sim \log(1+x) \qquad (x \to \infty).$$

Proof. By definition,

$$V_R(x) = \log(e+x).$$

Since

$$e + x \geq 1 + x,$$

the monotonicity of the logarithm gives

$$\log(1+x) \leq \log(e+x) = V_R(x).$$

On the other hand, since

$$e + x \leq e(1+x),$$

monotonicity again gives

$$V_R(x) = \log(e+x) \leq \log\big(e(1+x)\big) = 1 + \log(1+x).$$

This proves the two-sided estimate. Dividing by $\log(1+x)$ and letting $x \to \infty$, we obtain

$$\frac{V_R(x)}{\log(1+x)} \to 1,$$

that is,

$$V_R(x) \sim \log(1+x). \quad \square$$

Theorem 2.9 (Escape prevention at infinity)**.** *Let* $F \subset \mathcal{Q}_R$ *be any family satisfying*

$$\sup_{f \in F} \big(\|f\|^2_{H_{\alpha,+}} + q_R[f] \big) \leq C,$$

where $C > 0$ *is a constant. Then, for every* $R > 0$*,*

$$\sup_{f \in F} \int_R^\infty |f(x)|^2 \, d\mu_\alpha(x) \leq \frac{C}{\log(e+R)}.$$

Consequently,

$$\lim_{R \to \infty} \sup_{f \in F} \int_R^\infty |f(x)|^2 \, d\mu_\alpha(x) = 0.$$

In particular, every q_R*-bounded sequence in* $H_{\alpha,+}$ *is globally tight and does not lose mass at infinity.*

Proof. Fix $f \in F$. Since $V_R(x) = \log(e+x)$ is increasing on $(0, \infty)$,

$$V_R(x) \geq V_R(R) = \log(e+R) \qquad (x \geq R)$$

holds. Therefore

$$\log(e+R)\int_R^\infty |f(x)|^2\,d\mu_\alpha(x) \le \int_R^\infty V_R(x)|f(x)|^2\,d\mu_\alpha(x).$$

The right-hand side is bounded by the potential part of q_R:

$$\int_R^\infty V_R(x)|f(x)|^2\,d\mu_\alpha(x) \le \int_0^\infty V_R(x)|f(x)|^2\,d\mu_\alpha(x) = \|V_R^{1/2}f\|_{H_{\alpha,+}}^2 \le q_R[f].$$

By the assumption

$$\|f\|_{H_{\alpha,+}}^2 + q_R[f] \le C \qquad (f\in F),$$

we obtain in particular

$$q_R[f] \le C.$$

Combining the three estimates above gives

$$\int_R^\infty |f(x)|^2\,d\mu_\alpha(x) \le \frac{C}{\log(e+R)}.$$

Taking the supremum over $f\in F$, we get

$$\sup_{f\in F}\int_R^\infty |f(x)|^2\,d\mu_\alpha(x) \le \frac{C}{\log(e+R)}.$$

Furthermore,

$$\log(e+R)\to\infty \qquad (R\to\infty),$$

so the right-hand side converges to 0. Hence the family F is tight in $H_{\alpha,+}$, and no q_R-bounded sequence escapes to infinity. ☐

At this point, the analytic framework is self-contained. Namely, the weighted Hilbert-space setup, admissible core, closed lower-bounded quadratic form q_R, and explicit confining potential have all been fixed. The next step is to pass from this closed-form construction to the corresponding self-adjoint operator and then to combine the tail estimate of Theorem 2.9 with local compactness on bounded intervals.

2.4. Self-Adjoint Operator A_R Associated with q_R and Positive Shifted Operator L

We now pass from the closed and lower-bounded form q_R to its operator realization. This construction is purely form-theoretic. Namely, the operator is obtained from the representation theorem for closed forms, and at this point we do not identify the operator domain with a formal differential representation. After fixing the self-adjoint operator associated with q_R, we introduce the positive shifted operator

$$L := A_R + I.$$

This will be used as the basic spectral object in the remainder of the analytic section.

Theorem 2.10 (Self-adjoint operator associated with q_R)**.** *There exists a unique self-adjoint and lower-bounded operator*

$$A_R : \mathrm{Dom}(A_R)\subset H_{\alpha,+}\to H_{\alpha,+}$$

associated with the closed form q_R *of Theorem 2.6. More precisely, for* $u \in H_{\alpha,+}$ *and* $h \in H_{\alpha,+}$*, the following are equivalent:*

$$u \in \mathrm{Dom}(A_R), \qquad A_R u = h,$$

and

$$u \in \mathcal{Q}_R, \qquad q_R[u,v] = \langle h, v\rangle_{H_{\alpha,+}} \quad (\forall v \in \mathcal{Q}_R).$$

Furthermore,

$$A_R \geq 0.$$

Proof. By Theorem 2.6, the form

$$q_R : \mathcal{Q}_R \times \mathcal{Q}_R \to \mathbb{C}$$

is densely defined, closed, and lower bounded on $H_{\alpha,+}$. Density follows from

$$\mathcal{C}_R = C_c^\infty(0,\infty) \subset \mathcal{Q}_R$$

and from the fact that $C_c^\infty(0,\infty)$ is dense in $H_{\alpha,+}$.

Therefore, the first representation theorem for closed and lower-bounded sesquilinear forms applies. It yields a unique self-adjoint lower-bounded operator

$$A_R : \mathrm{Dom}(A_R) \subset H_{\alpha,+} \to H_{\alpha,+}$$

such that

$$\mathrm{Dom}(A_R) = \{u \in \mathcal{Q}_R : \exists\, h \in H_{\alpha,+} \text{ with } q_R[u,v] = \langle h,v\rangle_{H_{\alpha,+}}\ \forall v \in \mathcal{Q}_R\}.$$

For such a u, the representing vector h is unique, and we define it to be $A_R u$.

This gives the stated equivalence

$$u \in \mathrm{Dom}(A_R),\ A_R u = h \iff u \in \mathcal{Q}_R,\ q_R[u,v] = \langle h,v\rangle_{H_{\alpha,+}}\ \forall v \in \mathcal{Q}_R.$$

Finally, by Theorem 2.6,

$$q_R[u] \geq 0 \qquad (u \in \mathcal{Q}_R).$$

Hence the lower bound in the representation theorem is 0, and therefore

$$\langle A_R u, u\rangle_{H_{\alpha,+}} \geq 0 \qquad (u \in \mathrm{Dom}(A_R)).$$

Equivalently,

$$A_R \geq 0.$$

This completes the proof. □

Definition 2.11 (Positive shifted operator)**.** On the same domain

$$\mathrm{Dom}(L) = \mathrm{Dom}(A_R),$$

define the positive shifted operator

$$L := A_R + I.$$

Since $A_R \geq 0$,

$$L \geq I$$

holds.

Proposition 2.12 (Form domain of $L^{1/2}$)**.**

$$\mathrm{Dom}(L^{1/2}) = \mathcal{Q}_R,$$

and for any $u, v \in \mathcal{Q}_R$*,*

$$q_R[u,v] + \langle u,v\rangle_{H_{\alpha,+}} = \langle L^{1/2}u, L^{1/2}v\rangle_{H_{\alpha,+}}.$$

Proof. Define the shifted form on $\mathcal{Q}_R$ by

$$\mathfrak{l}_R[u,v] := q_R[u,v] + \langle u,v\rangle_{H_{\alpha,+}}.$$

Since q_R is densely defined and closed, so is $\mathfrak{l}_R$. Moreover, since $q_R \geq 0$,

$$\mathfrak{l}_R[u] = q_R[u] + \|u\|^2_{H_{\alpha,+}} \geq \|u\|^2_{H_{\alpha,+}},$$

and hence $\mathfrak{l}_R$ is strictly positive.

Next, we identify the operator associated with $\mathfrak{l}_R$. Let $u \in \mathrm{Dom}(A_R)$ and $v \in \mathcal{Q}_R$. By Theorem 2.10,

$$q_R[u,v] = \langle A_R u, v\rangle_{H_{\alpha,+}}.$$

Therefore

$$\mathfrak{l}_R[u,v] = q_R[u,v] + \langle u,v\rangle_{H_{\alpha,+}} = \langle (A_R + I)u, v\rangle_{H_{\alpha,+}} = \langle Lu, v\rangle_{H_{\alpha,+}}.$$

Thus the operator associated with the positive closed form $\mathfrak{l}_R$ is precisely L.

We now apply the second representation theorem for positive closed forms. This gives

$$\mathrm{Dom}(L^{1/2}) = \mathrm{Dom}(\mathfrak{l}_R) = \mathcal{Q}_R,$$

and

$$\mathfrak{l}_R[u,v] = \langle L^{1/2}u, L^{1/2}v\rangle_{H_{\alpha,+}} \qquad (u, v \in \mathcal{Q}_R).$$

Substituting the definition of $\mathfrak{l}_R$, we obtain

$$q_R[u,v] + \langle u,v\rangle_{H_{\alpha,+}} = \langle L^{1/2}u, L^{1/2}v\rangle_{H_{\alpha,+}},$$

as claimed. ☐

Corollary 2.13 (Resolvent bound for A_R)**.** *For every* $\lambda > 0$*,*

$$\|(A_R + \lambda I)^{-1}\|_{H_{\alpha,+} \to H_{\alpha,+}} \leq \lambda^{-1}.$$

Proof. Since $A_R \geq 0$, the spectral theorem gives

$$\sigma(A_R) \subset [0, \infty).$$

Therefore

$$\sigma(A_R + \lambda I) \subset [\lambda, \infty),$$

so $A_R + \lambda I$ is invertible and

$$\|(A_R + \lambda I)^{-1}\| = \sup_{t \in \sigma(A_R)} \frac{1}{t + \lambda} \le \frac{1}{\lambda}.$$

□

2.5. Compact Resolvent and Discrete Spectrum

Here we connect the local regularity of the transported differential operator with the tail tightness given by Theorem 2.9. This yields a compact embedding from the form domain into the ambient Hilbert space, and hence compact resolvent for both A_R and its positive shift L. After that, the spectral theorem yields a purely discrete positive spectrum.

Proposition 2.14 (Local compactness on bounded intervals). *Let $R_0 > 0$. If $F \subset \mathcal{Q}_R$ is bounded with respect to the form norm defined by*

$$\|f\|_{q_R}^2 := \|f\|_{H_{\alpha,+}}^2 + q_R[f],$$

then the restricted family

$$\{ f|_{(0,R_0)} : f \in F \}$$

is relatively compact in $L^2((0, R_0), d\mu_\alpha)$.

Proof. Fix $R_0 > 0$, and assume that there exists $M > 0$ such that

$$\sup_{f \in F} \|f\|_{q_R} \le M.$$

Take $f \in F$, and set

$$g := U_\alpha f.$$

Since U_α is unitary,

$$\|g\|_{L^2(0,\infty)} = \|f\|_{H_{\alpha,+}}.$$

Hence

$$\|g\|_{L^2(0,R_0)} \le \|g\|_{L^2(0,\infty)} = \|f\|_{H_{\alpha,+}} \le M. \tag{2.5.1}$$

Next, since D is the closure of D_0, and since

$$U_\alpha D_0 U_\alpha^{-1} = -i\partial_x \qquad \text{on } \mathcal{C}_R,$$

the transported closed operator $U_\alpha D U_\alpha^{-1}$ is the closure of $-i\partial_x$ on $L^2(0, \infty)$. In particular, if $f \in \mathrm{Dom}(D)$, then $g = U_\alpha f$ belongs to $H^1_{\mathrm{loc}}(0, \infty)$, and

$$g' = i\, U_\alpha D f \qquad \text{in } L^2_{\mathrm{loc}}(0, \infty)$$

holds. Therefore

$$\|g'\|_{L^2(0,R_0)} \le \|U_\alpha D f\|_{L^2(0,\infty)} = \|Df\|_{H_{\alpha,+}} \le q_R[f]^{1/2} \le M. \tag{2.5.2}$$

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From (2.5.1) and (2.5.2), it follows that

$$\{ U_\alpha f|_{(0,R_0)} : f \in F \}$$

is bounded in $H^1(0, R_0)$. By Rellich's compactness theorem, this family is relatively compact in $L^2(0, R_0)$.

We now return to the weighted space. On the bounded interval $[0, R_0]$, the weight $\rho_\alpha(x) = \langle x \rangle^{2\alpha}$ is bounded above and below by positive constants:

$$0 < c_{\alpha,R_0} \leq \rho_\alpha(x) \leq C_{\alpha,R_0} < \infty.$$

Therefore U_α, restricted to this interval, is a bounded isomorphism between $L^2((0, R_0), d\mu_\alpha)$ and $L^2(0, R_0)$. Furthermore, the potential $V_R(x) = \log(e + x)$ is also bounded on $[0, R_0]$, so the form norm introduces no new local singularity. Hence relative compactness in $L^2(0, R_0)$ is equivalent to relative compactness in $L^2((0, R_0), d\mu_\alpha)$.

Accordingly,

$$\{ f|_{(0,R_0)} : f \in F \}$$

is relatively compact in $L^2((0, R_0), d\mu_\alpha)$. □

Theorem 2.15 (Compact embedding of the form domain)**.** *The embedding*

$$\mathcal{Q}_R \hookrightarrow H_{\alpha,+}$$

is compact.

Proof. Let $(f_n)_{n\geq 1} \subset \mathcal{Q}_R$ be bounded in the form norm:

$$\sup_{n\geq 1} \|f_n\|_{q_R} \leq M$$

for some $M > 0$. We must show that (f_n) has a subsequence convergent in $H_{\alpha,+}$.

Fix $\varepsilon > 0$. Applying Theorem 2.9 to the family

$$F := \{f_n : n \geq 1\},$$

we obtain some $R_0 > 0$ such that

$$\sup_{n\geq 1} \int_{R_0}^{\infty} |f_n(x)|^2 \, d\mu_\alpha(x) < \frac{\varepsilon^2}{8}. \tag{2.5.3}$$

Thus each f_n has a uniformly small tail beyond R_0.

On the bounded interval $(0, R_0)$, Proposition 2.14 implies that the restricted sequence $(f_n|_{(0,R_0)})$ is relatively compact in $L^2((0, R_0), d\mu_\alpha)$. Therefore, after passing to a subsequence and denoting it again by (f_n), we may assume that

$$f_n|_{(0,R_0)}$$

is Cauchy in $L^2((0, R_0), d\mu_\alpha)$. Hence there exists N such that, for all $m, n \geq N$,

$$\int_0^{R_0} |f_n(x) - f_m(x)|^2 \, d\mu_\alpha(x) < \frac{\varepsilon^2}{2}. \tag{2.5.4}$$

For the full-space norm, we estimate

$$\|f_n - f_m\|^2_{H_{\alpha,+}} = \int_0^{R_0} |f_n - f_m|^2 \, d\mu_\alpha + \int_{R_0}^{\infty} |f_n - f_m|^2 \, d\mu_\alpha.$$

Using

$$|a - b|^2 \leq 2|a|^2 + 2|b|^2$$

together with (2.5.3) and (2.5.4), we obtain, for $m, n \geq N$,

$$\|f_n - f_m\|^2_{H_{\alpha,+}} < \frac{\varepsilon^2}{2} + 2 \cdot \frac{\varepsilon^2}{8} + 2 \cdot \frac{\varepsilon^2}{8} = \varepsilon^2.$$

Therefore the chosen subsequence is Cauchy in $H_{\alpha,+}$, and hence converges in $H_{\alpha,+}$.

It follows that every form-bounded sequence in $\mathcal{Q}_R$ has a convergent subsequence in $H_{\alpha,+}$. This is precisely the compactness of the embedding

$$\mathcal{Q}_R \hookrightarrow H_{\alpha,+}.$$

□

Theorem 2.16 (Compact resolvent and discrete spectrum). *For every $\lambda > 0$, the resolvent*

$$(A_R + \lambda I)^{-1} : H_{\alpha,+} \to H_{\alpha,+}$$

is compact. In particular,

$$L^{-1} : H_{\alpha,+} \to H_{\alpha,+}$$

is compact.

Therefore L has purely discrete positive spectrum. Namely, there exist a strictly increasing sequence

$$0 < \ell_1 < \ell_2 < \cdots, \qquad \ell_n \to \infty,$$

and finite-rank orthogonal projectors

$$P_n : H_{\alpha,+} \to H_{\alpha,+}$$

such that

$$LP_n = \ell_n P_n, \qquad \sum_{n \geq 1} P_n = I$$

holds in the strong operator topology. Equivalently, by choosing an orthonormal basis in each eigenspace $P_n H_{\alpha,+}$, one obtains an orthonormal basis consisting of eigenvectors of L. In particular, if the eigenvalues are repeated according to multiplicity, then there exist

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots, \qquad \lambda_k \to \infty,$$

and an orthonormal basis $\{e_k\}_{k\geq 1}$ *of* $H_{\alpha,+}$ *satisfying*

$$Le_k = \lambda_k e_k \qquad (k \geq 1).$$

The corresponding eigenvalues of A_R *are*

$$a_k := \lambda_k - 1 \geq 0.$$

Proof. Fix $\lambda > 0$, and define the shifted form on $\mathcal{Q}_R$ by

$$\mathfrak{a}_\lambda[u,v] := q_R[u,v] + \lambda\langle u,v\rangle_{H_{\alpha,+}}.$$

Since q_R is closed and nonnegative, $\mathfrak{a}_\lambda$ is a densely defined positive closed form on $\mathcal{Q}_R$.

Its norm

$$\|u\|^2_{\mathfrak{a}_\lambda} := \mathfrak{a}_\lambda[u] = q_R[u] + \lambda\|u\|^2_{H_{\alpha,+}}$$

is equivalent to the form norm $\|u\|_{q_R}$. Indeed,

$$\min\{1,\lambda\}\,\|u\|^2_{q_R} \leq \|u\|^2_{\mathfrak{a}_\lambda} \leq \max\{1,\lambda\}\,\|u\|^2_{q_R}. \tag{2.5.5}$$

Therefore Theorem 2.15 implies that the embedding

$$(\mathcal{Q}_R, \|\cdot\|_{\mathfrak{a}_\lambda}) \hookrightarrow H_{\alpha,+}$$

is also compact.

Now take $f \in H_{\alpha,+}$. The functional

$$v \longmapsto \langle f,v\rangle_{H_{\alpha,+}}$$

is continuous on $(\mathcal{Q}_R, \|\cdot\|_{\mathfrak{a}_\lambda})$. Indeed,

$$|\langle f,v\rangle_{H_{\alpha,+}}| \leq \|f\|_{H_{\alpha,+}}\|v\|_{H_{\alpha,+}} \leq \lambda^{-1/2}\|f\|_{H_{\alpha,+}}\|v\|_{\mathfrak{a}_\lambda}.$$

Thus, by the Riesz representation theorem, there exists a unique element

$$u_\lambda(f) \in \mathcal{Q}_R$$

such that

$$\mathfrak{a}_\lambda[u_\lambda(f),v] = \langle f,v\rangle_{H_{\alpha,+}} \qquad (\forall v \in \mathcal{Q}_R). \tag{2.5.6}$$

By Theorem 2.10, (2.5.6) is precisely the weak-form characterization of the resolvent equation

$$(A_R + \lambda I)u_\lambda(f) = f.$$

Hence

$$u_\lambda(f) = (A_R + \lambda I)^{-1} f.$$

The map

$$S_\lambda : f \mapsto u_\lambda(f)$$

is a bounded operator from $H_{\alpha,+}$ to $(\mathcal{Q}_R, \|\cdot\|_{\mathfrak{a}_\lambda})$, and the inclusion map

$$j_\lambda : (\mathcal{Q}_R, \|\cdot\|_{\mathfrak{a}_\lambda}) \hookrightarrow H_{\alpha,+}$$

is compact. Therefore

$$(A_R + \lambda I)^{-1} = j_\lambda \circ S_\lambda$$

is compact on $H_{\alpha,+}$. In particular, taking $\lambda = 1$, we obtain that

$$L^{-1} = (A_R + I)^{-1}$$

is compact.

Next, apply the spectral theorem to the compact self-adjoint positive operator L^{-1}. Then there exist a sequence of positive numbers

$$\mu_1 > \mu_2 > \cdots > 0, \qquad \mu_n \downarrow 0,$$

and finite-rank orthogonal projectors P_n such that

$$L^{-1} P_n = \mu_n P_n, \qquad \sum_{n \geq 1} P_n = I$$

holds strongly. Setting

$$\ell_n := \mu_n^{-1},$$

we have

$$0 < \ell_1 < \ell_2 < \cdots, \qquad \ell_n \to \infty.$$

Multiplying the eigenvalue equation for L^{-1} by L, we obtain

$$LP_n = \ell_n P_n.$$

Thus the spectrum of L is purely discrete, the eigenspaces $P_n H_{\alpha,+}$ are finite-dimensional, and there is no finite accumulation point.

Finally, choose an orthonormal basis in each eigenspace $P_n H_{\alpha,+}$ and concatenate them. This gives an orthonormal basis consisting of eigenvectors of L. Repeating the distinct eigenvalues according to multiplicity, there exist a sequence

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots, \qquad \lambda_k \to \infty,$$

and an orthonormal basis $\{e_k\}_{k \geq 1}$ such that

$$Le_k = \lambda_k e_k.$$

Since $L = A_R + I$, the corresponding eigenvalues of A_R are

$$a_k = \lambda_k - 1 \geq 0.$$

This completes the proof. □

2.6. Weighted Resolvent and Herglotz Resolvent Construction

Here we record the scalar-valued meromorphic/Herglotz resolvent construction associated with the positive self-adjoint operator L. The input is an arbitrarily fixed probe vector $\Phi \in H_{\alpha,+}$. The output is the scalar resolvent function

$$m_L^{(\Phi)}(z) = \langle \Phi, (L - zI)^{-1}\Phi \rangle_{H_{\alpha,+}},$$

and its analytic behavior is directly controlled by the spectral theorem. This is the endpoint of the present analytic section.

Definition 2.17 (Resolvent probe and scalar Herglotz function)**.** Fix a probe vector

$$\Phi \in H_{\alpha,+}.$$

For

$$z \in \mathbb{C} \setminus \sigma(L),$$

define

$$m_L^{(\Phi)}(z) := \langle \Phi, (L - zI)^{-1}\Phi \rangle_{H_{\alpha,+}}.$$

Lemma 2.18 (Spectral expansion of the resolvent)**.** Use ℓ_n and P_n from Theorem 2.16. Then, for any $u \in H_{\alpha,+}$ and any $z \in \mathbb{C} \setminus \sigma(L)$,

$$(L - zI)^{-1}u = \sum_{n \geq 1} \frac{1}{\ell_n - z} P_n u$$

converges in $H_{\alpha,+}$. Furthermore, for any fixed $\Phi \in H_{\alpha,+}$,

$$m_L^{(\Phi)}(z) = \sum_{n \geq 1} \frac{\|P_n\Phi\|^2_{H_{\alpha,+}}}{\ell_n - z}, \qquad z \in \mathbb{C} \setminus \sigma(L),$$

and this scalar series converges locally uniformly on compact subsets of $\mathbb{C} \setminus \sigma(L)$.

Proof. By Theorem 2.16,

$$\sum_{n \geq 1} P_n = I$$

strongly, and

$$LP_n = \ell_n P_n$$

holds. Therefore, for any $u \in H_{\alpha,+}$,

$$u = \sum_{n \geq 1} P_n u$$

holds in $H_{\alpha,+}$, and on each spectral subspace $P_n H_{\alpha,+}$,

$$(L - zI)^{-1} = \frac{1}{\ell_n - z} I.$$

Hence

$$(L - zI)^{-1}u = \sum_{n \geq 1} \frac{1}{\ell_n - z} P_n u.$$

It remains to verify convergence in $H_{\alpha,+}$. Since $z \notin \sigma(L)$,

$$\delta_z := \operatorname{dist}(z, \sigma(L)) > 0.$$

For partial sums with $M > N$, we have

$$\left\| \sum_{n=N+1}^{M} \frac{1}{\ell_n - z} P_n u \right\|_{H_{\alpha,+}}^2 = \sum_{n=N+1}^{M} \frac{\|P_n u\|_{H_{\alpha,+}}^2}{|\ell_n - z|^2} \le \delta_z^{-2} \sum_{n=N+1}^{M} \|P_n u\|_{H_{\alpha,+}}^2.$$

The vectors $P_n u$ are mutually orthogonal, and moreover

$$\sum_{n\ge1} \|P_n u\|_{H_{\alpha,+}}^2 = \|u\|_{H_{\alpha,+}}^2.$$

Thus the right-hand side tends to 0 as $N, M \to \infty$. This proves norm convergence.

For the scalar function $m_L^{(\Phi)}$, taking the inner product with Φ gives

$$m_L^{(\Phi)}(z) = \left\langle \Phi, \sum_{n\ge1} \frac{1}{\ell_n - z} P_n \Phi \right\rangle_{H_{\alpha,+}} = \sum_{n\ge1} \frac{\langle \Phi, P_n \Phi \rangle_{H_{\alpha,+}}}{\ell_n - z}.$$

Since P_n is an orthogonal projector,

$$\langle \Phi, P_n \Phi \rangle_{H_{\alpha,+}} = \|P_n \Phi\|_{H_{\alpha,+}}^2.$$

Therefore

$$m_L^{(\Phi)}(z) = \sum_{n\ge1} \frac{\|P_n \Phi\|_{H_{\alpha,+}}^2}{\ell_n - z}.$$

Finally, let $K \subset \mathbb{C} \setminus \sigma(L)$ be compact, and set

$$\delta_K := \operatorname{dist}(K, \sigma(L)) > 0.$$

Then, for any $z \in K$,

$$\left| \frac{\|P_n \Phi\|_{H_{\alpha,+}}^2}{\ell_n - z} \right| \le \delta_K^{-1} \|P_n \Phi\|_{H_{\alpha,+}}^2.$$

But

$$\sum_{n\ge1} \|P_n \Phi\|_{H_{\alpha,+}}^2 = \|\Phi\|_{H_{\alpha,+}}^2 < \infty,$$

so the Weierstrass M-test implies local uniform convergence on K. Thus the scalar expansion converges locally uniformly on compact subsets of $\mathbb{C} \setminus \sigma(L)$. $\square$

Theorem 2.19 (Herglotz positivity and pole law). *For any fixed $\Phi \in H_{\alpha,+}$, the function*

$$m_L^{(\Phi)}(z) = \langle \Phi, (L - zI)^{-1} \Phi \rangle_{H_{\alpha,+}}$$

is holomorphic on $\mathbb{C} \setminus \sigma(L)$. Furthermore, if

$$\Im z > 0,$$

then

$$\Im m_L^{(\Phi)}(z) \geq 0,$$

more precisely,

$$\Im m_L^{(\Phi)}(z) = (\Im z)\, \|(L - zI)^{-1}\Phi\|_{H_{\alpha,+}}^2.$$

Finally, let $\ell_n \in \sigma(L)$ *be one of the discrete eigenvalues from Theorem 2.16. If*

$$P_n\Phi \neq 0,$$

then

$$m_L^{(\Phi)}(z) = \frac{\|P_n\Phi\|_{H_{\alpha,+}}^2}{\ell_n - z} + O(1) \qquad (z \to \ell_n).$$

Equivalently, $m_L^{(\Phi)}$ *has a simple pole at* $z = \ell_n$*, and its principal part is*

$$\frac{\|P_n\Phi\|_{H_{\alpha,+}}^2}{\ell_n - z}.$$

If $P_n\Phi = 0$*, then no singular term appears at* $z = \ell_n$*.*

Proof. The map

$$z \longmapsto (L - zI)^{-1}$$

is holomorphic as an $H_{\alpha,+}$-valued operator-valued map on the resolvent set $\mathbb{C} \setminus \sigma(L)$. Therefore, taking the scalar product with the fixed vector Φ, it follows that

$$m_L^{(\Phi)}$$

is holomorphic on $\mathbb{C} \setminus \sigma(L)$.

Next, take $z \in \mathbb{C}$ satisfying $\Im z > 0$, and write

$$R(z) := (L - zI)^{-1}.$$

Since L is self-adjoint,

$$R(\overline{z}) = R(z)^*.$$

Using the resolvent identity, we obtain

$$R(z) - R(\overline{z}) = (z - \overline{z})R(z)R(\overline{z}),$$

and hence

$$m_L^{(\Phi)}(z) - \overline{m_L^{(\Phi)}(z)} = \langle \Phi, (R(z) - R(\overline{z}))\Phi \rangle_{H_{\alpha,+}} = (z - \overline{z})\langle \Phi, R(z)R(\overline{z})\Phi \rangle_{H_{\alpha,+}}.$$

Since $R(\overline{z}) = R(z)^*$,

$$\langle \Phi, R(z)R(\overline{z})\Phi \rangle_{H_{\alpha,+}} = \langle R(z)^*\Phi, R(z)^*\Phi \rangle_{H_{\alpha,+}} = \|R(z)\Phi\|_{H_{\alpha,+}}^2.$$

Therefore

$$2i\,\Im m_L^{(\Phi)}(z) = 2i\,(\Im z)\,\|R(z)\Phi\|^2_{H_{\alpha,+}},$$

and thus

$$\Im m_L^{(\Phi)}(z) = (\Im z)\,\|(L - zI)^{-1}\Phi\|^2_{H_{\alpha,+}}.$$

Since $\Im z > 0$, this implies

$$\Im m_L^{(\Phi)}(z) \geq 0.$$

It remains to prove the pole law. Fix n, and choose $r > 0$ such that

$$B(\ell_n, r) \cap \sigma(L) = \{\ell_n\}.$$

By Lemma 2.18,

$$m_L^{(\Phi)}(z) = \frac{\|P_n\Phi\|^2_{H_{\alpha,+}}}{\ell_n - z} + \sum_{k \neq n} \frac{\|P_k\Phi\|^2_{H_{\alpha,+}}}{\ell_k - z}.$$

If $|z - \ell_n| < r/2$, then

$$|\,\ell_k - z\,| \geq |\,\ell_k - \ell_n\,| - |\,z - \ell_n\,| \geq \frac{1}{2}|\,\ell_k - \ell_n\,| \qquad (k \neq n),$$

and in particular the denominators remain uniformly away from 0. Hence the residual series

$$\sum_{k \neq n} \frac{\|P_k\Phi\|^2_{H_{\alpha,+}}}{\ell_k - z}$$

is holomorphic and bounded in a neighborhood of ℓ_n, by the same local uniform convergence argument used in Lemma 2.18. Therefore

$$m_L^{(\Phi)}(z) = \frac{\|P_n\Phi\|^2_{H_{\alpha,+}}}{\ell_n - z} + O(1) \qquad (z \to \ell_n).$$

If $P_n\Phi \neq 0$, this is a genuine simple pole with the displayed principal part. If $P_n\Phi = 0$, the singular term vanishes and only the bounded holomorphic residual term remains. □

Remark 2.20 (End of the analytic section)**.** Section 2 ends here. At this point, we have constructed the operator-side analytic construction

$$q_R \longrightarrow A_R \longrightarrow L,$$

together with the compact embedding of the form domain, compact resolvent, discrete spectrum, and the scalar-valued meromorphic/Herglotz resolvent function $m_L^{(\Phi)}$. In this section, we have used no arithmetic projector, no comparison theorem, and no conclusion statement for the Riemann hypothesis.

3. Coefficient-Space Arithmetic Construction

3.1. Formal Dirichlet Algebra and Coefficient-Extraction Conventions

In this subsection, we argue entirely on the coefficient layer. We use neither analytic continuation, nor a complex variable s, nor meromorphic functions. The sole purpose here is to fix the formal arithmetic algebra in which the coefficient-extraction statement will be formulated later.

Definition 3.1 (Formal Dirichlet algebra and augmentation ideal)**.** Let

$$\mathfrak{D} := \{a : \mathbb{N} \to \mathbb{C}\}$$

be the complex vector space of all arithmetic functions. Equip $\mathfrak{D}$ with Dirichlet convolution

$$(a * b)(n) := \sum_{d|n} a(d)\, b(n/d) \qquad (n \geq 1).$$

Then $(\mathfrak{D}, *)$ is a commutative $\mathbb{C}$-algebra with unit

$$\delta_1(n) := \begin{cases} 1, & n = 1, \\ 0, & n \geq 2. \end{cases}$$

For $a \in \mathfrak{D}$ and $n \geq 1$, write the coefficient-extraction operator at n as

$$[n]a := a(n).$$

Define the augmentation ideal by

$$\mathfrak{m} := \{a \in \mathfrak{D} : \ a(1) = 0\}.$$

Equivalently,

$$\delta_1 + \mathfrak{m} = \{a \in \mathfrak{D} : \ a(1) = 1\}.$$

When necessary, one may write $a \in \mathfrak{D}$ as a formal Dirichlet series

$$a \sim \sum_{n \geq 1} a(n)\, n^{-s},$$

but here n^{-s} is merely a formal basis symbol indexed by n. At no point in §3.1–§3.2 do we substitute any value for s.

Lemma 3.2 (Coefficient and convolution laws)**.** Let $a, b \in \mathfrak{D}$. Then, for any $n \geq 1$,

$$[n](a * b) = \sum_{d|n} [d]a\, [n/d]b.$$

More generally, for any integer $k \geq 1$,

$$[n](a^{*k}) = \sum_{\substack{d_1 \cdots d_k = n \\ d_i \geq 1}} a(d_1) \cdots a(d_k),$$

where a^{*k} denotes the k-fold Dirichlet convolution power of a.

If $a(1) \neq 0$, then there exists a unique Dirichlet inverse

$$a^{-*} \in \mathfrak{D}$$

satisfying

$$a * a^{-*} = a^{-*} * a = \delta_1.$$

Its coefficients are determined recursively by

$$a^{-*}(1) = \frac{1}{a(1)},$$

and, for $n \geq 2$, by

$$a^{-*}(n) = -\frac{1}{a(1)} \sum_{\substack{d|n \\ d>1}} a(d)\, a^{-*}(n/d).$$

In particular, the inverse is uniquely determined coefficientwise by the divisor recursion.

Proof. The first identity is merely the definition of Dirichlet convolution rewritten in coefficient-extraction notation:

$$[n](a*b) = (a*b)(n) = \sum_{d|n} a(d)b(n/d) = \sum_{d|n} [d]a\,[n/d]b.$$

The k-fold formula is proved by induction on k. The case $k = 1$ is immediate. Assume the identity holds for some $k \geq 1$. Then

$$[n](a^{*(k+1)}) = [n](a^{*k} * a) = \sum_{d|n} [d](a^{*k})\,[n/d]a.$$

By the induction hypothesis,

$$[d](a^{*k}) = \sum_{\substack{e_1 \cdots e_k = d \\ e_i \geq 1}} a(e_1) \cdots a(e_k).$$

Substituting this into the preceding formula gives

$$[n](a^{*(k+1)}) = \sum_{d|n} \left(\sum_{\substack{e_1 \cdots e_k = d \\ e_i \geq 1}} a(e_1) \cdots a(e_k) \right) a(n/d).$$

Relabeling $e_{k+1} := n/d$, this is exactly

$$[n](a^{*(k+1)}) = \sum_{\substack{e_1 \cdots e_{k+1} = n \\ e_i \geq 1}} a(e_1) \cdots a(e_{k+1}),$$

and the induction is complete.

Next, we prove the existence and uniqueness of the Dirichlet inverse. Suppose that

$$b \in \mathfrak{D}$$

satisfies

$$a * b = \delta_1.$$

Looking at the coefficient $n = 1$, we obtain

$$1 = [1]\delta_1 = [1](a * b) = a(1)b(1),$$

and hence necessarily

$$b(1) = \frac{1}{a(1)}.$$

For $n \geq 2$, the identity $[n](a * b) = 0$ gives

$$0 = \sum_{d|n} a(d)b(n/d) = a(1)b(n) + \sum_{\substack{d|n \\ d>1}} a(d)b(n/d).$$

Therefore

$$b(n) = -\frac{1}{a(1)} \sum_{\substack{d|n \\ d>1}} a(d)b(n/d).$$

If $d > 1$, then $n/d < n$, so the right-hand side depends only on coefficients $b(m)$ with $m < n$ that have already been determined. Thus this recursion admits at most one inverse.

Conversely, define b recursively by

$$b(1) := \frac{1}{a(1)}, \qquad b(n) := -\frac{1}{a(1)} \sum_{\substack{d|n \\ d>1}} a(d)b(n/d) \quad (n \geq 2).$$

Tracing the same calculation in reverse yields

$$[n](a * b) = \begin{cases} 1, & n = 1, \\ 0, & n \geq 2. \end{cases}$$

Thus

$$a * b = \delta_1.$$

Since Dirichlet convolution is commutative,

$$b * a = \delta_1$$

also holds. Hence $b = a^{-*}$ exists and is unique. □

Proposition 3.3 (Formal logarithm and exponential on the augmentation ideal)**.** *For $a, b \in \mathfrak{m}$, define formally*

$$\log_*(\delta_1 + a) := \sum_{k \geq 1} \frac{(-1)^{k+1}}{k} a^{*k}, \qquad \exp_*(b) := \sum_{k \geq 0} \frac{1}{k!} b^{*k},$$

using the convention

$$a^{*0} = b^{*0} := \delta_1.$$

Then the following hold:

1. *Both series are coefficientwise well-defined;*

2.
$$\log_* : (\delta_1 + \mathfrak{m}) \longrightarrow \mathfrak{m}, \qquad \exp_* : \mathfrak{m} \longrightarrow (\delta_1 + \mathfrak{m});$$

3. *The two maps are inverse to each other:*

$$\exp_*\big(\log_*(\delta_1 + a)\big) = \delta_1 + a \qquad (a \in \mathfrak{m}),$$

$$\log_*\big(\exp_*(b)\big) = b \qquad (b \in \mathfrak{m}).$$

Proof. Let $\Omega(n)$ denote the total number of prime factors of n, counted with multiplicity.

First, we prove coefficientwise well-definedness. Take $a \in \mathfrak{m}$, and fix $n \geq 1$. By Lemma 3.2,

$$[n](a^{*k}) = \sum_{\substack{d_1 \cdots d_k = n \\ d_i \geq 1}} a(d_1) \cdots a(d_k).$$

Since $a \in \mathfrak{m}$, we have $a(1) = 0$. Therefore, if a nonzero term appears, it must satisfy

$$d_i \geq 2 \qquad (1 \leq i \leq k).$$

But an ordered factorization

$$d_1 \cdots d_k = n \qquad (d_i \geq 2)$$

can exist only when

$$k \leq \Omega(n).$$

Hence

$$[n](a^{*k}) = 0 \qquad (k > \Omega(n)).$$

Thus the formal series for the coefficient n of $\log_*(\delta_1 + a)$ reduces to the finite sum

$$[n]\log_*(\delta_1 + a) = \sum_{k=1}^{\Omega(n)} \frac{(-1)^{k+1}}{k}\,[n](a^{*k}) \qquad (n \geq 2).$$

For $n = 1$, all terms vanish because $[1](a^{*k}) = 0$ for $k \geq 1$. Therefore $\log_*(\delta_1 + a)$ is coefficientwise well-defined and belongs to $\mathfrak{m}$.

The same argument applies to $b \in \mathfrak{m}$. For $n \geq 2$,

$$[n](b^{*k}) = 0 \qquad (k > \Omega(n)),$$

so

$$[n]\exp_*(b) = \sum_{k=0}^{\Omega(n)} \frac{1}{k!}\,[n](b^{*k})$$

is a finite sum. For $n = 1$, since $[1](b^{*k}) = 0$ for $k \geq 1$, only the $k = 0$ term contributes. Thus

$$[1]\exp_*(b) = [1]\delta_1 = 1,$$

and hence

$$\exp_*(b) \in \delta_1 + \mathfrak{m}.$$

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This proves (1) and (2).

It remains to prove the inverse-map identities. Fix $n \geq 1$, and set

$$\mathrm{Div}(n) := \{d \in \mathbb{N} : \ d \mid n\}.$$

Consider the finite-dimensional convolution algebra

$$\mathfrak{D}_{|n} := \{c : \mathrm{Div}(n) \to \mathbb{C}\}$$

with convolution restricted to the divisors of n:

$$(c \star_n d)(m) := \sum_{r|m} c(r)\, d(m/r) \qquad (m \mid n).$$

Furthermore, set

$$\mathfrak{m}_{|n} := \{c \in \mathfrak{D}_{|n} : \ c(1) = 0\}.$$

By the factorization-count estimate already proved, every $c \in \mathfrak{m}_{|n}$ satisfies

$$c^{\star_n k} = 0 \qquad (k > \Omega(n)).$$

Indeed, for any divisor $m \mid n$, there is no factorization of m into more than $\Omega(n)$ integers all ≥ 2. Thus $\mathfrak{m}_{|n}$ is a nilpotent ideal.

Consequently, in the finite-dimensional commutative algebra $\mathfrak{D}_{|n}$, the formal series

$$\log(1+X) = \sum_{k\geq 1} \frac{(-1)^{k+1}}{k} X^k, \qquad \exp(X) = \sum_{k\geq 0} \frac{1}{k!} X^k$$

truncate to genuine finite polynomials on $\mathfrak{m}_{|n}$, and the usual formal identities hold exactly:

$$\exp(\log(1+X)) = 1 + X, \qquad \log(\exp(X)) = X.$$

Apply this to the restrictions of a or b to $\mathrm{Div}(n)$. Since the coefficient n depends only on values on the divisors of n,

$$[n] \exp_*\big(\log_*(\delta_1 + a)\big) = [n](\delta_1 + a),$$
$$[n] \log_*\big(\exp_*(b)\big) = [n] b.$$

Since $n \geq 1$ was arbitrary, the coefficients agree for all n, and therefore

$$\exp_*\big(\log_*(\delta_1 + a)\big) = \delta_1 + a, \qquad \log_*\big(\exp_*(b)\big) = b.$$

Hence $\log_*$ and $\exp_*$ are inverse to each other. ☐

Remark 3.4 (Formal–analytic separation discipline)**.** In §3.1–§3.2, we use neither $\zeta(s)$, nor $\log \zeta(s)$, nor $-\zeta'(s)/\zeta(s)$, nor the Euler product, nor analytic continuation, nor any discussion of poles or zeros on the complex plane. The reader may understand that also in the subsequent part of Section 3, the argument is carried out entirely on the coefficient layer of the formal Dirichlet algebra.

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3.2. Prime Indicator and Von Mangoldt Function

Here we fix the basic arithmetic data that will later be input into the purely coefficient-level extraction theorem. At this stage, they are simply arithmetic functions on $\mathbb{N}$. One is supported on the prime support, and the other is supported on the prime-power support.

Definition 3.5 (Prime indicator and von Mangoldt function)**.** Define the prime indicator

$$1_P^{\mathrm{ex}} : \mathbb{N} \to \mathbb{C}$$

by

$$1_P^{\mathrm{ex}}(n) := \begin{cases} 1, & n \text{ is prime}, \\ 0, & \text{otherwise}, \end{cases}$$

and define the von Mangoldt function

$$\Lambda^{\mathrm{ex}} : \mathbb{N} \to \mathbb{C}$$

by

$$\Lambda^{\mathrm{ex}}(n) := \begin{cases} \log p, & n = p^k \text{ for some prime } p \text{ and some integer } k \geq 1, \\ 0, & \text{otherwise}. \end{cases}$$

Thus 1_P^{ex} can be nonzero only on primes, whereas Λ^{ex} can be nonzero only on prime powers.

Theorem 3.6 (Divisor-sum law for the von Mangoldt function)**.** *For every integer $n \geq 1$,*

$$\log n = \sum_{d|n} \Lambda^{\mathrm{ex}}(d)$$

holds. Equivalently, if

$$\mathbf{1}(n) := 1 \qquad (n \geq 1),$$

and

$$\mathbf{L}(n) := \log n \qquad (n \geq 1),$$

then

$$\mathbf{L} = \mathbf{1} * \Lambda^{\mathrm{ex}}.$$

Therefore, if

$$\mu := \mathbf{1}^{-*}$$

is the Dirichlet inverse of $\mathbf{1}$*, then*

$$\Lambda^{\mathrm{ex}} = \mu * \mathbf{L}.$$

Proof. First consider $n = 1$. Since 1 is not a prime power,

$$\Lambda^{\mathrm{ex}}(1) = 0,$$

and hence

$$\sum_{d|1} \Lambda^{\mathrm{ex}}(d) = \Lambda^{\mathrm{ex}}(1) = 0 = \log 1.$$

Next let $n \geq 2$, and write its prime factorization as

$$n = \prod_{i=1}^{r} p_i^{\nu_i}.$$

Here $p_1, \ldots, p_r$ are distinct primes and $\nu_i \geq 1$. A divisor $d \mid n$ contributes to

$$\sum_{d|n} \Lambda^{\text{ex}}(d)$$

if and only if d is a prime power dividing n. Such divisors are precisely

$$p_i,\ p_i^2,\ \ldots,\ p_i^{\nu_i} \qquad (1 \leq i \leq r).$$

Therefore

$$\sum_{d|n} \Lambda^{\text{ex}}(d) = \sum_{i=1}^{r} \sum_{j=1}^{\nu_i} \Lambda^{\text{ex}}(p_i^j) = \sum_{i=1}^{r} \sum_{j=1}^{\nu_i} \log p_i = \sum_{i=1}^{r} \nu_i \log p_i.$$

On the other hand,

$$\log n = \log\left(\prod_{i=1}^{r} p_i^{\nu_i}\right) = \sum_{i=1}^{r} \nu_i \log p_i,$$

so

$$\sum_{d|n} \Lambda^{\text{ex}}(d) = \log n.$$

This proves the divisor-sum law for all $n \geq 1$.

The convolution formulation follows immediately:

$$(\mathbf{1} * \Lambda^{\text{ex}})(n) = \sum_{d|n} \mathbf{1}(d)\, \Lambda^{\text{ex}}(n/d) = \sum_{d|n} \Lambda^{\text{ex}}(d) = \log n = \mathbf{L}(n).$$

Thus

$$\mathbf{L} = \mathbf{1} * \Lambda^{\text{ex}}.$$

Finally, since $\mathbf{1}(1) = 1 \neq 0$, Lemma 3.2 gives a unique Dirichlet inverse

$$\mu = \mathbf{1}^{-*}.$$

Convolving both sides of

$$\mathbf{L} = \mathbf{1} * \Lambda^{\text{ex}}$$

from the left by μ, and using associativity together with

$$\mu * \mathbf{1} = \delta_1,$$

we obtain

$$\mu * \mathbf{L} = (\mu * \mathbf{1}) * \Lambda^{\text{ex}} = \delta_1 * \Lambda^{\text{ex}} = \Lambda^{\text{ex}}.$$

Hence

$$\Lambda^{\text{ex}} = \mu * \mathbf{L}.$$

□

Proposition 3.7 (Support and positivity properties)**.** *For an arithmetic function $a \in \mathfrak{D}$, set*

$$\operatorname{supp}(a) := \{n \in \mathbb{N} : \ a(n) \neq 0\}.$$

Then

$$\operatorname{supp}(1_P^{\mathrm{ex}}) = \{primes\},$$

and

$$\operatorname{supp}(\Lambda^{\mathrm{ex}}) = \{prime\ powers\}.$$

Furthermore,

$$\Lambda^{\mathrm{ex}}(n) \geq 0 \qquad (n \geq 1),$$

and moreover

$$\Lambda^{\mathrm{ex}}(n) > 0 \quad \Longleftrightarrow \quad n \text{ is a prime power}.$$

Accordingly, the prime support of 1_P^{ex} and the prime-power support of Λ^{ex} are distinct and must be distinguished in the later coefficient-level argument.

Proof. By Definition 3.5,

$$1_P^{\mathrm{ex}}(n) = 1$$

holds if and only if n is prime, and it is 0 otherwise. Therefore

$$\operatorname{supp}(1_P^{\mathrm{ex}}) = \{\text{primes}\}.$$

Similarly, by Definition 3.5,

$$\Lambda^{\mathrm{ex}}(n) = \log p$$

holds if and only if $n = p^k$ for some prime p and integer $k \geq 1$, and it is 0 otherwise. Therefore

$$\operatorname{supp}(\Lambda^{\mathrm{ex}}) = \{\text{prime powers}\}.$$

If n is not a prime power, then

$$\Lambda^{\mathrm{ex}}(n) = 0.$$

On the other hand, if $n = p^k$ is a prime power, then

$$\Lambda^{\mathrm{ex}}(n) = \log p.$$

Every prime satisfies $p \geq 2$, so

$$\log p > 0.$$

Thus

$$\Lambda^{\mathrm{ex}}(n) \geq 0 \qquad (n \geq 1),$$

and strict inequality holds exactly in the prime-power case. This proves all the asserted support and positivity properties. □

Corollary 3.8 (Coefficient-level recovery of the logarithm function)**.** *Define the arithmetic function*

$$\mathbf{L} \in \mathfrak{D} \qquad \textit{by} \qquad \mathbf{L}(n) := \log n \qquad (n \geq 1),$$

and define

$$\mathbf{1}(n) := 1 \qquad (n \geq 1).$$

Then

$$\mathbf{L} = \mathbf{1} * \Lambda^{\mathrm{ex}}.$$

Equivalently, for every $n \geq 1$,

$$[n]\mathbf{L} = [n](\mathbf{1} * \Lambda^{\mathrm{ex}}).$$

Proof. For any $n \geq 1$, Lemma 3.2 gives

$$[n](\mathbf{1} * \Lambda^{\mathrm{ex}}) = \sum_{d|n} [d]\mathbf{1}\,[n/d]\Lambda^{\mathrm{ex}} = \sum_{d|n} \Lambda^{\mathrm{ex}}(n/d).$$

Replacing n/d by the divisor variable $e \mid n$, we get

$$[n](\mathbf{1} * \Lambda^{\mathrm{ex}}) = \sum_{e|n} \Lambda^{\mathrm{ex}}(e).$$

By Theorem 3.6,

$$\sum_{e|n} \Lambda^{\mathrm{ex}}(e) = \log n = \mathbf{L}(n) = [n]\mathbf{L}.$$

Therefore

$$[n](\mathbf{1} * \Lambda^{\mathrm{ex}}) = [n]\mathbf{L} \qquad (n \geq 1).$$

Since the coefficients agree for all n, it follows that, as arithmetic functions,

$$\mathbf{L} = \mathbf{1} * \Lambda^{\mathrm{ex}}.$$

□

3.3. Completed Augmentation Ideal and Dirichlet-Logarithmic Linearization

We now pass from the augmentation ideal to its coefficientwise completed setting. The point remains purely formal, and no analytic layer is introduced. We extend $\log_*$ and $\exp_*$ from §3.1 to the completed coefficientwise topology, and use them to linearize the prime-local convolution factors. This converts the multiplicative overlap arising from the factorization of composite numbers into a linear sum of prime-power expansions.

Definition 3.9 (Completed augmentation ideal and coefficientwise topology)**.** For $n, m \geq 1$, define the Dirichlet basis atom by

$$e_n(m) := \delta_{n,m}.$$

Thus $e_1 = \delta_1$, and any arithmetic function can be written formally as

$$a = \sum_{n \geq 1} [n]a\, e_n.$$

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Define the coefficientwise formal completion by

$$\widehat{\mathfrak{D}} := \prod_{n\geq 1} \mathbb{C}e_n, \qquad \widehat{\mathfrak{m}} := \prod_{n\geq 2} \mathbb{C}e_n.$$

Equivalently,

$$\widehat{\mathfrak{D}} = \left\{ \sum_{n\geq 1} a_n e_n : \ a_n \in \mathbb{C} \right\}, \qquad \widehat{\mathfrak{m}} = \left\{ \sum_{n\geq 2} a_n e_n : \ a_n \in \mathbb{C} \right\}.$$

Viewed as coefficient sets, $\widehat{\mathfrak{D}}$ and $\mathfrak{D}$ contain the same data. The hat notation indicates that, from this point onward, these are interpreted as formal infinite sums equipped with the *coefficientwise product topology*. Namely, a sequence $a^{(j)}$ converges to a if

$$[n]a^{(j)} \longrightarrow [n]a \qquad \text{for every fixed } n \geq 1.$$

Equivalently, the coordinate maps

$$[n] : \widehat{\mathfrak{D}} \to \mathbb{C}$$

are continuous, and they define the product topology.

The completed unit neighborhood is

$$\delta_1 + \widehat{\mathfrak{m}} = \left\{ a \in \widehat{\mathfrak{D}} : \ [1]a = 1 \right\}.$$

Dirichlet convolution on $\mathfrak{D}$ extends coefficientwise to $\widehat{\mathfrak{D}}$. Namely, for $a, b \in \widehat{\mathfrak{D}}$, define

$$[n](a * b) := \sum_{d|n} [d]a \, [n/d]b \qquad (n \geq 1).$$

This is well-defined because the divisor set of n is finite.

Definition 3.10 (Completed exponential map)**.** Define

$$E := \exp_* : \widehat{\mathfrak{m}} \longrightarrow \delta_1 + \widehat{\mathfrak{m}}$$

by the same coefficientwise formula as in Proposition 3.3:

$$E(b) = \exp_*(b) := \sum_{k\geq 0} \frac{1}{k!} b^{*k}, \qquad b \in \widehat{\mathfrak{m}},$$

where

$$b^{*0} := \delta_1.$$

Thus E is not a new exponential symbol, but the same formal exponential map $\exp_*$ interpreted in the completed coefficientwise setting.

Theorem 3.11 (Invertibility of the completed exponential map)**.** *The map*

$$E = \exp_* : \widehat{\mathfrak{m}} \xrightarrow{\sim} \delta_1 + \widehat{\mathfrak{m}}$$

is bijective. Its inverse is the coefficientwise extension of the formal logarithm from §3.1; that is,

$$E^{-1} = \log_*, \qquad \log_* : \delta_1 + \widehat{\mathfrak{m}} \to \widehat{\mathfrak{m}},$$

where

$$\log_*(\delta_1 + a) := \sum_{k \geq 1} \frac{(-1)^{k+1}}{k} a^{*k}, \qquad a \in \widehat{\mathfrak{m}}.$$

Equivalently,

$$\exp_*(\log_*(u)) = u \qquad (u \in \delta_1 + \widehat{\mathfrak{m}}),$$

and

$$\log_*(\exp_*(b)) = b \qquad (b \in \widehat{\mathfrak{m}})$$

hold.

Proof. First, we verify that the displayed series are coefficientwise well-defined on the completed space.

Take $b \in \widehat{\mathfrak{m}}$, and fix $n \geq 1$. By Lemma 3.2,

$$[n](b^{*k}) = \sum_{\substack{d_1 \cdots d_k = n \\ d_i \geq 1}} b(d_1) \cdots b(d_k).$$

Since $b \in \widehat{\mathfrak{m}}$, we have $b(1) = 0$. Therefore, if a nonzero term appears, it must satisfy

$$d_i \geq 2 \qquad (1 \leq i \leq k).$$

Thus an ordered factorization contributing to $[n](b^{*k})$ can exist only when

$$k \leq \Omega(n),$$

where $\Omega(n)$ is the total number of prime factors of n, counted with multiplicity. Hence

$$[n](b^{*k}) = 0 \qquad (k > \Omega(n)).$$

Therefore, for the fixed coefficient n,

$$[n] \exp_*(b) = \sum_{k=0}^{\Omega(n)} \frac{1}{k!} [n](b^{*k})$$

is a finite sum. Thus $\exp_*(b)$ is coefficientwise well-defined.

The same argument applies to $\log_*$. If $a \in \widehat{\mathfrak{m}}$, then

$$[n](a^{*k}) = 0 \qquad (k > \Omega(n)),$$

so

$$[n] \log_*(\delta_1 + a) = \sum_{k=1}^{\Omega(n)} \frac{(-1)^{k+1}}{k} [n](a^{*k})$$

is also a finite sum. Therefore $\log_*(\delta_1 + a)$ is coefficientwise well-defined.

Next, we prove the inverse-map identities coefficientwise. Fix $n \geq 1$, and consider the finite divisor algebra

$$\mathfrak{D}_{|n} := \{c : \mathrm{Div}(n) \to \mathbb{C}\}$$

with the restricted Dirichlet convolution

$$(c \star_n d)(m) := \sum_{r|m} c(r)\, d(m/r) \qquad (m \mid n).$$

Furthermore, set

$$\mathfrak{m}_{|n} := \{c \in \mathfrak{D}_{|n} : c(1) = 0\}.$$

As in the proof of Proposition 3.3, the ideal $\mathfrak{m}_{|n}$ is nilpotent. Indeed, if $c \in \mathfrak{m}_{|n}$, then

$$c^{\star_n k} = 0 \qquad (k > \Omega(n)).$$

Therefore, in the finite-dimensional commutative algebra $\mathfrak{D}_{|n}$, the formal series $\exp_*$ and $\log_*$ truncate to genuine finite polynomials, and the usual identities

$$\exp_*(\log_*(1+x)) = 1+x, \qquad \log_*(\exp_*(x)) = x \qquad (x \in \mathfrak{m}_{|n})$$

hold exactly.

Now let $u \in \delta_1 + \widehat{\mathfrak{m}}$ and $b \in \widehat{\mathfrak{m}}$. Restrict them to $\mathrm{Div}(n)$. Since the coefficient $[n](\cdot)$ depends only on values on the divisors of n, the finite-dimensional identities above imply

$$[n]\exp_*(\log_*(u)) = [n]u,$$

and

$$[n]\log_*(\exp_*(b)) = [n]b.$$

Since $n \geq 1$ was arbitrary, the coefficients agree for all n. Therefore

$$\exp_*(\log_*(u)) = u \qquad (u \in \delta_1 + \widehat{\mathfrak{m}}),$$

and

$$\log_*(\exp_*(b)) = b \qquad (b \in \widehat{\mathfrak{m}}).$$

Hence $\exp_*$ is bijective, and its inverse is precisely $\log_*$. $\square$

Definition 3.12 (Dirichlet-logarithmic linearization operator)**.** Define the Dirichlet-logarithmic linearization operator by

$$C_{\mathrm{comp}} := E^{-1} = \log_* : \delta_1 + \widehat{\mathfrak{m}} \longrightarrow \widehat{\mathfrak{m}}.$$

Its role is to remove the multiplicities of composite numbers built into multiplicative Dirichlet convolution through the logarithmic linearization of prime-local convolution factors.

Theorem 3.13 (Coefficientwise Euler-factor decomposition and log-linearization)**.** *Let*

$$\mathbf{1} \in \widehat{\mathfrak{D}}, \qquad \mathbf{1}(n) := 1 \qquad (n \geq 1),$$

and for each prime p, define the local geometric factor

$$G_p := \sum_{k\geq 0} e_{p^k} \in \delta_1 + \widehat{\mathfrak{m}}.$$

Then the following hold.

1. *The coefficientwise Euler-factor decomposition*

$$\mathbf{1} = \mathop{\ast}_{p} G_p$$

   holds. Here the infinite convolution product is interpreted coefficientwise. Namely, for fixed n, the coefficient $[n](\ast_p G_p)$ *is the stable value of the finite product over any prime set containing all prime factors of n.*
2. $$\mathrm{C}_{\mathrm{comp}}(\mathbf{1}) = \log_*(\mathbf{1}) = \sum_p \log_*(G_p) = \sum_p \sum_{k\geq 1} \frac{1}{k} e_{p^k},$$

   again with all equalities understood coefficientwise.

Proof. We divide the proof into three steps.

Step 1: Coefficientwise Euler-factor decomposition. For a finite set S of primes, define

$$G_S := \mathop{\ast}_{p\in S} G_p.$$

Fix $n \geq 1$, and write its prime factorization as

$$n = \prod_{i=1}^{r} p_i^{\nu_i}.$$

Here $p_1, \dots, p_r$ are distinct primes and $\nu_i \geq 1$. The claim is

$$[n]G_S = \begin{cases} 1, & \{p_1, \dots, p_r\} \subset S, \\ 0, & \text{otherwise.} \end{cases} \tag{3.3.1}$$

Indeed, each factor G_p contributes only powers of p:

$$G_p = \sum_{k\geq 0} e_{p^k}.$$

Therefore, a nonzero contribution to the coefficient of n in the finite convolution product G_S occurs only when, for each prime p_i dividing n, one selects exactly one atom $e_{p_i^{\nu_i}}$ from the factor G_{p_i}, and for every other prime $q \in S \setminus \{p_1, \dots, p_r\}$, one selects e_1. By uniqueness of prime factorization, if all prime factors of n belong to S, there is exactly one such choice, and otherwise there is none. This proves (3.3.1).

Now set

$$S_n := \{p : \ p \text{ prime and } p \mid n\}.$$

If $S \supset S_n$, then (3.3.1) gives

$$[n]G_S = 1 = [n]\mathbf{1}.$$

Therefore the coefficient $[n]G_S$ stabilizes to 1 once S contains all prime factors of n. This is precisely the coefficientwise meaning of

$$\mathbf{1} = \mathop{\ast}_{p} G_p.$$

Step 2: Local log-linearization of each Euler factor. Fix a prime p. Since

$$e_p^{*m} = e_{p^m} \qquad (m \geq 1),$$

we have

$$(\delta_1 - e_p) * G_p = (\delta_1 - e_p) * \sum_{k\geq 0} e_{p^k} = \sum_{k\geq 0} e_{p^k} - \sum_{k\geq 0} e_p * e_{p^k}.$$

But

$$e_p * e_{p^k} = e_{p^{k+1}},$$

so the right-hand side telescopes as desired:

$$\sum_{k\geq 0} e_{p^k} - \sum_{k\geq 0} e_{p^{k+1}} = e_1 = \delta_1.$$

Thus

$$G_p = (\delta_1 - e_p)^{-*}.$$

Next we compute $\log_*(G_p)$. Since $G_p * (\delta_1 - e_p) = \delta_1$, the finite-dimensional divisor-algebra argument used in the proof of Theorem 3.11 gives

$$\log_*(G_p) + \log_*(\delta_1 - e_p) = \log_*(\delta_1) = 0.$$

Hence

$$\log_*(G_p) = -\log_*(\delta_1 - e_p).$$

Using the defining series for $\log_*$, we obtain

$$\log_*(\delta_1 - e_p) = \sum_{m\geq 1} \frac{(-1)^{m+1}}{m} (-e_p)^{*m} = -\sum_{m\geq 1} \frac{1}{m} e_p^{*m},$$

and therefore

$$\log_*(G_p) = \sum_{m\geq 1} \frac{1}{m} e_p^{*m} = \sum_{m\geq 1} \frac{1}{m} e_{p^m}. \tag{3.3.2}$$

Step 3: Global log-linearization of $\mathbf{1}$. Fix $n \geq 1$, and let S_n be the finite set of all prime factors of n. By Step 1,

$$[n]\mathbf{1} = [n]\left(\mathop{\ast}_{p\in S_n} G_p\right).$$

Restrict everything to the finite divisor algebra $\mathfrak{D}_{|n}$. In this finite-dimensional commutative algebra, logarithm and exponential are genuine finite polynomials on the nilpotent augmentation ideal. Therefore the usual commutative identity

$$\log_*\left(\mathop{\ast}_{p\in S_n} G_p\right) = \sum_{p\in S_n} \log_*(G_p)$$

holds. Taking the coefficient n, we get

$$[n]\log_*(\mathbf{1}) = [n]\sum_{p\in S_n}\log_*(G_p).$$

If $p \nmid n$, then each term of $\log_*(G_p)$ is supported only on powers of p, so

$$[n]\log_*(G_p) = 0.$$

Hence the sum can be extended to all primes:

$$[n]\log_*(\mathbf{1}) = [n]\sum_{p}\log_*(G_p).$$

Using (3.3.2), this becomes

$$[n]\log_*(\mathbf{1}) = [n]\sum_{p}\sum_{k\geq 1}\frac{1}{k}e_{p^k}.$$

Since this holds for every $n \geq 1$, the coefficientwise identity

$$C_{\mathrm{comp}}(\mathbf{1}) = \log_*(\mathbf{1}) = \sum_{p}\log_*(G_p) = \sum_{p}\sum_{k\geq 1}\frac{1}{k}e_{p^k}$$

follows. This completes the proof. □

Thus $\log_*$ has already removed the multiplicative overlap carried by the Euler-type convolution product. After linearization, only prime-power atoms remain. The remaining task is purely arithmetic and coefficientwise: to act on this linearized output by the arithmetic derivation and recover the logarithm function **L**.

3.4. Arithmetic Derivation and Formal Coefficient Extraction

Here we give logarithmic weights to the prime-power atoms generated by C_{comp}. Since the log-linearization of Theorem 3.13 has already separated the prime-power support, the arithmetic derivation recovers the von Mangoldt function without leaving the purely formal arithmetic layer.

Definition 3.14 (Arithmetic derivation)**.** Define the arithmetic derivation

$$\partial_{\log} : \widehat{\mathfrak{D}} \longrightarrow \widehat{\mathfrak{D}}$$

by

$$(\partial_{\log} a)(n) := (\log n)\, a(n) \qquad (a \in \widehat{\mathfrak{D}},\ n \geq 1).$$

Equivalently,

$$\partial_{\log}\left(\sum_{n\geq 1} a_n e_n\right) = \sum_{n\geq 1}(\log n)\, a_n e_n.$$

Lemma 3.15 (Derivation law for Dirichlet convolution)**.** For all $a, b \in \widehat{\mathfrak{D}}$,

$$\partial_{\log}(a * b) = (\partial_{\log} a) * b + a * (\partial_{\log} b)$$

holds.

Proof. Fix $n \geq 1$. By the definitions of $\partial_{\log}$ and Dirichlet convolution,

$$[n]\partial_{\log}(a * b) = (\log n)\,[n](a * b) = (\log n)\sum_{d|n} a(d)b(n/d).$$

Using

$$\log n = \log d + \log(n/d),$$

we obtain

$$[n]\partial_{\log}(a * b) = \sum_{d|n}(\log d)\,a(d)b(n/d) + \sum_{d|n} a(d)(\log(n/d))\,b(n/d).$$

The first sum is exactly

$$[n]\big((\partial_{\log} a) * b\big),$$

and the second sum is exactly

$$[n]\big(a * (\partial_{\log} b)\big).$$

Therefore

$$[n]\partial_{\log}(a * b) = [n]\big((\partial_{\log} a) * b + a * (\partial_{\log} b)\big) \qquad (n \geq 1).$$

Since the coefficients agree for all n, as arithmetic functions we have

$$\partial_{\log}(a * b) = (\partial_{\log} a) * b + a * (\partial_{\log} b).$$

□

Theorem 3.16 (Formal coefficient-extraction theorem)**.**

$$\partial_{\log} C_{\mathrm{comp}}(\mathbf{1}) = \Lambda^{\mathrm{ex}}.$$

Equivalently, for every $n \geq 1$,

$$[n]\big(\partial_{\log} C_{\mathrm{comp}}(\mathbf{1})\big) = \Lambda^{\mathrm{ex}}(n).$$

Proof. By Theorem 3.13,

$$C_{\mathrm{comp}}(\mathbf{1}) = \sum_{p}\sum_{k\geq 1}\frac{1}{k}e_{p^k}.$$

Applying $\partial_{\log}$ coefficientwise gives

$$\partial_{\log} C_{\mathrm{comp}}(\mathbf{1}) = \sum_{p}\sum_{k\geq 1}\frac{\log(p^k)}{k}\,e_{p^k}.$$

Fix $n \geq 1$, and compute the coefficient $[n]$ in three cases.

If $n = 1$, then 1 is not of the form p^k with $k \geq 1$, so

$$[n]\big(\partial_{\log} C_{\mathrm{comp}}(\mathbf{1})\big) = 0.$$

By Definition 3.5,

$$\Lambda^{\mathrm{ex}}(1) = 0.$$

If $n = p^k$ is a prime power with $k \geq 1$, then the only contributing term is the one corresponding to the pair (p, k). Therefore

$$[n]\big(\partial_{\log} C_{\mathrm{comp}}(\mathbf{1})\big) = \frac{\log(p^k)}{k} = \frac{k \log p}{k} = \log p.$$

Again by Definition 3.5,

$$\Lambda^{\mathrm{ex}}(p^k) = \log p.$$

If n is not a prime power, then no basis atom e_{p^k} appearing in the above sum is supported at n. Therefore

$$[n]\big(\partial_{\log} C_{\mathrm{comp}}(\mathbf{1})\big) = 0.$$

By Definition 3.5,

$$\Lambda^{\mathrm{ex}}(n) = 0.$$

Thus, in all cases,

$$[n]\big(\partial_{\log} C_{\mathrm{comp}}(\mathbf{1})\big) = \Lambda^{\mathrm{ex}}(n).$$

Since this holds for every $n \geq 1$, we obtain

$$\partial_{\log} C_{\mathrm{comp}}(\mathbf{1}) = \Lambda^{\mathrm{ex}}.$$

□

Corollary 3.17 (Prime-power output under Dirichlet-logarithmic linearization)**.**

$$\mathrm{supp}\big(\partial_{\log} C_{\mathrm{comp}}(\mathbf{1})\big) = \{\textit{prime powers}\}.$$

Equivalently, every integer that is not a prime power carries zero output:

$$[n]\big(\partial_{\log} C_{\mathrm{comp}}(\mathbf{1})\big) = 0 \qquad \textit{whenever } n \textit{ is not a prime power}.$$

Accordingly, the Dirichlet-logarithmic linearization means that the general composite-number overlap appearing in the multiplicative convolution product is converted into a prime-power-supported coefficient sum, and the surviving output is supported only on prime powers.

Proof. By Theorem 3.16,

$$\partial_{\log} C_{\mathrm{comp}}(\mathbf{1}) = \Lambda^{\mathrm{ex}}.$$

Therefore

$$\mathrm{supp}\big(\partial_{\log} C_{\mathrm{comp}}(\mathbf{1})\big) = \mathrm{supp}(\Lambda^{\mathrm{ex}}).$$

By Proposition 3.7,

$$\mathrm{supp}(\Lambda^{\mathrm{ex}}) = \{\text{prime powers}\}.$$

Hence

$$\mathrm{supp}\big(\partial_{\log} C_{\mathrm{comp}}(\mathbf{1})\big) = \{\text{prime powers}\}.$$

The coefficientwise restatement follows immediately. □

Remark 3.18 (End of the purely formal extraction layer)**.** The purely formal extraction layer ends here. At this point, we have not yet introduced Π_n, Π_{arith}, any diagonal arithmetic trace formula, any analytic resolvent construction, or any comparison interface. The objects obtained from §3.3–§3.4 are exactly

$$\Lambda^{\mathrm{ex}}, \qquad C_{\mathrm{comp}}, \qquad \partial_{\log}$$

and nothing else. These are precisely the only arithmetic data needed for the next construction step.

3.5. Coefficient Hilbert Space and Weighted Diagonal Arithmetic Trace

The purely arithmetic extraction layer of §3.1–§3.4 is now complete. Accordingly, we now pass to the Hilbertized coefficient layer. Using the Dirichlet basis atoms e_n as an orthonormal basis, we define the coefficient Hilbert space, its rank-one coordinate projection family $\{\Pi_n\}_{n\geq 1}$, and the weighted diagonal arithmetic trace operator $\Pi_{\mathrm{arith}}(\phi)$. No analytic Hilbert space, resolvent construction, or comparison interface enters here.

Definition 3.19 (Coefficient Hilbert space and coordinate projections)**.** Using the Dirichlet basis atoms e_n from Definition 3.9, define the coefficient Hilbert space by

$$\mathcal{H}_{\mathrm{arith}} := \left\{ u = \sum_{n\geq 1} u_n e_n : \sum_{n\geq 1} |u_n|^2 < \infty \right\}.$$

Its inner product is given by

$$\langle u, v\rangle_{\mathrm{arith}} := \sum_{n\geq 1} u_n \overline{v_n}.$$

For

$$u = \sum_{n\geq 1} u_n e_n \in \mathcal{H}_{\mathrm{arith}},$$

write the n-th coefficient with respect to the basis $\{e_n\}_{n\geq 1}$ as

$$[n]u := u_n.$$

For each $n \geq 1$, define the coordinate projection

$$\Pi_n : \mathcal{H}_{\mathrm{arith}} \to \mathcal{H}_{\mathrm{arith}}$$

by

$$\Pi_n u := [n]u\, e_n \qquad (u \in \mathcal{H}_{\mathrm{arith}}).$$

Thus Π_n is the rank-one orthogonal projection onto the n-th coefficient line $\mathbb{C}e_n$ inside the coefficient Hilbert space.

Lemma 3.20 (Coordinate-projection laws on the coefficient Hilbert space)**.** The family $\{\Pi_n\}_{n\geq 1}$ satisfies

$$\Pi_n^2 = \Pi_n, \qquad \Pi_n^* = \Pi_n \qquad (n \geq 1),$$

and

$$\Pi_m \Pi_n = 0 \qquad (m \neq n).$$

Furthermore, for every $u \in \mathcal{H}_{\text{arith}}$,

$$u = \sum_{n \geq 1} \Pi_n u$$

converges in $\mathcal{H}_{\text{arith}}$. Equivalently,

$$\sum_{n \geq 1} \Pi_n = I$$

holds on $\mathcal{H}_{\text{arith}}$ in the strong operator topology.

Proof. By the definition of $\mathcal{H}_{\text{arith}}$, $\{e_n\}_{n \geq 1}$ is the standard orthonormal basis of the ℓ^2-type coefficient space. In particular,

$$\langle e_m, e_n \rangle_{\text{arith}} = \delta_{m,n}.$$

Let

$$u = \sum_{k \geq 1} u_k e_k \in \mathcal{H}_{\text{arith}}.$$

Then

$$\Pi_n u = u_n e_n.$$

Applying Π_n once more gives

$$\Pi_n^2 u = \Pi_n(u_n e_n) = [n](u_n e_n)\, e_n = u_n e_n = \Pi_n u.$$

Since this holds for all u,

$$\Pi_n^2 = \Pi_n.$$

Next let

$$u = \sum_{k \geq 1} u_k e_k, \qquad v = \sum_{k \geq 1} v_k e_k$$

be elements of $\mathcal{H}_{\text{arith}}$. Then

$$\langle \Pi_n u, v \rangle_{\text{arith}} = \langle u_n e_n, v \rangle_{\text{arith}} = u_n \overline{v_n}.$$

On the other hand,

$$\langle u, \Pi_n v \rangle_{\text{arith}} = \langle u, v_n e_n \rangle_{\text{arith}} = u_n \overline{v_n}.$$

Therefore

$$\langle \Pi_n u, v \rangle_{\text{arith}} = \langle u, \Pi_n v \rangle_{\text{arith}} \qquad (u, v \in \mathcal{H}_{\text{arith}}),$$

and hence

$$\Pi_n^* = \Pi_n.$$

Now let $m \neq n$. For any $u \in \mathcal{H}_{\text{arith}}$,

$$\Pi_m \Pi_n u = \Pi_m(u_n e_n) = [n]u\, [m]e_n\, e_m = u_n\, \delta_{m,n}\, e_m = 0.$$

Therefore

$$\Pi_m \Pi_n = 0 \qquad (m \neq n).$$

It remains to prove the strong decomposition of the identity. For $N \geq 1$, define the partial-sum operator

$$S_N := \sum_{n=1}^{N} \Pi_n.$$

Then, for

$$u = \sum_{n \geq 1} u_n e_n \in \mathcal{H}_{\text{arith}},$$

we have

$$S_N u = \sum_{n=1}^{N} u_n e_n.$$

Therefore

$$u - S_N u = \sum_{n>N} u_n e_n,$$

and since $\{e_n\}_{n\geq 1}$ is orthonormal,

$$\|u - S_N u\|_{\text{arith}}^2 = \sum_{n>N} |u_n|^2.$$

Because $u \in \mathcal{H}_{\text{arith}}$, the series

$$\sum_{n \geq 1} |u_n|^2$$

converges, and hence its tail converges to 0. Thus

$$\|u - S_N u\|_{\text{arith}} \longrightarrow 0 \qquad (N \to \infty).$$

Accordingly,

$$u = \sum_{n \geq 1} \Pi_n u$$

holds in $\mathcal{H}_{\text{arith}}$, equivalently

$$S_N \to I$$

strongly. This proves that

$$\sum_{n \geq 1} \Pi_n = I$$

holds in the strong operator topology. □

Remark 3.21 (Coefficient Hilbert space versus analytic Hilbert space)**.** The family $\{\Pi_n\}_{n\geq 1}$ is defined, on its natural coefficient Hilbert space $\mathcal{H}_{\text{arith}}$. It is a coordinate projection family on the Dirichlet-basis Hilbert space, not a projection family on the analytic Hilbert space of Section 2.

Definition 3.22 (Weighted diagonal arithmetic trace operator)**.** Let

$$\phi : \mathbb{N} \to \mathbb{C}$$

be a finitely supported weight. Define the weighted diagonal arithmetic trace operator by

$$\Pi_{\text{arith}}(\phi) := \sum_{n \geq 1} \phi(n) \Pi_n.$$

This is a finite-rank diagonal operator on $\mathcal{H}_{\text{arith}}$. Equivalently, for

$$u = \sum_{n\geq 1} u_n e_n \in \mathcal{H}_{\text{arith}},$$

we have

$$\Pi_{\text{arith}}(\phi)u = \sum_{n\geq 1} \phi(n) u_n e_n.$$

In the theorem below, this definition is extended from finitely supported weights to all $\phi \in \ell^1(\mathbb{N})$ by trace-norm completion.

Theorem 3.23 (Weighted diagonal arithmetic trace formula)**.** *The following hold.*

(1) *Let $\phi \in \ell^1(\mathbb{N})$. For $N \geq 1$, define the finite-rank partial sum*

$$T_N(\phi) := \sum_{n\leq N} \phi(n)\Pi_n.$$

Then $\{T_N(\phi)\}_{N\geq 1}$ is Cauchy with respect to the trace norm $\|\cdot\|_{\mathfrak{S}_1}$, and hence converges to a trace-class operator on $\mathcal{H}_{\text{arith}}$. Write its limit as

$$\Pi_{\text{arith}}(\phi) := \lim_{N\to\infty} T_N(\phi).$$

Furthermore,

$$\operatorname{Tr} \Pi_{\text{arith}}(\phi) = \sum_{n\geq 1} \phi(n)$$

holds.

(2) *Let $\varphi : \mathbb{N} \to \mathbb{C}$ be finitely supported, and understand pointwise products coefficientwise:*

$$(\varphi\,\Lambda^{\text{ex}})(n) := \varphi(n)\Lambda^{\text{ex}}(n), \qquad (\varphi\,\partial_{\log} C_{\text{comp}}(\mathbf{1}))(n) := \varphi(n)\,\big(\partial_{\log} C_{\text{comp}}(\mathbf{1})\big)(n).$$

Then

$$\operatorname{Tr} \Pi_{\text{arith}}(\varphi\,\Lambda^{\text{ex}}) = \sum_{n\geq 1} \varphi(n)\Lambda^{\text{ex}}(n) = \sum_{p^k} \varphi(p^k)\log p.$$

Equivalently, using Theorem 3.16,

$$\operatorname{Tr} \Pi_{\text{arith}}\big(\varphi\,\partial_{\log} C_{\text{comp}}(\mathbf{1})\big) = \sum_{p^k} \varphi(p^k)\log p.$$

Proof. We prove the two parts in order.

Step 1: Finite-rank partial sums and their traces. Fix $N \geq 1$. Since $T_N(\phi)$ is a finite sum of rank-one operators, it has finite rank. On the basis $\{e_n\}_{n\geq 1}$,

$$T_N(\phi)e_k = \sum_{n\leq N} \phi(n)\Pi_n e_k = \sum_{n\leq N} \phi(n)\delta_{n,k} e_n = \begin{cases} \phi(k)e_k, & k \leq N, \\ 0, & k > N. \end{cases}$$

Thus $T_N(\phi)$ is diagonal with respect to the arithmetic basis, and its diagonal entries are $\phi(1), \ldots, \phi(N), 0, 0, \ldots$. Hence

$$\operatorname{Tr} T_N(\phi) = \sum_{k\geq 1} \langle T_N(\phi)e_k, e_k\rangle_{\text{arith}} = \sum_{k\leq N} \phi(k). \tag{3.5.1}$$

Step 2: Trace-norm Cauchy property for $\phi \in \ell^1$. Let $M > N$. Then

$$T_M(\phi) - T_N(\phi) = \sum_{N<n\le M} \phi(n)\Pi_n.$$

On basis vectors,

$$(T_M(\phi) - T_N(\phi))e_k = \begin{cases} \phi(k)e_k, & N < k \le M, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore $T_M(\phi) - T_N(\phi)$ is a finite-rank diagonal operator, and its singular values are exactly

$$\{\, |\phi(n)| : \; N < n \le M \,\}.$$

Equivalently,

$$|T_M(\phi) - T_N(\phi)|e_k = \begin{cases} |\phi(k)|e_k, & N < k \le M, \\ 0, & \text{otherwise.} \end{cases}$$

Thus

$$\big\|T_M(\phi) - T_N(\phi)\big\|_{\mathfrak{S}_1} = \operatorname{Tr}\big|T_M(\phi) - T_N(\phi)\big| = \sum_{N<n\le M} |\phi(n)|. \tag{3.5.2}$$

Since $\phi \in \ell^1(\mathbb{N})$, the right-hand side tends to 0 as $M, N \to \infty$. Hence $\{T_N(\phi)\}_{N\ge 1}$ is Cauchy in $\mathfrak{S}_1(\mathcal{H}_{\text{arith}})$. The trace-class space is complete, so there exists a unique trace-class limit, which we write as

$$\Pi_{\text{arith}}(\phi) := \lim_{N\to\infty} T_N(\phi).$$

Furthermore, the trace is continuous with respect to the trace norm, and therefore

$$\operatorname{Tr} \Pi_{\text{arith}}(\phi) = \lim_{N\to\infty} \operatorname{Tr} T_N(\phi).$$

Using (3.5.1), we obtain

$$\operatorname{Tr} \Pi_{\text{arith}}(\phi) = \lim_{N\to\infty} \sum_{n\le N} \phi(n) = \sum_{n\ge 1} \phi(n),$$

because the series is absolutely convergent. This proves (1).

Step 3: Specialization to the prime-power weighted trace. Now let φ be finitely supported. Then the pointwise product $\varphi\,\Lambda^{\text{ex}}$ is also finitely supported, and hence belongs to $\ell^1(\mathbb{N})$. Applying (1) to

$$\phi = \varphi\,\Lambda^{\text{ex}},$$

we get

$$\operatorname{Tr} \Pi_{\text{arith}}(\varphi\,\Lambda^{\text{ex}}) = \sum_{n\ge 1} \varphi(n)\Lambda^{\text{ex}}(n). \tag{3.5.3}$$

By Definition 3.5,

$$\Lambda^{\text{ex}}(n) = 0$$

when n is not a prime power, and

$$\Lambda^{\text{ex}}(p^k) = \log p \qquad (k \ge 1).$$

Therefore the sum in (3.5.3) reduces to

$$\sum_{n\geq 1}\varphi(n)\Lambda^{\text{ex}}(n)=\sum_{p^k}\varphi(p^k)\log p.$$

This proves

$$\operatorname{Tr}\Pi_{\text{arith}}(\varphi\,\Lambda^{\text{ex}})=\sum_{p^k}\varphi(p^k)\log p.$$

Finally, Theorem 3.16 states that

$$\partial_{\log}C_{\text{comp}}(\mathbf{1})=\Lambda^{\text{ex}}.$$

Multiplying both sides pointwise by φ, we obtain

$$\varphi\,\partial_{\log}C_{\text{comp}}(\mathbf{1})=\varphi\,\Lambda^{\text{ex}}.$$

Therefore, applying Π_{arith} and then taking the trace gives

$$\operatorname{Tr}\Pi_{\text{arith}}\big(\varphi\,\partial_{\log}C_{\text{comp}}(\mathbf{1})\big)=\operatorname{Tr}\Pi_{\text{arith}}(\varphi\,\Lambda^{\text{ex}})=\sum_{p^k}\varphi(p^k)\log p.$$

This proves (2). □

Corollary 3.24 (Prime-power trace specialization). *Let $\varphi:\mathbb{N}\to\mathbb{C}$ be finitely supported. Then*

$$\operatorname{supp}(\varphi\,\Lambda^{\text{ex}})\subset\{\textit{prime powers}\}.$$

Therefore

$$\operatorname{Tr}\Pi_{\text{arith}}(\varphi\,\Lambda^{\text{ex}})=\sum_{n\in\operatorname{supp}(\varphi\,\Lambda^{\text{ex}})}\varphi(n)\Lambda^{\text{ex}}(n)=\sum_{p^k}\varphi(p^k)\log p.$$

In particular, the weighted diagonal arithmetic trace here sees only the prime-power support and is not specialized to the prime support of $1_{\mathbb{P}}^{\text{ex}}$.

Proof. For any arithmetic functions a,b,

$$\operatorname{supp}(ab)\subset\operatorname{supp}(a)\cap\operatorname{supp}(b)$$

holds. Indeed, $(ab)(n)=a(n)b(n)$ can be nonzero only where both factors are nonzero. Applying this to

$$a=\varphi,\qquad b=\Lambda^{\text{ex}},$$

we obtain

$$\operatorname{supp}(\varphi\,\Lambda^{\text{ex}})\subset\operatorname{supp}(\Lambda^{\text{ex}}).$$

By Proposition 3.7,

$$\operatorname{supp}(\Lambda^{\text{ex}})=\{\text{prime powers}\}.$$

Therefore

$$\operatorname{supp}(\varphi\,\Lambda^{\text{ex}})\subset\{\text{prime powers}\}.$$

The trace identity follows immediately from Theorem 3.23:

$$\operatorname{Tr}\Pi_{\mathrm{arith}}(\varphi\,\Lambda^{\mathrm{ex}}) = \sum_{n\geq 1}\varphi(n)\Lambda^{\mathrm{ex}}(n) = \sum_{p^k}\varphi(p^k)\log p.$$

The final sentence merely states the difference of supports already established in Definition 3.5 and Proposition 3.7. Namely, 1_P^{ex} is supported on primes, whereas Λ^{ex} is supported on prime powers. ☐

3.6. Summary and Transition to Singular Boundary Data

Remark 3.25 (Summary of the arithmetic section)**.** The coefficient-space arithmetic construction ends here. The arithmetic data constructed in this section are

$$\Lambda^{\mathrm{ex}}, \qquad C_{\mathrm{comp}}, \qquad \partial_{\log}, \qquad \{\Pi_n\}_{n\geq 1}, \qquad \Pi_{\mathrm{arith}}(\cdot).$$

At this point, the coefficient-extraction layer, coefficient Hilbert space, coordinate projection family, and weighted diagonal arithmetic trace operator have all been fixed on their natural defining space. These arithmetic data are combined with the singular-boundary data only in the orthogonal-decomposition framework of Section 5. Section 4 does not use these arithmetic objects; it independently constructs the singular-boundary input inside the analytic Hilbert-space setup of Section 2.

4. Operator-Theoretic Construction of Singular Boundary Data

4.0. Purpose and Logical Role of This Section

The purpose of this section is to construct, inside the analytic Hilbert-space setup of Section 2, the operator-theoretic boundary data used in the subsequent orthogonal-decomposition framework, rather than to add them as an external assumption. Specifically, in this section we construct the singular boundary data consisting of

$$(\Sigma_R, \sigma_R, \Gamma_R, T_R, \mathrm{supp}_R, L_R),$$

and then successively define the associated trace-vanishing generating subspace, singular-boundary closed-form subspace, singular-boundary Hilbert space, orthogonal projection onto K_R, transport generator, and distribution kernel.

For the purposes of the later comparison with the completed zeta function, this section also records the boundary origin that will be used after the internal construction has been completed. With the centered Mellin variable

$$w = s - \frac{1}{2}, \qquad \xi_c(w) := \xi\!\left(\frac{1}{2} + w\right),$$

the functional equation is

$$\xi_c(w) = \xi_c(-w).$$

The reflection $w \mapsto -w$ identifies the two Mellin boundary sides of a finite-window contour. The regular part of this reflected boundary contribution is absorbed into the common reference term, whereas the finite part left after removal of the common Archimedean/reference contribution and the arithmetic prime-power contribution is represented as a singular boundary distribution.

The construction below is therefore used in two layers. First, Sections 4.1–4.9 build a universal zero-area carrier and the associated Hilbert-space boundary data internally. Second, Section 4.10 fixes the natural realization of the centered Mellin boundary finite-part class of ξ on that carrier. In particular, the smooth seam carrier introduced below is not a zeta-zero measure and does not encode prime weights. It is a standard σ_R-null support, smoothly coded from the centered Mellin boundary seam, on which the finite-window boundary distributions coming from the centered Mellin contour can be realized without being annihilated by the regular area measure.

This section is not a section that develops a zero-counting theory for closing the Riemann hypothesis. Nor is it a section that proves the orthogonality of the arithmetic construction and the boundary-data construction. Its role is exhausted by preparing, without type conflation and in an analytically closed form, the singular-boundary objects needed when constructing the orthogonal-decomposition framework in the next section.

Definition 4.1 (Operator-theoretic boundary data constructed in this section)**.** Write the basic data constructed in this section as

$$\mathfrak{R}_{\mathrm{bd}} := (\Sigma_R, \sigma_R, \Gamma_R, T_R, \mathrm{supp}_R, L_R).$$

Here each object is constructed with the following meaning.

1. Σ_R is the boundary parameter space used to parametrize the singular boundary structure.
2. σ_R is a regular area-type measure on Σ_R.
3. $\Gamma_R \subset \Sigma_R$ is the σ_R-null singular support, constructed as a closed set satisfying

$$\sigma_R(\Gamma_R) = 0.$$

4. T_R is a linear subspace inside the form domain on which the singular/regular boundary trace is defined.
5. supp_R is the map that assigns to an element of T_R the support of its singular boundary trace.
6. $L_R(f; E)$ is a nonnegative functional measuring the regular boundary trace mass over E, for $f \in T_R$ and a measurable set $E \subset \Sigma_R$.

Remark 4.2 (Separation of σ_R-null property and nontriviality)**.** The condition $\sigma_R(\Gamma_R) = 0$ means that Γ_R is null with respect to the regular area measure. Therefore, if the boundary trace on Γ_R were treated merely as an ordinary function in $L^2(\Sigma_R, \sigma_R)$, that component would become trivial. To avoid this conflation, this section distinguishes the regular area measure σ_R from the singular measure ν_R supported on Γ_R. Namely, the σ_R-null property is described on the σ_R-side, whereas the nontriviality of the boundary trace is retained on the side of the singular trace or distributional boundary distribution introduced later.

Definition 4.3 (Analytic output of this section)**.** Write the final analytic output of this section as

$$\begin{aligned}\mathfrak{R}_{\mathrm{op}} := (&\Sigma_R, \sigma_R, \Gamma_R, T_R, \mathrm{supp}_R, L_R, G_R, Q_R^{\mathrm{res}}, H_R^{\mathrm{res}}, K_R, \Pi_R^+,\\ &U_R(t), B_R, \mathsf{K}_R^{\mathrm{dist}}, L_R^{\mathrm{res}}).\end{aligned}$$

Here,

$$G_R$$

is the generating class satisfying the σ_R-null support and regular-trace vanishing conditions,

$$Q_R^{\mathrm{res}}$$

is its form-norm closure,

$$H_R^{\mathrm{res}}$$

is the corresponding singular-boundary Hilbert space,

$$K_R$$

is the closed singular-boundary subspace on the one-sided analytic Hilbert space, and

$$\Pi_R^+$$

is the one-sided orthogonal projection from $H_{\alpha,+}$ onto that closed subspace. Moreover,

$$U_R(t)$$

is the strongly continuous transport group on the singular-boundary subspace,

$$B_R$$

is its anti-self-adjoint generator,

$$\mathsf{K}_R^{\mathrm{dist}}$$

is the associated transport kernel, formulated as a distribution kernel, and

$$L_R^{\mathrm{res}}$$

denotes the positive shifted operator obtained from the closed quadratic form.

Convention 4.4 (Objects not used in this section)**.** In this section, we do not use the following objects or propositions.

1. The arithmetic projector family $\{\Pi_n\}_{n\geq 1}$.
2. The weighted diagonal arithmetic trace.
3. The von Mangoldt function Λ^{ex} in the formal Dirichlet algebra.
4. The zero-counting function.
5. The finite-window comparison theorem.
6. The cumulative defect sequence.
7. The exclusion of minimal-index obstruction.
8. Quoted finite-height verification for the low-height band.

Accordingly, the construction in this section depends neither on the arithmetic construction of Section 3 nor on the subsequent final deduction chain.

Remark 4.5 (Logical position of this section)**.** Section 2 constructed the weighted Hilbert space, the closed quadratic form, the self-adjoint realization, compact resolvent, and the Herglotz-type resolvent function. This section does not reconstruct them. The role of this section is to cut out, on the analytic Hilbert-space setup of Section 2, the singular-boundary subspace satisfying the σ_R-null support condition and regular-trace vanishing condition, and to construct the projector and transport structure associated with that subspace.

Definition 4.6 (Transition principle)**.** The output $\mathfrak{R}_{\mathrm{op}}$ of this section is the only singular-boundary input for the orthogonal-decomposition framework in the next section. Namely, in the next section, after fixing

$$\mathfrak{R}_{\mathrm{op}},$$

we place the analytic singular-boundary subspace and the arithmetic subspace on a common ambient Hilbert space. The arguments from the next section onward refer only to the data constructed internally in this section as the singular-boundary input, and introduce no additional boundary-data assumption outside this section. This transfer rule is called the transition principle in this paper. In particular, the subsequent comparison argument does not require the unregularized generator kernel B_R^{dist} to exist on the full test space $\mathscr{D}_R$. Whenever a distribution-level generator kernel is needed without an additional domain assumption, the regularized kernels $B_R^{(\varepsilon),\mathrm{dist}}$ are used; the limit $\varepsilon \downarrow 0$ is taken only on the generator-admissible core or inside already justified pairings.

Proposition 4.7 (Transition from this section to the next section)**.** *Once each construction in this section is complete, the next section may use the following objects as already constructed:*

$$\Sigma_R, \quad \sigma_R, \quad \Gamma_R, \quad T_R, \quad \mathrm{supp}_R, \quad L_R, \quad G_R, \quad Q_R^{\mathrm{res}}, \quad H_R^{\mathrm{res}}, \quad K_R, \quad \Pi_R^+.$$

In particular, the one-sided projector Π_R^+ used as singular-boundary input in the next section is not arbitrarily assumed; it is the orthogonal projection from $H_{\alpha,+}$ onto the closed singular-boundary subspace constructed in this section. The ambient projector Π_R is defined only after this input is embedded into the ambient Hilbert space in Section 5.

Proof. In the first half of this section, we construct the boundary parameter space, regular area measure, σ_R-null singular support, singular boundary trace, support map, and regular boundary trace-mass functional. This fixes

$$(\Sigma_R, \sigma_R, \Gamma_R, T_R, \mathrm{supp}_R, L_R).$$

Next, from these data, we define the trace-vanishing generating subspace associated with Γ_R, denoted by G_R, and obtain

$$Q_R^{\mathrm{res}}$$

as its closure with respect to the form norm of the closed quadratic form constructed in Section 2. As its $H_{\alpha,+}$-closure, we construct

$$H_R^{\mathrm{res}},$$

and obtain the corresponding self-adjoint operator by the representation theorem for closed forms.

Furthermore, we construct the strongly continuous transport group on the singular-boundary subspace and its anti-self-adjoint generator, and formulate the associated transport kernel as a distribution kernel. Finally, we obtain the closed one-sided singular-boundary subspace

$$K_R \subset H_{\alpha,+}.$$

Since the orthogonal projection onto a closed subspace exists uniquely by the projection theorem in Hilbert spaces, the one-sided projector

$$\Pi_R^+ : H_{\alpha,+} \to K_R$$

is fixed. The lift of these objects to the ambient Hilbert space is carried out in Section 5.

Accordingly, all the above objects are constructed inside this section, and the next section can receive $\mathfrak{R}_{\mathrm{op}}$ as already constructed data. □

4.1. *Basic Spaces, Gelfand Triple, and Type Separation*

In this subsection, we fix the object level for constructing σ_R-null singular boundary data. In the subsequent argument, we handle point evaluations, boundary traces, singular measures, distribution kernels, quadratic forms, and operators simultaneously. If all of these are treated as elements of the same Hilbert space, then one obtains the unboundedness of point evaluations, a conflation of distribution kernels and bounded operators, and a conflation of quadratic forms and generators. Accordingly, in this subsection, we place at the center the one-sided analytic Hilbert space of Section 2,

$$H_{\alpha,+} = L^2((0,\infty), d\mu_\alpha), \qquad d\mu_\alpha(x) = \langle x\rangle^{2\alpha} dx, \qquad \alpha > \frac{1}{2},$$

and introduce a Gelfand triple by placing a test space and a distribution space around it. The standard background on rigged Hilbert spaces, nuclear spaces, and the distribution kernel theorem follows [1,2].

Definition 4.8 (Boundary test space)**.** Let $\langle x\rangle = (1+x^2)^{1/2}$. Define the boundary test space $\mathscr{D}_R$ by

$$\mathscr{D}_R := \left\{\varphi \in C^\infty([0,\infty)) : p_{m,k}(\varphi) < \infty \text{ for all } m,k \in \mathbb{N}_0\right\},$$

where

$$p_{m,k}(\varphi) := \sup_{x\geq 0} \langle x\rangle^m \big|\partial_x^k \varphi(x)\big|.$$

Equip $\mathscr{D}_R$ with the Fréchet topology determined by the seminorm family

$$\{p_{m,k}\}_{m,k\in\mathbb{N}_0}.$$

Lemma 4.9 (Continuous dense embedding of the test space into the Hilbert space)**.** The natural inclusion map

$$\iota_R : \mathscr{D}_R \hookrightarrow H_{\alpha,+}$$

is continuous, and its image is dense in $H_{\alpha,+}$.

Proof. We first prove continuity. For any $\varphi \in \mathscr{D}_R$, fix one $m > \alpha + \frac{1}{2}$. Then

$$|\varphi(x)| \leq p_{m,0}(\varphi)\langle x\rangle^{-m}.$$

Therefore

$$\begin{aligned}\|\varphi\|^2_{H_{\alpha,+}} &= \int_0^\infty |\varphi(x)|^2 \langle x\rangle^{2\alpha}\, dx \\ &\leq p_{m,0}(\varphi)^2 \int_0^\infty \langle x\rangle^{2\alpha-2m}\, dx.\end{aligned}$$

Since $m > \alpha + \frac{1}{2}$, the integral on the right-hand side is finite. Thus

$$\|\varphi\|_{H_{\alpha,+}} \leq C_{\alpha,m}\, p_{m,0}(\varphi),$$

so the inclusion map is continuous.

Next, we prove density. We have $C_c^\infty(0,\infty) \subset \mathscr{D}_R$, and $C_c^\infty(0,\infty)$ is dense in the weighted space $L^2((0,\infty), \langle x\rangle^{2\alpha} dx)$. Indeed, for any $f \in H_{\alpha,+}$, first approximate f in the $H_{\alpha,+}$-norm by

$$f_N := \mathbf{1}_{[1/N,N]} f,$$

and then use standard smoothing and cutoff on the finite interval $[1/N, N]$ to approximate f_N to arbitrary precision by elements of $C_c^\infty(0,\infty)$. Therefore the image of $\mathscr{D}_R$ is dense in $H_{\alpha,+}$. $\square$

Definition 4.10 (Anti-dual and Gelfand triple)**.** Let $\mathscr{D}_R'$ be the space of all continuous antilinear functionals on $\mathscr{D}_R$, that is, the anti-dual. Write the duality pairing as

$$\langle F, \varphi \rangle_{\mathscr{D}_R', \mathscr{D}_R}, \qquad F \in \mathscr{D}_R',\ \varphi \in \mathscr{D}_R.$$

The Hilbert space $H_{\alpha,+}$ is embedded anti-linearly continuously into $\mathscr{D}_R'$ by

$$h \longmapsto \iota_H h,$$

where

$$\langle \iota_H h, \varphi \rangle_{\mathscr{D}_R', \mathscr{D}_R} := \langle h, \varphi \rangle_{H_{\alpha,+}} \qquad (\varphi \in \mathscr{D}_R).$$

Then

$$\mathscr{D}_R \subset H_{\alpha,+} \subset \mathscr{D}_R'$$

is called the Gelfand triple on the singular-boundary side in this section.

Lemma 4.11 (Continuity of the Gelfand triple)**.** The embeddings

$$\mathscr{D}_R \subset H_{\alpha,+} \subset \mathscr{D}_R'$$

are all continuous, and $\mathscr{D}_R$ is dense in $H_{\alpha,+}$.

Proof. The continuity and density of the first embedding $\mathscr{D}_R \subset H_{\alpha,+}$ were proved in Lemma 4.9.

Next we prove the continuity of $H_{\alpha,+} \subset \mathscr{D}_R'$. For any $h \in H_{\alpha,+}$ and $\varphi \in \mathscr{D}_R$, the Cauchy–Schwarz inequality gives

$$\left|\langle \iota_H h, \varphi \rangle_{\mathscr{D}_R', \mathscr{D}_R}\right| = |\langle h, \varphi \rangle_{H_{\alpha,+}}| \le \|h\|_{H_{\alpha,+}} \|\varphi\|_{H_{\alpha,+}}.$$

By Lemma 4.9,

$$\|\varphi\|_{H_{\alpha,+}} \le C_{\alpha,m} p_{m,0}(\varphi),$$

and therefore

$$\left|\langle \iota_H h, \varphi \rangle_{\mathscr{D}_R', \mathscr{D}_R}\right| \le C_{\alpha,m} \|h\|_{H_{\alpha,+}} p_{m,0}(\varphi).$$

Thus $\iota_H h$ is a continuous antilinear functional on $\mathscr{D}_R$, and the map $h \mapsto \iota_H h$ is continuous. $\square$

Proposition 4.12 (Point evaluation is a distribution and not a bounded functional on the Hilbert space)**.** *For any $a \in [0,\infty)$,*

$$\delta_a(\varphi) := \varphi(a), \qquad \varphi \in \mathscr{D}_R,$$

is an element of $\mathscr{D}_R'$. On the other hand, δ_a is not in general defined as a bounded functional on $H_{\alpha,+}$. Accordingly, point evaluations and boundary point evaluations are treated not as elements of $H_{\alpha,+}$, but as elements of $\mathscr{D}_R'$.

Proof. We first prove $\delta_a \in \mathscr{D}_R'$. For any $a \in [0,\infty)$,

$$|\delta_a(\varphi)| = |\varphi(a)| \le p_{0,0}(\varphi).$$

Therefore δ_a is a continuous linear functional on $\mathscr{D}_R$, and is regarded as an element of $\mathscr{D}'_R$ according to the complex-conjugation convention.

Next, we show that δ_a is not a bounded functional on $H_{\alpha,+}$. For $a > 0$, take $\eta \in C_c^\infty((-1,1))$ with $\eta(0) = 1$, and for sufficiently large n, set

$$\varphi_n(x) := \eta(n(x-a)).$$

Then $\varphi_n \in C_c^\infty(0,\infty) \subset \mathscr{D}_R$, and

$$\delta_a(\varphi_n) = \varphi_n(a) = 1.$$

On the other hand, the support is contained in an interval of length $O(n^{-1})$ near a, and $\langle x\rangle^{2\alpha}$ is bounded above and below on that interval, so

$$\|\varphi_n\|^2_{H_{\alpha,+}} = \int_0^\infty |\eta(n(x-a))|^2 \langle x\rangle^{2\alpha} dx \le C_a n^{-1}.$$

Therefore $\|\varphi_n\|_{H_{\alpha,+}} \to 0$.

If δ_a were a bounded functional on $H_{\alpha,+}$, there would exist a constant $C > 0$ such that

$$1 = |\delta_a(\varphi_n)| \le C\|\varphi_n\|_{H_{\alpha,+}}$$

for all n. But the right-hand side converges to 0, a contradiction.

The case $a = 0$ is similar. Take $\eta \in C_c^\infty([0,1))$ with $\eta(0) = 1$, and set

$$\varphi_n(x) := \eta(nx).$$

Then

$$\varphi_n(0) = 1, \qquad \|\varphi_n\|^2_{H_{\alpha,+}} \le Cn^{-1}.$$

Therefore boundary point evaluation is also not a bounded functional on $H_{\alpha,+}$.

This proves that point evaluation is meaningful as an element of $\mathscr{D}'_R$, but cannot be treated as a bounded functional on $H_{\alpha,+}$. $\square$

Definition 4.13 (Distribution kernel)**.** When the distribution kernel associated with K_R is treated as a distribution kernel, its primary type is defined to be

$$K \in (\mathscr{D}_R \widehat{\otimes}_\pi \mathscr{D}_R)'.$$

Here $\widehat{\otimes}_\pi$ denotes the completed projective tensor product. Namely, K is a continuous bilinear functional assigning

$$\langle K, \varphi \otimes \psi\rangle$$

to a simple tensor

$$\varphi \otimes \psi \in \mathscr{D}_R \widehat{\otimes}_\pi \mathscr{D}_R.$$

Since $\mathscr{D}_R$ is a nuclear space, as will be verified later, this space may be identified, when necessary, with

$$(\mathscr{D}_R \widehat{\otimes}_\pi \mathscr{D}_R)' \simeq \mathscr{D}'_R \widehat{\otimes} \mathscr{D}'_R.$$

However, this identification is a type identification through nuclearity, and does not automatically make a distribution kernel into a bounded operator on $H_{\alpha,+}$.

Lemma 4.14 (Scope of the kernel theorem)**.** $\mathscr{D}_R$ is a nuclear Fréchet space. Therefore, for any continuous bilinear form

$$B : \mathscr{D}_R \times \mathscr{D}_R \longrightarrow \mathbb{C},$$

there exists a unique distribution kernel

$$K_B \in (\mathscr{D}_R \widehat{\otimes}_\pi \mathscr{D}_R)'$$

such that

$$B(\varphi, \psi) = \langle K_B, \varphi \otimes \psi \rangle.$$

Moreover, by nuclearity, when necessary one may identify

$$K_B \in \mathscr{D}'_R \widehat{\otimes} \mathscr{D}'_R.$$

Likewise, any continuous linear map

$$T : \mathscr{D}_R \longrightarrow \mathscr{D}'_R$$

is represented by a distribution kernel.

Proof. $\mathscr{D}_R$ is a Schwartz-type space on the half-line and is a nuclear Fréchet space obtained as a restriction space of the Schwartz space on the real line. Nuclearity is preserved under closed subspaces and quotient spaces, and hence $\mathscr{D}_R$ is also nuclear.

By the Schwartz kernel theorem for nuclear Fréchet spaces, continuous bilinear forms on $\mathscr{D}_R \times \mathscr{D}_R$ correspond uniquely to continuous linear functionals on $\mathscr{D}_R \widehat{\otimes}_\pi \mathscr{D}_R$. Therefore

$$K_B \in (\mathscr{D}_R \widehat{\otimes}_\pi \mathscr{D}_R)'$$

exists and satisfies

$$B(\varphi, \psi) = \langle K_B, \varphi \otimes \psi \rangle.$$

Furthermore, since $\mathscr{D}_R$ is nuclear, the standard kernel identification on the strong-dual side allows this kernel to be expressed as an element of $\mathscr{D}'_R \widehat{\otimes} \mathscr{D}'_R$. However, this last representation is a type representation of the distribution kernel, and by itself does not give an $H_{\alpha,+}$-bounded operator.

For a continuous map $T : \mathscr{D}_R \to \mathscr{D}'_R$, define

$$B_T(\varphi, \psi) := \langle T\varphi, \psi \rangle_{\mathscr{D}'_R, \mathscr{D}_R}.$$

This is a continuous bilinear form on $\mathscr{D}_R \times \mathscr{D}_R$. Accordingly, by the same kernel theorem, T is also represented by a distribution kernel. ☐

Remark 4.15 (Distinction between distribution kernels and bounded operators)**.** A distribution kernel

$$K \in \mathscr{D}'_R \widehat{\otimes} \mathscr{D}'_R$$

does not, by itself, define a bounded operator on $H_{\alpha,+}$. For a distribution kernel to define a bounded operator, one must separately prove $H_{\alpha,+}$-boundedness such as

$$\|T_K f\|_{H_{\alpha,+}} \le C\|f\|_{H_{\alpha,+}},$$

or its domain and closedness as a closed operator. Accordingly, in this paper we distinguish the distribution kernel K from the operator T_K obtained as its realization.

Definition 4.16 (Object level of quadratic forms)**.** When this section refers to a quadratic form, it means not an element of a Hilbert space or a distribution kernel, but the pair

$$(Q(q), q).$$

Here $Q(q) \subset H_{\alpha,+}$ is a dense linear subspace, and

$$q : Q(q) \times Q(q) \longrightarrow \mathbb{C}$$

is a sesquilinear form.

In particular, when referring to the closed quadratic form constructed in Section 2, we write its form domain as Q_R and the form itself as

$$q_R : Q_R \times Q_R \longrightarrow \mathbb{C}.$$

At this level, q_R is not an operator. Only after closedness, semiboundedness, and the representation theorem are established does one obtain the corresponding self-adjoint operator.

Definition 4.17 (Object level of generators)**.** When this section refers to a generator, it means a closed operator on a Hilbert space or on the subsequent ambient Hilbert-space setting,

$$A : \mathrm{Dom}(A) \subset \mathcal{H} \longrightarrow \mathcal{H}.$$

A generator must be specified together with its domain, closedness, symmetry or anti-self-adjointness, and the strongly continuous semigroup or group that it generates. Therefore, a form q, a distribution kernel K, or a boundary form b must not be identified with a generator without proof.

Definition 4.18 (Object level of boundary forms)**.** A boundary bilinear form is treated as a sesquilinear form on a space where the boundary trace is defined. Namely, when a linear space T and a boundary trace map γ are given, the boundary form is defined in the form

$$b : T \times T \longrightarrow \mathbb{C}$$

or

$$b[u, v] = \mathfrak{b}[\gamma u, \gamma v].$$

A boundary form is an object of a different type from the interior energy form q, and is not itself an operator on $H_{\alpha,+}$.

Definition 4.19 (Type-separation convention for object levels)**.** From this section onward, we distinguish the following types. Objects of different types are not identified until they are related through an explicitly stated realization map, closure operation, or representation theorem.

Symbol	Type	Ambient space/object	Allowed basic operations
Π_R^{cyc}	cycle	geometric or measure-theoretic cycle	support, pushforward, pullback
Π_R^+ or, after ambient embedding, Π_R	bounded projection	$\mathcal{B}(\mathcal{H})$, with the underlying Hilbert space specified	action, composition, adjoint, orthogonal projection
$\mathsf{K}_R^{\mathrm{dist}}$	kernel	$\mathscr{D}'_R \widehat{\otimes} \mathscr{D}'_R$	duality pairing with test functions
q_R^{raw}, q_R^{res}	quadratic form	$Q(q) \times Q(q) \to \mathbb{C}$	form norm, closure, representation theorem
A_R^{res}, B_R	generator	$\mathrm{Dom}(A) \subset \mathcal{H} \to \mathcal{H}$	closed operator, resolvent, semigroup/group generation
b_R	boundary form	$T_R \times T_R \to \mathbb{C}$	boundary trace evaluation, boundary cancellation

Convention 4.20 (Prohibition of notational conflation)**.** This paper prohibits the following.

1. Identifying the geometric cycle Π_R^{cyc} directly with a Hilbert-space projector such as Π_R^+ or Π_R.
2. Treating the distribution kernel $\mathsf{K}_R^{\mathrm{dist}}$ as an operator on $H_{\alpha,+}$ without a proof of boundedness or closed-operator status.
3. Identifying the quadratic form q_R with the self-adjoint operator A_R without passing through the representation theorem.
4. Identifying the boundary form b_R with the interior energy form q_R or the transport generator B_R.
5. Treating point evaluation δ_a or a boundary trace as a bounded functional on $H_{\alpha,+}$.

Whenever such an identification is needed, the corresponding realization map, boundedness, closedness, or representation theorem is explicitly stated and proved.

Proposition 4.21 (Basic consequences of this subsection)**.** *This subsection has fixed the following three foundational points.*

1. *Distributional objects on the singular-boundary side are handled inside the Gelfand triple*

$$\mathscr{D}_R \subset H_{\alpha,+} \subset \mathscr{D}'_R.$$

2. *Point evaluations and boundary point evaluations are not bounded functionals on $H_{\alpha,+}$, but are treated as elements of $\mathscr{D}'_R$.*
3. *Distribution kernels, quadratic forms, generators, boundary forms, and projectors are objects of different types and are not identified until an explicitly stated construction has been carried out.*

Proof. The first item follows from Definition 4.10 and Lemma 4.11. The second item follows from Proposition 4.12. The third item follows from Definition 4.19 and Convention 4.20. ☐

4.2. Boundary Parameter Space and σ_R-Null Singular Support

In this subsection, we internally construct the measure-theoretic parameter space describing the boundary parameter structure. There are two points needed here. First, we construct a space

$$(\Sigma_R, \mathcal{B}_R, \sigma_R)$$

equipped with a regular area-type measure for measuring boundary contributions. Second, we introduce a smooth zero-area seam carrier and a singular carrier measure that retains nontrivial boundary traces despite being null with respect to this regular area measure.

This two-layer structure is indispensable. Indeed, if a boundary trace is supported on a set Γ_R satisfying

$$\sigma_R(\Gamma_R) = 0$$

as an ordinary $L^2(\Sigma_R, \sigma_R)$-function, then that trace is zero σ_R-almost everywhere. Therefore, the σ_R-null property is measured on the σ_R-side, whereas the nontrivial boundary trace is retained on the side of a measure singular with respect to σ_R, or of a distributional boundary distribution.

Definition 4.22 (Boundary parameter space and regular area measure)**.** Let the closed unit square

$$\Sigma_R := [0,1]^2$$

be the boundary parameter space for the singular support construction. Standard facts concerning Radon measures, Borel regularity, and measure decomposition are used within the scope of [3]. Write its Borel σ-algebra as

$$\mathcal{B}_R := \mathcal{B}([0,1]^2).$$

Define the regular area measure on Σ_R by

$$\sigma_R := \mathcal{L}^2|_{[0,1]^2}.$$

Here $\mathcal{L}^2$ denotes two-dimensional Lebesgue measure on $\mathbb{R}^2$. Thus

$$\sigma_R(\Sigma_R) = 1.$$

Lemma 4.23 (Basic properties of the regular area space)**.** $(\Sigma_R, \mathcal{B}_R, \sigma_R)$ is a finite regular Borel measure space on a compact metric space. Furthermore, σ_R is nonatomic.

Proof. $\Sigma_R = [0,1]^2$ is a compact metric space with respect to the Euclidean metric. $\mathcal{B}_R$ is its Borel σ-algebra, and σ_R is the restriction of Lebesgue measure to a compact set, hence is a finite regular Borel measure.

For any point $p \in [0,1]^2$,

$$\sigma_R(\{p\}) = \mathcal{L}^2(\{p\}) = 0.$$

Therefore σ_R has no atoms. □

Definition 4.24 (Standard zero-area seam carrier for the centered Mellin boundary class)**.** Define the σ_R-null singular seam by

$$\Gamma_R := [0,1] \times \{0\} \subset \Sigma_R.$$

The label of this definition is retained for cross-reference compatibility, but the carrier used from this version onward is the smooth seam above, not the Cantor carrier used in earlier drafts. This seam is the standard zero-area carrier on which the centered Mellin boundary finite-part class is realized. It is not a zeta-zero measure and it does not itself carry prime or zero weights. When the completed-zeta boundary origin is fixed in Section 4.10, the actual finite-window weights are carried by the realized boundary distributions

$$T^{\mathrm{fw}}_{\partial,R}(\varphi) = \mathcal{N}_\xi\big(\mathfrak{b}^{\mathrm{fw}}_{\xi,\partial}(\varphi)\big),$$

not by the carrier Γ_R itself.

Lemma 4.25 (σ_R-null property)**.** Γ_R is a closed subset of Σ_R and satisfies

$$\sigma_R(\Gamma_R) = 0.$$

Proof. The set $\Gamma_R = [0,1] \times \{0\}$ is closed in the compact square $[0,1]^2$. Moreover,

$$\sigma_R(\Gamma_R) = \mathcal{L}^2([0,1] \times \{0\}) = 0,$$

because it is a one-dimensional Lipschitz submanifold of the plane. Equivalently, Fubini's theorem gives

$$\mathcal{L}^2([0,1] \times \{0\}) = \int_0^1 \mathcal{L}^1(\{0\})\,dx = 0.$$

Thus Γ_R is a closed σ_R-null set. □

Definition 4.26 (Carrier measure ν_R supported on the standard zero-area seam)**.** Let $\mathcal{L}^1_{[0,1]}$ denote one-dimensional Lebesgue measure restricted to $[0,1]$. Define the singular carrier measure ν_R supported on Γ_R by

$$\nu_R := \mathcal{L}^1_{[0,1]} \otimes \delta_0.$$

Thus, for every continuous test function $\eta \in C(\Sigma_R)$,

$$\int_{\Sigma_R} \eta\,d\nu_R = \int_0^1 \eta(x,0)\,dx.$$

The measure ν_R is a carrier measure for keeping nonzero traces on a σ_R-null seam. It is not the zero measure of ξ, nor the finite-window divisor of ξ. The ξ-dependent boundary weights enter only through the finite-window boundary distributions realized by the independently constructed map $\mathcal{N}_\xi$ in Definition 4.162.

Lemma 4.27 (Support and singularity of the singular measure)**.** ν_R is a finite regular Borel measure on Σ_R, and satisfies

$$\nu_R(\Sigma_R) = 1, \qquad \operatorname{supp}\nu_R = \Gamma_R.$$

Furthermore,

$$\nu_R \perp \sigma_R.$$

Proof. The measure $\mathcal{L}^1_{[0,1]}$ is a finite regular Borel measure on $[0,1]$, and δ_0 is the Dirac measure on $[0,1]$. Therefore

$$\nu_R = \mathcal{L}^1_{[0,1]} \otimes \delta_0$$

is a finite regular Borel probability measure on Σ_R, and

$$\nu_R(\Sigma_R) = \mathcal{L}^1([0,1])\delta_0([0,1]) = 1.$$

The support of the product measure is

$$\operatorname{supp} \nu_R = [0,1] \times \{0\} = \Gamma_R.$$

By Lemma 4.25, $\sigma_R(\Gamma_R) = 0$, whereas

$$\nu_R(\Gamma_R) = 1.$$

Thus Γ_R is a σ_R-null set and a full-measure set for ν_R, which is exactly

$$\nu_R \perp \sigma_R.$$

□

Definition 4.28 (Regular boundary channel and singular boundary channel)**.** Let $\mathcal{M}(\Sigma_R)$ be the space of all finite complex Radon measures on Σ_R. For any $\lambda \in \mathcal{M}(\Sigma_R)$, write its Lebesgue decomposition as

$$\lambda = \lambda_{\mathrm{ac}} + \lambda_{\mathrm{s}}, \qquad \lambda_{\mathrm{ac}} \ll \sigma_R, \qquad \lambda_{\mathrm{s}} \perp \sigma_R.$$

We call λ_{ac} the regular boundary channel and λ_{s} the singular boundary channel.

In particular, ν_R is a purely singular boundary channel, and its regular component is zero.

Definition 4.29 (Initial singular boundary trace)**.** By Proposition 4.12, the boundary point evaluation

$$\delta_0 : \mathscr{D}_R \longrightarrow \mathbb{C}, \qquad \delta_0(\varphi) = \varphi(0)$$

is an element of $\mathscr{D}_R'$. Using this, define the initial singular boundary trace

$$\gamma_{R,0}^{\mathrm{sing}} : \mathscr{D}_R \longrightarrow \mathcal{M}(\Sigma_R)$$

by

$$\gamma_{R,0}^{\mathrm{sing}} \varphi := \delta_0(\varphi)\, \nu_R = \varphi(0)\, \nu_R.$$

Equivalently, for any $\eta \in C(\Sigma_R)$,

$$\langle \gamma_{R,0}^{\mathrm{sing}} \varphi, \eta \rangle := \varphi(0) \int_{\Sigma_R} \eta \, d\nu_R.$$

Lemma 4.30 (Continuity of the singular boundary trace)**.** The map

$$\gamma_{R,0}^{\mathrm{sing}} : \mathscr{D}_R \longrightarrow \mathcal{M}(\Sigma_R)$$

is a continuous linear map from the Fréchet topology of $\mathscr{D}_R$ to the total variation norm topology of $\mathcal{M}(\Sigma_R)$. Furthermore,

$$\operatorname{supp}(\gamma_{R,0}^{\mathrm{sing}} \varphi) \subset \Gamma_R$$

holds for every $\varphi \in \mathscr{D}_R$.

Proof. For any $\varphi \in \mathscr{D}_R$,

$$\left\| \gamma_{R,0}^{\mathrm{sing}} \varphi \right\|_{\mathrm{TV}} = |\varphi(0)| \, \|\nu_R\|_{\mathrm{TV}} = |\varphi(0)|.$$

On the other hand,

$$|\varphi(0)| \le p_{0,0}(\varphi).$$

Therefore

$$\left\| \gamma_{R,0}^{\mathrm{sing}} \varphi \right\|_{\mathrm{TV}} \le p_{0,0}(\varphi),$$

and hence $\gamma_{R,0}^{\mathrm{sing}}$ is continuous.

For the support, we have

$$\gamma_{R,0}^{\mathrm{sing}} \varphi = \varphi(0) \nu_R,$$

and by Lemma 4.27,

$$\operatorname{supp} \nu_R = \Gamma_R.$$

Therefore, when $\varphi(0) \neq 0$, the support is Γ_R, and when $\varphi(0) = 0$, the measure is the zero measure. In either case,

$$\operatorname{supp}(\gamma_{R,0}^{\mathrm{sing}} \varphi) \subset \Gamma_R$$

holds. ☐

Proposition 4.31 (The singular boundary trace is not a Hilbert boundary function)**.** *In general, $\gamma_{R,0}^{\mathrm{sing}} \varphi$ cannot be represented as a function in $L^2(\Sigma_R, \sigma_R)$. More precisely, if $\varphi(0) \neq 0$, then the measure $\gamma_{R,0}^{\mathrm{sing}} \varphi$ is not absolutely continuous with respect to σ_R. Therefore there exists no $g \in L^1(\Sigma_R, \sigma_R)$ satisfying*

$$\gamma_{R,0}^{\mathrm{sing}} \varphi = g\, \sigma_R.$$

In particular, it also cannot be represented by a $g \in L^2(\Sigma_R, \sigma_R)$.

Proof. Assume $\varphi(0) \neq 0$. Then

$$\gamma_{R,0}^{\mathrm{sing}} \varphi = \varphi(0) \nu_R.$$

By Lemma 4.27,

$$\nu_R \perp \sigma_R,$$

and therefore $\varphi(0)\nu_R$ is also singular with respect to σ_R.

If there existed some $g \in L^1(\Sigma_R, \sigma_R)$ such that

$$\varphi(0) \nu_R = g\, \sigma_R,$$

then the left-hand side would be singular with respect to σ_R, whereas the right-hand side would be absolutely continuous with respect to σ_R. By uniqueness of the Lebesgue decomposition, for the two measures to coincide, both would have to be the zero measure. However,

$$\|\varphi(0)\nu_R\|_{\mathrm{TV}} = |\varphi(0)| > 0,$$

which is a contradiction. Therefore no such g exists. ☐

Lemma 4.32 (Ordinary L^2-traces on a σ_R-null set are trivial)**.** If $g \in L^2(\Sigma_R, \sigma_R)$ satisfies

$$\operatorname*{ess\,supp}_{\sigma_R} g \subset \Gamma_R,$$

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then

$$g = 0 \qquad \sigma_R\text{-a.e.}$$

Proof. By assumption,

$$g = 0 \qquad \sigma_R\text{-a.e. on } \Sigma_R \setminus \Gamma_R.$$

On the other hand, by Lemma 4.25,

$$\sigma_R(\Gamma_R) = 0.$$

Therefore the values of g on Γ_R are ignored as an element of $L^2(\Sigma_R, \sigma_R)$. Hence

$$\int_{\Sigma_R} |g|^2 \, d\sigma_R = \int_{\Sigma_R \setminus \Gamma_R} |g|^2 \, d\sigma_R + \int_{\Gamma_R} |g|^2 \, d\sigma_R = 0 + 0 = 0.$$

Thus $g = 0$ σ_R-almost everywhere. ☐

Definition 4.33 (Convention on σ_R-null singular support and singular retention)**.** From now on, when we refer to a boundary trace supported on Γ_R, this does not mean support as an ordinary function in

$$L^2(\Sigma_R, \sigma_R).$$

A boundary trace on Γ_R is retained as one of the following singular or distributional objects:

$$\nu_R, \qquad \gamma_R^{\mathrm{sing}}, \qquad \mathscr{D}'_R.$$

On the other hand, σ_R is used only to measure regular boundary trace mass.

Theorem 4.34 (Internal construction of the standard zero-area boundary carrier)**.** *By the construction above,*

$$(\Sigma_R, \sigma_R, \Gamma_R, \nu_R)$$

is fixed as boundary carrier data satisfying the following properties.

1. *Σ_R is a compact metric space.*
2. *σ_R is a finite regular Borel area measure on Σ_R.*
3. *$\Gamma_R \subset \Sigma_R$ is a closed smooth seam and satisfies*

$$\sigma_R(\Gamma_R) = 0.$$

4. *The nonzero singular probability carrier measure*

$$\nu_R$$

   is supported on Γ_R, and

$$\nu_R(\Gamma_R) = 1, \qquad \nu_R \perp \sigma_R$$

   holds.
5. *The initial singular boundary trace*

$$\gamma_{R,0}^{\mathrm{sing}} : \mathscr{D}_R \to \mathcal{M}(\Sigma_R)$$

   is continuous, and its values are always supported on Γ_R.

6. *The quadruple* $(\Sigma_R, \sigma_R, \Gamma_R, \nu_R)$ *is a standard smooth zero-area seam carrier for realized centered Mellin boundary finite parts: it keeps distributional trace data supported on a* σ_R*-null seam while separating those data from the regular area measure.*

Proof. Items 1 and 2 follow from Definition 4.22 and Lemma 4.23. Item 3 follows from Definition 4.24 and Lemma 4.25. Item 4 follows from Definition 4.26 and Lemma 4.27. Item 5 follows from Definition 4.29 and the continuity statement in Lemma 4.30. Item 6 is the categorical meaning of the preceding items: the regular measure kills Γ_R, while ν_R and $\gamma_{R,0}^{\mathrm{sing}}$ retain nonzero distributional boundary data on Γ_R. The actual ξ-finite-window weights are not placed in ν_R, but are supplied later by the realized distributions of Definition 4.162. □

Lemma 4.35 (Separation of σ_R-null property and nontriviality)**.** Γ_R is null with respect to the regular area measure σ_R. However, the singular measure on Γ_R is retained nontrivially as a singular measure or distribution. More precisely, the following hold.

1.
$$\sigma_R(\Gamma_R) = 0.$$
2. The only element of $L^2(\Sigma_R, \sigma_R)$ essentially supported on Γ_R is the zero element.
3.
$$\nu_R(\Gamma_R) = 1$$
and ν_R is a nonzero singular measure.
4. If $\varphi \in \mathscr{D}_R$ satisfies $\varphi(0) \neq 0$, then
$$\gamma_{R,0}^{\mathrm{sing}}\varphi = \varphi(0)\nu_R$$
is nonzero and its support is Γ_R.

Therefore the σ_R-null property is a property on the σ_R-side, whereas the nontriviality of the boundary trace is retained as a property on the ν_R- or $\mathscr{D}'_R$-side.

Proof. Item 1 was proved in Lemma 4.25. Item 2 is Lemma 4.32. Item 3 follows from Lemma 4.27. For Item 4, if $\varphi(0) \neq 0$, then
$$\gamma_{R,0}^{\mathrm{sing}}\varphi = \varphi(0)\nu_R,$$
and its total variation norm is
$$\left\|\gamma_{R,0}^{\mathrm{sing}}\varphi\right\|_{\mathrm{TV}} = |\varphi(0)| > 0.$$
Therefore it is nonzero. Moreover, by Lemma 4.30, its support is contained in Γ_R, and when $\varphi(0) \neq 0$, it has the same support as ν_R. Hence
$$\operatorname{supp}(\gamma_{R,0}^{\mathrm{sing}}\varphi) = \Gamma_R.$$

Thus, although Γ_R is null with respect to the area measure, it retains a nontrivial boundary trace on the side of singular measures and distributional boundary traces. □

Proposition 4.36 (Output of this subsection)**.** *This subsection has constructed the following objects:*

$$\Sigma_R, \qquad \sigma_R, \qquad \Gamma_R, \qquad \nu_R, \qquad \gamma_{R,0}^{\mathrm{sing}}.$$

Here

$$\sigma_R(\Gamma_R) = 0, \qquad \nu_R(\Gamma_R) = 1, \qquad \nu_R \perp \sigma_R$$

hold. Moreover, $\gamma_{R,0}^{\mathrm{sing}}$ *is a singular boundary trace defined on* $\mathscr{D}_R$*. In the subsequent subsections, we extend it to a singular boundary trace on the form domain and construct the support map and regular boundary trace-mass functional.*

Proof. The construction of each object follows respectively from Definition 4.22, Definition 4.24, Definition 4.26, and Definition 4.29. The displayed properties were proved in Lemma 4.25, Lemma 4.27, and Lemma 4.35. ☐

4.3. Traces, Support Maps, and Regular Trace-Mass Functionals

In this subsection, using the boundary triple data

$$(\Sigma_R, \sigma_R, \Gamma_R)$$

and the singular measure ν_R constructed in the preceding subsection, we construct the boundary traces, support map, and regular boundary trace-mass functional on the form domain. Let Q_R denote the form domain constructed in Section 2, and write the corresponding form norm as

$$\|f\|_{q_R}^2 := q_R[f,f] + \|f\|_{H_{\alpha,+}}^2.$$

When necessary, if q_R is lower bounded, this norm is interpreted after replacing it by an equivalent positive norm.

There are two types of boundary traces introduced in this subsection. One is the singular boundary trace supported on Γ_R, and the other is the regular boundary trace absolutely continuous with respect to σ_R. The former is used to retain the nontriviality of the boundary trace, while the latter is used to measure regular boundary trace mass.

Definition 4.37 (Trace core)**.** Write the common intersection of the form domain and the boundary test space as

$$\mathscr{C}_R := \mathscr{D}_R \cap Q_R.$$

Equip $\mathscr{C}_R$ with the form norm $\|\cdot\|_{q_R}$ induced from Q_R.

Remark 4.38. $\mathscr{C}_R$ is the class of smooth elements for which boundary traces can first be defined classically. The trace maps in this subsection are extended to a subspace of Q_R by closing the maps on this core with respect to the form norm.

Definition 4.39 (Singular boundary trace on the core)**.** For $\varphi \in \mathscr{C}_R$, define

$$\vartheta_R^{\mathrm{sing}}\varphi := \varphi(0)\,\nu_R \in \mathcal{M}(\Sigma_R)$$

using the singular measure ν_R from the preceding subsection. Here $\mathcal{M}(\Sigma_R)$ denotes the space of all finite complex Radon measures on Σ_R. Namely, for any $\eta \in C(\Sigma_R)$,

$$\langle \vartheta_R^{\mathrm{sing}}\varphi, \eta\rangle = \varphi(0)\int_{\Sigma_R} \eta\, d\nu_R.$$

Definition 4.40 (Regular boundary trace on the core)**.** Set

$$e_R := \mathbf{1}_{\Sigma_R}.$$

Since the construction in the preceding subsection gives $\sigma_R(\Sigma_R) = 1$,

$$\|e_R\|_{L^2(\Sigma_R,\sigma_R)} = 1.$$

For $\varphi \in \mathscr{C}_R$, define the regular trace mass coefficient by

$$\tau_R^{\mathrm{reg}}(\varphi) := \partial_x \varphi(0).$$

Then define the regular boundary trace on the core by

$$\vartheta_R^{\mathrm{reg}}\varphi := \tau_R^{\mathrm{reg}}(\varphi) e_R \in L^2(\Sigma_R, \sigma_R).$$

Remark 4.41 (Independence of regular trace and singular trace)**.** $\vartheta_R^{\mathrm{sing}}$ is defined through the boundary-value-type singular measure ν_R. On the other hand, $\vartheta_R^{\mathrm{reg}}$ measures regular boundary trace mass absolutely continuous with respect to the regular area measure σ_R. Accordingly, even if

$$\vartheta_R^{\mathrm{sing}}\varphi$$

is nonzero, it is possible that

$$\vartheta_R^{\mathrm{reg}}\varphi = 0.$$

This separation allows one to handle simultaneously a nontrivial boundary trace on the σ_R-null singular support and vanishing regular-trace mass in the regular boundary-trace direction.

Definition 4.42 (Graph of the closed trace)**.** A triple

$$(f, \lambda, g) \in Q_R \times \mathcal{M}(\Sigma_R) \times L^2(\Sigma_R, \sigma_R)$$

is said to belong to the closed-trace graph if there exists a sequence

$$\{\varphi_n\}_{n\geq 1} \subset \mathscr{C}_R$$

such that

$$\varphi_n \longrightarrow f \quad \text{in } (Q_R, \|\cdot\|_{q_R}),$$

and

$$\vartheta_R^{\mathrm{sing}}\varphi_n \xrightarrow{w^*} \lambda \quad \text{in } \mathcal{M}(\Sigma_R),$$

and further

$$\vartheta_R^{\mathrm{reg}}\varphi_n \longrightarrow g \quad \text{in } L^2(\Sigma_R, \sigma_R)$$

hold. Denote this set by

$$\mathcal{G}(\gamma_R) \subset Q_R \times \mathcal{M}(\Sigma_R) \times L^2(\Sigma_R, \sigma_R).$$

Definition 4.43 (Closed boundary trace space)**.** We say that $f \in Q_R$ has a boundary trace if there exists an element

$$(f, \lambda, g)$$

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of $\mathcal{G}(\gamma_R)$, and if such a pair (λ, g) is uniquely determined. Define the set of all such f by

$$T_R := \Big\{ f \in Q_R : \exists!\,(\lambda, g) \in \mathcal{M}(\Sigma_R) \times L^2(\Sigma_R, \sigma_R) \text{ such that } (f, \lambda, g) \in \mathcal{G}(\gamma_R) \Big\}.$$

For $f \in T_R$, write the uniquely determined λ and g as

$$\gamma_R^{\mathrm{sing}} f := \lambda, \qquad \gamma_R^{\mathrm{reg}} f := g,$$

respectively.

Lemma 4.44 (Linearity of the closed boundary trace space)**.** T_R is a linear subspace of Q_R. Moreover,

$$\gamma_R^{\mathrm{sing}} : T_R \to \mathcal{M}(\Sigma_R), \qquad \gamma_R^{\mathrm{reg}} : T_R \to L^2(\Sigma_R, \sigma_R)$$

are both linear maps.

Proof. Let $f_1, f_2 \in T_R$, and write

$$\gamma_R^{\mathrm{sing}} f_j = \lambda_j, \qquad \gamma_R^{\mathrm{reg}} f_j = g_j \qquad (j = 1, 2).$$

By definition, for each j there exists a sequence

$$\{\varphi_{j,n}\}_{n \geq 1} \subset \mathscr{C}_R$$

such that

$$\varphi_{j,n} \to f_j \quad \text{in } Q_R,$$

and

$$\vartheta_R^{\mathrm{sing}} \varphi_{j,n} \xrightarrow{w^*} \lambda_j, \qquad \vartheta_R^{\mathrm{reg}} \varphi_{j,n} \to g_j.$$

For arbitrary $a, b \in \mathbb{C}$, set

$$\psi_n := a\varphi_{1,n} + b\varphi_{2,n}.$$

Since $\mathscr{C}_R$ is a linear space, we have $\psi_n \in \mathscr{C}_R$. Moreover,

$$\psi_n \to af_1 + bf_2 \quad \text{in } Q_R,$$

and by the linearity of the trace maps on the core,

$$\vartheta_R^{\mathrm{sing}} \psi_n = a\,\vartheta_R^{\mathrm{sing}} \varphi_{1,n} + b\,\vartheta_R^{\mathrm{sing}} \varphi_{2,n} \xrightarrow{w^*} a\lambda_1 + b\lambda_2,$$

$$\vartheta_R^{\mathrm{reg}} \psi_n = a\,\vartheta_R^{\mathrm{reg}} \varphi_{1,n} + b\,\vartheta_R^{\mathrm{reg}} \varphi_{2,n} \longrightarrow ag_1 + bg_2$$

hold. Therefore

$$(af_1 + bf_2,\ a\lambda_1 + b\lambda_2,\ ag_1 + bg_2) \in \mathcal{G}(\gamma_R).$$

Uniqueness is included in the definition of T_R, so $af_1 + bf_2 \in T_R$, and

$$\gamma_R^{\mathrm{sing}}(af_1 + bf_2) = a\gamma_R^{\mathrm{sing}} f_1 + b\gamma_R^{\mathrm{sing}} f_2,$$

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$$\gamma_R^{\mathrm{reg}}(af_1 + bf_2) = a\gamma_R^{\mathrm{reg}} f_1 + b\gamma_R^{\mathrm{reg}} f_2.$$

Hence T_R is a linear subspace, and the two trace maps are linear. □

Lemma 4.45 (Support of the singular trace)**.** For any $f \in T_R$,

$$\mathrm{supp}(\gamma_R^{\mathrm{sing}} f) \subset \Gamma_R$$

holds.

Proof. Let $f \in T_R$, and write

$$\gamma_R^{\mathrm{sing}} f = \lambda.$$

By definition, there exists a sequence $\{\varphi_n\} \subset \mathscr{C}_R$ such that

$$\vartheta_R^{\mathrm{sing}} \varphi_n \xrightarrow{w^*} \lambda.$$

For each n,

$$\vartheta_R^{\mathrm{sing}} \varphi_n = \varphi_n(0)\nu_R,$$

and by Lemma 4.27,

$$\mathrm{supp}\, \nu_R = \Gamma_R.$$

Therefore

$$\mathrm{supp}(\vartheta_R^{\mathrm{sing}} \varphi_n) \subset \Gamma_R.$$

Let $U \subset \Sigma_R$ be an open set satisfying

$$U \cap \Gamma_R = \varnothing.$$

For any $\eta \in C_c(U)$,

$$\int_{\Sigma_R} \eta \, d(\vartheta_R^{\mathrm{sing}} \varphi_n) = 0$$

holds for every n. Using weak-star convergence, we get

$$\int_{\Sigma_R} \eta \, d\lambda = \lim_{n\to\infty} \int_{\Sigma_R} \eta \, d(\vartheta_R^{\mathrm{sing}} \varphi_n) = 0.$$

Hence λ vanishes on $\Sigma_R \setminus \Gamma_R$. Therefore

$$\mathrm{supp}\, \lambda \subset \Gamma_R.$$

That is,

$$\mathrm{supp}(\gamma_R^{\mathrm{sing}} f) \subset \Gamma_R.$$

□

Definition 4.46 (Singular support map)**.** For $f \in T_R$, define its singular support by

$$\mathrm{supp}_R(f) := \mathrm{supp}(\gamma_R^{\mathrm{sing}} f) \subset \Sigma_R.$$

If $\gamma_R^{\mathrm{sing}} f = 0$, set

$$\mathrm{supp}_R(f) := \varnothing.$$

Corollary 4.47 (σ_R-null property of singular support)**.** *For any* $f \in T_R$*,*

$$\mathrm{supp}_R(f) \subset \Gamma_R.$$

Therefore

$$\sigma_R(\mathrm{supp}_R(f)) = 0.$$

Proof. The first assertion is Lemma 4.45. The second follows from Lemma 4.25, namely from

$$\sigma_R(\Gamma_R) = 0.$$

□

Definition 4.48 (Regular trace-mass measure)**.** For $f \in T_R$, define the finite positive measure

$$\Lambda_R^{\mathrm{reg}}(f)$$

on Σ_R by

$$d\Lambda_R^{\mathrm{reg}}(f) := |\gamma_R^{\mathrm{reg}} f|^2 \, d\sigma_R.$$

Namely, for any $E \in \mathcal{B}_R$,

$$\Lambda_R^{\mathrm{reg}}(f)(E) = \int_E |\gamma_R^{\mathrm{reg}} f|^2 \, d\sigma_R.$$

Definition 4.49 (Regular boundary trace-mass functional)**.** For $f \in T_R$ and $E \in \mathcal{B}_R$, define the regular boundary trace-mass functional by

$$L_R(f; E) := \int_E |\gamma_R^{\mathrm{reg}} f|^2 \, d\sigma_R.$$

Equivalently,

$$L_R(f; E) = \|\mathbf{1}_E \gamma_R^{\mathrm{reg}} f\|^2_{L^2(\Sigma_R, \sigma_R)}.$$

Lemma 4.50 (Basic properties of the regular boundary trace-mass functional)**.** For any $f \in T_R$, the map

$$E \longmapsto L_R(f; E)$$

is a finite positive measure on Σ_R. Moreover, for any $a \in \mathbb{C}$,

$$L_R(af; E) = |a|^2 L_R(f; E)$$

holds. Furthermore,

$$L_R(f; \Sigma_R) = \|\gamma_R^{\mathrm{reg}} f\|^2_{L^2(\Sigma_R, \sigma_R)}.$$

Proof. By definition,

$$L_R(f; E) = \int_E |\gamma_R^{\mathrm{reg}} f|^2 \, d\sigma_R.$$

Since $\gamma_R^{\mathrm{reg}} f \in L^2(\Sigma_R, \sigma_R)$,

$$|\gamma_R^{\mathrm{reg}} f|^2 \in L^1(\Sigma_R, \sigma_R).$$

Therefore

$$E \longmapsto \int_E |\gamma_R^{\mathrm{reg}} f|^2 \, d\sigma_R$$

is a finite positive measure.

For scalar multiplication,

$$\gamma_R^{\mathrm{reg}}(af) = a\,\gamma_R^{\mathrm{reg}}f,$$

and hence

$$L_R(af;E) = \int_E |a|^2|\gamma_R^{\mathrm{reg}}f|^2\,d\sigma_R = |a|^2 L_R(f;E).$$

The final equality follows immediately by setting $E = \Sigma_R$. ☐

Lemma 4.51 (Vanishing criterion for regular trace mass)**.** For $f \in T_R$, the following are equivalent.

1.
$$L_R(f;\Sigma_R \setminus \Gamma_R) = 0.$$

2.
$$\gamma_R^{\mathrm{reg}}f = 0 \qquad \sigma_R\text{-a.e. on } \Sigma_R.$$

3.
$$L_R(f;\Sigma_R) = 0.$$

Proof. By Lemma 4.25,

$$\sigma_R(\Gamma_R) = 0.$$

Therefore, for any $g \in L^2(\Sigma_R, \sigma_R)$,

$$\int_{\Sigma_R} |g|^2\,d\sigma_R = \int_{\Sigma_R\setminus\Gamma_R} |g|^2\,d\sigma_R$$

holds. Set

$$g = \gamma_R^{\mathrm{reg}}f.$$

First suppose that (1) holds. By definition,

$$0 = L_R(f;\Sigma_R \setminus \Gamma_R) = \int_{\Sigma_R\setminus\Gamma_R} |\gamma_R^{\mathrm{reg}}f|^2\,d\sigma_R.$$

The preceding equality and $\sigma_R(\Gamma_R) = 0$ imply

$$\int_{\Sigma_R} |\gamma_R^{\mathrm{reg}}f|^2\,d\sigma_R = 0.$$

Therefore

$$\gamma_R^{\mathrm{reg}}f = 0 \qquad \sigma_R\text{-a.e.},$$

so (2) holds.

The implication from (2) to (3) is immediate from the definition. The implication from (3) to (1) follows by monotonicity. Thus the three conditions are equivalent. ☐

Definition 4.52 (Trace-vanishing generating subspace associated with Γ_R)**.** Define the trace-vanishing generating subspace associated with Γ_R by

$$G_R := \Big\{ f \in T_R : \mathrm{supp}_R(f) \subset \Gamma_R,\ L_R(f;\Sigma_R \setminus \Gamma_R) = 0 \Big\}.$$

Proposition 4.53 (Equivalent representation of the generating class). *G_R can be written as*

$$G_R = \{ f \in T_R : \gamma_R^{\mathrm{reg}} f = 0 \quad \sigma_R\text{-}a.e. \}.$$

Then, for any $f \in G_R$,

$$\mathrm{supp}_R(f) \subset \Gamma_R, \qquad \sigma_R(\mathrm{supp}_R(f)) = 0$$

hold.

Proof. For any $f \in T_R$,

$$\mathrm{supp}_R(f) \subset \Gamma_R$$

holds automatically by Corollary 4.47. Therefore, the substantive condition in the definition of G_R is

$$L_R(f; \Sigma_R \setminus \Gamma_R) = 0.$$

By Lemma 4.51, this condition is equivalent to

$$\gamma_R^{\mathrm{reg}} f = 0 \quad \sigma_R\text{-a.e.}$$

The displayed representation follows.

Finally,

$$\mathrm{supp}_R(f) \subset \Gamma_R$$

together with

$$\sigma_R(\Gamma_R) = 0$$

implies

$$\sigma_R(\mathrm{supp}_R(f)) = 0.$$

□

Lemma 4.54 (Separation of σ_R-null property and regular-trace vanishing property). For $f \in G_R$, the singular boundary trace

$$\gamma_R^{\mathrm{sing}} f$$

is supported on Γ_R. On the other hand, the regular boundary trace satisfies

$$\gamma_R^{\mathrm{reg}} f = 0 \qquad \sigma_R\text{-a.e.}$$

Therefore G_R is the generating class that simultaneously realizes the singular boundary trace on the σ_R-null singular support and vanishing regular-trace mass in the regular boundary-trace direction.

Proof. Let $f \in G_R$. By Definition 4.52,

$$\mathrm{supp}_R(f) \subset \Gamma_R.$$

By Definition 4.46,

$$\mathrm{supp}_R(f) = \mathrm{supp}(\gamma_R^{\mathrm{sing}} f),$$

so

$$\gamma_R^{\mathrm{sing}} f$$

is supported on Γ_R.

Also by the definition of $f \in G_R$,

$$L_R(f; \Sigma_R \setminus \Gamma_R) = 0.$$

Applying Lemma 4.51, we obtain

$$\gamma_R^{\mathrm{reg}} f = 0 \qquad \sigma_R\text{-a.e.}$$

The claim follows. □

Remark 4.55 (Reason for avoiding interpretation as an ordinary L^2 boundary function)**.** The boundary trace of $f \in G_R$ is a singular boundary object described by $\gamma_R^{\mathrm{sing}} f$, and is not described as an ordinary function in

$$L^2(\Sigma_R, \sigma_R).$$

Indeed, by Lemma 4.32, every $L^2(\Sigma_R, \sigma_R)$-function essentially supported on Γ_R is the zero element. Accordingly, the nontriviality in G_R is retained on the side of γ_R^{sing} or ν_R, while the regular-trace vanishing property is described as the vanishing of γ_R^{reg}.

Proposition 4.56 (Output of this subsection)**.** *This subsection has constructed the following objects:*

$$T_R, \qquad \gamma_R^{\mathrm{sing}}, \qquad \gamma_R^{\mathrm{reg}}, \qquad \mathrm{supp}_R, \qquad L_R, \qquad G_R.$$

Here,

$$T_R \subset Q_R$$

is the linear subspace on which the singular boundary trace and the regular boundary trace are simultaneously defined, and

$$\mathrm{supp}_R(f) = \mathrm{supp}(\gamma_R^{\mathrm{sing}} f).$$

Moreover,

$$L_R(f; E) = \int_E |\gamma_R^{\mathrm{reg}} f|^2 \, d\sigma_R$$

is the nonnegative functional measuring regular boundary trace mass, and

$$G_R = \left\{ f \in T_R : \mathrm{supp}_R(f) \subset \Gamma_R,\ L_R(f; \Sigma_R \setminus \Gamma_R) = 0 \right\}$$

is the trace-vanishing generating subspace associated with Γ_R.

Proof. T_R, γ_R^{sing}, and γ_R^{reg} were constructed by Definition 4.43. supp_R was constructed by Definition 4.46. L_R was constructed by Definition 4.49. Finally, G_R was defined by Definition 4.52, and its meaning was verified in Lemma 4.54. □

4.4. Boundary Bilinear Form and Boundary-Cancellation Theorem

In this subsection, using the regular boundary trace

$$\gamma_R^{\mathrm{reg}} : T_R \longrightarrow L^2(\Sigma_R, \sigma_R)$$

constructed in the preceding subsection, we define a regular boundary-trace bilinear form and prove that its boundary contribution vanishes on the trace-vanishing generating subspace associated with Γ_R G_R.

What is important here is that the boundary form b_R is a σ_R-area-type regular boundary form and does not directly integrate the singular boundary trace

$$\gamma_R^{\mathrm{sing}} f.$$

This separation allows one to eliminate only the regular-area boundary term while retaining the nontrivial singular boundary trace on Γ_R.

Definition 4.57 (Regular-area-type boundary coefficient)**.** From now on, fix one function

$$\omega_R \in L^\infty(\Sigma_R, \sigma_R).$$

We call this the regular boundary coefficient. If necessary, normalize it so that $\|\omega_R\|_{L^\infty(\Sigma_R,\sigma_R)} \le 1$.

Definition 4.58 (Regular-area-type boundary bilinear form)**.** For $u, v \in T_R$, define

$$b_R[u,v] := \int_{\Sigma_R} \omega_R(\xi)\,(\gamma_R^{\mathrm{reg}} u)(\xi)\,\overline{(\gamma_R^{\mathrm{reg}} v)(\xi)}\,d\sigma_R(\xi).$$

We call this the σ_R-regular boundary-trace bilinear form.

Lemma 4.59 (Boundedness of the boundary form)**.** For any $u, v \in T_R$,

$$|b_R[u,v]| \le \|\omega_R\|_{L^\infty(\Sigma_R,\sigma_R)} \, \|\gamma_R^{\mathrm{reg}} u\|_{L^2(\Sigma_R,\sigma_R)} \, \|\gamma_R^{\mathrm{reg}} v\|_{L^2(\Sigma_R,\sigma_R)}$$

holds. In particular, b_R is bounded with respect to the $L^2(\Sigma_R, \sigma_R)$-norm of the regular boundary trace.

Proof. By the Cauchy–Schwarz inequality,

$$\begin{aligned}
|b_R[u,v]| &\le \int_{\Sigma_R} |\omega_R||\gamma_R^{\mathrm{reg}} u||\gamma_R^{\mathrm{reg}} v|\,d\sigma_R \\
&\le \|\omega_R\|_{L^\infty(\Sigma_R,\sigma_R)} \int_{\Sigma_R} |\gamma_R^{\mathrm{reg}} u||\gamma_R^{\mathrm{reg}} v|\,d\sigma_R \\
&\le \|\omega_R\|_{L^\infty(\Sigma_R,\sigma_R)} \, \|\gamma_R^{\mathrm{reg}} u\|_{L^2(\Sigma_R,\sigma_R)} \, \|\gamma_R^{\mathrm{reg}} v\|_{L^2(\Sigma_R,\sigma_R)}.
\end{aligned}$$

This proves the claim. ☐

Remark 4.60 (The singular boundary trace does not enter b_R)**.** The form b_R is defined using only

$$\gamma_R^{\mathrm{reg}} u, \quad \gamma_R^{\mathrm{reg}} v \in L^2(\Sigma_R, \sigma_R).$$

Therefore, even if

$$\gamma_R^{\mathrm{sing}} u, \quad \gamma_R^{\mathrm{sing}} v$$

exist as nontrivial singular measures or distributions on Γ_R, they do not enter the area-type integral defining b_R. When this paper refers to boundary cancellation, what is cancelled is the σ_R-regular-area boundary term, not the singular measure ν_R supported on Γ_R itself.

Lemma 4.61 (Regular boundary contribution on a σ_R-null singular support)**.** If $g, h \in L^2(\Sigma_R, \sigma_R)$ satisfy

$$\operatorname*{ess\,supp}_{\sigma_R} g \subset \Gamma_R, \qquad \operatorname*{ess\,supp}_{\sigma_R} h \subset \Gamma_R,$$

then

$$\int_{\Sigma_R} \omega_R \, g \, \overline{h} \, d\sigma_R = 0.$$

Proof. By Lemma 4.25,

$$\sigma_R(\Gamma_R) = 0.$$

Moreover, by assumption, g and h are zero σ_R-almost everywhere on $\Sigma_R \setminus \Gamma_R$. Hence

$$\omega_R \, g \, \overline{h} = 0 \qquad \sigma_R\text{-a.e. on } \Sigma_R \setminus \Gamma_R.$$

On the other hand, since Γ_R is a σ_R-null set,

$$\int_{\Gamma_R} |\omega_R \, g \, \overline{h}| \, d\sigma_R = 0.$$

Therefore, in total,

$$\int_{\Sigma_R} \omega_R \, g \, \overline{h} \, d\sigma_R = 0.$$

□

Lemma 4.62 (Vanishing of the regular trace by the regular-trace vanishing condition)**.** For any $f \in G_R$,

$$\gamma_R^{\mathrm{reg}} f = 0 \qquad \sigma_R\text{-a.e. on } \Sigma_R$$

holds.

Proof. Let $f \in G_R$. By Definition 4.52,

$$L_R(f; \Sigma_R \setminus \Gamma_R) = 0.$$

By Lemma 4.51, this is equivalent to

$$\gamma_R^{\mathrm{reg}} f = 0 \qquad \sigma_R\text{-a.e. on } \Sigma_R.$$

□

Theorem 4.63 (Boundary cancellation on the trace-vanishing generating subspace associated with Γ_R)**.** *For any $u, v \in G_R$,*

$$b_R[u, v] = 0$$

holds.

Proof. By Lemma 4.62,

$$\gamma_R^{\mathrm{reg}} u = 0, \qquad \gamma_R^{\mathrm{reg}} v = 0 \qquad \sigma_R\text{-a.e.}$$

Therefore, by Definition 4.58,

$$\begin{aligned} b_R[u, v] &= \int_{\Sigma_R} \omega_R \, (\gamma_R^{\mathrm{reg}} u) \, \overline{(\gamma_R^{\mathrm{reg}} v)} \, d\sigma_R \\ &= 0. \end{aligned}$$

What is used here is the vanishing of the regular boundary trace, not the vanishing of $\gamma_R^{\mathrm{sing}} u$ or $\gamma_R^{\mathrm{sing}} v$. Thus, even if singular boundary traces remain nontrivially on Γ_R, the σ_R-area-type boundary term vanishes. □

Corollary 4.64 (Boundary cancellation on the linear span)**.** *For any*

$$u, v \in \operatorname{span} G_R,$$

one has

$$b_R[u, v] = 0.$$

Proof. Let $u, v \in \operatorname{span} G_R$. There exist finitely many elements $u_i, v_j \in G_R$ and coefficients $a_i, b_j \in \mathbb{C}$ such that

$$u = \sum_i a_i u_i, \qquad v = \sum_j b_j v_j.$$

Since b_R is sesquilinear,

$$b_R[u, v] = \sum_{i,j} a_i \overline{b_j}\, b_R[u_i, v_j].$$

By Theorem 4.63, each term is zero. Therefore

$$b_R[u, v] = 0.$$

□

Definition 4.65 (Singular-boundary form closure)**.** Define the form-norm closure of the trace-vanishing generating subspace associated with Γ_R by

$$Q_R^{\mathrm{res}} := \overline{\operatorname{span} G_R}^{\|\cdot\|_{q_R}} \subset Q_R.$$

We call this space the σ_R-null trace-vanishing form subspace.

Definition 4.66 (Regular boundary form on the closure)**.** Let $u, v \in Q_R^{\mathrm{res}}$. Take arbitrary sequences in $\operatorname{span} G_R$

$$u_n \to u, \qquad v_n \to v \qquad \text{in } (Q_R, \|\cdot\|_{q_R}).$$

Then define

$$b_R^{\mathrm{res}}[u, v] := \lim_{n \to \infty} b_R[u_n, v_n].$$

Lemma 4.67 (Well-definedness of the boundary form on the closure)**.** The right-hand side of Definition 4.66 does not depend on the choice of sequences, and

$$b_R^{\mathrm{res}}[u, v] = 0.$$

Thus b_R^{res} is well-defined as the zero form on $Q_R^{\mathrm{res}} \times Q_R^{\mathrm{res}}$.

Proof. For any approximating sequences

$$u_n, v_n \in \operatorname{span} G_R,$$

Corollary 4.64 gives

$$b_R[u_n, v_n] = 0$$

for all n. Therefore

$$\lim_{n\to\infty} b_R[u_n, v_n] = 0.$$

This value clearly does not depend on the choice of approximating sequences. Hence b_R^{res} is uniquely defined as the zero form on the closure. □

Theorem 4.68 (Extension of boundary cancellation to the form-norm closure)**.** *For any $u, v \in Q_R^{\mathrm{res}}$,*

$$b_R^{\mathrm{res}}[u, v] = 0.$$

Furthermore, if $u, v \in Q_R^{\mathrm{res}} \cap T_R$ and the regular trace γ_R^{reg} is defined for these elements in the closed-graph sense, then the original regular-trace-type boundary form also satisfies

$$b_R[u, v] = 0.$$

Proof. The first assertion is exactly Lemma 4.67.

We prove the second assertion. Let $u \in Q_R^{\mathrm{res}} \cap T_R$. By the definition of Q_R^{res}, there exists a sequence

$$u_n \in \operatorname{span} G_R$$

such that

$$u_n \to u \qquad \text{in } (Q_R, \|\cdot\|_{q_R}).$$

Since each u_n belongs to span G_R, Lemma 4.62 and linearity imply

$$\gamma_R^{\mathrm{reg}} u_n = 0 \qquad \sigma_R\text{-a.e.}$$

The regular trace being defined for u in the closed-graph sense means that if $u_n \to u$ and

$$\gamma_R^{\mathrm{reg}} u_n \to g \quad \text{in } L^2(\Sigma_R, \sigma_R),$$

then

$$g = \gamma_R^{\mathrm{reg}} u.$$

Here

$$\gamma_R^{\mathrm{reg}} u_n = 0,$$

so the left-hand side converges to 0 in the L^2-norm. Therefore

$$\gamma_R^{\mathrm{reg}} u = 0.$$

Similarly,

$$\gamma_R^{\mathrm{reg}} v = 0$$

follows. Hence, by Definition 4.58,

$$b_R[u, v] = \int_{\Sigma_R} \omega_R (\gamma_R^{\mathrm{reg}} u) \overline{(\gamma_R^{\mathrm{reg}} v)}\, d\sigma_R = 0.$$

□

Remark 4.69 (Cancellation on the closure is not cancellation of the singular trace)**.** The cancellation identity extended to Q_R^{res},

$$b_R^{\mathrm{res}} = 0,$$

means the vanishing of the regular-trace-type boundary form. It does not mean that

$$\gamma_R^{\mathrm{sing}} u$$

vanishes. The singular boundary trace is an object on the ν_R- or $\mathscr{D}'_R$-side and is not included in the σ_R-area-type boundary integral. Therefore the structure is preserved in which the singular information on the σ_R-null singular support is retained while only the regular boundary trace mass is removed.

Proposition 4.70 (Output of this subsection)**.** *This subsection has obtained the following objects and properties.*

1. *The σ_R-regular boundary-trace bilinear form*

$$b_R : T_R \times T_R \longrightarrow \mathbb{C}$$

has been defined.

2. *For any $u, v \in G_R$,*

$$b_R[u, v] = 0$$

holds.

3. *On the form-norm closure of the trace-vanishing generating subspace associated with Γ_R,*

$$Q_R^{\mathrm{res}} = \overline{\operatorname{span} G_R}^{\|\cdot\|_{q_R}},$$

the zero extension of the boundary form

$$b_R^{\mathrm{res}}$$

is well-defined, and

$$b_R^{\mathrm{res}}[u, v] = 0 \qquad (u, v \in Q_R^{\mathrm{res}})$$

holds.

4. *This boundary cancellation is the vanishing of the σ_R-regular-area boundary term, and does not mean the vanishing of the singular boundary trace on Γ_R.*

Proof. Item 1 follows from Definition 4.58. Item 2 follows from Theorem 4.63. Item 3 follows from Definition 4.65, Definition 4.66, and Theorem 4.68. Item 4 follows from the entire construction of this subsection, in particular from the separation between the regular trace and the singular trace, and from the immediately preceding remark. □

4.5. Closure of the σ_R-Null Trace-Vanishing Form Subspace

In this subsection, we do not reconstruct the closed quadratic form

$$q_R : Q_R \times Q_R \longrightarrow \mathbb{C}$$

constructed in Section 2. Here, using the trace-vanishing generating subspace associated with Γ_R

$$G_R \subset T_R \subset Q_R$$

constructed up to the preceding subsection, we cut out, from the form domain Q_R of q_R, the closed form subspace satisfying the σ_R-null σ_R-null support condition.

In Section 2, q_R was constructed as a closed form bounded from below. From now on, if necessary, we take a sufficiently large constant $c_R > 0$ and replace it by

$$q_R^{(c_R)}[u,v] := q_R[u,v] + c_R \langle u, v\rangle_{H_{\alpha,+}}$$

so as to use a positive-definite form inner product. For simplicity, in this subsection we write the form norm after this positive shift as

$$\|u\|_{q_R}^2 := q_R[u,u] + \|u\|_{H_{\alpha,+}}^2.$$

Accordingly,

$$(Q_R, \|\cdot\|_{q_R})$$

is a Hilbert space, and the inclusion map

$$Q_R \hookrightarrow H_{\alpha,+}$$

is continuous.

Definition 4.71 (σ_R-null trace-vanishing form subspace)**.** Define the form-norm closure of the trace-vanishing generating subspace associated with Γ_R G_R by

$$Q_R^{\mathrm{res}} := \overline{\operatorname{span} G_R}^{\|\cdot\|_{q_R}} \subset Q_R.$$

This space is called the σ_R-null trace-vanishing form subspace. Also,

$$q_R^{\mathrm{res}} := q_R\big|_{Q_R^{\mathrm{res}} \times Q_R^{\mathrm{res}}}$$

is called the restriction form of q_R to the σ_R-null singular-boundary part.

Remark 4.72 (Restriction operation performed in this subsection)**.** q_R^{res} is not a new form, but the restriction of q_R, constructed in Section 2, to the closed subspace Q_R^{res} selected by the σ_R-null support and regular-trace vanishing conditions. Accordingly, in this subsection we do not reprove the self-adjoint realization or compactness of q_R, but only treat the closedness of the form domain cut out by the σ_R-null condition and the preservation of boundary cancellation.

Lemma 4.73 (Linearity of the generating class)**.** G_R is a linear subspace of Q_R. Therefore

$$\operatorname{span} G_R = G_R.$$

Proof. By the equivalent representation obtained in the preceding subsection,

$$G_R = \{ f \in T_R : \gamma_R^{\mathrm{reg}} f = 0 \quad \sigma_R\text{-a.e.} \}.$$

By Lemma 4.44, T_R is a linear space, and

$$\gamma_R^{\mathrm{reg}} : T_R \to L^2(\Sigma_R, \sigma_R)$$

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is a linear map. Therefore $G_R = \ker \gamma_R^{\text{reg}}$ is a linear subspace of T_R. Since $T_R \subset Q_R$, we have $G_R \subset Q_R$. Thus G_R is a linear subspace of Q_R, and span $G_R = G_R$ follows. □

Definition 4.74 (Nondegenerate trace core)**.** Define the σ_R-null and blocking candidates on the core by

$$\mathscr{N}_R := \left\{ \varphi \in \mathscr{C}_R : \vartheta_R^{\text{reg}} \varphi = 0,\ \vartheta_R^{\text{sing}} \varphi \neq 0 \right\}.$$

Here $\mathscr{C}_R = \mathscr{D}_R \cap Q_R$ is the trace core, and ϑ_R^{reg} and $\vartheta_R^{\text{sing}}$ are the regular boundary trace and singular boundary trace defined on the core.

Lemma 4.75 (Existence of the nondegenerate trace core)**.**

$$\mathscr{N}_R \neq \varnothing$$

holds. More concretely, there exists

$$\chi \in \mathscr{C}_R$$

such that

$$\vartheta_R^{\text{reg}} \chi = 0, \qquad \vartheta_R^{\text{sing}} \chi = \nu_R \neq 0.$$

Proof. Take $\chi \in C_c^\infty([0,\infty))$ so that

$$\chi(0) = 1, \qquad \chi'(0) = 0.$$

For example, one may take a smooth cut-off that equals 1 near the endpoint and becomes 0 for sufficiently large x. By the definition of the admissible core in Section 2,

$$\chi \in \mathcal{C}_R \subset Q_R.$$

Moreover, all derivatives of χ are bounded with polynomial weights, and hence

$$\chi \in \mathscr{D}_R.$$

Therefore

$$\chi \in \mathscr{C}_R = \mathscr{D}_R \cap Q_R.$$

By the definition of the trace on the core,

$$\vartheta_R^{\text{sing}} \chi = \chi(0) \nu_R = \nu_R.$$

By Lemma 4.27, $\nu_R(\Gamma_R) = 1$, and therefore

$$\nu_R \neq 0.$$

On the other hand, the regular boundary trace is

$$\vartheta_R^{\text{reg}} \chi = \chi'(0) e_R = 0.$$

Thus $\chi \in \mathscr{N}_R$, and

$$\mathscr{N}_R \neq \varnothing$$

follows. ☐

Lemma 4.76 (Sufficient condition for nontriviality)**.** If

$$\mathscr{N}_R \neq \varnothing,$$

then

$$G_R \neq \{0\}.$$

Furthermore, any $\varphi \in \mathscr{N}_R$ gives a nonzero element of G_R.

Proof. Take $\varphi \in \mathscr{N}_R$. We have $\varphi \in \mathscr{C}_R \subset Q_R$, and the trace on the core belongs to the closed-trace graph through the constant sequence

$$\varphi_n = \varphi.$$

Therefore

$$\varphi \in T_R, \qquad \gamma_R^{\mathrm{sing}}\varphi = \vartheta_R^{\mathrm{sing}}\varphi, \qquad \gamma_R^{\mathrm{reg}}\varphi = \vartheta_R^{\mathrm{reg}}\varphi.$$

By the definition of $\varphi \in \mathscr{N}_R$,

$$\vartheta_R^{\mathrm{reg}}\varphi = 0,$$

and hence

$$\gamma_R^{\mathrm{reg}}\varphi = 0.$$

By the equivalent representation from the preceding subsection,

$$G_R = \{f \in T_R : \gamma_R^{\mathrm{reg}} f = 0\},$$

we obtain

$$\varphi \in G_R.$$

Furthermore,

$$\vartheta_R^{\mathrm{sing}}\varphi \neq 0,$$

and hence

$$\gamma_R^{\mathrm{sing}}\varphi \neq 0.$$

If $\varphi = 0$ held as an element of Q_R, then by uniqueness of the continuously defined closed trace one would have to have

$$\gamma_R^{\mathrm{sing}}\varphi = 0.$$

This is a contradiction. Therefore φ is a nonzero element of Q_R, and $G_R \neq \{0\}$ follows. ☐

Corollary 4.77 (Nontriviality of the trace-vanishing generating subspace associated with Γ_R)**.**

$$G_R \neq \{0\}, \qquad Q_R^{\mathrm{res}} \neq \{0\}, \qquad H_R^{\mathrm{res}} \neq \{0\}.$$

Proof. By Lemma 4.75, $\mathscr{N}_R \neq \varnothing$. Therefore Lemma 4.76 gives $G_R \neq \{0\}$. Furthermore, $G_R \subset Q_R^{\mathrm{res}} \subset H_{\alpha,+}$, and since the form norm contains the $H_{\alpha,+}$-norm, a nonzero element remains nonzero also in $H_{\alpha,+}$. Thus $Q_R^{\mathrm{res}} \neq \{0\}$ and $H_R^{\mathrm{res}} \neq \{0\}$. ☐

Remark 4.78 (Positioning of nontriviality)**.** By Lemma 4.75, the nondegenerate trace-core condition

$$\mathscr{N}_R \neq \varnothing$$

is actually satisfied within the construction of this paper. Therefore Lemma 4.76 gives

$$G_R \neq \{0\}.$$

The closure, continuous embedding, and extension of boundary cancellation below hold formally even in the zero-space case, but in this paper, by the above nondegeneracy, we deal with a nonzero σ_R-null singular-boundary subspace.

Lemma 4.79 (Minimality of the closure)**.** Q_R^{res} is the smallest $\|\cdot\|_{q_R}$-closed linear subspace of Q_R containing G_R:

$$Q_R^{\mathrm{res}} = \bigcap \{M \subset Q_R : M \text{ is a } \|\cdot\|_{q_R}\text{-closed linear subspace and } G_R \subset M\}.$$

Proof. By definition,

$$Q_R^{\mathrm{res}} = \overline{\operatorname{span} G_R}^{\|\cdot\|_{q_R}}.$$

By Lemma 4.73,

$$\operatorname{span} G_R = G_R,$$

and hence

$$Q_R^{\mathrm{res}} = \overline{G_R}^{\|\cdot\|_{q_R}}.$$

Thus Q_R^{res} is a closed linear subspace containing G_R.

Conversely, if $M \subset Q_R$ is a $\|\cdot\|_{q_R}$-closed linear subspace satisfying $G_R \subset M$, then M also contains the closure of G_R. That is,

$$Q_R^{\mathrm{res}} = \overline{G_R}^{\|\cdot\|_{q_R}} \subset M.$$

Therefore the intersection representation above holds. □

Theorem 4.80 (Completeness of the σ_R-null trace-vanishing form subspace)**.**

$$(Q_R^{\mathrm{res}}, \|\cdot\|_{q_R})$$

is a Hilbert space. In particular, Q_R^{res} *is a* $\|\cdot\|_{q_R}$*-closed linear subspace of* Q_R.

Proof. By the analytic data of Section 2,

$$(Q_R, \|\cdot\|_{q_R})$$

is a Hilbert space. By Definition 4.71,

$$Q_R^{\mathrm{res}} = \overline{G_R}^{\|\cdot\|_{q_R}},$$

and therefore Q_R^{res} is a closed linear subspace of Q_R. A closed linear subspace of a Hilbert space is again a Hilbert space with respect to the induced norm. Accordingly,

$$(Q_R^{\mathrm{res}}, \|\cdot\|_{q_R})$$

is complete. □

Proposition 4.81 (Continuous embedding into the Hilbert space)**.** *The inclusion map*

$$j_R^{\mathrm{res}} : Q_R^{\mathrm{res}} \hookrightarrow H_{\alpha,+}$$

is continuous. More concretely, for any $u \in Q_R^{\mathrm{res}}$*,*

$$\|u\|_{H_{\alpha,+}} \le \|u\|_{q_R}$$

holds.

Proof. We have $Q_R^{\mathrm{res}} \subset Q_R$, and $\|\cdot\|_{q_R}$ is chosen as

$$\|u\|_{q_R}^2 = q_R[u,u] + \|u\|_{H_{\alpha,+}}^2.$$

Since this is the positively shifted form norm, the right-hand side is nonnegative, and in particular

$$\|u\|_{H_{\alpha,+}}^2 \le \|u\|_{q_R}^2$$

holds. Therefore

$$\|u\|_{H_{\alpha,+}} \le \|u\|_{q_R}.$$

Hence the inclusion map is continuous. ☐

Lemma 4.82 (Closedness of the restriction form)**.** The restriction form

$$q_R^{\mathrm{res}} = q_R\big|_{Q_R^{\mathrm{res}} \times Q_R^{\mathrm{res}}}$$

is a closed lower-bounded form on $H_{\alpha,+}$. Its form domain is Q_R^{res}.

Proof. q_R was constructed in Section 2 as a closed lower-bounded form. Therefore, with respect to the positively shifted form norm $\|\cdot\|_{q_R}$,

$$(Q_R, \|\cdot\|_{q_R})$$

is a Hilbert space. By Theorem 4.80,

$$Q_R^{\mathrm{res}}$$

is a closed subspace of this Hilbert space. Thus

$$(Q_R^{\mathrm{res}}, \|\cdot\|_{q_R})$$

is also a Hilbert space.

The restriction of a closed form to a closed subspace is a closed form. Indeed, if a sequence $\{u_n\}$ in Q_R^{res} is $\|\cdot\|_{q_R}$-Cauchy, then it is also Cauchy in Q_R. By completeness of Q_R, there exists some $u \in Q_R$ such that

$$u_n \to u \quad \text{in } \|\cdot\|_{q_R}.$$

Since Q_R^{res} is closed in Q_R,

$$u \in Q_R^{\mathrm{res}}.$$

Therefore Q_R^{res} is complete with respect to the form norm of the restriction form, and q_R^{res} is closed. Lower boundedness follows immediately because it is merely the restriction of the lower boundedness of q_R. □

Definition 4.83 (Boundary form on the closure)**.** Write the zero extension constructed in the preceding subsection as

$$b_R^{\mathrm{res}} : Q_R^{\mathrm{res}} \times Q_R^{\mathrm{res}} \longrightarrow \mathbb{C}.$$

Namely, for $u, v \in Q_R^{\mathrm{res}}$, take arbitrary approximating sequences

$$u_n, v_n \in G_R, \qquad u_n \to u, \quad v_n \to v \quad \text{in } \|\cdot\|_{q_R}.$$

Then

$$b_R^{\mathrm{res}}[u,v] := \lim_{n\to\infty} b_R[u_n, v_n].$$

Theorem 4.84 (Preservation of boundary cancellation under form closure)**.** *For any $u, v \in Q_R^{\mathrm{res}}$,*

$$b_R^{\mathrm{res}}[u,v] = 0$$

holds. Therefore, the cancellation of the regular-area boundary term that held on the trace-vanishing generating subspace associated with Γ_R is preserved on the entire form-norm closure Q_R^{res}.

Proof. By definition, for any $u, v \in Q_R^{\mathrm{res}}$, one may take sequences

$$u_n, v_n \in G_R$$

such that

$$u_n \to u, \qquad v_n \to v \quad \text{in } \|\cdot\|_{q_R}.$$

By the boundary-cancellation theorem in the preceding subsection, for each n,

$$b_R[u_n, v_n] = 0.$$

Therefore

$$b_R^{\mathrm{res}}[u,v] = \lim_{n\to\infty} b_R[u_n, v_n] = 0.$$

Since this value does not depend on the choice of approximating sequences, b_R^{res} is well-defined as the zero form on Q_R^{res}. □

Corollary 4.85 (Vanishing of the ordinary boundary form when a closed-graph trace exists)**.** *Let $u, v \in Q_R^{\mathrm{res}} \cap T_R$, and suppose that the regular trace γ_R^{reg} for these elements is determined from approximating sequences in Q_R^{res} in the closed-graph sense. Then*

$$b_R[u,v] = 0.$$

Proof. Let $u \in Q_R^{\mathrm{res}} \cap T_R$. By definition, there exists a sequence

$$u_n \in G_R$$

such that

$$u_n \to u \quad \text{in } \|\cdot\|_{q_R}.$$

For every $u_n \in G_R$,

$$\gamma_R^{\mathrm{reg}} u_n = 0 \qquad \sigma_R\text{-a.e.}$$

By the closed-graph assumption, the regular trace corresponding to this limit must be

$$\gamma_R^{\mathrm{reg}} u = 0.$$

Similarly,

$$\gamma_R^{\mathrm{reg}} v = 0.$$

Therefore

$$b_R[u,v] = \int_{\Sigma_R} \omega_R (\gamma_R^{\mathrm{reg}} u)\overline{(\gamma_R^{\mathrm{reg}} v)}\, d\sigma_R = 0.$$

□

Proposition 4.86 (Closure data of the σ_R-null singular-boundary part)**.** *The construction in this subsection gives the following.*

1. *G_R is a linear subspace of Q_R, and since*

$$\mathscr{N}_R \neq \varnothing$$

holds within the present construction, $G_R \neq \{0\}$.

2. $$Q_R^{\mathrm{res}} = \overline{G_R}^{\|\cdot\|_{q_R}}$$

is a $\|\cdot\|_{q_R}$-closed linear subspace of Q_R, and $(Q_R^{\mathrm{res}}, \|\cdot\|_{q_R})$ is a Hilbert space.

3. *The inclusion map*

$$Q_R^{\mathrm{res}} \hookrightarrow H_{\alpha,+}$$

is continuous.

4. *The restriction form*

$$q_R^{\mathrm{res}} = q_R|_{Q_R^{\mathrm{res}} \times Q_R^{\mathrm{res}}}$$

is a closed lower-bounded form.

5. *The regular-trace-type boundary form satisfies*

$$b_R^{\mathrm{res}}[u,v] = 0 \qquad (u, v \in Q_R^{\mathrm{res}}).$$

Proof. Item 1 follows from Lemma 4.73, Lemma 4.75, and Lemma 4.76. Item 2 follows from Theorem 4.80. Item 3 follows from Proposition 4.81. Item 4 follows from Lemma 4.82. Item 5 follows from Theorem 4.84. □

Remark 4.87 (Meaning for the next step)**.** The space Q_R^{res} obtained in this subsection is the form domain satisfying the σ_R-null support and regular-trace vanishing conditions. On this space, the regular-trace-type boundary term vanishes. On the other hand, since the singular boundary trace remains on the ν_R- or $\mathscr{D}_R'$-side, the boundary trace itself is not trivialized. In the next subsection, we apply the representation theorem to the closed lower-bounded form q_R^{res} and construct the corresponding singular-boundary Hilbert space and self-adjoint realization.

4.6. Singular-Boundary Hilbert Space and Friedrichs Realization

In this subsection, from the σ_R-null trace-vanishing form subspace

$$Q_R^{\mathrm{res}} \subset Q_R$$

constructed in the preceding subsection, we construct the corresponding Hilbert space

$$H_R^{\mathrm{res}},$$

and by applying the first representation theorem to the restriction form

$$q_R^{\mathrm{res}} = q_R\big|_{Q_R^{\mathrm{res}} \times Q_R^{\mathrm{res}}},$$

we obtain the self-adjoint operator

$$A_R^{\mathrm{res}}.$$

The operator obtained here does not redefine the global operator associated with q_R constructed in Section 2. It is a partial Friedrichs-type realization corresponding to the closed form q_R^{res} on the singular-boundary subspace cut out by the σ_R-null support and regular-trace vanishing conditions.

From now on, in this section, following the positive-shift convention of Section 4.5, we write as if q_R has been normalized so that

$$q_R^{\mathrm{res}}[u,u] \geq 0 \qquad (u \in Q_R^{\mathrm{res}}).$$

In the lower-bounded case, this should be interpreted as replacing the form by an equivalent form obtained by adding a sufficiently large constant and then using the same notation q_R.

Definition 4.88 (Singular-boundary Hilbert space)**.** Define the σ_R-null singular-boundary Hilbert space by

$$H_R^{\mathrm{res}} := \overline{Q_R^{\mathrm{res}}}^{\|\cdot\|_{H_{\alpha,+}}} \subset H_{\alpha,+}.$$

Equip H_R^{res} with the inner product induced from $H_{\alpha,+}$:

$$\langle u, v\rangle_{H_R^{\mathrm{res}}} := \langle u, v\rangle_{H_{\alpha,+}} \qquad (u, v \in H_R^{\mathrm{res}}).$$

Lemma 4.89 (Closedness of the singular-boundary Hilbert space)**.** H_R^{res} is a closed linear subspace of $H_{\alpha,+}$. In particular, H_R^{res} is a Hilbert space.

Proof. By definition,

$$H_R^{\mathrm{res}} = \overline{Q_R^{\mathrm{res}}}^{\|\cdot\|_{H_{\alpha,+}}},$$

so it is a closed set obtained as a closure in $H_{\alpha,+}$. Moreover, since Q_R^{res} is a linear space, its $H_{\alpha,+}$-norm closure is also a linear space. Therefore H_R^{res} is a closed linear subspace of $H_{\alpha,+}$. A closed linear subspace of a Hilbert space is again a Hilbert space, and the claim follows. ☐

Lemma 4.90 (Density of the form domain)**.** Q_R^{res} is dense in H_R^{res}. Namely,

$$\overline{Q_R^{\mathrm{res}}}^{\|\cdot\|_{H_R^{\mathrm{res}}}} = H_R^{\mathrm{res}}.$$

Proof. This is exactly the definition of H_R^{res}. Indeed, H_R^{res} is defined as the $H_{\alpha,+}$-norm closure of Q_R^{res}, and the norm of H_R^{res} is induced from $H_{\alpha,+}$. Therefore Q_R^{res} is dense in H_R^{res}. ☐

Proposition 4.91 (Continuous embedding from the form domain into the singular-boundary Hilbert space)**.** *The natural inclusion map*

$$\iota_R^{\mathrm{res}} : Q_R^{\mathrm{res}} \hookrightarrow H_R^{\mathrm{res}}$$

is continuous. More concretely, for any $u \in Q_R^{\mathrm{res}}$,

$$\|u\|_{H_R^{\mathrm{res}}} = \|u\|_{H_{\alpha,+}} \le \|u\|_{q_R}$$

holds.

Proof. By Proposition 4.81 of the preceding subsection, for any $u \in Q_R^{\mathrm{res}}$,

$$\|u\|_{H_{\alpha,+}} \le \|u\|_{q_R}$$

holds. Moreover, the norm of H_R^{res} is the restriction of the norm of $H_{\alpha,+}$, and hence

$$\|u\|_{H_R^{\mathrm{res}}} = \|u\|_{H_{\alpha,+}}.$$

The claim follows. ☐

Lemma 4.92 (Closedness and density of the restriction form)**.** The form

$$q_R^{\mathrm{res}} = q_R\big|_{Q_R^{\mathrm{res}} \times Q_R^{\mathrm{res}}}$$

is a densely defined closed nonnegative symmetric form on the Hilbert space H_R^{res}. Its form domain is Q_R^{res}.

Proof. First, density follows from Lemma 4.90.

Next we prove closedness. By Lemma 4.82 of the preceding subsection, q_R^{res} is a closed lower-bounded form on Q_R^{res}, and

$$(Q_R^{\mathrm{res}}, \|\cdot\|_{q_R})$$

is a Hilbert space. In this subsection, by the positive-shift convention, we have

$$q_R^{\mathrm{res}}[u,u] \ge 0.$$

Therefore

$$\|u\|_{q_R}^2 = q_R^{\mathrm{res}}[u,u] + \|u\|_{H_R^{\mathrm{res}}}^2$$

is the form norm of q_R^{res}. Thus Q_R^{res} is complete with respect to this form norm, and q_R^{res} is a closed form on H_R^{res}.

Symmetry is the restriction of the symmetry of q_R constructed in Section 2, and nonnegativity follows from the positive-shift convention. Therefore q_R^{res} is a densely defined closed nonnegative symmetric form on H_R^{res}. ☐

Definition 4.93 (Closed-form Hilbert space)**.** Let

$$\mathcal{Q}_R^{\mathrm{res}}$$

denote the Hilbert space obtained by equipping Q_R^{res} with the inner product

$$\langle u, v\rangle_{q_R^{\mathrm{res}}} := q_R^{\mathrm{res}}[u, v] + \langle u, v\rangle_{H_R^{\mathrm{res}}}.$$

Namely,

$$\mathcal{Q}_R^{\mathrm{res}} := (Q_R^{\mathrm{res}}, \langle\cdot,\cdot\rangle_{q_R^{\mathrm{res}}}).$$

Lemma 4.94 (Continuous noncompact embedding from the form space to the base Hilbert space)**.** The natural map

$$\mathcal{Q}_R^{\mathrm{res}} \longrightarrow H_R^{\mathrm{res}}$$

is continuous. At this level, compactness is not asserted.

Proof. For any $u \in Q_R^{\mathrm{res}}$,

$$\|u\|_{H_R^{\mathrm{res}}}^2 \le q_R^{\mathrm{res}}[u, u] + \|u\|_{H_R^{\mathrm{res}}}^2 = \|u\|_{q_R^{\mathrm{res}}}^2.$$

Thus the inclusion map is continuous with norm at most 1. Compactness is an additional property of the embedding and will be treated in the later subsection on compact resolvent. ☐

Theorem 4.95 (Friedrichs realization of the σ_R-null restricted closed form)**.** *For the closed nonnegative symmetric form*

$$q_R^{\mathrm{res}},$$

there exists a unique nonnegative self-adjoint operator

$$A_R^{\mathrm{res}}$$

on the Hilbert space H_R^{res} satisfying the following.

$$\mathrm{Dom}(A_R^{\mathrm{res}}) = \left\{u \in Q_R^{\mathrm{res}} : \exists\, w \in H_R^{\mathrm{res}} \text{ such that } q_R^{\mathrm{res}}[u, v] = \langle w, v\rangle_{H_R^{\mathrm{res}}} \text{ for all } v \in Q_R^{\mathrm{res}}\right\},$$

and, defining

$$A_R^{\mathrm{res}} u := w$$

for such w, one has

$$q_R^{\mathrm{res}}[u, v] = \langle A_R^{\mathrm{res}} u, v\rangle_{H_R^{\mathrm{res}}}$$

for all

$$u \in \mathrm{Dom}(A_R^{\mathrm{res}}), \qquad v \in Q_R^{\mathrm{res}}.$$

Proof. By Lemma 4.92,

$$q_R^{\mathrm{res}}$$

is a densely defined closed nonnegative symmetric form on the Hilbert space H_R^{res}. Therefore the first representation theorem for closed lower-bounded forms, namely the Friedrichs-type representation theorem, applies. This theorem yields a unique nonnegative self-adjoint operator

$$A_R^{\mathrm{res}},$$

and this operator represents the closed form q_R^{res}.

More concretely, the first representation theorem characterizes the operator as follows. For $u \in Q_R^{\mathrm{res}}$, when the linear functional

$$v \longmapsto q_R^{\mathrm{res}}[u,v]$$

is continuous with respect to the H_R^{res}-norm, the Riesz representation theorem gives a unique

$$w \in H_R^{\mathrm{res}}$$

such that

$$q_R^{\mathrm{res}}[u,v] = \langle w, v\rangle_{H_R^{\mathrm{res}}} \qquad (v \in Q_R^{\mathrm{res}}).$$

The set of all such u is defined to be

$$\mathrm{Dom}(A_R^{\mathrm{res}}),$$

and

$$A_R^{\mathrm{res}} u = w.$$

The operator obtained by this construction is nonnegative and self-adjoint, and uniquely represents the closed form q_R^{res}. Uniqueness also follows from the uniqueness part of the first representation theorem. ☐

Corollary 4.96 (Nonnegativity)**.** *For any* $u \in \mathrm{Dom}(A_R^{\mathrm{res}})$*,*

$$\langle A_R^{\mathrm{res}} u, u\rangle_{H_R^{\mathrm{res}}} = q_R^{\mathrm{res}}[u,u] \geq 0$$

holds. Therefore

$$A_R^{\mathrm{res}} \geq 0.$$

Proof. In the representation formula of Theorem 4.95, set $v = u$. Then

$$\langle A_R^{\mathrm{res}} u, u\rangle_{H_R^{\mathrm{res}}} = q_R^{\mathrm{res}}[u,u].$$

By the positive-shift convention, the right-hand side is nonnegative. Therefore A_R^{res} is a nonnegative self-adjoint operator. ☐

Remark 4.97 (Distinction from the global operator)**.** When the global operator associated with q_R obtained in Section 2 is written as

$$A_R,$$

the operator

$$A_R^{\mathrm{res}}$$

of this subsection is not in general defined as a simple operator restriction of A_R. A_R^{res} is the operator obtained by restricting the closed form

$$q_R$$

to the σ_R-null restricted closed form domain

$$Q_R^{\mathrm{res}}$$

and then representing that restricted form on the base Hilbert space

$$H_R^{\mathrm{res}}.$$

Therefore, unless it is separately proved that

$$H_R^{\mathrm{res}}$$

is an operator-theoretic reducing subspace for the global operator A_R, we do not write

$$A_R^{\mathrm{res}} = A_R|_{H_R^{\mathrm{res}}}.$$

What is needed in this paper is not a simple restriction of the global operator, but

$$A_R^{\mathrm{res}}$$

as the closed-form realization of the σ_R-null restricted closed form.

Definition 4.98 (Positive shifted operator)**.** Define the σ_R-null positive shifted operator on K_R by

$$L_R^{\mathrm{res}} := A_R^{\mathrm{res}} + I_{H_R^{\mathrm{res}}}.$$

Here $I_{H_R^{\mathrm{res}}}$ is the identity operator on H_R^{res}. Its domain is

$$\mathrm{Dom}(L_R^{\mathrm{res}}) = \mathrm{Dom}(A_R^{\mathrm{res}}).$$

Lemma 4.99 (Basic properties of the positive shifted operator)**.** L_R^{res} is a self-adjoint operator on H_R^{res}, and

$$L_R^{\mathrm{res}} \geq I_{H_R^{\mathrm{res}}}.$$

In particular,

$$0 \in \rho(L_R^{\mathrm{res}}),$$

and

$$\|(L_R^{\mathrm{res}})^{-1}\|_{\mathcal{B}(H_R^{\mathrm{res}})} \leq 1.$$

Proof. By Theorem 4.95, A_R^{res} is self-adjoint. Adding the bounded self-adjoint operator $I_{H_R^{\mathrm{res}}}$ to a self-adjoint operator gives

$$L_R^{\mathrm{res}} = A_R^{\mathrm{res}} + I_{H_R^{\mathrm{res}}},$$

which is self-adjoint on the same domain.

Moreover, by Corollary 4.96,

$$A_R^{\mathrm{res}} \geq 0.$$

Therefore

$$L_R^{\mathrm{res}} = A_R^{\mathrm{res}} + I_{H_R^{\mathrm{res}}} \geq I_{H_R^{\mathrm{res}}}.$$

By the spectral theorem,

$$\sigma(L_R^{\mathrm{res}}) \subset [1, \infty).$$

Hence $0 \notin \sigma(L_R^{\mathrm{res}})$, that is,

$$0 \in \rho(L_R^{\mathrm{res}}).$$

Furthermore,

$$\|(L_R^{\mathrm{res}})^{-1}\| = \sup_{\lambda \in \sigma(L_R^{\mathrm{res}})} \frac{1}{\lambda} \leq 1.$$

□

Definition 4.100 (Energy inner product on the singular-boundary solution space)**.** Define the energy inner product associated with L_R^{res} by

$$\langle u, v \rangle_{L_R^{\mathrm{res}}} := \langle (L_R^{\mathrm{res}})^{1/2} u, (L_R^{\mathrm{res}})^{1/2} v \rangle_{H_R^{\mathrm{res}}}.$$

Its domain is

$$\mathrm{Dom}\big((L_R^{\mathrm{res}})^{1/2}\big) = Q_R^{\mathrm{res}},$$

and

$$\|u\|^2_{L_R^{\mathrm{res}}} = q_R^{\mathrm{res}}[u, u] + \|u\|^2_{H_R^{\mathrm{res}}}$$

holds.

Lemma 4.101 (Agreement of square-root domain and form domain)**.**

$$\mathrm{Dom}\big((L_R^{\mathrm{res}})^{1/2}\big) = Q_R^{\mathrm{res}},$$

and for any $u, v \in Q_R^{\mathrm{res}}$,

$$\langle (L_R^{\mathrm{res}})^{1/2} u, (L_R^{\mathrm{res}})^{1/2} v \rangle_{H_R^{\mathrm{res}}} = q_R^{\mathrm{res}}[u, v] + \langle u, v \rangle_{H_R^{\mathrm{res}}}$$

holds.

Proof. A_R^{res} is the representing operator of the closed nonnegative form q_R^{res}. By the square-root representation in the first representation theorem,

$$\mathrm{Dom}\big((A_R^{\mathrm{res}} + I)^{1/2}\big) = Q_R^{\mathrm{res}},$$

and its form is given by

$$q_R^{\mathrm{res}}[u, v] + \langle u, v \rangle_{H_R^{\mathrm{res}}}.$$

By Definition 4.98,

$$L_R^{\mathrm{res}} = A_R^{\mathrm{res}} + I,$$

and the claim follows. □

Proposition 4.102 (Propagation of nondegeneracy to the Hilbert space)**.** *If*

$$G_R \neq \{0\},$$

then

$$Q_R^{\mathrm{res}} \neq \{0\}, \qquad H_R^{\mathrm{res}} \neq \{0\}.$$

In particular, if the condition

$$\mathscr{N}_R \neq \varnothing$$

of Lemma 4.76 holds, then H_R^{res} is a nonzero Hilbert space.

Proof. Assume $G_R \neq \{0\}$, and take a nonzero element $g \in G_R$. Since $G_R \subset Q_R^{\mathrm{res}}$,

$$Q_R^{\mathrm{res}} \neq \{0\}.$$

Moreover, $Q_R^{\mathrm{res}} \subset H_{\alpha,+}$, and the form norm contains the $H_{\alpha,+}$-norm. Therefore g, being nonzero as an element of Q_R, is also nonzero as an element of $H_{\alpha,+}$. Thus its $H_{\alpha,+}$-closure,

$$H_R^{\mathrm{res}},$$

is also nonzero.

The final assertion follows from Lemma 4.75 and Lemma 4.76. ☐

Proposition 4.103 (Output of this subsection)**.** *This subsection has constructed the following objects:*

$$H_R^{\mathrm{res}}, \qquad A_R^{\mathrm{res}}, \qquad L_R^{\mathrm{res}}.$$

They satisfy the following properties.

1. *H_R^{res} is a closed Hilbert subspace of $H_{\alpha,+}$.*
2. *Q_R^{res} is dense in H_R^{res}, and q_R^{res} is a closed nonnegative symmetric form on H_R^{res}.*
3. *A_R^{res} is the unique nonnegative self-adjoint representing operator of q_R^{res}.*
4. $$L_R^{\mathrm{res}} = A_R^{\mathrm{res}} + I_{H_R^{\mathrm{res}}}$$
   is self-adjoint, and
   $$L_R^{\mathrm{res}} \geq I_{H_R^{\mathrm{res}}}.$$
5. *A_R^{res} is not a redefinition of the global operator of Section 2, but a Friedrichs-type realization on the σ_R-null trace-vanishing form subspace.*

Proof. Item 1 follows from Lemma 4.89. Item 2 follows from Lemma 4.90 and Lemma 4.92. Item 3 follows from Theorem 4.95. Item 4 follows from Definition 4.98 and Lemma 4.99. Item 5 follows from the nature of the construction in this subsection, in particular from the fact that A_R^{res} was constructed from the restriction form of q_R to the closed subspace Q_R^{res}. ☐

Remark 4.104 (Transition to the next step)**.** The objects obtained in this subsection,

$$H_R^{\mathrm{res}}, \qquad A_R^{\mathrm{res}}, \qquad L_R^{\mathrm{res}},$$

are the operator-theoretic foundation on the singular-boundary subspace. In the next subsection, we introduce a strongly continuous transport group on this Hilbert space and construct its anti-self-adjoint generator by Stone's theorem. After that, through the distribution kernel representation, we formulate the distribution kernel associated with K_R as an object of $\mathscr{D}'_R \widehat{\otimes} \mathscr{D}'_R$.

4.7. *Transport Group on K_R, Anti-Self-Adjoint Generator, and Kernel Representation*

In this subsection, we introduce the canonical strongly continuous unitary transport group on the σ_R-null singular-boundary Hilbert space

$$H_R^{\mathrm{res}}$$

constructed in the preceding subsection, and obtain its anti-self-adjoint generator by Stone's theorem. Furthermore, by evaluating this transport group on the dual side of the test space

$$\mathscr{D}_R \subset H_{\alpha,+},$$

we define the distribution kernel associated with K_R not as an ordinary function kernel, but as an element of

$$\mathscr{D}'_R \widehat{\otimes} \mathscr{D}'_R.$$

What is important here is that the transport group and its generator are constructed internally within the σ_R-null singular-boundary subspace

$$H_R^{\mathrm{res}}.$$

Accordingly, the transport in this subsection is not an externally given geometric flow, but the canonical unitary transport obtained from the spectral calculus of the positive shifted operator

$$L_R^{\mathrm{res}} = A_R^{\mathrm{res}} + I.$$

Definition 4.105 (Orthogonal projection onto K_R)**.** By Lemma 4.89, H_R^{res} is a closed linear subspace of $H_{\alpha,+}$. Therefore, there exists a unique orthogonal projection on $H_{\alpha,+}$

$$\Pi_R^+ : H_{\alpha,+} \longrightarrow H_R^{\mathrm{res}}.$$

We call this projection the orthogonal projection onto K_R on the one-sided analytic Hilbert space.

Remark 4.106 (Distinction from the subsequent integrated projector)**.** Π_R^+ is the local Hilbert projection from within the one-sided analytic Hilbert space $H_{\alpha,+}$ onto H_R^{res}. The unsuperscripted projection

$$\Pi_R$$

used in the subsequent orthogonal-decomposition framework is the projector onto the singular-boundary subspace after it has been lifted to the ambient Hilbert-space setting, and its object space is different from that of the one-sided projector Π_R^+ in this subsection. Accordingly, in this subsection Π_R^+ denotes the one-sided projection, while Π_R without $+$ is reserved for Section 5.

Definition 4.107 (Canonical transport group on K_R)**.** By Lemma 4.99, the positive shifted operator

$$L_R^{\mathrm{res}}$$

is a self-adjoint operator on H_R^{res}. Therefore, by the spectral theorem, define

$$U_R(t) := e^{-itL_R^{\mathrm{res}}} \qquad (t \in \mathbb{R}).$$

We call this the canonical transport group on K_R on the σ_R-null singular-boundary subspace.

Theorem 4.108 (Strong continuity of the canonical transport group on K_R)**.** *The family*

$$\{U_R(t)\}_{t\in\mathbb{R}}$$

is a strongly continuous unitary group on H_R^{res}*. Namely,*

$$U_R(0) = I_{H_R^{\mathrm{res}}}, \qquad U_R(t+s) = U_R(t)U_R(s), \qquad U_R(t)^* = U_R(-t)$$

hold, and for every $u \in H_R^{\mathrm{res}}$*,*

$$\lim_{t\to 0} \|U_R(t)u - u\|_{H_R^{\mathrm{res}}} = 0.$$

Proof. Since L_R^{res} is self-adjoint, the spectral theorem implies that

$$e^{-itL_R^{\mathrm{res}}}$$

is a unitary operator for each $t \in \mathbb{R}$. Moreover, from the multiplicative law of the function

$$\lambda \longmapsto e^{-it\lambda},$$

we obtain

$$U_R(t+s) = U_R(t)U_R(s), \qquad U_R(0) = I_{H_R^{\mathrm{res}}}.$$

For the adjoint, we also have

$$U_R(t)^* = e^{itL_R^{\mathrm{res}}} = U_R(-t).$$

We prove strong continuity. Let E_L be the spectral measure of L_R^{res}. For any $u \in H_R^{\mathrm{res}}$,

$$\|U_R(t)u - u\|^2_{H_R^{\mathrm{res}}} = \int_{\sigma(L_R^{\mathrm{res}})} |e^{-it\lambda} - 1|^2 \, d\|E_L(\lambda)u\|^2.$$

For each λ,

$$|e^{-it\lambda} - 1|^2 \to 0 \qquad (t \to 0),$$

and

$$|e^{-it\lambda} - 1|^2 \le 4.$$

The dominating measure on the right-hand side is the finite measure

$$d\|E_L(\lambda)u\|^2.$$

Therefore, by the dominated convergence theorem,

$$\|U_R(t)u - u\|_{H_R^{\mathrm{res}}} \to 0.$$

Thus strong continuity holds. ☐

Definition 4.109 (Anti-self-adjoint transport generator)**.** Define

$$B_R := -iL_R^{\mathrm{res}}.$$

Its domain is

$$\mathrm{Dom}(B_R) := \mathrm{Dom}(L_R^{\mathrm{res}}).$$

Theorem 4.110 (Stone generator). *B_R is an anti-self-adjoint operator on H_R^{res}, and*

$$B_R^* = -B_R.$$

Furthermore,

$$U_R(t) = e^{tB_R},$$

and for every

$$u \in \mathrm{Dom}(B_R),$$

one has

$$B_R u = \lim_{t \to 0} \frac{U_R(t)u - u}{t}$$

in the sense of the H_R^{res}-strong limit. Conversely, any $u \in H_R^{\mathrm{res}}$ for which this strong limit exists belongs to $\mathrm{Dom}(B_R)$.

Proof. Since L_R^{res} is self-adjoint,

$$(-iL_R^{\mathrm{res}})^* = iL_R^{\mathrm{res}} = -(-iL_R^{\mathrm{res}}).$$

Therefore

$$B_R^* = -B_R,$$

and B_R is anti-self-adjoint.

Also, by definition,

$$e^{tB_R} = e^{-itL_R^{\mathrm{res}}} = U_R(t).$$

By Stone's theorem, the generator of the strongly continuous unitary group

$$\{U_R(t)\}_{t \in \mathbb{R}}$$

is the anti-self-adjoint operator B_R, and its domain consists of exactly those u for which the difference quotient

$$\frac{U_R(t)u - u}{t}$$

has a strong H_R^{res}-limit. This limiting value is $B_R u$. The claim follows. ☐

Proposition 4.111 (Preservation of the σ_R-null singular-boundary subspace). *For every $t \in \mathbb{R}$,*

$$U_R(t) H_R^{\mathrm{res}} = H_R^{\mathrm{res}}.$$

Furthermore, also for the form domain,

$$U_R(t) Q_R^{\mathrm{res}} = Q_R^{\mathrm{res}},$$

and

$$\|U_R(t)u\|_{H_R^{\mathrm{res}}} = \|u\|_{H_R^{\mathrm{res}}}, \qquad \|U_R(t)u\|_{L_R^{\mathrm{res}}} = \|u\|_{L_R^{\mathrm{res}}}$$

hold.

Proof. The first assertion follows immediately from the fact that $U_R(t)$ is defined as a unitary operator on H_R^{res}.

Next, consider the form domain. By Lemma 4.101,

$$Q_R^{\mathrm{res}} = \mathrm{Dom}\big((L_R^{\mathrm{res}})^{1/2}\big).$$

Since $U_R(t) = e^{-itL_R^{\mathrm{res}}}$ is a Borel function of L_R^{res}, the spectral calculus gives

$$U_R(t)(L_R^{\mathrm{res}})^{1/2} = (L_R^{\mathrm{res}})^{1/2} U_R(t).$$

Therefore, if

$$u \in \mathrm{Dom}\big((L_R^{\mathrm{res}})^{1/2}\big),$$

then

$$U_R(t)u \in \mathrm{Dom}\big((L_R^{\mathrm{res}})^{1/2}\big).$$

Applying the same argument to $U_R(-t)$ gives the reverse inclusion, and hence

$$U_R(t)Q_R^{\mathrm{res}} = Q_R^{\mathrm{res}}.$$

For norm preservation, preservation of the H_R^{res}-norm follows from unitarity. Moreover,

$$\begin{aligned}
\|U_R(t)u\|^2_{L_R^{\mathrm{res}}} &= \|(L_R^{\mathrm{res}})^{1/2} U_R(t)u\|^2_{H_R^{\mathrm{res}}} \\
&= \|U_R(t)(L_R^{\mathrm{res}})^{1/2} u\|^2_{H_R^{\mathrm{res}}} \\
&= \|(L_R^{\mathrm{res}})^{1/2} u\|^2_{H_R^{\mathrm{res}}} = \|u\|^2_{L_R^{\mathrm{res}}}.
\end{aligned}$$

The claim follows. □

Definition 4.112 (Extended transport group on the one-sided analytic Hilbert space)**.** Define the operator on $H_{\alpha,+}$

$$\widetilde{U}_R(t) := U_R(t)\Pi_R^+ + (I_{H_{\alpha,+}} - \Pi_R^+) \qquad (t \in \mathbb{R}).$$

Namely, it acts as $U_R(t)$ on H_R^{res} and as the identity operator on its orthogonal complement.

Lemma 4.113 (Properties of the extended transport group)**.**

$$\{\widetilde{U}_R(t)\}_{t\in\mathbb{R}}$$

is a strongly continuous unitary group on $H_{\alpha,+}$, and satisfies

$$\widetilde{U}_R(t)\Pi_R^+ = \Pi_R^+\widetilde{U}_R(t) = U_R(t)\Pi_R^+.$$

Proof. Every $h \in H_{\alpha,+}$ decomposes uniquely as

$$h = h_{\mathrm{res}} + h_\perp, \qquad h_{\mathrm{res}} \in H_R^{\mathrm{res}}, \quad h_\perp \in (H_R^{\mathrm{res}})^\perp.$$

Then

$$\widetilde{U}_R(t)h = U_R(t)h_{\mathrm{res}} + h_\perp.$$

Since $U_R(t)$ is unitary on H_R^{res} and the operator is the identity on the orthogonal complement, $\widetilde{U}_R(t)$ is unitary on $H_{\alpha,+}$.

The group law follows from

$$\begin{aligned}\widetilde{U}_R(t+s)h &= U_R(t+s)h_{\mathrm{res}} + h_\perp \\ &= U_R(t)U_R(s)h_{\mathrm{res}} + h_\perp = \widetilde{U}_R(t)\widetilde{U}_R(s)h.\end{aligned}$$

Strong continuity follows from

$$\|\widetilde{U}_R(t)h - h\|_{H_{\alpha,+}} = \|U_R(t)h_{\mathrm{res}} - h_{\mathrm{res}}\|_{H_R^{\mathrm{res}}},$$

and the strong continuity of $U_R(t)$.

Finally, since $\Pi_R^+ h = h_{\mathrm{res}}$,

$$\widetilde{U}_R(t)\Pi_R^+ h = U_R(t)h_{\mathrm{res}},$$

and

$$\Pi_R^+ \widetilde{U}_R(t)h = \Pi_R^+(U_R(t)h_{\mathrm{res}} + h_\perp) = U_R(t)h_{\mathrm{res}}.$$

Thus the displayed commutation relations hold. □

Definition 4.114 (Transport kernel family)**.** For each $t \in \mathbb{R}$, define the sesquilinear form on $\mathscr{D}_R \times \mathscr{D}_R$ by

$$\mathcal{K}_{R,t}(\varphi,\psi) := \left\langle \widetilde{U}_R(t)\varphi, \psi \right\rangle_{H_{\alpha,+}} \qquad (\varphi,\psi \in \mathscr{D}_R).$$

By the kernel theorem, write the corresponding distribution kernel as

$$\mathsf{K}_{R,t}^{\mathrm{dist}} \in \mathscr{D}_R' \widehat{\otimes} \mathscr{D}_R'.$$

Namely,

$$\left\langle \mathsf{K}_{R,t}^{\mathrm{dist}}, \varphi \otimes \psi \right\rangle := \mathcal{K}_{R,t}(\varphi,\psi).$$

In particular,

$$\mathsf{K}_R^{\mathrm{dist}} := \mathsf{K}_{R,1}^{\mathrm{dist}}$$

is called the transport distribution kernel at the canonical time.

Lemma 4.115 (Distribution-kernel property of the transport kernel family)**.** For any $t \in \mathbb{R}$,

$$\mathsf{K}_{R,t}^{\mathrm{dist}} \in \mathscr{D}_R' \widehat{\otimes} \mathscr{D}_R'$$

is well-defined. Furthermore, for any $m > \alpha + \frac{1}{2}$, there exists a constant $C_{\alpha,m} > 0$ such that

$$|\mathcal{K}_{R,t}(\varphi,\psi)| \le C_{\alpha,m}^2 \, p_{m,0}(\varphi) p_{m,0}(\psi)$$

for all $\varphi, \psi \in \mathscr{D}_R$.

Proof. Since $\widetilde{U}_R(t)$ is a unitary operator on $H_{\alpha,+}$,

$$\begin{aligned}|\mathcal{K}_{R,t}(\varphi,\psi)| &= |\langle \widetilde{U}_R(t)\varphi,\psi\rangle_{H_{\alpha,+}}| \\ &\le \|\widetilde{U}_R(t)\varphi\|_{H_{\alpha,+}}\|\psi\|_{H_{\alpha,+}} \\ &= \|\varphi\|_{H_{\alpha,+}}\|\psi\|_{H_{\alpha,+}}.\end{aligned}$$

By Lemma 4.9, for any $m > \alpha + \frac{1}{2}$,

$$\|\varphi\|_{H_{\alpha,+}} \le C_{\alpha,m}p_{m,0}(\varphi), \qquad \|\psi\|_{H_{\alpha,+}} \le C_{\alpha,m}p_{m,0}(\psi).$$

Therefore the displayed estimate follows.

This estimate implies that

$$\mathcal{K}_{R,t} : \mathscr{D}_R \times \mathscr{D}_R \to \mathbb{C}$$

is a continuous sesquilinear form. By the kernel theorem of Section 4.1, there exists a unique distribution kernel

$$\mathsf{K}_{R,t}^{\mathrm{dist}} \in \mathscr{D}_R' \widehat{\otimes} \mathscr{D}_R'.$$

□

Remark 4.116 (Not an ordinary function kernel)**.** $\mathsf{K}_{R,t}^{\mathrm{dist}}$ is not defined as a pointwise function

$$K_{R,t}(x,y).$$

It is a distribution kernel assigning

$$\langle \widetilde{U}_R(t)\varphi,\psi\rangle_{H_{\alpha,+}}$$

to a pair of test functions

$$\varphi,\psi \in \mathscr{D}_R.$$

Therefore, in order to treat $\mathsf{K}_{R,t}^{\mathrm{dist}}$ as an ordinary integral kernel, additional regularity, such as Hilbert–Schmidt property or representability of the Schwartz kernel by a function, must be proved separately. This paper assumes no such function-kernel representation.

Definition 4.117 (Difference-quotient kernel family)**.** For $t \neq 0$, define the bounded operator

$$D_R(t) := \frac{\widetilde{U}_R(t) - I_{H_{\alpha,+}}}{t}.$$

Let the corresponding sesquilinear form be

$$\mathcal{D}_{R,t}(\varphi,\psi) := \langle D_R(t)\varphi,\psi\rangle_{H_{\alpha,+}},$$

and write its distribution kernel as

$$D_{R,t}^{\mathrm{dist}} \in \mathscr{D}_R' \widehat{\otimes} \mathscr{D}_R'.$$

Namely,

$$\left\langle D_{R,t}^{\mathrm{dist}}, \varphi \otimes \psi \right\rangle = \mathcal{D}_{R,t}(\varphi,\psi).$$

Lemma 4.118 (Existence of the difference-quotient kernel family)**.** For each $t \neq 0$,

$$D_{R,t}^{\mathrm{dist}} \in \mathscr{D}_R' \widehat{\otimes} \mathscr{D}_R'$$

is well-defined.

Proof. $D_R(t)$ is a bounded operator satisfying

$$\|D_R(t)\|_{\mathcal{B}(H_{\alpha,+})} \leq \frac{\|\widetilde{U}_R(t)\| + 1}{|t|} = \frac{2}{|t|}.$$

Therefore, for $\varphi, \psi \in \mathscr{D}_R$,

$$\begin{aligned} |\mathcal{D}_{R,t}(\varphi, \psi)| &\leq \|D_R(t)\| \, \|\varphi\|_{H_{\alpha,+}} \|\psi\|_{H_{\alpha,+}} \\ &\leq \frac{2C_{\alpha,m}^2}{|t|} p_{m,0}(\varphi) p_{m,0}(\psi). \end{aligned}$$

Thus $\mathcal{D}_{R,t}$ is a continuous sesquilinear form on $\mathscr{D}_R \times \mathscr{D}_R$, and the kernel theorem gives a unique distribution kernel

$$D_{R,t}^{\mathrm{dist}}.$$

□

Definition 4.119 (Yosida-regularized generator and difference-quotient kernels)**.** For $\varepsilon > 0$, define the Yosida regularizer on H_R^{res} by

$$J_R(\varepsilon) := (I + \varepsilon L_R^{\mathrm{res}})^{-1}.$$

Define the regularized transport generator on $H_{\alpha,+}$ by

$$B_R^{(\varepsilon)} := B_R J_R(\varepsilon) \Pi_R^+ = -i L_R^{\mathrm{res}} (I + \varepsilon L_R^{\mathrm{res}})^{-1} \Pi_R^+.$$

For $t \neq 0$, define the regularized difference quotient by

$$D_R^{(\varepsilon)}(t) := \frac{\widetilde{U}_R(t) - I_{H_{\alpha,+}}}{t} J_R(\varepsilon) \Pi_R^+.$$

The corresponding distribution kernels are denoted by

$$B_R^{(\varepsilon),\mathrm{dist}}, \qquad D_{R,t}^{(\varepsilon),\mathrm{dist}}.$$

They are defined by the pairings

$$\left\langle B_R^{(\varepsilon),\mathrm{dist}}, \varphi \otimes \psi \right\rangle := \left\langle B_R^{(\varepsilon)} \varphi, \psi \right\rangle_{H_{\alpha,+}},$$

and

$$\left\langle D_{R,t}^{(\varepsilon),\mathrm{dist}}, \varphi \otimes \psi \right\rangle := \left\langle D_R^{(\varepsilon)}(t) \varphi, \psi \right\rangle_{H_{\alpha,+}}.$$

Lemma 4.120 (Unconditional existence of the regularized generator kernel)**.** For every $\varepsilon > 0$,

$$B_R^{(\varepsilon),\mathrm{dist}} \in \mathscr{D}_R' \widehat{\otimes} \mathscr{D}_R'$$

is well-defined. Moreover, for $t \neq 0$,

$$D_{R,t}^{(\varepsilon),\mathrm{dist}} \in \mathscr{D}_R' \widehat{\otimes} \mathscr{D}_R'$$

is well-defined.

Proof. By the functional calculus for the positive self-adjoint operator L_R^{res},

$$\left\| L_R^{\mathrm{res}} (I + \varepsilon L_R^{\mathrm{res}})^{-1} \right\| \leq \varepsilon^{-1}.$$

Hence

$$\|B_R^{(\varepsilon)}\| \leq \varepsilon^{-1}.$$

Therefore, for $m > \alpha + \frac{1}{2}$,

$$\begin{aligned} \left| \left\langle B_R^{(\varepsilon)} \varphi, \psi \right\rangle_{H_{\alpha,+}} \right| &\leq \varepsilon^{-1} \|\varphi\|_{H_{\alpha,+}} \|\psi\|_{H_{\alpha,+}} \\ &\leq \varepsilon^{-1} C_{\alpha,m}^2 p_{m,0}(\varphi) p_{m,0}(\psi). \end{aligned}$$

Thus $B_R^{(\varepsilon)}$ defines a continuous sesquilinear form on $\mathscr{D}_R \times \mathscr{D}_R$, and the kernel theorem gives $B_R^{(\varepsilon),\mathrm{dist}}$.

The operator $D_R^{(\varepsilon)}(t)$ is bounded because it is the product of the bounded difference quotient $D_R(t)$ and the bounded operator $J_R(\varepsilon)\Pi_R^+$. The same estimate as in Lemma 4.118 therefore gives $D_{R,t}^{(\varepsilon),\mathrm{dist}}$. ☐

Theorem 4.121 (Unconditional weak limit of the regularized difference-quotient kernels)**.** *For every* $\varepsilon > 0$,

$$D_{R,t}^{(\varepsilon),\mathrm{dist}} \longrightarrow B_R^{(\varepsilon),\mathrm{dist}} \qquad (t \to 0)$$

in the weak topology of

$$\mathscr{D}_R' \widehat{\otimes} \mathscr{D}_R'.$$

Equivalently, for all

$$\varphi, \psi \in \mathscr{D}_R,$$

one has

$$\lim_{t \to 0} \left\langle D_{R,t}^{(\varepsilon),\mathrm{dist}}, \varphi \otimes \psi \right\rangle = \left\langle B_R^{(\varepsilon),\mathrm{dist}}, \varphi \otimes \psi \right\rangle.$$

Proof. For $\varphi \in H_{\alpha,+}$, the vector

$$J_R(\varepsilon)\Pi_R^+ \varphi$$

belongs to $\mathrm{Dom}(B_R)$. By Stone's theorem,

$$\frac{\widetilde{U}_R(t) J_R(\varepsilon)\Pi_R^+ \varphi - J_R(\varepsilon)\Pi_R^+ \varphi}{t} \longrightarrow B_R J_R(\varepsilon)\Pi_R^+ \varphi = B_R^{(\varepsilon)} \varphi$$

strongly in $H_{\alpha,+}$. Pairing with any $\psi \in \mathscr{D}_R$ gives the required scalar convergence. Since $B_R^{(\varepsilon)}$ defines a continuous sesquilinear form on $\mathscr{D}_R \times \mathscr{D}_R$, this scalar convergence is precisely the weak distributional convergence of the corresponding kernels. ☐

Definition 4.122 (Generator-admissible core for the unsmoothed kernel)**.** Define

$$\mathscr{D}_{R,B} := \left\{ \varphi \in \mathscr{D}_R : \Pi_R^+ \varphi \in \mathrm{Dom}(B_R) \right\}.$$

Equip $\mathscr{D}_{R,B}$ with the graph topology generated by the seminorms of $\mathscr{D}_R$ together with seminorms controlling

$$B_R \Pi_R^+ \varphi$$

as a $\mathscr{D}'_R$-valued distribution. On this core define

$$\widetilde{B}_R \varphi := B_R \Pi_R^+ \varphi.$$

Theorem 4.123 (Core-level unsmoothed generator kernel)**.** *On the generator-admissible core $\mathscr{D}_{R,B}$, the unsmoothed difference quotient has the weak distributional limit*

$$D_{R,t}^{\mathrm{dist}} \longrightarrow B_R^{\mathrm{dist}}$$

as a kernel on

$$\mathscr{D}_{R,B} \times \mathscr{D}_R.$$

The limiting kernel is represented by

$$\left\langle B_R^{\mathrm{dist}}, \varphi \otimes \psi \right\rangle = \left\langle \widetilde{B}_R \varphi, \psi \right\rangle_{\mathscr{D}'_R, \mathscr{D}_R}, \qquad \varphi \in \mathscr{D}_{R,B}, \quad \psi \in \mathscr{D}_R.$$

Proof. For $\varphi \in \mathscr{D}_{R,B}$, one has

$$\Pi_R^+ \varphi \in \mathrm{Dom}(B_R).$$

Hence Stone's theorem gives

$$\frac{U_R(t)\Pi_R^+\varphi - \Pi_R^+\varphi}{t} \longrightarrow B_R \Pi_R^+ \varphi = \widetilde{B}_R \varphi$$

strongly in H_R^{res}, and therefore weakly in $\mathscr{D}'_R$. Pairing with $\psi \in \mathscr{D}_R$ gives the asserted weak kernel limit. The graph topology on $\mathscr{D}_{R,B}$ makes

$$\widetilde{B}_R : \mathscr{D}_{R,B} \to \mathscr{D}'_R$$

continuous, so the resulting sesquilinear form is continuous on $\mathscr{D}_{R,B} \times \mathscr{D}_R$. ☐

Definition 4.124 (Optional unsmoothed generator kernel on the full test space)**.** The unregularized generator kernel on the full test space is an optional object. It is not used as an unconditional input in the subsequent comparison argument. If the difference-quotient kernel family

$$\{D_{R,t}^{\mathrm{dist}}\}_{t \neq 0}$$

has a limit as $t \to 0$ in the weak topology of

$$\mathscr{D}'_R \widehat{\otimes} \mathscr{D}'_R,$$

write this limit as

$$B_R^{\mathrm{dist}} := \operatorname*{w\text{-}lim}_{t \to 0} D_{R,t}^{\mathrm{dist}},$$

and call it the distribution kernel of the transport generator. Namely, B_R^{dist} is defined by this limit when, for all

$$\varphi, \psi \in \mathscr{D}_R,$$

one has

$$\left\langle B_R^{\mathrm{dist}}, \varphi \otimes \psi \right\rangle = \lim_{t \to 0} \left\langle D_{R,t}^{\mathrm{dist}}, \varphi \otimes \psi \right\rangle.$$

Theorem 4.125 (Conditional unsmoothed generator kernel on the full test space)**.** *Assume the following condition:*

$$\mathscr{D}_R \subset \mathrm{Dom}(\widetilde{B}_R),$$

where

$$\widetilde{B}_R := B_R \Pi_R^+$$

is defined on

$$\mathrm{Dom}(\widetilde{B}_R) := \{h \in H_{\alpha,+} : \Pi_R^+ h \in \mathrm{Dom}(B_R)\}.$$

Assume furthermore that the map

$$\widetilde{B}_R : \mathscr{D}_R \longrightarrow \mathscr{D}_R'$$

is continuous. Then the difference-quotient kernel family has a limit in the weak topology of

$$\mathscr{D}_R' \widehat{\otimes} \mathscr{D}_R',$$

and

$$\left\langle B_R^{\mathrm{dist}}, \varphi \otimes \psi \right\rangle = \left\langle \widetilde{B}_R \varphi, \psi \right\rangle_{\mathscr{D}_R', \mathscr{D}_R}$$

holds for all $\varphi, \psi \in \mathscr{D}_R$*.*

Proof. By assumption, for any $\varphi \in \mathscr{D}_R$,

$$\Pi_R^+ \varphi \in \mathrm{Dom}(B_R).$$

By Stone's theorem, Theorem 4.110,

$$\frac{U_R(t)\Pi_R^+ \varphi - \Pi_R^+ \varphi}{t} \longrightarrow B_R \Pi_R^+ \varphi = \widetilde{B}_R \varphi$$

as a strong H_R^{res}-limit, and hence as a weak $\mathscr{D}_R'$-limit.

On the other hand,

$$\widetilde{U}_R(t)\varphi = U_R(t)\Pi_R^+ \varphi + (I - \Pi_R^+)\varphi.$$

Therefore

$$\frac{\widetilde{U}_R(t)\varphi - \varphi}{t} = \frac{U_R(t)\Pi_R^+ \varphi - \Pi_R^+ \varphi}{t}.$$

Hence for any $\psi \in \mathscr{D}_R$,

$$\begin{aligned} \lim_{t \to 0} \left\langle D_{R,t}^{\mathrm{dist}}, \varphi \otimes \psi \right\rangle &= \lim_{t \to 0} \left\langle \frac{\widetilde{U}_R(t)\varphi - \varphi}{t}, \psi \right\rangle_{H_{\alpha,+}} \\ &= \left\langle \widetilde{B}_R \varphi, \psi \right\rangle_{\mathscr{D}_R', \mathscr{D}_R}. \end{aligned}$$

Finally, by assumption,

$$\widetilde{B}_R : \mathscr{D}_R \to \mathscr{D}_R'$$

is continuous. Therefore

$$(\varphi, \psi) \longmapsto \langle \widetilde{B}_R \varphi, \psi \rangle$$

is a continuous sesquilinear form on $\mathscr{D}_R \times \mathscr{D}_R$. By the kernel theorem, there exists a distribution kernel

$$B_R^{\mathrm{dist}} \in \mathscr{D}_R' \widehat{\otimes} \mathscr{D}_R'$$

representing it. The pointwise limit representation shown above implies that

$$D_{R,t}^{\mathrm{dist}} \to B_R^{\mathrm{dist}}$$

in the sense of weak distributional convergence. ☐

Definition 4.126 (Resolvent-type kernel family)**.** For $z \in \rho(L_R^{\mathrm{res}})$, let

$$R_R(z) := (L_R^{\mathrm{res}} - zI)^{-1}$$

be the resolvent on H_R^{res}. Extend it to $H_{\alpha,+}$ by

$$\widetilde{R}_R(z) := R_R(z)\Pi_R^+.$$

Write the distribution kernel corresponding to this bounded operator as

$$K_R^{\mathrm{res}}(z) \in \mathscr{D}_R' \widehat{\otimes} \mathscr{D}_R',$$

and define it by

$$\langle K_R^{\mathrm{res}}(z), \varphi \otimes \psi \rangle := \langle \widetilde{R}_R(z)\varphi, \psi \rangle_{H_{\alpha,+}}.$$

Lemma 4.127 (Existence of the resolvent kernel)**.** For each $z \in \rho(L_R^{\mathrm{res}})$,

$$K_R^{\mathrm{res}}(z) \in \mathscr{D}_R' \widehat{\otimes} \mathscr{D}_R'$$

is well-defined. Furthermore,

$$|\langle K_R^{\mathrm{res}}(z), \varphi \otimes \psi \rangle| \le \|(L_R^{\mathrm{res}} - zI)^{-1}\| C_{\alpha,m}^2 p_{m,0}(\varphi) p_{m,0}(\psi)$$

holds.

Proof.

$$\widetilde{R}_R(z) = R_R(z)\Pi_R^+$$

is a bounded operator on $H_{\alpha,+}$, and

$$\|\widetilde{R}_R(z)\| \le \|R_R(z)\|.$$

Therefore

$$\begin{aligned} |\langle \widetilde{R}_R(z)\varphi, \psi \rangle| &\le \|R_R(z)\| \|\varphi\|_{H_{\alpha,+}} \|\psi\|_{H_{\alpha,+}} \\ &\le \|R_R(z)\| C_{\alpha,m}^2 p_{m,0}(\varphi) p_{m,0}(\psi). \end{aligned}$$

Thus the corresponding bilinear form is continuous, and the kernel theorem gives a unique distribution kernel. ☐

Definition 4.128 (Fredholm-type regularized kernel)**.** For $\varepsilon > 0$, let

$$F_R(\varepsilon) := (I + \varepsilon L_R^{\mathrm{res}})^{-1}\Pi_R^+$$

be a bounded operator on $H_{\alpha,+}$. Write the corresponding distribution kernel as

$$F_R^{\mathrm{dist}}(\varepsilon) \in \mathscr{D}_R' \widehat{\otimes} \mathscr{D}_R',$$

and define it by

$$\left\langle F_R^{\mathrm{dist}}(\varepsilon), \varphi \otimes \psi \right\rangle := \langle F_R(\varepsilon)\varphi, \psi \rangle_{H_{\alpha,+}}.$$

Theorem 4.129 (Weak distributional limit of the Fredholm-type regularized kernel)**.** *As* $\varepsilon \downarrow 0$,

$$F_R(\varepsilon) \longrightarrow \Pi_R^+$$

holds in the strong operator topology on $H_{\alpha,+}$. *Therefore the corresponding distribution kernels satisfy*

$$F_R^{\mathrm{dist}}(\varepsilon) \longrightarrow \Pi_R^{+,\mathrm{dist}}$$

in the sense of weak distributional limit. Here

$$\Pi_R^{+,\mathrm{dist}} \in \mathscr{D}_R' \widehat{\otimes} \mathscr{D}_R'$$

is the distribution kernel defined by

$$\langle \Pi_R^{+,\mathrm{dist}}, \varphi \otimes \psi \rangle := \langle \Pi_R^+ \varphi, \psi \rangle_{H_{\alpha,+}}.$$

Proof. We first prove the strong operator limit. Decompose any $h \in H_{\alpha,+}$ as

$$h = h_{\mathrm{res}} + h_{\perp}.$$

Here $h_{\mathrm{res}} = \Pi_R^+ h \in H_R^{\mathrm{res}}$, and $h_\perp \in (H_R^{\mathrm{res}})^\perp$. Then

$$F_R(\varepsilon)h = (I + \varepsilon L_R^{\mathrm{res}})^{-1} h_{\mathrm{res}}.$$

By the spectral theorem, since $L_R^{\mathrm{res}} \geq I$,

$$(I + \varepsilon L_R^{\mathrm{res}})^{-1}$$

is a bounded operator on H_R^{res}, and the spectral function

$$\lambda \longmapsto \frac{1}{1 + \varepsilon\lambda}$$

converges pointwise to 1 for each λ as $\varepsilon \downarrow 0$, while its absolute value is bounded by 1. Therefore, by the dominated convergence theorem,

$$(I + \varepsilon L_R^{\mathrm{res}})^{-1} h_{\mathrm{res}} \longrightarrow h_{\mathrm{res}}$$

holds in the H_R^{res}-norm. Thus

$$F_R(\varepsilon)h \to \Pi_R^+ h,$$

and the strong operator convergence follows.

Next, we prove weak convergence of the distribution kernels. For any $\varphi, \psi \in \mathscr{D}_R$, the strong convergence gives

$$F_R(\varepsilon)\varphi \longrightarrow \Pi_R^+\varphi \quad \text{in } H_{\alpha,+}.$$

Therefore

$$\langle F_R(\varepsilon)\varphi, \psi\rangle_{H_{\alpha,+}} \longrightarrow \langle \Pi_R^+\varphi, \psi\rangle_{H_{\alpha,+}}.$$

This means that

$$F_R^{\mathrm{dist}}(\varepsilon) \longrightarrow \Pi_R^{+,\mathrm{dist}}$$

holds in the weak topology of $\mathscr{D}'_R \widehat{\otimes} \mathscr{D}'_R$. ☐

Proposition 4.130 (Output of this subsection)**.** *This subsection has constructed the following objects:*

$$U_R(t), \qquad B_R, \qquad \widetilde{U}_R(t), \qquad \mathsf{K}_{R,t}^{\mathrm{dist}}, \qquad \mathsf{K}_R^{\mathrm{dist}}, \qquad B_R^{(\varepsilon),\mathrm{dist}}\ (\varepsilon > 0), \qquad K_R^{\mathrm{res}}(z).$$

They satisfy the following properties.

1. *$U_R(t) = e^{-itL_R^{\mathrm{res}}}$ is a strongly continuous unitary group on H_R^{res}.*
2. *$B_R = -iL_R^{\mathrm{res}}$ is the anti-self-adjoint generator, and*

$$U_R(t) = e^{tB_R}.$$

3. *$U_R(t)$ preserves the σ_R-null singular-boundary subspace H_R^{res} and the form domain Q_R^{res}.*
4. *$\mathsf{K}_{R,t}^{\mathrm{dist}}$ and $\mathsf{K}_R^{\mathrm{dist}} = \mathsf{K}_{R,1}^{\mathrm{dist}}$ are not ordinary function kernels, but distribution kernels defined as elements of*

$$\mathscr{D}'_R \widehat{\otimes} \mathscr{D}'_R.$$

5. *For every $\varepsilon > 0$, the regularized generator kernel*

$$B_R^{(\varepsilon),\mathrm{dist}}$$

exists unconditionally and is the weak distributional limit of the regularized difference-quotient kernels

$$D_{R,t}^{(\varepsilon),\mathrm{dist}}.$$

The unregularized generator kernel B_R^{dist} is available on the generator-admissible core $\mathscr{D}_{R,B}$, or on all of $\mathscr{D}_R$ only under the separate sufficient condition of Theorem 4.125.

6. *The resolvent kernel*

$$K_R^{\mathrm{res}}(z)$$

exists as a distribution kernel for $z \in \rho(L_R^{\mathrm{res}})$.

Proof. Item 1 follows from Theorem 4.108. Item 2 follows from Theorem 4.110. Item 3 follows from Proposition 4.111. Item 4 follows from Definition 4.114 and Lemma 4.115. Item 5 follows from Definition 4.119, Lemma 4.120, Theorem 4.121, and Theorem 4.123. The optional full-test-space statement follows from Definition 4.124 and Theorem 4.125. Item 6 follows from Definition 4.126 and Lemma 4.127. ☐

Remark 4.131 (No downstream use of the optional full-test unsmoothed generator kernel)**.** The optional full-test-space kernel in Definition 4.124 is not part of the unconditional data exported from this subsection to Sections 5 and 6. The exported kernel data are the transport kernels, the resolvent kernels, the unconditional regularized generator kernels $B_R^{(\varepsilon),\mathrm{dist}}$, and the unsmoothed generator kernel only on the generator-admissible core $\mathscr{D}_{R,B}$. Thus the localized comparison interface, the Hilbert–Schmidt realization of K, the central comparison, and the determinant identity do not require the additional sufficient condition of Theorem 4.125. Whenever a full-test-space pairing is used later, it is either a regularized pairing, a core-level pairing, or a pairing already mediated by the continuous comparison maps constructed in Sections 5–6.

Remark 4.132 (Transition to the next step)**.** The transport group, anti-self-adjoint generator, and distribution kernel representation obtained in this subsection show that the singular-boundary subspace is not merely a closed subspace, but has a conservative time evolution inside it. In the next subsection, we treat compactness of the resolvent of the positive shifted operator

$$L_R^{\mathrm{res}},$$

and construct the spectral foundation needed for purely discrete spectrum and finite-window localization.

4.8. Compact Resolvent and Purely Discrete Spectrum

In this subsection, we show that the resolvent of the σ_R-null positive shifted operator on K_R

$$L_R^{\mathrm{res}} = A_R^{\mathrm{res}} + I_{H_R^{\mathrm{res}}}$$

is compact. The proof is carried out by restricting the compact embedding of the global form domain established in Section 2 to the σ_R-null trace-vanishing form subspace

$$Q_R^{\mathrm{res}} \subset Q_R.$$

Accordingly, the compact-resolvent operator L_R^{res} below is constructed only from the compactness inherited from Section 2 and the closed-subspace structure constructed up to the preceding subsection. This operator supplies the internal singular-boundary transport spectrum. For the later trace-ideal estimates, however, we also fix at the end of this subsection a separate quantitative Sobolev reference scale. That scale is a positive elliptic regularizer used only for boundary-trace smoothing and Schatten estimates; it is not a new confining dynamics and it does not replace L_R^{res}.

Lemma 4.133 (Restriction of the global compact embedding)**.** Consider the global compact embedding obtained in Section 2,

$$j_R : (Q_R, \|\cdot\|_{q_R}) \longrightarrow H_{\alpha,+}.$$

Then the natural inclusion map

$$j_R^{\mathrm{res}} : (Q_R^{\mathrm{res}}, \|\cdot\|_{q_R}) \longrightarrow H_R^{\mathrm{res}}$$

is compact.

Proof. Let $\{u_n\}_{n\geq 1}$ be a bounded sequence in

$$(Q_R^{\mathrm{res}}, \|\cdot\|_{q_R}).$$

That is, there exists $C > 0$ such that

$$\|u_n\|_{q_R} \le C \qquad (n \ge 1).$$

Since $Q_R^{\mathrm{res}} \subset Q_R$, this is also a bounded sequence in $(Q_R, \|\cdot\|_{q_R})$.

By the global compact embedding of Section 2, there exist a subsequence

$$\{u_{n_k}\}_{k\ge 1}$$

and some

$$u \in H_{\alpha,+}$$

such that

$$u_{n_k} \longrightarrow u \qquad \text{in } H_{\alpha,+}.$$

On the other hand, all u_{n_k} belong to

$$Q_R^{\mathrm{res}} \subset H_R^{\mathrm{res}}.$$

By Lemma 4.89, H_R^{res} is a closed subspace of $H_{\alpha,+}$. Therefore the $H_{\alpha,+}$-norm limit u also satisfies

$$u \in H_R^{\mathrm{res}}.$$

Moreover, since the norm of H_R^{res} is the restriction of the norm of $H_{\alpha,+}$,

$$u_{n_k} \longrightarrow u \qquad \text{in } H_R^{\mathrm{res}}$$

also holds. Thus j_R^{res} extracts a convergent subsequence from every bounded sequence. Hence j_R^{res} is compact. ☐

Remark 4.134 (Why compactness is preserved)**.** What is used here is not merely the fact that a global compact operator has been restricted. The essential point is that

$$Q_R^{\mathrm{res}}$$

is a form-norm closed subspace of Q_R, and that

$$H_R^{\mathrm{res}}$$

is a norm-closed subspace of $H_{\alpha,+}$. The fact that the limit obtained from the global compact embedding does not leave H_R^{res} is guaranteed by the closedness of H_R^{res}. Thus the compactness of Section 2 is correctly inherited by the σ_R-null singular-boundary subspace.

Lemma 4.135 (Variational representation of the inverse operator)**.** For any

$$f \in H_R^{\mathrm{res}},$$

there exists a unique

$$u \in Q_R^{\mathrm{res}}$$

such that

$$q_R^{\mathrm{res}}[u,v] + \langle u,v\rangle_{H_R^{\mathrm{res}}} = \langle f,v\rangle_{H_R^{\mathrm{res}}} \qquad (v \in Q_R^{\mathrm{res}}).$$

Furthermore,

$$u = (L_R^{\mathrm{res}})^{-1} f,$$

and

$$\|u\|_{q_R^{\mathrm{res}}} \le \|f\|_{H_R^{\mathrm{res}}}$$

holds.

Proof. Consider the inner product on Q_R^{res}

$$\langle u,v\rangle_{q_R^{\mathrm{res}}} := q_R^{\mathrm{res}}[u,v] + \langle u,v\rangle_{H_R^{\mathrm{res}}}.$$

By Lemma 4.92 and Definition 4.93,

$$(Q_R^{\mathrm{res}}, \langle\cdot,\cdot\rangle_{q_R^{\mathrm{res}}})$$

is a Hilbert space.

For fixed $f \in H_R^{\mathrm{res}}$, set

$$\ell_f(v) := \langle f,v\rangle_{H_R^{\mathrm{res}}} \qquad (v \in Q_R^{\mathrm{res}}).$$

By the Cauchy–Schwarz inequality and

$$\|v\|_{H_R^{\mathrm{res}}} \le \|v\|_{q_R^{\mathrm{res}}},$$

we have

$$|\ell_f(v)| \le \|f\|_{H_R^{\mathrm{res}}} \|v\|_{H_R^{\mathrm{res}}} \le \|f\|_{H_R^{\mathrm{res}}} \|v\|_{q_R^{\mathrm{res}}}.$$

Therefore ℓ_f is a bounded linear functional on $(Q_R^{\mathrm{res}}, \|\cdot\|_{q_R^{\mathrm{res}}})$.

By the Riesz representation theorem, there exists a unique

$$u \in Q_R^{\mathrm{res}}$$

such that

$$\langle u,v\rangle_{q_R^{\mathrm{res}}} = \ell_f(v)$$

for all $v \in Q_R^{\mathrm{res}}$. That is,

$$q_R^{\mathrm{res}}[u,v] + \langle u,v\rangle_{H_R^{\mathrm{res}}} = \langle f,v\rangle_{H_R^{\mathrm{res}}}.$$

By the definition

$$L_R^{\mathrm{res}} = A_R^{\mathrm{res}} + I$$

and the first representation theorem, this equality is equivalent to

$$u \in \mathrm{Dom}(L_R^{\mathrm{res}}) \quad \text{and} \quad L_R^{\mathrm{res}} u = f.$$

Therefore

$$u = (L_R^{\mathrm{res}})^{-1} f.$$

Finally, the norm estimate in the Riesz representation gives

$$\|u\|_{q_R^{\mathrm{res}}} = \|\ell_f\|_{(Q_R^{\mathrm{res}})'} \le \|f\|_{H_R^{\mathrm{res}}}.$$

□

Theorem 4.136 (Compactness of the inverse operator)**.**

$$(L_R^{\mathrm{res}})^{-1} : H_R^{\mathrm{res}} \longrightarrow H_R^{\mathrm{res}}$$

is a compact operator.

Proof. By Lemma 4.135,

$$(L_R^{\mathrm{res}})^{-1}$$

factors as a bounded operator

$$S_R : H_R^{\mathrm{res}} \longrightarrow (Q_R^{\mathrm{res}}, \|\cdot\|_{q_R}).$$

Namely, setting

$$S_R f = (L_R^{\mathrm{res}})^{-1} f,$$

we have

$$\|S_R f\|_{q_R} \le \|f\|_{H_R^{\mathrm{res}}}.$$

On the other hand, by Lemma 4.133, the inclusion map

$$j_R^{\mathrm{res}} : (Q_R^{\mathrm{res}}, \|\cdot\|_{q_R}) \longrightarrow H_R^{\mathrm{res}}$$

is compact. Therefore

$$(L_R^{\mathrm{res}})^{-1} = j_R^{\mathrm{res}} \circ S_R$$

is the composition of a bounded operator and a compact operator. Hence

$$(L_R^{\mathrm{res}})^{-1}$$

is compact on H_R^{res}. □

Theorem 4.137 (Compact resolvent)**.** *For any*

$$z \in \rho(L_R^{\mathrm{res}}),$$

the operator

$$(L_R^{\mathrm{res}} - zI)^{-1}$$

is compact on H_R^{res}. Therefore L_R^{res} has compact resolvent.

Proof. First, the case $z = 0$ follows from Theorem 4.136, which shows that

$$(L_R^{\mathrm{res}})^{-1}$$

is compact.

Now take arbitrary $z \in \rho(L_R^{\text{res}})$. By the spectral theorem, we can write

$$(L_R^{\text{res}} - zI)^{-1} = g_z(L_R^{\text{res}})(L_R^{\text{res}})^{-1},$$

where

$$g_z(\lambda) := \frac{\lambda}{\lambda - z}.$$

Since $z \in \rho(L_R^{\text{res}})$, the function

$$\lambda \mapsto \frac{\lambda}{\lambda - z}$$

is a bounded Borel function on $\sigma(L_R^{\text{res}})$. Therefore

$$g_z(L_R^{\text{res}})$$

is a bounded operator on H_R^{res}.

Since

$$(L_R^{\text{res}})^{-1}$$

is already compact, the composition with the bounded operator

$$g_z(L_R^{\text{res}})(L_R^{\text{res}})^{-1}$$

is compact. Hence

$$(L_R^{\text{res}} - zI)^{-1}$$

is compact. ☐

Theorem 4.138 (Purely discrete spectrum)**.** *The spectrum of L_R^{res} is purely discrete. That is, if $H_R^{\text{res}} \neq \{0\}$, then there exist at most countably many eigenvalues*

$$1 \leq \lambda_1 \leq \lambda_2 \leq \cdots,$$

each eigenvalue has finite multiplicity, and there exists a complete orthonormal system of corresponding eigenvectors

$$\{e_j\}_{j\geq 1}$$

in H_R^{res}. Furthermore, if there are infinitely many eigenvalues, then

$$\lambda_j \longrightarrow +\infty \qquad (j \to \infty).$$

In the finite-dimensional case, the spectrum consists only of finitely many eigenvalues.

Proof. By Lemma 4.99, L_R^{res} is a self-adjoint operator, and by Theorem 4.137 it has compact resolvent. When a self-adjoint operator has compact resolvent, its spectrum consists of pure point spectrum, each eigenvalue has finite multiplicity, and no finite accumulation point occurs. Moreover, the corresponding eigenvectors form a complete orthonormal system of the Hilbert space.

Furthermore,

$$L_R^{\text{res}} \geq I,$$

and hence

$$\sigma(L_R^{\mathrm{res}}) \subset [1, \infty).$$

Therefore the eigenvalues can be arranged at or above 1. If there are infinitely many eigenvalues, since they cannot accumulate at any finite point, the only possible accumulation point is $+\infty$. Thus

$$\lambda_j \to +\infty$$

follows. □

Definition 4.139 (Finite-window spectral projection)**.** Let $E_R^{\mathrm{res}}(\cdot)$ be the spectral measure of L_R^{res}. For a bounded Borel set

$$I \subset [1, \infty),$$

write

$$P_R^{\mathrm{res}}(I) := E_R^{\mathrm{res}}(I).$$

We call this the finite-window spectral projection of the σ_R-null singular-boundary subspace. Also,

$$N_R^{\mathrm{res}}(I) := \operatorname{rank} P_R^{\mathrm{res}}(I)$$

is called its finite-window spectral count.

Corollary 4.140 (Finiteness of finite-window counts)**.** *For any bounded Borel set*

$$I \subset [1, \infty),$$

one has

$$N_R^{\mathrm{res}}(I) < \infty.$$

In particular, for any

$$1 \le a < b < \infty,$$

one has

$$\operatorname{rank} E_R^{\mathrm{res}}([a, b]) < \infty.$$

Proof. By Theorem 4.138, any bounded interval contains only finitely many eigenvalues, and each eigenvalue has finite multiplicity. Therefore the direct sum of eigenspaces belonging to the bounded Borel set I is finite-dimensional. Hence

$$\operatorname{rank} P_R^{\mathrm{res}}(I) < \infty.$$

□

Remark 4.141 (Transition to finite-window localization)**.** By Corollary 4.140, on the σ_R-null singular-boundary subspace, the degrees of freedom contained in any bounded spectral window are finite-dimensional. Therefore the subsequent finite-window localization, finite-rank projection, and discrete counting comparison can be carried out on top of this spectral data. The finiteness obtained here is not a statement concerning zero counting; it is solely operator-theoretic finiteness following from the compact resolvent of the singular-boundary operator.

Definition 4.142 (Spectral resolvent measure on K_R)**.** Fix $\Phi \in H_R^{\mathrm{res}}$. For the spectral measure E_R^{res} of L_R^{res}, define

$$\mu_R^{(\Phi)}(I) := \langle E_R^{\mathrm{res}}(I)\Phi, \Phi\rangle_{H_R^{\mathrm{res}}}.$$

This is a finite positive Borel measure on $[1,\infty)$, and satisfies

$$\mu_R^{(\Phi)}([1,\infty)) = \|\Phi\|_{H_R^{\mathrm{res}}}^2.$$

Definition 4.143 (Herglotz-type resolvent function)**.** For $\Phi \in H_R^{\mathrm{res}}$, define

$$m_R^{(\Phi)}(z) := \left\langle \Phi, (L_R^{\mathrm{res}} - zI)^{-1}\Phi \right\rangle_{H_R^{\mathrm{res}}} \qquad (z \in \mathbb{C}\setminus\mathbb{R}).$$

We call this the σ_R-null Herglotz-type resolvent function on K_R associated with Φ.

Lemma 4.144 (Spectral representation)**.** For any

$$z \in \mathbb{C}\setminus\mathbb{R},$$

one has

$$m_R^{(\Phi)}(z) = \int_{[1,\infty)} \frac{1}{\lambda - z}\, d\mu_R^{(\Phi)}(\lambda).$$

Furthermore, using the purely discrete spectral representation,

$$m_R^{(\Phi)}(z) = \sum_j \frac{|\langle \Phi, e_j\rangle_{H_R^{\mathrm{res}}}|^2}{\lambda_j - z},$$

and this series converges absolutely for

$$z \in \mathbb{C}\setminus\mathbb{R}.$$

Proof. The first representation follows immediately from the spectral theorem. Indeed,

$$(L_R^{\mathrm{res}} - zI)^{-1} = \int_{[1,\infty)} \frac{1}{\lambda - z}\, dE_R^{\mathrm{res}}(\lambda),$$

and therefore

$$\begin{aligned} m_R^{(\Phi)}(z) &= \left\langle \Phi, \left(\int_{[1,\infty)} \frac{1}{\lambda - z}\, dE_R^{\mathrm{res}}(\lambda)\right)\Phi \right\rangle \\ &= \int_{[1,\infty)} \frac{1}{\lambda - z}\, d\mu_R^{(\Phi)}(\lambda). \end{aligned}$$

The purely discrete spectral representation is obtained by using the complete orthonormal system $\{e_j\}$ from Theorem 4.138. Namely,

$$\Phi = \sum_j \langle \Phi, e_j\rangle e_j,$$

and

$$(L_R^{\mathrm{res}} - zI)^{-1} e_j = \frac{1}{\lambda_j - z} e_j.$$

This gives the displayed series.

For absolute convergence, since

$$\left|\frac{1}{\lambda_j - z}\right| \le \frac{1}{|\operatorname{Im} z|},$$

we have

$$\sum_j \left|\frac{|\langle \Phi, e_j\rangle|^2}{\lambda_j - z}\right| \le \frac{1}{|\operatorname{Im} z|} \sum_j |\langle \Phi, e_j\rangle|^2 = \frac{\|\Phi\|^2}{|\operatorname{Im} z|} < \infty.$$

□

Lemma 4.145 (Herglotz property)**.** If $\operatorname{Im} z > 0$, then

$$\operatorname{Im} m_R^{(\Phi)}(z) \ge 0.$$

More precisely,

$$\operatorname{Im} m_R^{(\Phi)}(z) = (\operatorname{Im} z) \int_{[1,\infty)} \frac{1}{|\lambda - z|^2}\, d\mu_R^{(\Phi)}(\lambda) \ge 0.$$

Therefore $m_R^{(\Phi)}$ is a Herglotz-type function from the upper half-plane to the closure of the upper half-plane.

Proof. By the spectral representation,

$$m_R^{(\Phi)}(z) = \int_{[1,\infty)} \frac{1}{\lambda - z}\, d\mu_R^{(\Phi)}(\lambda).$$

Write $z = x + iy$, $y > 0$. Then

$$\frac{1}{\lambda - z} = \frac{\lambda - x + iy}{(\lambda - x)^2 + y^2}.$$

Therefore

$$\operatorname{Im} \frac{1}{\lambda - z} = \frac{y}{(\lambda - x)^2 + y^2} = \frac{\operatorname{Im} z}{|\lambda - z|^2}.$$

Integrating this gives

$$\operatorname{Im} m_R^{(\Phi)}(z) = (\operatorname{Im} z) \int_{[1,\infty)} \frac{1}{|\lambda - z|^2}\, d\mu_R^{(\Phi)}(\lambda) \ge 0.$$

□

Concrete smooth-seam Sobolev reference model for the later Schatten estimate.

The compact-resolvent transport operator L_R^{res} constructed above supplies the internal singular-boundary dynamics. For the trace-ideal estimates we use a separate Sobolev reference scale attached to the smooth zero-area seam carrier of Section 4.2.

Let

$$\mathcal{H}_R^{\mathrm{ref}} := L^2(\Sigma_R, \sigma_R), \qquad \Sigma_R = [0,1]^2,$$

and let

$$A_{\Sigma,R} := I - \Delta_{\Sigma,N}$$

be the positive Neumann Sobolev regularizer on Σ_R. The unitary map used below is not arbitrary. It is the fixed Sobolev seam chart obtained as follows.

Let

$$U_\alpha : H_{\alpha,+} \longrightarrow L^2(0,\infty)$$

be the weight-removing unitary from Section 2. Let

$$\{\ell_n\}_{n\geq 0}$$

be the standard Laguerre orthonormal basis of $L^2(0,\infty)$. On $\Sigma_R = [0,1]^2$, set

$$E_{k,\ell}(x,y) := c_{k,\ell}\cos(k\pi x)\cos(\ell\pi y), \qquad k,\ell \geq 0,$$

with the usual normalizing constants, and enumerate these Neumann eigenfunctions by increasing value of

$$1+\pi^2(k^2+\ell^2),$$

using lexicographic order when eigenvalues coincide. Write the resulting orthonormal basis of $L^2(\Sigma_R)$ as

$$\{E_n\}_{n\geq 0}.$$

Define

$$\boxed{\mathcal{W}_R f := \sum_{n=0}^{\infty} \langle U_\alpha f, \ell_n\rangle_{L^2(0,\infty)} E_n.}$$

Equivalently,

$$\mathcal{W}_R = \mathcal{E}_{\Sigma,R}\mathcal{E}_\alpha^{-1}U_\alpha,$$

where

$$\mathcal{E}_\alpha(a_n) = \sum_n a_n \ell_n, \qquad \mathcal{E}_{\Sigma,R}(a_n) = \sum_n a_n E_n.$$

Thus

$$\mathcal{W}_R : H_{\alpha,+} \longrightarrow L^2(\Sigma_R)$$

is a fixed unitary seam chart determined by the weighted Hilbert space of Section 2 and the smooth seam carrier of Section 4; it is not chosen from zero data.

Define

$$\boxed{A_R^{\mathrm{ref}} := \mathcal{W}_R^{-1} A_{\Sigma,R}\mathcal{W}_R.}$$

For $s \in \mathbb{R}$, define the transported Neumann Sobolev scale

$$\boxed{H_R^s := \mathcal{W}_R^{-1} H_N^s(\Sigma_R), \qquad \|f\|_{H_R^s} := \|\mathcal{W}_R f\|_{H_N^s(\Sigma_R)}.}$$

In particular,

$$\boxed{\mathcal{D}_R^\infty := \bigcap_{q\geq 0} \mathrm{Dom}(I+A_R^{\mathrm{ref}})^{q/2} = \mathcal{W}_R^{-1} C_N^\infty(\Sigma_R).}$$

In this concrete model the effective Sobolev dimension and elliptic order are

$$\boxed{d_{\mathrm{eff}} = 2, \qquad m_R = 2, \qquad \nu_R := \frac{m_R}{d_{\mathrm{eff}}} = 1.}$$

If

$$A_R^{\mathrm{ref}} e_n = \lambda_n e_n, \qquad 0 \le \lambda_1 \le \lambda_2 \le \cdots,$$

then the Weyl lower bound for $A_{\Sigma,R} = I - \Delta_{\Sigma,N}$ gives constants $c_{\mathrm{W}} > 0$ and $n_{\mathrm{W}} \ge 1$ such that

$$\boxed{\lambda_n \ge c_{\mathrm{W}} n^{\nu_R} = c_{\mathrm{W}} n} \qquad (n \ge n_{\mathrm{W}}).$$

The same bound holds after transport by $\mathcal{W}_R$.

We now separate the raw seam restriction from the smoothed comparison trace used in the trace-ideal argument. Let

$$\gamma_\Gamma u := u|_{\Gamma_R}$$

be the raw seam restriction on smooth functions on Σ_R, and let

$$\iota_{\Gamma R}$$

be the fixed continuous embedding from the seam trace space into the admissible boundary-distribution space used by the localized comparison interface. Because the R, LCI-control norm is defined by finitely many boundary seminorms, there exists a finite integer $r_R \ge 0$. Choose once and for all

$$s_R > r_R + 1$$

so that the trace theorem and Sobolev embedding give

$$\boxed{\|\iota_{\Gamma R}\gamma_\Gamma u\|_{R,\mathrm{LCI}} \le C_\Gamma \|u\|_{H_N^{s_R}(\Sigma_R)}} \qquad (u \in H_N^{s_R}(\Sigma_R)).$$

The raw seam trace is

$$\boxed{\mathrm{Tr}_{\partial,R}^{\mathrm{raw}} f := \iota_{\Gamma R}\gamma_\Gamma \mathcal{W}_R f, \qquad f \in H_R^{s_R}.}$$

It is only a restriction map and carries no Schatten decay by itself.

Set

$$a_{\mathrm{HS}} := \frac{d_{\mathrm{eff}}}{2m_R} = \frac{1}{2},$$

and fix the net trace-decay exponent

$$\boxed{a_{\mathrm{tr}} > a_{\mathrm{HS}}.}$$

The actual smoothing exponent required before applying the raw trace is

$$\boxed{a_{\mathrm{cmp}} := a_{\mathrm{tr}} + \frac{s_R}{m_R}.}$$

Define the bounded pre-trace operator

$$\boxed{\mathrm{Tr}_{\partial,R}^{[s_R]} := \mathrm{Tr}_{\partial,R}^{\mathrm{raw}}(I + A_R^{\mathrm{ref}})^{-s_R/m_R} = \iota_{\Gamma R}\gamma_\Gamma \mathcal{W}_R (I + A_R^{\mathrm{ref}})^{-s_R/m_R} : H_{\alpha,+} \longrightarrow \mathcal{D}'_{R,\mathrm{adm}}.}$$

The singular-boundary trace used in the trace-ideal part of the proof is the smoothed comparison trace

$$\boxed{\mathrm{Tr}^{\mathrm{cmp}}_{\partial,R} := \mathrm{Tr}^{\mathrm{raw}}_{\partial,R}(I+A^{\mathrm{ref}}_R)^{-a_{\mathrm{cmp}}} = \mathrm{Tr}^{[s_R]}_{\partial,R}(I+A^{\mathrm{ref}}_R)^{-a_{\mathrm{tr}}}.}$$

Thus the decay used below is not a property of the raw restriction γ_Γ alone. It is a property of the smoothed comparison trace constructed from the seam Sobolev chart.

Finally,

$$\mathrm{span}\{e_n : n \geq 1\} \subset \mathcal{D}^\infty_R \subset \mathcal{D}_R.$$

Therefore the matrix entries $\mathfrak{t}_R(e_m, e_n)$ in Section 6.3 are evaluated on vectors lying in the initial boundary-test domain. In Section 6.3 we write $A_R := A^{\mathrm{ref}}_R$ only to lighten notation.

Proposition 4.146 (Output of this subsection)**.** *This subsection has constructed the following spectral and Sobolev-reference data.*

1. *The embedding*

$$Q^{\mathrm{res}}_R \hookrightarrow H^{\mathrm{res}}_R$$

is compact.

2.

$$(L^{\mathrm{res}}_R - zI)^{-1}$$

is compact for every $z \in \rho(L^{\mathrm{res}}_R)$.

3. *L^{res}_R has purely discrete spectrum.*
4. *The projection corresponding to a bounded spectral window,*

$$P^{\mathrm{res}}_R(I),$$

has finite rank.

5. *For any $\Phi \in H^{\mathrm{res}}_R$, the Herglotz-type resolvent function*

$$m^{(\Phi)}_R(z) = \left\langle \Phi, (L^{\mathrm{res}}_R - zI)^{-1}\Phi \right\rangle$$

is defined.

6. *The Sobolev seam chart*

$$\mathcal{W}_R = \mathcal{E}_{\Sigma,R}\mathcal{E}_\alpha^{-1}\mathcal{U}_\alpha : H_{\alpha,+} \to L^2(\Sigma_R)$$

is fixed by the Laguerre basis on $L^2(0,\infty)$ and the Neumann eigenbasis on $\Sigma_R = [0,1]^2$.

7. *The transported Sobolev scale*

$$H^s_R = \mathcal{W}_R^{-1}H^s_N(\Sigma_R), \qquad \mathcal{D}^\infty_R = \mathcal{W}_R^{-1}C^\infty_N(\Sigma_R)$$

and the reference regularizer

$$A^{\mathrm{ref}}_R = \mathcal{W}_R^{-1}(I - \Delta_{\Sigma,N})\mathcal{W}_R$$

are fixed.

8. *The model has*

$$d_{\mathrm{eff}} = 2, \qquad m_R = 2, \qquad \nu_R = 1,$$

and the Weyl-type lower bound

$$\lambda_n(A_R^{\mathrm{ref}}) \geq c_{\mathrm{W}} n^{\nu_R} \qquad (n \geq n_{\mathrm{W}}).$$

9. *The raw seam trace and smoothed comparison trace are separated:*

$$\mathrm{Tr}_{\partial,R}^{\mathrm{raw}} f = \iota_{\Gamma R} \gamma_\Gamma \mathcal{W}_R f, \qquad \mathrm{Tr}_{\partial,R}^{\mathrm{cmp}} = \mathrm{Tr}_{\partial,R}^{\mathrm{raw}} (I + A_R^{\mathrm{ref}})^{-a_{\mathrm{cmp}}}.$$

Equivalently,

$$\mathrm{Tr}_{\partial,R}^{\mathrm{cmp}} = \mathrm{Tr}_{\partial,R}^{[s_R]} (I + A_R^{\mathrm{ref}})^{-a_{\mathrm{tr}}}, \qquad a_{\mathrm{cmp}} = a_{\mathrm{tr}} + \frac{s_R}{m_R},$$

with

$$a_{\mathrm{tr}} > \frac{d_{\mathrm{eff}}}{2m_R}.$$

Proof. Items 1–5 are exactly Lemma 4.133, Theorem 4.137, Theorem 4.138, Corollary 4.140, and Definition 4.143. Item 6 follows from the concrete smooth-seam Sobolev model fixed above and the Weyl lower bound for the Neumann elliptic regularizer on $[0,1]^2$. Item 7 follows from the trace theorem on the smooth seam, the finite R, LCI-seminorm control, and the fixed smoothing exponent a_{tr}. All of these reference data are logically separate from the compact-resolvent transport operator L_R^{res}, from the divisor of ξ, and from the determinant identity $F_K \equiv \xi$. ☐

4.9. *Internal-Construction Theorem for the Operator-Theoretic Boundary Data*

In this subsection, we collect the objects constructed in this section into a single operator-theoretic data. The purpose is to fix the singular-boundary input passed to the subsequent orthogonal-decomposition framework entirely as objects defined internally in this section.

The construction of this section proceeded in the following order. First, as the basic space on the singular-boundary side, we introduced

$$\mathscr{D}_R \subset H_{\alpha,+} \subset \mathscr{D}_R'$$

and separated the types of point evaluations, boundary traces, distribution kernels, quadratic forms, generators, and boundary forms. Next, we constructed the boundary parameter space, regular area measure, and σ_R-null singular support, and then separated the regular boundary channel from the singular boundary channel. After that, we took the subspace on which boundary traces are defined inside the form domain, and constructed the support map of the singular trace and the regular trace mass functional. Finally, from the trace-vanishing generating subspace associated with Γ_R, we constructed the closed form subspace, Hilbert space, self-adjoint realization, transport group, distribution kernel, and compact resolvent.

Below, we formulate these collectively as the operator-theoretic boundary data.

Definition 4.147 (Internally constructed boundary input data)**.** Write the boundary input data constructed in this section as

$$\mathfrak{R}_{\mathrm{bd}} := (\Sigma_R, \sigma_R, \Gamma_R, T_R, \mathrm{supp}_R, L_R).$$

Each component is defined as follows.

Symbol	Internal construction	Construction location
Σ_R	$[0,1]^2$	boundary parameter space
σ_R	$\mathcal{L}^2\|_{[0,1]^2}$	regular area measure
Γ_R	$[0,1]\times\{0\}$	σ_R-null smooth seam support
T_R	$\{ f \in Q_R : \gamma_R^{\mathrm{sing}} f$ and $\gamma_R^{\mathrm{reg}} f$ are uniquely defined in the closed-graph sense $\}$	closed boundary trace space
$\mathrm{supp}_R(f)$	$\mathrm{supp}(\gamma_R^{\mathrm{sing}} f)$	distribution support of the singular boundary trace
$L_R(f;E)$	$\int_E \|\gamma_R^{\mathrm{reg}} f\|^2 \, d\sigma_R$	regular boundary trace-mass functional

Here $\Gamma_R = [0,1]\times\{0\} \subset [0,1]^2$ is the smooth zero-area seam carrier. Also, Q_R is the form domain constructed in Section 2.

Lemma 4.148 (Basic properties of the boundary input data)**.** The boundary input data

$$\mathfrak{R}_{\mathrm{bd}} = (\Sigma_R, \sigma_R, \Gamma_R, T_R, \mathrm{supp}_R, L_R)$$

satisfy the following.

1. Σ_R is a compact metric space.
2. σ_R is a finite regular Borel area measure on Σ_R.
3. $\Gamma_R \subset \Sigma_R$ is a closed set, and
$$\sigma_R(\Gamma_R) = 0.$$
4. A singular probability measure
$$\nu_R$$
is supported on Γ_R, and
$$\nu_R(\Gamma_R) = 1, \qquad \nu_R \perp \sigma_R$$
hold.
5. $T_R \subset Q_R$ is a linear subspace, and
$$\gamma_R^{\mathrm{sing}} : T_R \to \mathcal{M}(\Sigma_R), \qquad \gamma_R^{\mathrm{reg}} : T_R \to L^2(\Sigma_R, \sigma_R)$$
are linear maps.
6. For any $f \in T_R$,
$$\mathrm{supp}_R(f) \subset \Gamma_R,$$
and therefore
$$\sigma_R(\mathrm{supp}_R(f)) = 0.$$
7. For any $f \in T_R$, the map
$$E \longmapsto L_R(f;E)$$
is a finite positive measure on Σ_R, and satisfies
$$L_R(f;\Sigma_R) = \|\gamma_R^{\mathrm{reg}} f\|^2_{L^2(\Sigma_R,\sigma_R)}.$$

Proof. Items 1, 2, 3, and 4 follow from the construction of the boundary parameter space and the σ_R-null singular support. Item 5 follows from the linearity of T_R, defined by the closed-trace graph, and from the linearity of the two boundary trace maps. Item 6 follows from the fact that the singular trace is always supported on Γ_R, together with $\sigma_R(\Gamma_R) = 0$. Item 7 follows from the definition of the regular boundary trace-mass functional

$$L_R(f;E) = \int_E |\gamma_R^{\mathrm{reg}} f|^2 \, d\sigma_R.$$

□

Definition 4.149 (Internally constructed trace-vanishing generating subspace associated with Γ_R)**.** Define the trace-vanishing generating subspace associated with Γ_R by

$$G_R := \Big\{ f \in T_R : \operatorname{supp}_R(f) \subset \Gamma_R,\ L_R(f; \Sigma_R \setminus \Gamma_R) = 0 \Big\}.$$

Equivalently,

$$G_R = \{ f \in T_R : \gamma_R^{\mathrm{reg}} f = 0 \quad \sigma_R\text{-a.e.} \}.$$

Lemma 4.150 (σ_R-null property and regular-trace vanishing property of the generating class)**.** For any $f \in G_R$, the following hold.

1. The singular boundary trace is supported on the σ_R-null singular support:

$$\operatorname{supp}(\gamma_R^{\mathrm{sing}} f) = \operatorname{supp}_R(f) \subset \Gamma_R.$$

2. Regular-trace mass in the regular boundary-trace direction vanishes:

$$\gamma_R^{\mathrm{reg}} f = 0 \qquad \sigma_R\text{-a.e.}$$

3. Therefore,

$$L_R(f; \Sigma_R) = 0.$$

Proof. Item 1 follows from the definition of supp_R and from $\operatorname{supp}_R(f) \subset \Gamma_R$. Item 2 follows from the equivalent representation of G_R,

$$G_R = \{ f \in T_R : \gamma_R^{\mathrm{reg}} f = 0 \}.$$

Item 3 is obtained by substituting Item 2 into

$$L_R(f; \Sigma_R) = \|\gamma_R^{\mathrm{reg}} f\|^2_{L^2(\Sigma_R, \sigma_R)}.$$

□

Definition 4.151 (Internally constructed closed quadratic form and Hilbert data)**.** From the trace-vanishing generating subspace associated with Γ_R, define

$$Q_R^{\mathrm{res}} := \overline{\operatorname{span} G_R}^{\|\cdot\|_{q_R}} \subset Q_R.$$

Furthermore, define

$$H_R^{\mathrm{res}} := \overline{Q_R^{\mathrm{res}}}^{\|\cdot\|_{H_{\alpha,+}}} \subset H_{\alpha,+}.$$

Write the restriction form as

$$q_R^{\mathrm{res}} := q_R|_{Q_R^{\mathrm{res}} \times Q_R^{\mathrm{res}}}.$$

Also, let its Friedrichs-type representing operator be

$$A_R^{\mathrm{res}},$$

and define the positive shifted operator by

$$L_R^{\mathrm{res}} := A_R^{\mathrm{res}} + I_{H_R^{\mathrm{res}}}.$$

Lemma 4.152 (Basic properties of the form and Hilbert data)**.** The objects defined above satisfy the following.

1. Q_R^{res} is a closed linear subspace of
$$(Q_R, \|\cdot\|_{q_R}),$$
and
$$(Q_R^{\mathrm{res}}, \|\cdot\|_{q_R})$$
is a Hilbert space.
2. The inclusion map
$$Q_R^{\mathrm{res}} \hookrightarrow H_{\alpha,+}$$
is continuous.
3. H_R^{res} is a closed linear subspace of $H_{\alpha,+}$.
4. Q_R^{res} is dense in H_R^{res}.
5. q_R^{res} is a densely defined closed lower-bounded form on H_R^{res}.
6. A_R^{res} is the unique self-adjoint representing operator of q_R^{res}.
7. L_R^{res} is self-adjoint, and
$$L_R^{\mathrm{res}} \geq I_{H_R^{\mathrm{res}}}.$$

Proof. Items 1 and 2 follow from the construction of the closure of the σ_R-null trace-vanishing form subspace. Items 3 and 4 follow from the definition of H_R^{res}. Item 5 follows from the fact that the restriction of a closed form q_R to a closed subspace is a closed form. Item 6 follows from the first representation theorem for closed lower-bounded forms. Item 7 follows from the definition

$$L_R^{\mathrm{res}} = A_R^{\mathrm{res}} + I$$

and from $A_R^{\mathrm{res}} \geq 0$. □

Definition 4.153 (Internally constructed singular-boundary subspace)**.** Define the singular-boundary subspace on the one-sided analytic Hilbert space by

$$K_R := H_R^{\mathrm{res}} \subset H_{\alpha,+}.$$

Also, write the canonical kernel associated with its distribution kernel representation as

$$\mathsf{K}_R^{\mathrm{dist}} := \mathsf{K}_{R,1}^{\mathrm{dist}} \in \mathscr{D}_R' \widehat{\otimes} \mathscr{D}_R'.$$

Here $\mathsf{K}^{\mathrm{dist}}_{R,t}$ is the transport kernel family determined from the canonical transport group on K_R

$$U_R(t) = e^{-itL_R^{\mathrm{res}}}.$$

Remark 4.154 (Distinction between the subspace and the distribution kernel)**.** The symbol K_R denotes a closed subspace inside a Hilbert space. On the other hand,

$$\mathsf{K}_R^{\mathrm{dist}}$$

is a distribution kernel belonging to

$$\mathscr{D}'_R \widehat{\otimes} \mathscr{D}'_R.$$

They are objects of different types and are not identified, in accordance with the type-separation convention.

Lemma 4.155 (Closedness of the singular-boundary subspace)**.**

$$K_R = H_R^{\mathrm{res}}$$

is a closed linear subspace of $H_{\alpha,+}$.

Proof. This follows immediately from Lemma 4.89 and Definition 4.153. ☐

Lemma 4.156 (Transport and spectral structure on the singular-boundary subspace)**.** The following structures exist on K_R.

1. There exists a strongly continuous unitary group
$$U_R(t) = e^{-itL_R^{\mathrm{res}}} \qquad (t \in \mathbb{R}).$$
2. Its anti-self-adjoint generator
$$B_R = -iL_R^{\mathrm{res}}$$
exists, and
$$U_R(t) = e^{tB_R}.$$
3. L_R^{res} has compact resolvent.
4. L_R^{res} has purely discrete spectrum, and the spectral projection on a bounded spectral window has finite rank.

Proof. Items 1 and 2 follow from the construction of the transport group on K_R and the Stone generator. Item 3 follows from the compact-resolvent theorem. Item 4 follows from the purely discrete spectrum theorem and the finiteness of finite-window counts. ☐

Theorem 4.157 (Internal construction of the operator-theoretic boundary data)**.** *By the construction of this section, the operator-theoretic boundary data*

$$\mathfrak{R}_{\mathrm{op}} := (\Sigma_R, \sigma_R, \Gamma_R, T_R, \mathrm{supp}_R, L_R, G_R, Q_R^{\mathrm{res}}, H_R^{\mathrm{res}}, K_R, U_R(t), B_R, \mathsf{K}_R^{\mathrm{dist}}, L_R^{\mathrm{res}})$$

are internally fixed. Together with the standard zero-area carrier of Theorem 4.34, these data provide the boundary model on which the centered Mellin boundary finite-part class of ξ is realized in Section 4.10. They satisfy the following properties.

1. Γ_R *is null with respect to the regular area measure:*

$$\sigma_R(\Gamma_R) = 0.$$

2. *A nonzero singular carrier measure*

$$\nu_R$$

is supported on Γ_R.

3. *The singular boundary trace*

$$\gamma_R^{\text{sing}}$$

and the regular boundary trace

$$\gamma_R^{\text{reg}}$$

are defined on $T_R \subset Q_R$.

4. *The support map is defined by*

$$\text{supp}_R(f) = \text{supp}(\gamma_R^{\text{sing}} f).$$

5. *The regular boundary trace-mass functional is defined by*

$$L_R(f; E) = \int_E |\gamma_R^{\text{reg}} f|^2 \, d\sigma_R.$$

6. *The trace-vanishing generating subspace associated with* Γ_R

$$G_R = \left\{ f \in T_R : \text{supp}_R(f) \subset \Gamma_R,\ L_R(f; \Sigma_R \setminus \Gamma_R) = 0 \right\}$$

is defined.

7.

$$Q_R^{\text{res}} = \overline{\text{span}\, G_R}^{\|\cdot\|_{q_R}}$$

is a closed form domain.

8.

$$H_R^{\text{res}} = \overline{Q_R^{\text{res}}}^{\|\cdot\|_{H_{\alpha,+}}}$$

is a Hilbert space, and

$$K_R = H_R^{\text{res}}$$

is a closed subspace of $H_{\alpha,+}$.

9. *On* K_R*, there exists the positive shifted operator*

$$L_R^{\text{res}} = A_R^{\text{res}} + I,$$

which is self-adjoint and has compact resolvent.

10. *On K_R, there exist the conservative transport group*
$$U_R(t) = e^{-itL_R^{\mathrm{res}}}$$
and the anti-self-adjoint generator
$$B_R = -iL_R^{\mathrm{res}}.$$
11. *The distribution kernel associated with K_R is not an ordinary function kernel, but is defined as the distribution kernel*
$$\mathsf{K}_R^{\mathrm{dist}} \in \mathscr{D}'_R \widehat{\otimes} \mathscr{D}'_R.$$
12. *The package is ready to receive realized centered Mellin boundary finite parts: after the completed-zeta origin is fixed, the finite-window distributions*
$$\mathcal{N}_\xi\big(\mathfrak{b}^{\mathrm{fw}}_{\xi,\partial}(\varphi)\big) \in \mathcal{D}'_R$$
are inserted through the same singular boundary trace and boundary-pairing structure, without altering the internally constructed Hilbert-space data.

Proof. Items 1 and 2 follow from the construction of the boundary parameter space and the σ_R-null singular support. Items 3, 4, and 5 follow from the construction of traces, support maps, and regular boundary trace-mass functionals. Item 6 follows from Definition 4.149. Item 7 follows from the definition

$$Q_R^{\mathrm{res}} = \overline{\operatorname{span} G_R}^{\|\cdot\|_{q_R}}$$

and from completeness of the σ_R-null trace-vanishing form subspace. Item 8 follows from the definition of H_R^{res} and from the fact that it is a closed linear subspace of $H_{\alpha,+}$. Item 9 follows from the Friedrichs-type realization of q_R^{res}, which gives

$$A_R^{\mathrm{res}},$$

from the self-adjointness of its positive shift

$$L_R^{\mathrm{res}} = A_R^{\mathrm{res}} + I,$$

and from the compact-resolvent theorem. Item 10 follows from the construction

$$U_R(t) = e^{-itL_R^{\mathrm{res}}}$$

by the spectral theorem and the construction

$$B_R = -iL_R^{\mathrm{res}}$$

by Stone's theorem. Item 11 follows from the distribution kernel representation of the transport kernel family. Item 12 follows from Theorem 4.34: the smooth seam carrier keeps σ_R-null distributional boundary data nontrivial, so the completed-zeta boundary finite parts can be realized later by the transpose map $\mathcal{N}_\xi = \mathcal{C}'_{\xi R}$ without changing the construction of K_R, Π_R^+, or the transport kernel.

Thus all displayed objects are defined internally in this section and satisfy the listed properties. □

Corollary 4.158 (Existence of the orthogonal projection onto K_R)**.** *K_R is a closed linear subspace of $H_{\alpha,+}$. Therefore, by the projection theorem for Hilbert spaces, there exists a unique orthogonal projection*

$$\Pi_R^+ : H_{\alpha,+} \longrightarrow K_R.$$

Namely,

$$(\Pi_R^+)^* = \Pi_R^+, \qquad (\Pi_R^+)^2 = \Pi_R^+, \qquad \operatorname{Ran}\Pi_R^+ = K_R.$$

Proof. By Lemma 4.155,

$$K_R \subset H_{\alpha,+}$$

is a closed linear subspace. By the projection theorem for Hilbert spaces, any

$$h \in H_{\alpha,+}$$

decomposes uniquely as

$$h = k + r, \qquad k \in K_R, \quad r \in K_R^\perp.$$

Defining

$$\Pi_R^+ h := k,$$

this is the orthogonal projection onto K_R. By the general properties of orthogonal projections,

$$(\Pi_R^+)^* = \Pi_R^+, \qquad (\Pi_R^+)^2 = \Pi_R^+,$$

and its range is K_R. Uniqueness is also included in the projection theorem. □

Definition 4.159 (Canonical one-sided projection onto K_R)**.** The orthogonal projection

$$\Pi_R^+ : H_{\alpha,+} \to K_R$$

constructed on the one-sided analytic Hilbert space is called the canonical one-sided projection onto K_R. The unsuperscripted symbol

$$\Pi_R$$

is reserved for the ambient Hilbert-space projector defined in Section 5 after K_R has been embedded into X.

Proposition 4.160 (Output of this subsection)**.** *This subsection fixes the singular-boundary input as the following internally constructed data:*

$$\mathfrak{R}_{\mathrm{op}} = (\Sigma_R, \sigma_R, \Gamma_R, T_R, \mathrm{supp}_R, L_R, G_R, Q_R^{\mathrm{res}}, H_R^{\mathrm{res}}, K_R, \Pi_R^+, U_R(t), B_R, \mathsf{K}_R^{\mathrm{dist}}, L_R^{\mathrm{res}}).$$

These data include the σ_R-null property, regular-trace vanishing property, closed-form property, Hilbert closedness, self-adjoint realization, conservative transport, distribution kernel representation, compact resolvent, and existence of the orthogonal projection.

Proof. The boundary data were constructed by Definition 4.147. The trace-vanishing generating subspace associated with Γ_R was constructed by Definition 4.149. The form and Hilbert data were constructed by Definition 4.151. The singular-boundary subspace K_R was constructed by Definition 4.153, and its closedness was proved by Lemma 4.155. The orthogonal projection Π_R^+ exists by Corollary 4.158. The transport group, generator, distribution kernel, and compact resolvent are given by Lemma 4.156 and Theorem 4.157. Therefore the displayed data is fixed as internally constructed data. ☐

4.10. Fixing the Operator-Theoretic Boundary Data and Transition to the Next Section

In this subsection, we fix the singular-boundary objects constructed in this section as a single set of input data. In the next section, the data fixed here are received as the singular-boundary input and integrated with the arithmetic construction of Section 3 on a common ambient Hilbert space. Accordingly, the role of this subsection is not to add a new analytic construction, but to record the output of this section and fix the types and dependencies of the objects referred to in the next section.

Definition 4.161 (Operator-theoretic boundary data)**.** Define the singular-boundary input data constructed in this section by

$$\begin{aligned}\mathfrak{R}_{\mathrm{op}} := (&\Sigma_R, \sigma_R, \Gamma_R, T_R, \mathrm{supp}_R, L_R, G_R, Q_R^{\mathrm{res}}, H_R^{\mathrm{res}}, K_R, \Pi_R^+, \\ &U_R(t), B_R, \mathsf{K}_R^{\mathrm{dist}}, L_R^{\mathrm{res}}).\end{aligned}$$

Here each component has the following meaning.

1. $(\Sigma_R, \sigma_R, \Gamma_R)$ is the boundary triple data equipped with a regular area measure and a σ_R-null singular support.
2. T_R is a linear subspace inside the form domain on which the singular boundary trace and the regular boundary trace are defined.
3. supp_R is the support map assigning the distribution support of the singular boundary trace.
4. L_R is the regular boundary trace-mass functional with respect to the regular area measure σ_R.
5. G_R is the generating class satisfying the σ_R-null property and the regular-trace vanishing condition simultaneously.
6. Q_R^{res} is the q_R-form-norm closure of G_R.
7. H_R^{res} is the $H_{\alpha,+}$-closure of Q_R^{res}.
8. $K_R = H_R^{\mathrm{res}}$ is the closed singular-boundary subspace inside the one-sided analytic Hilbert space.
9. Π_R^+ is the orthogonal projection from $H_{\alpha,+}$ onto K_R.
10. L_R^{res} is the positive shifted self-adjoint operator obtained from the Friedrichs-type realization of the σ_R-null restricted closed form.
11. $U_R(t) = e^{-itL_R^{\mathrm{res}}}$ is the strongly continuous unitary transport group on K_R.
12. $B_R = -iL_R^{\mathrm{res}}$ is the anti-self-adjoint generator of $U_R(t)$.
13. $\mathsf{K}_R^{\mathrm{dist}} \in \mathscr{D}_R' \widehat{\otimes} \mathscr{D}_R'$ is the distribution kernel representation of the transport distribution kernel.

Definition 4.162 (Completed-zeta centered Mellin boundary realization)**.** Set

$$\xi_c(w) := \xi\left(\frac{1}{2} + w\right).$$

The functional equation is written as

$$\xi_c(w) = \xi_c(-w).$$

For each finite window M, let

$$\boxed{\mathfrak{S}_M^{\mathrm{cfw}}}$$

denote the universal completed finite-window contour-coordinate space. This space records only the finite-window contour convention, the functional-equation symmetric centered contour coordinate, the central and endpoint jet convention, and the contour finite-part probes. It is not a zeta-zero space and it is fixed before any seam realization, K_R-readout, determinant construction, or central comparison is applied. We identify the finite readout-probe space with this finite-window contour-coordinate space by writing

$$\mathcal{D}^{\mathrm{rd}}_{\xi,\partial,M} := \mathfrak{S}^{\mathrm{cfw}}_M.$$

The subscript ξ in this notation records the centered Mellin finite-window contour-coordinate convention; it does not denote any dependence on the location of the zeros of ξ.

Let $\mathcal{D}^{\mathrm{rd}}_{\xi,\partial,M}$ denote the finite-dimensional contour finite-part probe space attached to the centered Mellin finite window M. Its elements are the fixed contour-side probes used to read finite parts before any realization on the Section 4 seam carrier is applied. For

$$\psi \in \mathcal{T}_{\log,M}, \qquad \eta \in \mathcal{D}^{\mathrm{rd}}_{\xi,\partial,M},$$

define the contour finite-part bilinear form by

$$\boxed{\mathcal{Z}^{\mathrm{fw}}_{\xi,\partial}(\psi,\eta) := \mathrm{Fp}_{\partial,\xi}\left[\frac{1}{2\pi i}\int_{\partial\Omega_{\mathrm{fw}}} -\frac{\xi_c'(z)}{\xi_c(z)}\,\mathcal{M}_{\mathrm{fw}}(\psi,\eta)(z)\,dz\right].}$$

Here $\mathcal{M}_{\mathrm{fw}}(\psi,\eta)$ is the standard finite-window Mellin contour test obtained from the logarithmic representative ψ and the contour finite-part probe η. The operation $\mathrm{Fp}_{\partial,\xi}$ is the finite-part extraction in the classical centered finite-window contour formula after the common Archimedean/reference part and the prime-power part have been separated.

Equivalently, the completed zeta function defines a ξ-finite-part evaluation functional

$$\boxed{\mathfrak{e}_{\xi,M}(\psi) \in (\mathfrak{S}^{\mathrm{cfw}}_M)'}$$

by

$$\langle \mathfrak{e}_{\xi,M}(\psi), \eta\rangle := \mathcal{Z}^{\mathrm{fw}}_{\xi,\partial}(\psi,\eta), \qquad \eta \in \mathfrak{S}^{\mathrm{cfw}}_M.$$

This is the only ξ-specific finite-part object used in the finite-window boundary comparison. The seam carrier, the LCI synthesis, the Gram-inverse reconstruction, and the determinant construction are not part of the definition of $\mathfrak{e}_{\xi,M}$.

Let $\mathfrak{B}^{\mathrm{fw}}_{\xi,\partial}$ be the linear span of the distributions represented by this ξ-finite-part evaluation functional. For each logarithmic finite-window representative ψ, define

$$\mathfrak{b}^{\mathrm{fw}}_{\xi,\partial}(\psi) \in \mathfrak{B}^{\mathrm{fw}}_{\xi,\partial}$$

by the rule

$$\boxed{\left\langle \mathfrak{b}^{\mathrm{fw}}_{\xi,\partial}(\psi), \eta\right\rangle_{\partial,\xi} := \mathcal{Z}^{\mathrm{fw}}_{\xi,\partial}(\psi,\eta) \qquad (\eta \in \mathcal{D}^{\mathrm{rd}}_{\xi,\partial,M}).}$$

Thus the zeta-side boundary finite-part distribution is obtained from the classical contour finite-part bilinear form before any K_R-side readout is mentioned. It is not defined using F_K, the spectral operator K, the central equality, or any assumption on the real parts of the zeros of ξ.

For each finite window M, fix once and for all the finite-part probe

$$\eta_M^{\mathrm{fp}} \in \mathcal{D}^{\mathrm{rd}}_{\xi,\partial,M}.$$

This probe is determined by the oriented centered Mellin finite-window contour, the finite-part convention, and the window cutoff. For $\tau = [\psi]_{\mathrm{fw}}$ represented in $\mathcal{T}_{\log,M}$, define the scalar contour finite part by

$$\boxed{Z^{\mathrm{fw}}_{\xi,\partial}(\tau) := \left\langle \mathfrak{b}^{\mathrm{fw}}_{\xi,\partial}(\psi), \eta_M^{\mathrm{fp}} \right\rangle_{\partial,\xi}.}$$

If the same representative is read in a larger window, the fixed probes are chosen compatibly with the window-refinement maps, so the displayed scalar depends only on the class $\tau = [\psi]_{\mathrm{fw}}$. Let

$$\mathscr{S}_\xi := \mathbb{R}_u$$

be the centered Mellin boundary seam. The fixed smooth coding map from the centered Mellin seam to the Section 4 zero-area seam carrier is

$$\kappa : \mathbb{R} \longrightarrow (0,1) \times \{0\}, \qquad \kappa(u) := \left(\frac{1}{2} + \frac{1}{\pi} \arctan u,\ 0\right).$$

It is a C^∞-diffeomorphism from $\mathbb{R}$ onto the open seam $(0,1) \times \{0\}$, with inverse

$$\kappa^{-1}(x,0) = \tan\!\left(\pi\left(x - \frac{1}{2}\right)\right).$$

This coding map is fixed before any zeta-side or K-side pairing is evaluated.

For a boundary test representative ϕ on the Γ_R-seam, write

$$\gamma_\Gamma \phi(x) := \phi(x,0).$$

Define the pullback from the Section 4 seam to the centered Mellin boundary seam by

$$\mathcal{C}_{\xi R}\phi(u) := (\gamma_\Gamma \phi)(\kappa(u)).$$

Thus

$$\mathcal{C}_{\xi R} : \mathcal{D}_R^\Gamma \longrightarrow \mathcal{D}_{\xi,\partial}$$

is a fixed continuous linear map between the seam-test spaces. Here $\mathcal{D}_R^\Gamma$ denotes the seam trace test layer of the Section 4 boundary-test space, and $\mathcal{D}_{\xi,\partial}$ denotes the centered Mellin boundary test space.

The natural realization map of the completed-zeta boundary finite part on the Section 4 carrier is defined as the transpose of this fixed pullback:

$$\boxed{\mathcal{N}_\xi := \mathcal{C}'_{\xi R}.}$$

Equivalently, for

$$b \in \mathfrak{B}^{\mathrm{fw}}_{\xi,\partial}, \qquad \phi \in \mathcal{D}_R^\Gamma,$$

we define

$$\langle \mathcal{N}_\xi b, \phi \rangle_{\partial,R} := \langle b, \mathcal{C}_{\xi R}\phi \rangle_{\partial,\xi}.$$

This is the definition of $\mathcal{N}_\xi$. In particular, $\mathcal{N}_\xi$ is not characterized by requiring agreement of zeta-side and K_R-side readouts; it is constructed from κ, seam trace, and transposition.

For a logarithmic finite-window representative ψ, write

$$T_\psi := T^{\mathrm{fw}}_{\partial,R}(\psi) := \mathcal{N}_\xi\big(\mathfrak{b}^{\mathrm{fw}}_{\xi,\partial}(\psi)\big).$$

If $\tau = [\psi]_{\mathrm{fw}}$, we also write $T_\tau := T_\psi$.

More generally, for any completed finite-window canonical finite-part functional

$$\ell \in (\mathfrak{S}^{\mathrm{cfw}}_M)'$$

let $b_\ell \in \mathfrak{B}^{\mathrm{fw}}_{\xi,\partial}$ denote its boundary-distribution representative, defined by

$$\langle b_\ell, \eta \rangle_{\partial,\xi} := \langle \ell, \eta \rangle, \qquad \eta \in \mathfrak{S}^{\mathrm{cfw}}_M.$$

The boundary-level comparison-independent realization machine is

$$\boxed{\mathfrak{R}_{\partial,R,M}(\ell) := \mathcal{N}_\xi b_\ell.}$$

Although the notation $\mathcal{N}_\xi$ records the centered Mellin convention, the map $\mathfrak{R}_{\partial,R,M}$ uses only the fixed seam coding map, the seam pullback, and transposition. It does not use the zeros of ξ, F_K, the central comparison equality, or spectral localization. For the completed-zeta canonical finite-part functional determined by $\mathfrak{e}_{\xi,M}$,

$$T_\psi = \mathfrak{R}_{\partial,R,M}\big(\mathfrak{e}_{\xi,M}(\psi)\big).$$

Finally, define a fixed readout lifting operator

$$\mathcal{U}_{\xi R} : \mathcal{D}^{\mathrm{fw}}_{\xi,\partial} \longrightarrow \mathcal{D}^{\Gamma}_R$$

as follows. Choose once and for all a smooth transverse cutoff $\rho_\Gamma \in C^\infty_c((-1,1))$ with $\rho_\Gamma(0) = 1$, and choose an interior cutoff $\chi_{\mathrm{int}} \in C^\infty_c((0,1))$ which is identically 1 on the compact x-interval corresponding to the finite window under κ. For a contour finite-part readout ω supported in that finite window, set

$$(\mathcal{U}_{\xi R}\omega)(x,y) := \rho_\Gamma(y)\,\chi_{\mathrm{int}}(x)\,\omega\bigg(\tan\bigg(\pi\bigg(x - \frac{1}{2}\bigg)\bigg)\bigg).$$

On the corresponding finite-window support,

$$\boxed{\mathcal{C}_{\xi R}\mathcal{U}_{\xi R}\omega = \omega.}$$

The construction uses the divisor of ξ only through the classical contour identity for $-\xi'/\xi$, and uses no zero-location hypothesis.

Lemma 4.163 (canonical finite-part audit: no zero-location input)**.** For every finite window M and every logarithmic representative $\psi \in \mathcal{T}_{\log,M}$, the ξ-finite-part evaluation functional

$$\mathfrak{e}_{\xi,M}(\psi) \in (\mathfrak{S}^{\mathrm{cfw}}_{M})'$$

is constructed only from the completed function ξ_c, the classical finite-window contour identity for $-\xi_c'/\xi_c$, the functional equation $\xi_c(w) = \xi_c(-w)$, and the finite-window contour-coordinate convention. Its construction uses neither

$$F_K \equiv \xi, \qquad \operatorname{Re}\rho = \frac{1}{2}, \qquad \mu_L = \mu_\xi, \qquad \langle \mu_L, \Psi^{\mathrm{cen}}_w \rangle = \langle \mu_\xi, \Psi^{\mathrm{cen}}_w \rangle,$$

nor any spectral localization conclusion.

Proof. The definition of $\mathfrak{e}_{\xi,M}(\psi)$ is the bilinear finite-part contour formula

$$\langle \mathfrak{e}_{\xi,M}(\psi), \eta \rangle = \mathcal{Z}^{\mathrm{fw}}_{\xi,\partial}(\psi,\eta),$$

where the integrand is $-\xi_c'/\xi_c$ tested against the fixed finite-window Mellin contour representative $\mathcal{M}_{\mathrm{fw}}(\psi,\eta)$. No operator K, determinant F_K, central equality, or assertion on zero locations enters this formula. $\square$

Proposition 4.164 (Independent realization, pairing theorem, and Mellin-boundary naturality)**.** *Each component of $\mathfrak{R}_{\mathrm{op}}$ is determined by the analytic Hilbert-space setup of Section 2 and the construction of this section. After the completed-zeta boundary origin is fixed by Definition 4.162, the realized boundary finite parts are attached to $\mathfrak{R}_{\mathrm{op}}$ by the already constructed transpose map*

$$\mathcal{N}_\xi = \mathcal{C}'_{\xi R},$$

without changing any internally constructed Hilbert-space object. In particular, the following hold.

1. $$\sigma_R(\Gamma_R) = 0, \qquad \nu_R(\Gamma_R) = 1, \qquad \nu_R \perp \sigma_R.$$
2. *For any $f \in G_R$,*
   $$\operatorname{supp}_R(f) \subset \Gamma_R, \qquad L_R(f; \Sigma_R \setminus \Gamma_R) = 0.$$
3. *Q^{res}_R is a closed form subspace of Q_R, and*
   $$q^{\mathrm{res}}_R = q_R|_{Q^{\mathrm{res}}_R \times Q^{\mathrm{res}}_R}$$
   is a closed lower-bounded form on H^{res}_R.
4. *A^{res}_R is the unique self-adjoint representing operator of q^{res}_R, and*
   $$L^{\mathrm{res}}_R = A^{\mathrm{res}}_R + I$$
   is self-adjoint and positive.
5. *L^{res}_R has compact resolvent and purely discrete spectrum.*
6. *K_R is a closed subspace of $H_{\alpha,+}$, and therefore*
   $$\Pi^+_R : H_{\alpha,+} \to K_R$$

exists uniquely.

7. *For every finite-window finite-part boundary distribution $b \in \mathfrak{B}^{\mathrm{fw}}_{\xi,\partial}$ and every finite-window contour finite-part readout ω,*
$$\langle \mathcal{N}_\xi b, \mathcal{U}_{\xi R}\omega \rangle_{\partial,R} = \langle b, \omega \rangle_{\partial,\xi}.$$

8. *The centered Mellin reflection Θ_ξ induced by $u \mapsto -u$ is realized on the Section 4 seam by*
$$\theta_R(x,0) = (1-x,0).$$
If Θ_R denotes the distributional pushforward induced by θ_R, then
$$\Theta_R \mathcal{N}_\xi = \mathcal{N}_\xi \Theta_\xi$$
on finite-window realized boundary distributions.

Proof. Items 1–6 follow exactly as in the internal construction of the boundary parameter space, trace-vanishing form subspace, Friedrichs realization, compact-resolvent theorem, and Hilbert projection theorem.

For Item 7, use the definition
$$\mathcal{N}_\xi = \mathcal{C}'_{\xi R}.$$
Then
$$\begin{aligned}\langle \mathcal{N}_\xi b, \mathcal{U}_{\xi R}\omega \rangle_{\partial,R} &= \langle b, \mathcal{C}_{\xi R}\mathcal{U}_{\xi R}\omega \rangle_{\partial,\xi} \\ &= \langle b, \omega \rangle_{\partial,\xi},\end{aligned}$$
because $\mathcal{C}_{\xi R}\mathcal{U}_{\xi R} = I$ on the finite-window support. Thus the pairing preservation is a theorem following from the explicitly constructed pullback and lifting maps, not a defining property of $\mathcal{N}_\xi$.

For Item 8, compute the seam reflection. Since
$$\kappa(u) = \left(\frac{1}{2} + \frac{1}{\pi}\arctan u,\ 0\right),$$
we have
$$\kappa(-u) = \left(\frac{1}{2} - \frac{1}{\pi}\arctan u,\ 0\right) = (1-x,0) \quad \text{if } \kappa(u) = (x,0).$$
Hence
$$\theta_R \circ \kappa = \kappa \circ \Theta_\xi.$$
Taking pushforwards on distributions and using $\mathcal{N}_\xi = \mathcal{C}'_{\xi R}$ gives
$$\Theta_R \mathcal{N}_\xi = \mathcal{N}_\xi \Theta_\xi.$$
Accordingly, all components of $\mathfrak{R}_{\mathrm{op}}$ and the attached completed-zeta realization data are fixed without circular use of the determinant comparison or of any zero-location assertion. ☐

Definition 4.165 (Fixing as the singular-boundary input)**.** To fix the singular-boundary input from the next section onward means first to fix the data
$$\mathfrak{R}_{\mathrm{op}}$$

of Definition 4.161. When the completed-zeta boundary origin is used, the realized package is written

$$\mathfrak{R}_{\mathrm{op}}^{\xi} := (\mathfrak{R}_{\mathrm{op}}, \kappa, \mathcal{C}_{\xi R}, \mathcal{N}_{\xi}, \mathcal{U}_{\xi R}, \Theta_{\xi}, \theta_R, \Theta_R).$$

The Hilbert-space and operator-theoretic components passed to the orthogonal-decomposition framework are still precisely those contained in $\mathfrak{R}_{\mathrm{op}}$:

$$\Sigma_R, \sigma_R, \Gamma_R, T_R, \mathrm{supp}_R, L_R, G_R, Q_R^{\mathrm{res}}, H_R^{\mathrm{res}}, K_R, \Pi_R^{+}, U_R(t), B_R, \mathsf{K}_R^{\mathrm{dist}}, L_R^{\mathrm{res}}.$$

The extra symbols $\kappa, \mathcal{C}_{\xi R}, \mathcal{N}_{\xi}, \mathcal{U}_{\xi R}, \Theta_{\xi}, \theta_R, \Theta_R$ specify only how the centered Mellin boundary finite parts of ξ are transported to these already constructed data.

Proposition 4.166 (Transition to the next section)**.** *In the next section, $\mathfrak{R}_{\mathrm{op}}$ is fixed as the already constructed singular-boundary input and is placed, together with the arithmetic-side data constructed in Section 3, on an ambient Hilbert space. When a finite-window zeta-side boundary term is used, it enters only through the realized distribution*

$$T_{\partial,R}^{\mathrm{fw}}(\varphi) = \mathcal{N}_{\xi}\big(\mathfrak{b}_{\xi,\partial}^{\mathrm{fw}}(\varphi)\big) \in \mathcal{D}_R'.$$

In this transition, the Hilbert-space information passed from the singular-boundary side is limited to what is contained in

$$\mathfrak{R}_{\mathrm{op}}.$$

In particular, the singular-boundary subspace and orthogonal projection onto K_R used in the next section are obtained by lifting

$$K_R \quad \textit{and} \quad \Pi_R^{+}$$

constructed in this section to the ambient Hilbert-space setting.

Proof. By Definition 4.165, the singular-boundary input fixed in the next section is $\mathfrak{R}_{\mathrm{op}}$, and the optional completed-zeta realization data only identify which finite-window boundary distributions are inserted into $\mathcal{D}_R'$. Moreover, by Proposition 4.164, each component of $\mathfrak{R}_{\mathrm{op}}$ has already been constructed in this section and the realization map $\mathcal{N}_{\xi} = \mathcal{C}_{\xi R}'$ is a fixed transpose construction whose boundary-pairing identity is proved from $\mathcal{C}_{\xi R}\mathcal{U}_{\xi R} = I$.

In the orthogonal-decomposition framework of the next section, K_R and Π_R^{+} are first lifted through the analytic embedding into the ambient Hilbert space. The isometric image of the closed subspace is closed, so the orthogonal projection onto that image exists. This lifted projection is denoted by Π_R in Section 5 and acts on X, not on $H_{\alpha,+}$. Therefore, the singular-boundary subspace and projector required in the next section are derived from the one-sided objects K_R and Π_R^{+} of this section. It follows that the transition on the singular-boundary side is completed by $\mathfrak{R}_{\mathrm{op}}$, while $\mathcal{N}_{\xi}$ supplies only the fixed seam-pushforward origin of the finite-window boundary distributions. □

Remark 4.167 (Separation of dependencies)**.** The construction in this section is based on the analytic Hilbert-space setup of Section 2 and the measure-theoretic and operator-theoretic constructions inside this section. The arithmetic projector family, arithmetic trace, and subsequent zero-counting machinery of Section 3 are integrated only from the next section onward. Therefore this section is the section that constructs the singular-boundary input, and it is not a section that proves orthogonality with the arithmetic side or closure of finite-window counting.

Convention 4.168 (Meaning of fixing at the beginning of the next section)**.** When the beginning of the next section says that "the auxiliary operator-theoretic boundary data is fixed," this does not mean assuming externally and arbitrarily given unconstructed data. It means fixing the singular boundary data

$$\mathfrak{R}_{\mathrm{op}}$$

constructed by Theorem 4.157 of this section. From now on, the singular-boundary input is used in this sense.

Proposition 4.169 (Completion of Section 4)**.** *This section fixes all singular-boundary data, form data, Hilbert-space data, projection data, transport data, distribution-kernel data, and spectral data required in the next section as*

$$\mathfrak{R}_{\mathrm{op}}.$$

Proof. The boundary data have been fixed as

$$(\Sigma_R, \sigma_R, \Gamma_R, T_R, \mathrm{supp}_R, L_R).$$

The form data have been fixed as

$$G_R, \quad Q_R^{\mathrm{res}}, \quad q_R^{\mathrm{res}}.$$

The Hilbert-space data have been fixed as

$$H_R^{\mathrm{res}}, \quad K_R.$$

The projection data have been fixed as

$$\Pi_R^+.$$

The transport data have been fixed as

$$U_R(t), \quad B_R.$$

The distribution-kernel data have been fixed as

$$\mathsf{K}_R^{\mathrm{dist}}.$$

The spectral data have been fixed as

$$L_R^{\mathrm{res}},$$

together with its compact resolvent, purely discrete spectrum, and finite-window spectral projections. Therefore all singular-boundary objects required in the next section are contained in

$$\mathfrak{R}_{\mathrm{op}}.$$

□

Preparation for the analytic comparison to Section 6.

The type separation on the singular-boundary side constructed in this section is the foundation for the analytic comparison used in Section 6. In that comparison, distributional objects on the singular-boundary side are handled inside a Gelfand triple

$$\mathcal{D}_R \subset H_{\alpha,+} \subset \mathcal{D}'_R,$$

and point evaluations and singular boundary traces are treated not as ordinary bounded functionals on $H_{\alpha,+}$, but as distributional boundary distributions belonging to $\mathcal{D}'_R$. Furthermore, the distribution kernels, quadratic forms, generators, boundary forms, and projectors distinguished in this section are preparation for defining the tempered distribution pairings and $\mathfrak{S}_2$-approximations appearing in Section 6,

$$\langle \mu, \varphi \rangle, \qquad \varphi \in \mathcal{S}(\mathbb{R}), \quad \mu \in \mathcal{S}'(\mathbb{R}).$$

In particular, the boundary-distribution comparison kernel K introduced later and its finite-rank cutoffs $K_N = P_N K P_N$ are interpreted within the framework of distribution kernels, projectors, and compactness established up to this section. This comparison is not a new assumption, but a notational bridge for using the operator-theoretic objects already constructed from Sections 2 through 5 in the distributional evaluations, central regularization, and $\mathfrak{S}_2$-approximations of Section 6.

5. Orthogonal-Decomposition Spectral Comparison Framework

5.1. Embedding the Constructed Singular-Boundary Subspace into the Ambient Hilbert Space

Section 2 fixed the analytic Hilbert-space data

$$H_{\alpha,+}, \qquad q_R, \qquad A_R, \qquad L, \qquad m_L^{(\Phi)},$$

and Section 3 fixed the coefficient-space arithmetic construction

$$\Lambda^{\mathrm{ex}}, \qquad C_{\mathrm{comp}}, \qquad \partial_{\log}, \qquad \{\Pi_n\}_{n\geq 1}, \qquad \Pi_{\mathrm{arith}}(\cdot).$$

Section 4 constructed, independently of these, the singular-boundary input data

$$\mathfrak{R}_{\mathrm{op}}$$

operator-theoretically. In this section, we lift this constructed input to the ambient Hilbert space and place it on the same orthogonal-decomposition stage as the coefficient-space arithmetic construction of Section 3.

What is done in this subsection is not a reconstruction of the operator-theoretic singular-boundary data. Namely,

$$(\Sigma_R, \sigma_R, \Gamma_R, T_R, \mathrm{supp}_R, L_R)$$

is fixed as data already constructed in Section 4. In this subsection, among its outputs,

$$K_R, \qquad \Pi_R^+, \qquad Q_R^{\mathrm{res}}, \qquad H_R^{\mathrm{res}}$$

are lifted to the ambient stage X, and the singular-boundary projector used below is defined.

Definition 5.1 (Ambient Hilbert space and canonical embeddings)**.** Define the ambient Hilbert space for the orthogonal-decomposition comparison by

$$X := H_{\alpha,+} \oplus \mathcal{H}_{\mathrm{arith}}.$$

Its inner product is given by

$$\langle (f,u),(g,v)\rangle_X := \langle f,g\rangle_{H_{\alpha,+}} + \langle u,v\rangle_{\mathrm{arith}}.$$

Write its norm as

$$\|x\|_X := \langle x,x\rangle_X^{1/2}, \qquad |x|_X := \|x\|_X.$$

Define the canonical embeddings

$$J_{\mathrm{an}} : H_{\alpha,+} \to X, \qquad J_{\mathrm{an}} f := (f,0),$$

and

$$J_{\mathrm{arith}} : \mathcal{H}_{\mathrm{arith}} \to X, \qquad J_{\mathrm{arith}} u := (0,u).$$

Then

$$J_{\mathrm{an}} H_{\alpha,+} \perp J_{\mathrm{arith}} \mathcal{H}_{\mathrm{arith}},$$

and

$$X = J_{\mathrm{an}} H_{\alpha,+} \oplus J_{\mathrm{arith}} \mathcal{H}_{\mathrm{arith}}.$$

At this stage, no further operator on X is introduced.

Definition 5.2 (Fixing and lifting the constructed singular-boundary input)**.** Fix the singular-boundary input data

$$\mathfrak{R}_{\mathrm{op}}$$

constructed in Section 4. Among its components, write the closed one-sided singular-boundary subspace as

$$K_R^+ := K_R = H_R^{\mathrm{res}} \subset H_{\alpha,+}.$$

Define the singular-boundary component on the ambient stage by

$$\mathcal{K}_R := J_{\mathrm{an}} K_R^+ \subset X.$$

Also, let

$$\Pi_R^+ : H_{\alpha,+} \to K_R^+$$

be the orthogonal projection on the one-sided analytic stage constructed in Section 4. Define the singular-boundary projector on the ambient stage by

$$\Pi_R := J_{\mathrm{an}} \Pi_R^+ J_{\mathrm{an}}^*.$$

Convention 5.3 (One-sided and ambient singular-boundary notation)**.** In this section and in Section 6, $K_R^+ \subset H_{\alpha,+}$ denotes the one-sided closed subspace constructed in Section 4, while

$$\mathcal{K}_R = J_{\mathrm{an}} K_R^+ \subset X$$

denotes its ambient image. Likewise, Π_R^+ denotes the one-sided projector on $H_{\alpha,+}$, whereas

$$\Pi_R = J_{\mathrm{an}} \Pi_R^+ J_{\mathrm{an}}^*$$

denotes the ambient projector on X. Thus every occurrence of Π_R from this point onward is an X-operator, while Π_R^+ is reserved for the one-sided analytic stage.

Lemma 5.4 (Closedness of the lifted singular-boundary component)**.** $\mathcal{K}_R$ is a closed linear subspace of X, and

$$\mathcal{K}_R \subset J_{\mathrm{an}} H_{\alpha,+}.$$

Proof. By Section 4, K_R^+ is a closed linear subspace of $H_{\alpha,+}$. The map J_{an} is an isometric isomorphism from $H_{\alpha,+}$ onto $J_{\mathrm{an}} H_{\alpha,+} \subset X$. Therefore the image of the closed set $K_R^+ \subset H_{\alpha,+}$,

$$J_{\mathrm{an}} K_R^+,$$

is closed in $J_{\mathrm{an}} H_{\alpha,+}$. Furthermore, $J_{\mathrm{an}} H_{\alpha,+}$ is a closed subspace of X, so

$$\mathcal{K}_R = J_{\mathrm{an}} K_R^+$$

is closed in X. The inclusion

$$\mathcal{K}_R \subset J_{\mathrm{an}} H_{\alpha,+}$$

follows immediately from the definition. ☐

Theorem 5.5 (Singular-boundary projector on the ambient stage)**.** *The operator*

$$\Pi_R := J_{\mathrm{an}} \Pi_R^+ J_{\mathrm{an}}^*$$

is an orthogonal projector on X, and satisfies

$$\Pi_R^2 = \Pi_R, \qquad \Pi_R^* = \Pi_R, \qquad \operatorname{Ran} \Pi_R = \mathcal{K}_R, \qquad \ker \Pi_R = \mathcal{K}_R^\perp.$$

Equivalently, every $x \in X$ has a unique orthogonal decomposition

$$x = x_R + x_\perp, \qquad x_R \in \mathcal{K}_R, \quad x_\perp \in \mathcal{K}_R^\perp,$$

and

$$\Pi_R x = x_R.$$

Proof. First we show that Π_R is a projection. For the canonical embedding,

$$J_{\mathrm{an}}^* J_{\mathrm{an}} = I_{H_{\alpha,+}}.$$

Therefore

$$\Pi_R^2 = J_{\mathrm{an}} \Pi_R^+ J_{\mathrm{an}}^* J_{\mathrm{an}} \Pi_R^+ J_{\mathrm{an}}^* = J_{\mathrm{an}} (\Pi_R^+)^2 J_{\mathrm{an}}^* = J_{\mathrm{an}} \Pi_R^+ J_{\mathrm{an}}^* = \Pi_R.$$

Also,

$$\Pi_R^* = \left(J_{\mathrm{an}} \Pi_R^+ J_{\mathrm{an}}^*\right)^* = J_{\mathrm{an}} (\Pi_R^+)^* J_{\mathrm{an}}^* = J_{\mathrm{an}} \Pi_R^+ J_{\mathrm{an}}^* = \Pi_R.$$

Thus Π_R is a self-adjoint projector.

Next we identify its range. For any $x = (f, u) \in X$,

$$J_{\text{an}}^* x = f,$$

so

$$\Pi_R x = J_{\text{an}} \Pi_R^+ f \in J_{\text{an}} K_R^+ = \mathcal{K}_R.$$

Therefore

$$\operatorname{Ran} \Pi_R \subset \mathcal{K}_R.$$

Conversely, any $k \in \mathcal{K}_R$ can be written as

$$k = J_{\text{an}} h \qquad (h \in K_R^+).$$

Then $\Pi_R^+ h = h$, and hence

$$\Pi_R k = J_{\text{an}} \Pi_R^+ J_{\text{an}}^* J_{\text{an}} h = J_{\text{an}} \Pi_R^+ h = J_{\text{an}} h = k.$$

Thus

$$\mathcal{K}_R \subset \operatorname{Ran} \Pi_R.$$

Therefore

$$\operatorname{Ran} \Pi_R = \mathcal{K}_R.$$

The kernel of a self-adjoint projector is the orthogonal complement of its range, so

$$\ker \Pi_R = (\operatorname{Ran} \Pi_R)^\perp = \mathcal{K}_R^\perp.$$

The final orthogonal decomposition follows from the projection theorem for Hilbert spaces. ☐

Remark 5.6 (Status of the constructed input)**.** The $\mathcal{K}_R$-side input used in this section is

$$\mathfrak{R}_{\text{op}},$$

constructed in Section 4. The role of this section is to lift these constructed data to the ambient Hilbert space X and place them on the same ambient stage as the coefficient-space arithmetic construction of Section 3. Therefore, this section does not reconstruct the operator-theoretic singular-boundary data.

Theorem 5.7 (Lifting boundary cancellation to the ambient stage)**.** *For the zero extension of the boundary form constructed in Section 4,*

$$b_R^{\text{res}},$$

one has

$$b_R^{\text{res}}[u, v] = 0$$

for any

$$u, v \in Q_R^{\text{res}}.$$

Therefore, for the lifts

$$J_{\text{an}} u,\ J_{\text{an}} v \in \mathcal{K}_R$$

on the ambient stage,

$$b^{\mathrm{res}}_{R,X}[J_{\mathrm{an}}u, J_{\mathrm{an}}v] := b^{\mathrm{res}}_R[u,v] = 0$$

holds.

Proof. By the boundary cancellation theorem of Section 4,

$$b^{\mathrm{res}}_R[u,v] = 0 \qquad (u, v \in Q^{\mathrm{res}}_R).$$

Since J_{an} is an isometric embedding and $Q^{\mathrm{res}}_R \subset K^+_R$, we have

$$J_{\mathrm{an}}u,\ J_{\mathrm{an}}v \in J_{\mathrm{an}}K^+_R = \mathcal{K}_R.$$

The boundary form on the ambient stage is defined by

$$b^{\mathrm{res}}_{R,X}[J_{\mathrm{an}}u, J_{\mathrm{an}}v] := b^{\mathrm{res}}_R[u,v],$$

so immediately

$$b^{\mathrm{res}}_{R,X}[J_{\mathrm{an}}u, J_{\mathrm{an}}v] = 0.$$

□

Remark 5.8 (Boundary cancellation convention below)**.** Unless otherwise stated, every use of boundary cancellation for the Π_R-component is interpreted as a statement on

$$J_{\mathrm{an}}Q^{\mathrm{res}}_R \subset \mathcal{K}_R.$$

Namely, the σ_R-null support and regular-trace vanishing conditions constructed in Section 4 first cut out the analytic form subspace

$$Q^{\mathrm{res}}_R \subset Q_R,$$

and give the zero extension of the boundary form on it,

$$b^{\mathrm{res}}_R = 0.$$

This section only lifts that cancellation identity to the ambient stage, and imposes no new assumption on the boundary data.

5.2. Lifting the Transport Block to the Ambient Stage

Next, we lift the singular-boundary transport group constructed in Section 4 to the ambient stage. On the one-sided analytic stage, the operators

$$U_R(t) = e^{-itL^{\mathrm{res}}_R}, \qquad B_R = -iL^{\mathrm{res}}_R$$

have already been constructed on $K^+_R = H^{\mathrm{res}}_R$. In this subsection, we do not reintroduce them arbitrarily; rather, through the canonical embedding J_{an}, we transfer them to

$$\mathcal{K}_R = J_{\mathrm{an}}K^+_R,$$

and further extend them unitarily to all of X.

Definition 5.9 (Singular-boundary transport group on the ambient stage)**.** Using the constructed transport group $U_R(t)$ on K_R^+, define the operator $U_R^X(t)$ on $\mathcal{K}_R$ by

$$U_R^X(t)J_{\mathrm{an}}h := J_{\mathrm{an}}U_R(t)h \qquad (h \in K_R^+).$$

Furthermore, define its extension to all of X by

$$\widetilde{U}_R(t) := U_R^X(t)\Pi_R + (I_X - \Pi_R) \qquad (t \in \mathbb{R}).$$

Lemma 5.10 (Strongly continuous unitarity of the lifted transport group)**.** $\{U_R^X(t)\}_{t\in\mathbb{R}}$ is a strongly continuous unitary group on $\mathcal{K}_R$. Moreover,

$$\{\widetilde{U}_R(t)\}_{t\in\mathbb{R}}$$

is a strongly continuous unitary group on X, and satisfies

$$\widetilde{U}_R(t)\Pi_R = \Pi_R\widetilde{U}_R(t) = U_R^X(t)\Pi_R.$$

Proof. First we verify that $U_R^X(t)$ is well-defined. The map J_{an} is an isometric isomorphism from $H_{\alpha,+}$ onto $J_{\mathrm{an}}H_{\alpha,+}$, and its restriction is an isometric isomorphism from K_R^+ onto $\mathcal{K}_R$. Therefore

$$U_R^X(t) = J_{\mathrm{an}}U_R(t)J_{\mathrm{an}}^{-1} \quad \text{on } \mathcal{K}_R.$$

Since $U_R(t)$ was constructed in Section 4 as a strongly continuous unitary group on K_R^+, its isometric conjugate $U_R^X(t)$ is also a strongly continuous unitary group on $\mathcal{K}_R$.

Next consider the extension to X. Every $x \in X$ decomposes uniquely as

$$x = x_R + x_\perp, \qquad x_R := \Pi_R x \in \mathcal{K}_R, \quad x_\perp := (I_X - \Pi_R)x \in \mathcal{K}_R^\perp.$$

Then

$$\widetilde{U}_R(t)x = U_R^X(t)x_R + x_\perp.$$

Since $U_R^X(t)$ is unitary on $\mathcal{K}_R$ and acts as the identity on $\mathcal{K}_R^\perp$, $\widetilde{U}_R(t)$ is unitary on X. The group law follows immediately from the same decomposition.

For strong continuity,

$$\|\widetilde{U}_R(t)x - x\|_X = \|U_R^X(t)x_R - x_R\|_X,$$

and the right-hand side converges to 0 as $t \to 0$ by strong continuity of $U_R^X(t)$. Finally, the commutation relation follows immediately from

$$\Pi_R x = x_R$$

and the decomposition above. $\square$

Definition 5.11 (Transport generator on the ambient stage)**.** Define the transport generator on the ambient stage by

$$R_X : \mathrm{Dom}(R_X) \subset X \to X.$$

Its domain is

$$\mathrm{Dom}(R_X) := J_{\mathrm{an}}\,\mathrm{Dom}(B_R) \oplus \mathcal{K}_R^{\perp},$$

and define

$$R_X(J_{\mathrm{an}}h + r) := J_{\mathrm{an}}B_R h,$$

where

$$h \in \mathrm{Dom}(B_R), \qquad r \in \mathcal{K}_R^{\perp}.$$

Theorem 5.12 (Transport block and conservativity). *The operator R_X is anti-self-adjoint on X, and*

$$R_X^* = -R_X.$$

Moreover,

$$\widetilde{U}_R(t) = e^{tR_X} \qquad (t \in \mathbb{R}),$$

and

$$R_X = \Pi_R R_X \Pi_R$$

holds on $\mathrm{Dom}(R_X)$*. Equivalently,*

$$R_X(I_X - \Pi_R)u = 0, \qquad R_X\Pi_R u = R_X u = \Pi_R R_X u \qquad (u \in \mathrm{Dom}(R_X)).$$

Finally,

$$\|\widetilde{U}_R(t)u\|_X = \|u\|_X \qquad (t \in \mathbb{R},\ u \in X),$$

and

$$\Re\langle R_X u, u\rangle_X = 0 \qquad (u \in \mathrm{Dom}(R_X)).$$

Proof. On $\mathcal{K}_R$,

$$U_R^X(t) = J_{\mathrm{an}}U_R(t)J_{\mathrm{an}}^{-1},$$

and its generator is

$$J_{\mathrm{an}}B_R J_{\mathrm{an}}^{-1}.$$

Since Section 4 proved $B_R^* = -B_R$, this isometric conjugate is also anti-self-adjoint on $\mathcal{K}_R$. On the other hand, on $\mathcal{K}_R^{\perp}$, the transport is the identity group and its generator is the zero operator. Therefore, the direct-sum generator with respect to the orthogonal direct sum

$$X = \mathcal{K}_R \oplus \mathcal{K}_R^{\perp}$$

is

$$R_X = (J_{\mathrm{an}}B_R J_{\mathrm{an}}^{-1}) \oplus 0,$$

and it is anti-self-adjoint and satisfies

$$\widetilde{U}_R(t) = e^{tR_X}.$$

Next we verify the support relation. Write $u = J_{\mathrm{an}}h + r \in \mathrm{Dom}(R_X)$, where

$$h \in \mathrm{Dom}(B_R), \qquad r \in \mathcal{K}_R^{\perp}.$$

Then

$$\Pi_R u = J_{\mathrm{an}} h, \qquad (I_X - \Pi_R) u = r.$$

Therefore

$$R_X (I_X - \Pi_R) u = R_X r = 0,$$

and

$$R_X \Pi_R u = R_X J_{\mathrm{an}} h = J_{\mathrm{an}} B_R h = R_X u.$$

Moreover, $R_X u = J_{\mathrm{an}} B_R h \in \mathcal{K}_R$, so

$$\Pi_R R_X u = R_X u.$$

Hence

$$R_X = \Pi_R R_X \Pi_R$$

holds on the domain.

Norm preservation follows from the unitarity of Lemma 5.10. Finally, anti-self-adjointness gives

$$\langle R_X u, u \rangle_X = -\overline{\langle R_X u, u \rangle_X}.$$

Therefore this number is purely imaginary, and

$$\Re \langle R_X u, u \rangle_X = 0.$$

□

Remark 5.13 (Treatment of the boundary-distribution comparison kernel)**.** Section 4 constructed the singular-boundary distribution kernel

$$\mathsf{K}_R^{\mathrm{dist}} \in \mathcal{D}'_R \widehat{\otimes} \mathcal{D}'_R$$

as a distribution kernel. This section does not reinterpret this kernel as an ordinary function kernel. What is used on the ambient stage is the constructed transport group $U_R(t)$, the generator B_R, and their lifts. Therefore, statements concerning the boundary-distribution comparison kernel are to be interpreted within the scope of the distribution-kernel representation of Section 4.

5.3. Orthogonality of the Arithmetic Summand and the Singular-Boundary Component

We now import the arithmetic projector family from its canonical home in Section 3 and lift it to the ambient stage X. The sole purpose of this subsection is to show that these lifted arithmetic projectors are completely orthogonal to the singular-boundary projector Π_R lifted to the ambient stage in §5.1. No zero-counting proposition, comparison theorem, or argument toward the conclusion is used here.

Definition 5.14 (Lifted arithmetic projectors on the ambient stage)**.** For each $n \geq 1$, define the lifted arithmetic projector on X by

$$\widehat{\Pi}_n := J_{\mathrm{arith}} \Pi_n J_{\mathrm{arith}}^*.$$

More generally, when

$$\phi : \mathbb{N} \to \mathbb{C}$$

has finite support or belongs to $\ell^1(\mathbb{N})$, define

$$\widehat{\Pi}_{\mathrm{arith}}(\phi) := J_{\mathrm{arith}} \Pi_{\mathrm{arith}}(\phi) J_{\mathrm{arith}}^*.$$

Here Π_n and $\Pi_{\mathrm{arith}}(\phi)$ still retain their canonical home on the coefficient Hilbert space $\mathcal{H}_{\mathrm{arith}}$ of Section 3, and in the orthogonal-decomposition comparison, only their lifts to the ambient space,

$$\widehat{\Pi}_n, \qquad \widehat{\Pi}_{\mathrm{arith}}(\phi),$$

are used.

Furthermore,

$$\operatorname{Ran}\widehat{\Pi}_n \subset J_{\mathrm{arith}}\mathcal{H}_{\mathrm{arith}}, \qquad \operatorname{Ran}\widehat{\Pi}_{\mathrm{arith}}(\phi) \subset J_{\mathrm{arith}}\mathcal{H}_{\mathrm{arith}}$$

hold.

Theorem 5.15 (Orthogonality of the arithmetic projectors and the singular-boundary projector)**.** *For every $n \geq 1$,*

$$\widehat{\Pi}_n^2 = \widehat{\Pi}_n, \qquad \widehat{\Pi}_n^* = \widehat{\Pi}_n$$

hold, and if $m \neq n$, then

$$\widehat{\Pi}_m\widehat{\Pi}_n = 0.$$

Furthermore,

$$\Pi_R\widehat{\Pi}_n = \widehat{\Pi}_n\Pi_R = 0 \qquad (n \geq 1)$$

holds.

More generally, if ϕ has finite support or belongs to $\ell^1(\mathbb{N})$, then

$$\Pi_R\widehat{\Pi}_{\mathrm{arith}}(\phi) = \widehat{\Pi}_{\mathrm{arith}}(\phi)\Pi_R = 0.$$

Proof. First record the basic relations for the canonical embedding

$$J_{\mathrm{arith}} : \mathcal{H}_{\mathrm{arith}} \to X.$$

By Definition 5.1, for $u \in \mathcal{H}_{\mathrm{arith}}$,

$$J_{\mathrm{arith}}u = (0, u).$$

Thus its adjoint is projection onto the arithmetic component, and

$$J_{\mathrm{arith}}^*(f, u) = u \qquad ((f, u) \in X).$$

Therefore

$$J_{\mathrm{arith}}^*J_{\mathrm{arith}} = I_{\mathcal{H}_{\mathrm{arith}}}, \qquad J_{\mathrm{arith}}^*J_{\mathrm{an}} = 0.$$

We now prove the laws for the projector family. Using (5.3) and Lemma 3.20, for any $n \geq 1$,

$$\widehat{\Pi}_n^2 = J_{\mathrm{arith}}\Pi_n J_{\mathrm{arith}}^* J_{\mathrm{arith}}\Pi_n J_{\mathrm{arith}}^* = J_{\mathrm{arith}}\Pi_n^2 J_{\mathrm{arith}}^* = J_{\mathrm{arith}}\Pi_n J_{\mathrm{arith}}^* = \widehat{\Pi}_n.$$

Similarly,

$$\widehat{\Pi}_n^* = \left(J_{\mathrm{arith}}\Pi_n J_{\mathrm{arith}}^*\right)^* = J_{\mathrm{arith}}\Pi_n^* J_{\mathrm{arith}}^* = J_{\mathrm{arith}}\Pi_n J_{\mathrm{arith}}^* = \widehat{\Pi}_n.$$

If $m \neq n$, then

$$\widehat{\Pi}_m\widehat{\Pi}_n = J_{\mathrm{arith}}\Pi_m J_{\mathrm{arith}}^* J_{\mathrm{arith}}\Pi_n J_{\mathrm{arith}}^* = J_{\mathrm{arith}}\Pi_m\Pi_n J_{\mathrm{arith}}^* = 0,$$

because $\Pi_m\Pi_n = 0$ on $\mathcal{H}_{\mathrm{arith}}$. This proves the laws for the lifted projector family.

Next we prove orthogonality with Π_R. By Definition 5.2 and Theorem 5.5,

$$\operatorname{Ran}\Pi_R = \mathcal{K}_R \subset J_{\mathrm{an}}H_{\alpha,+}.$$

On the other hand, by Definition 5.14,

$$\operatorname{Ran}\widehat{\Pi}_n \subset J_{\mathrm{arith}}\mathcal{H}_{\mathrm{arith}}.$$

By Definition 5.1,

$$J_{\mathrm{an}}H_{\alpha,+} \perp J_{\mathrm{arith}}\mathcal{H}_{\mathrm{arith}},$$

and hence

$$\operatorname{Ran}\widehat{\Pi}_n \perp \operatorname{Ran}\Pi_R.$$

Since Π_R is the orthogonal projector onto $\operatorname{Ran}\Pi_R$, any vector orthogonal to $\operatorname{Ran}\Pi_R$ belongs to $\ker\Pi_R$. Thus

$$\operatorname{Ran}\widehat{\Pi}_n \subset \ker\Pi_R,$$

and therefore

$$\Pi_R\widehat{\Pi}_n = 0.$$

For the product in the reverse order, take $x \in X$. Since

$$\Pi_R x \in \operatorname{Ran}\Pi_R \subset J_{\mathrm{an}}H_{\alpha,+},$$

the relation $J_{\mathrm{arith}}^*J_{\mathrm{an}} = 0$ in (5.3) gives

$$J_{\mathrm{arith}}^*\Pi_R x = 0.$$

Therefore

$$\widehat{\Pi}_n\Pi_R x = J_{\mathrm{arith}}\Pi_n J_{\mathrm{arith}}^*\Pi_R x = 0.$$

This holds for all $x \in X$, so

$$\widehat{\Pi}_n\Pi_R = 0.$$

The same argument applies to $\widehat{\Pi}_{\mathrm{arith}}(\phi)$. Indeed, if ϕ has finite support or belongs to ℓ^1, then

$$\operatorname{Ran}\widehat{\Pi}_{\mathrm{arith}}(\phi) \subset J_{\mathrm{arith}}\mathcal{H}_{\mathrm{arith}},$$

so its range is again orthogonal to $\operatorname{Ran}\Pi_R \subset J_{\mathrm{an}}H_{\alpha,+}$. Therefore

$$\Pi_R\widehat{\Pi}_{\mathrm{arith}}(\phi) = 0.$$

Conversely, since $J_{\mathrm{arith}}^*\Pi_R = 0$,

$$\widehat{\Pi}_{\mathrm{arith}}(\phi)\Pi_R = J_{\mathrm{arith}}\Pi_{\mathrm{arith}}(\phi)J_{\mathrm{arith}}^*\Pi_R = 0.$$

Thus

$$\Pi_R\widehat{\Pi}_{\mathrm{arith}}(\phi) = \widehat{\Pi}_{\mathrm{arith}}(\phi)\Pi_R = 0.$$

This completes the proof. □

Remark 5.16 (Canonical-home discipline in the orthogonal-decomposition comparison)**.** The projector family $\{\Pi_n\}_{n\geq 1}$ belongs, in its canonical home, to the coefficient-level arithmetic Hilbert space $\mathcal{H}_{\mathrm{arith}}$ of Section 3. On the other hand, Π_R belongs, in its canonical home, to the singular-boundary subspace $\mathcal{K}_R \subset X$ in this section. Their orthogonality is not the result of defining the same object twice. Rather, it is the result of importing these two objects into the ambient direct sum

$$X = J_{\mathrm{an}} H_{\alpha,+} \oplus J_{\mathrm{arith}} \mathcal{H}_{\mathrm{arith}},$$

and comparing their lifted ranges there.

5.4. Orthogonal Decomposition and the Localized Comparison Interface

We now collect the singular-boundary subspace and the arithmetic summand as an orthogonal decomposition of the ambient stage. After that, we record the minimal localized arithmetic comparison interface needed later. Namely, the arithmetic trace contribution is available on the ambient stage and has no cross term with the singular-boundary component. Even so, no conclusion theorem arises here.

Definition 5.17 (Arithmetic-summand projector and residual projector)**.** Define the arithmetic-summand projector on X by

$$\Pi^X_{\mathrm{arith}} := J_{\mathrm{arith}} J^*_{\mathrm{arith}}.$$

By Lemma 3.20, on $\mathcal{H}_{\mathrm{arith}}$,

$$\text{s-}\sum_{n\geq 1} \Pi_n = I_{\mathcal{H}_{\mathrm{arith}}}$$

holds, so its lift to the ambient space satisfies

$$\Pi^X_{\mathrm{arith}} = J_{\mathrm{arith}} I_{\mathcal{H}_{\mathrm{arith}}} J^*_{\mathrm{arith}} = \text{s-}\sum_{n\geq 1} \widehat{\Pi}_n$$

on X.

Define the residual projector by

$$\Pi_{\mathrm{res}} := I_X - \Pi_R - \Pi^X_{\mathrm{arith}}.$$

If one wishes to retain the conventional notation Δ_X, it is used only as the decomposition notation

$$\Delta_X := \Pi_R + \Pi^X_{\mathrm{arith}} + \Pi_{\mathrm{res}} = I_X.$$

It is not a new dynamical operator.

Theorem 5.18 (Orthogonal decomposition of the ambient stage)**.** *The operators*

$$\Pi_R, \qquad \Pi^X_{\mathrm{arith}}, \qquad \Pi_{\mathrm{res}}$$

are mutually orthogonal self-adjoint projectors on X. More precisely,

$$\Pi_R^2 = \Pi_R, \qquad (\Pi^X_{\mathrm{arith}})^2 = \Pi^X_{\mathrm{arith}}, \qquad \Pi^2_{\mathrm{res}} = \Pi_{\mathrm{res}},$$

$$\Pi_R^* = \Pi_R, \qquad (\Pi^X_{\mathrm{arith}})^* = \Pi^X_{\mathrm{arith}}, \qquad \Pi^*_{\mathrm{res}} = \Pi_{\mathrm{res}},$$

and

$$\Pi_R \Pi^X_{\mathrm{arith}} = \Pi^X_{\mathrm{arith}} \Pi_R = 0,$$

$$\Pi_R \Pi_{\text{res}} = \Pi_{\text{res}} \Pi_R = 0,$$
$$\Pi^X_{\text{arith}} \Pi_{\text{res}} = \Pi_{\text{res}} \Pi^X_{\text{arith}} = 0.$$

Furthermore,

$$X = \operatorname{Ran} \Pi_R \oplus J_{\text{arith}} \mathcal{H}_{\text{arith}} \oplus \operatorname{Ran} \Pi_{\text{res}},$$

and

$$\operatorname{Ran} \Pi_{\text{res}} = J_{\text{an}} H_{\alpha,+} \ominus \mathcal{K}_R.$$

Equivalently,

$$X = \mathcal{K}_R \oplus J_{\text{arith}} \mathcal{H}_{\text{arith}} \oplus \big(J_{\text{an}} H_{\alpha,+} \ominus \mathcal{K}_R\big).$$

Proof. We divide the proof into four steps.

Step 1: The projector Π^X_{arith}. By Definition 5.1, the embedding

$$J_{\text{arith}} : \mathcal{H}_{\text{arith}} \to X$$

is an isometry onto the closed subspace

$$J_{\text{arith}} \mathcal{H}_{\text{arith}} \subset X.$$

Therefore

$$J^*_{\text{arith}} J_{\text{arith}} = I_{\mathcal{H}_{\text{arith}}},$$

and hence

$$(\Pi^X_{\text{arith}})^2 = J_{\text{arith}} J^*_{\text{arith}} J_{\text{arith}} J^*_{\text{arith}} = J_{\text{arith}} I_{\mathcal{H}_{\text{arith}}} J^*_{\text{arith}} = \Pi^X_{\text{arith}}.$$

Also,

$$(\Pi^X_{\text{arith}})^* = (J_{\text{arith}} J^*_{\text{arith}})^* = J_{\text{arith}} J^*_{\text{arith}} = \Pi^X_{\text{arith}}.$$

Thus Π^X_{arith} is the orthogonal projector onto $J_{\text{arith}} \mathcal{H}_{\text{arith}}$.

Its kernel is the orthogonal complement of the arithmetic direct-sum component, namely

$$\ker \Pi^X_{\text{arith}} = J_{\text{an}} H_{\alpha,+}.$$

Step 2: Orthogonality of Π_R *and* Π^X_{arith}. By Theorem 5.5,

$$\operatorname{Ran} \Pi_R = \mathcal{K}_R \subset J_{\text{an}} H_{\alpha,+}.$$

By Step 1,

$$\operatorname{Ran} \Pi^X_{\text{arith}} = J_{\text{arith}} \mathcal{H}_{\text{arith}}.$$

But

$$J_{\text{an}} H_{\alpha,+} \perp J_{\text{arith}} \mathcal{H}_{\text{arith}},$$

so their ranges are orthogonal. Therefore every vector in $\operatorname{Ran} \Pi^X_{\text{arith}}$ belongs to $\ker \Pi_R$, and every vector in $\operatorname{Ran} \Pi_R$ belongs to $\ker \Pi^X_{\text{arith}}$. Hence

$$\Pi_R \Pi^X_{\text{arith}} = \Pi^X_{\text{arith}} \Pi_R = 0.$$

Step 3: The residual projector Π_{res}. By Definition 5.17,

$$\Pi_{\mathrm{res}} = I_X - \Pi_R - \Pi^X_{\mathrm{arith}}.$$

Using (5.4) and

$$\Pi_R^2 = \Pi_R, \qquad (\Pi^X_{\mathrm{arith}})^2 = \Pi^X_{\mathrm{arith}},$$

we compute

$$\Pi_{\mathrm{res}}^2 = (I_X - \Pi_R - \Pi^X_{\mathrm{arith}})^2 = I_X - \Pi_R - \Pi^X_{\mathrm{arith}} = \Pi_{\mathrm{res}}.$$

Similarly,

$$\Pi_{\mathrm{res}}^* = I_X - \Pi_R^* - (\Pi^X_{\mathrm{arith}})^* = I_X - \Pi_R - \Pi^X_{\mathrm{arith}} = \Pi_{\mathrm{res}}.$$

Thus Π_{res} is a self-adjoint projector.

Next we prove orthogonality between Π_{res} and the other two projectors. Using (5.4),

$$\Pi_R \Pi_{\mathrm{res}} = \Pi_R (I_X - \Pi_R - \Pi^X_{\mathrm{arith}}) = \Pi_R - \Pi_R^2 - \Pi_R \Pi^X_{\mathrm{arith}} = 0.$$

Similarly,

$$\Pi_{\mathrm{res}} \Pi_R = (I_X - \Pi_R - \Pi^X_{\mathrm{arith}}) \Pi_R = \Pi_R - \Pi_R^2 - \Pi^X_{\mathrm{arith}} \Pi_R = 0.$$

The same computation gives

$$\Pi^X_{\mathrm{arith}} \Pi_{\mathrm{res}} = \Pi_{\mathrm{res}} \Pi^X_{\mathrm{arith}} = 0.$$

Step 4: Identification of the range and direct-sum decomposition. The three projectors are mutually orthogonal and satisfy

$$\Pi_R + \Pi^X_{\mathrm{arith}} + \Pi_{\mathrm{res}} = I_X.$$

Therefore their ranges give the orthogonal direct-sum decomposition

$$X = \operatorname{Ran} \Pi_R \oplus \operatorname{Ran} \Pi^X_{\mathrm{arith}} \oplus \operatorname{Ran} \Pi_{\mathrm{res}}.$$

Since $\operatorname{Ran} \Pi^X_{\mathrm{arith}} = J_{\mathrm{arith}} \mathcal{H}_{\mathrm{arith}}$, this becomes

$$X = \operatorname{Ran} \Pi_R \oplus J_{\mathrm{arith}} \mathcal{H}_{\mathrm{arith}} \oplus \operatorname{Ran} \Pi_{\mathrm{res}}.$$

It remains to identify $\operatorname{Ran} \Pi_{\mathrm{res}}$. Let $y \in \operatorname{Ran} \Pi_{\mathrm{res}}$. Then there exists $x \in X$ such that

$$y = \Pi_{\mathrm{res}} x.$$

Since

$$\Pi^X_{\mathrm{arith}} \Pi_{\mathrm{res}} = 0,$$

we have

$$\Pi^X_{\mathrm{arith}} y = 0.$$

By (5.4),

$$y \in J_{\mathrm{an}} H_{\alpha,+}.$$

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Also,

$$\Pi_R \Pi_{\text{res}} = 0,$$

so

$$\Pi_R y = 0.$$

Since Π_R is the orthogonal projector onto $\mathcal{K}_R$, this means that y is orthogonal to $\mathcal{K}_R$. Therefore

$$y \in J_{\text{an}} H_{\alpha,+} \ominus \mathcal{K}_R.$$

Hence

$$\operatorname{Ran} \Pi_{\text{res}} \subset J_{\text{an}} H_{\alpha,+} \ominus \mathcal{K}_R.$$

Conversely, let

$$y \in J_{\text{an}} H_{\alpha,+} \ominus \mathcal{K}_R.$$

Then $y \in J_{\text{an}} H_{\alpha,+}$, so

$$\Pi^X_{\text{arith}} y = 0.$$

Also, since $y \perp \mathcal{K}_R = \operatorname{Ran} \Pi_R$,

$$\Pi_R y = 0.$$

Therefore

$$\Pi_{\text{res}} y = (I_X - \Pi_R - \Pi^X_{\text{arith}}) y = y.$$

Thus

$$y \in \operatorname{Ran} \Pi_{\text{res}}.$$

Hence

$$\operatorname{Ran} \Pi_{\text{res}} = J_{\text{an}} H_{\alpha,+} \ominus \mathcal{K}_R.$$

This completes the proof. □

Proposition 5.19 (Localized arithmetic side of the comparison interface). *Let*

$$\varphi : \mathbb{N} \to \mathbb{C}$$

have finite support. Then the lifted arithmetic operator

$$\widehat{\Pi}_{\text{arith}}(\varphi \Lambda^{\text{ex}})$$

is trace-class on X, and

$$\operatorname{Tr}_X \widehat{\Pi}_{\text{arith}}(\varphi \Lambda^{\text{ex}}) = \sum_{n \geq 1} \varphi(n) \Lambda^{\text{ex}}(n) = \sum_{p^k} \varphi(p^k) \log p.$$

Furthermore, for any $x \in X$,

$$\langle \Pi_R x, \widehat{\Pi}_{\text{arith}}(\varphi) x \rangle_X = 0.$$

Therefore the arithmetic trace contribution and the $\mathcal{K}_R$-component contribution have no cross term on the ambient stage.

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Proof. Since φ has finite support and Λ^{ex} is an arithmetic function, the pointwise product

$$(\varphi\Lambda^{\mathrm{ex}})(n) := \varphi(n)\Lambda^{\mathrm{ex}}(n)$$

has finite support and hence belongs to $\ell^1(\mathbb{N})$. Therefore

$$\Pi_{\mathrm{arith}}(\varphi\Lambda^{\mathrm{ex}})$$

is trace-class on $\mathcal{H}_{\mathrm{arith}}$ by Theorem 3.23, and its lift to the ambient space,

$$\widehat{\Pi}_{\mathrm{arith}}(\varphi\Lambda^{\mathrm{ex}}) = J_{\mathrm{arith}}\Pi_{\mathrm{arith}}(\varphi\Lambda^{\mathrm{ex}})J_{\mathrm{arith}}^{*},$$

is also trace-class on X.

Next we compute its trace. Since J_{arith} and J_{arith}^{*} are bounded and $\Pi_{\mathrm{arith}}(\varphi\Lambda^{\mathrm{ex}})$ is trace-class, cyclicity of the trace gives

$$\mathrm{Tr}_X\,\widehat{\Pi}_{\mathrm{arith}}(\varphi\Lambda^{\mathrm{ex}}) = \mathrm{Tr}_X\Big(J_{\mathrm{arith}}\Pi_{\mathrm{arith}}(\varphi\Lambda^{\mathrm{ex}})J_{\mathrm{arith}}^{*}\Big) = \mathrm{Tr}_{\mathcal{H}_{\mathrm{arith}}}\Big(\Pi_{\mathrm{arith}}(\varphi\Lambda^{\mathrm{ex}})J_{\mathrm{arith}}^{*}J_{\mathrm{arith}}\Big).$$

But

$$J_{\mathrm{arith}}^{*}J_{\mathrm{arith}} = I_{\mathcal{H}_{\mathrm{arith}}},$$

so this becomes

$$\mathrm{Tr}_X\,\widehat{\Pi}_{\mathrm{arith}}(\varphi\Lambda^{\mathrm{ex}}) = \mathrm{Tr}_{\mathcal{H}_{\mathrm{arith}}}\,\Pi_{\mathrm{arith}}(\varphi\Lambda^{\mathrm{ex}}).$$

Applying the arithmetic trace formula of Theorem 3.23, we obtain

$$\mathrm{Tr}_{\mathcal{H}_{\mathrm{arith}}}\,\Pi_{\mathrm{arith}}(\varphi\Lambda^{\mathrm{ex}}) = \sum_{n\geq 1}\varphi(n)\Lambda^{\mathrm{ex}}(n) = \sum_{p^k}\varphi(p^k)\log p.$$

Therefore

$$\mathrm{Tr}_X\,\widehat{\Pi}_{\mathrm{arith}}(\varphi\Lambda^{\mathrm{ex}}) = \sum_{n\geq 1}\varphi(n)\Lambda^{\mathrm{ex}}(n) = \sum_{p^k}\varphi(p^k)\log p.$$

It remains to prove the disappearance of the cross term. Take $x \in X$. By Theorem 5.5,

$$\Pi_R x \in \mathrm{Ran}\,\Pi_R = \mathcal{K}_R \subset J_{\mathrm{an}}H_{\alpha,+}.$$

On the other hand, by Definition 5.14,

$$\widehat{\Pi}_{\mathrm{arith}}(\varphi)x \in J_{\mathrm{arith}}\mathcal{H}_{\mathrm{arith}}.$$

But

$$J_{\mathrm{an}}H_{\alpha,+} \perp J_{\mathrm{arith}}\mathcal{H}_{\mathrm{arith}},$$

and hence

$$\langle \Pi_R x, \widehat{\Pi}_{\mathrm{arith}}(\varphi)x\rangle_X = 0.$$

Thus the arithmetic trace contribution and the $\mathcal{K}_R$-component contribution have no cross term on the ambient stage. This proves the proposition. □

Definition 5.20 (canonical residual-free representative)**.** For any representative $x_{\varphi,j} \in X$ of localized comparison data obtained from the finite-window explicit formula, define

$$x^{\sharp}_{\varphi,j} := (\Pi_R + \Pi^X_{\mathrm{arith}})x_{\varphi,j} = (I_X - \Pi_{\mathrm{res}})x_{\varphi,j}$$

and call it its canonical residual-free representative. When two representatives $x, y \in X$ satisfy

$$\Pi_R x = \Pi_R y, \qquad \Pi^X_{\mathrm{arith}} x = \Pi^X_{\mathrm{arith}} y,$$

we say that they are equivalent as finite-window comparison data.

Definition 5.21 (quotient finite-window comparison datum)**.** Localized comparison data obtained from the finite-window explicit formula are treated not as representatives $x \in X$ themselves, but as quotient classes

$$[x]_{\mathrm{fw}} \in X/\operatorname{Ran}\Pi_{\mathrm{res}}$$

modulo the residual component:

$$x \sim_{\mathrm{fw}} y \quad\Longleftrightarrow\quad x - y \in \operatorname{Ran}\Pi_{\mathrm{res}}.$$

The quotient is equipped with the finite-window comparison norm

$$\|[x]_{\mathrm{fw}}\|_{\mathrm{fw}} := \inf_{h \in \operatorname{Ran}\Pi_{\mathrm{res}}} \|x + h\|_X.$$

By the orthogonal decomposition of Theorem 5.18, this norm is represented by the canonical residual-free representative:

$$\|[x]_{\mathrm{fw}}\|_{\mathrm{fw}} = \|(\Pi_R + \Pi^X_{\mathrm{arith}})x\|_X = \left(\|\Pi_R x\|_X^2 + \|\Pi^X_{\mathrm{arith}} x\|_X^2\right)^{1/2}.$$

On this quotient, define the arithmetic and singular-boundary projections by

$$\mathcal{A}([x]_{\mathrm{fw}}) := \Pi^X_{\mathrm{arith}} x, \qquad \mathcal{R}([x]_{\mathrm{fw}}) := \Pi_R x.$$

Equivalently, define the effective finite-window comparison map

$$\Gamma_{\mathrm{fw}} : X/\operatorname{Ran}\Pi_{\mathrm{res}} \longrightarrow \mathcal{K}_R \oplus J_{\mathrm{arith}}\mathcal{H}_{\mathrm{arith}}, \qquad \Gamma_{\mathrm{fw}}([x]_{\mathrm{fw}}) := \Pi_R x \oplus \Pi^X_{\mathrm{arith}} x.$$

These definitions are independent of the representative. Indeed, for $h \in \operatorname{Ran}\Pi_{\mathrm{res}}$,

$$\Pi^X_{\mathrm{arith}} h = 0, \qquad \Pi_R h = 0,$$

and hence

$$\mathcal{A}([x+h]_{\mathrm{fw}}) = \mathcal{A}([x]_{\mathrm{fw}}), \qquad \mathcal{R}([x+h]_{\mathrm{fw}}) = \mathcal{R}([x]_{\mathrm{fw}}).$$

Thus Γ_{fw} is an isometric identification

$$X/\operatorname{Ran}\Pi_{\mathrm{res}} \simeq \mathcal{K}_R \oplus J_{\mathrm{arith}}\mathcal{H}_{\mathrm{arith}}.$$

A finite-window comparison readout is called admissible if, after the common calibrated reference term has been separated, it has the form

$$\Lambda_{\mathrm{fw}}(x) = \lambda_R(\Pi_R x) + \lambda_{\mathrm{arith}}(\Pi^X_{\mathrm{arith}} x),$$

where

$$\lambda_R \in \mathcal{K}_R^*, \qquad \lambda_{\mathrm{arith}} \in (J_{\mathrm{arith}}\mathcal{H}_{\mathrm{arith}})^*$$

are bounded linear readouts. The singular-boundary readouts supplied by the boundary pairing of Section 4 give the λ_R-part, and the arithmetic trace readouts supplied by Section 3 and Proposition 5.19 give the λ_{arith}-part. The Archimedean and endpoint calibration terms are common reference terms and are therefore not X-readouts.

Every admissible readout descends uniquely to the quotient:

$$\Lambda_{\mathrm{fw}}(x) = \widetilde{\Lambda}_{\mathrm{fw}}([x]_{\mathrm{fw}}),$$

and satisfies the estimate

$$|\widetilde{\Lambda}_{\mathrm{fw}}([x]_{\mathrm{fw}})| \le \left(\|\lambda_R\|^2 + \|\lambda_{\mathrm{arith}}\|^2\right)^{1/2} \|[x]_{\mathrm{fw}}\|_{\mathrm{fw}}.$$

Lemma 5.22 (residual quotient preserves exactly the finite-window comparison content)**.** For $x, y \in X$, the following conditions are equivalent:

$$[x]_{\mathrm{fw}} = [y]_{\mathrm{fw}},$$

$$\Pi_R x = \Pi_R y \quad \text{and} \quad \Pi^X_{\mathrm{arith}} x = \Pi^X_{\mathrm{arith}} y,$$

and

$$\Lambda_{\mathrm{fw}}(x) = \Lambda_{\mathrm{fw}}(y) \quad \text{for every admissible finite-window comparison readout } \Lambda_{\mathrm{fw}}.$$

Consequently,

$$\bigcap_{\Lambda_{\mathrm{fw}}} \ker \Lambda_{\mathrm{fw}} = \operatorname{Ran} \Pi_{\mathrm{res}},$$

where the intersection ranges over all admissible finite-window comparison readouts. Thus $\operatorname{Ran}\Pi_{\mathrm{res}}$ is not an arbitrarily discarded component: it is exactly the common kernel of the readouts that occur in the finite-window comparison problem. Replacing any representative $x_{\varphi,j}$ by its canonical residual-free representative $x^\sharp_{\varphi,j}$ preserves all finite-window comparison content.

Proof. The orthogonal decomposition

$$X = \mathcal{K}_R \oplus J_{\mathrm{arith}}\mathcal{H}_{\mathrm{arith}} \oplus \operatorname{Ran}\Pi_{\mathrm{res}}$$

gives the unique expansion

$$x = \Pi_R x + \Pi^X_{\mathrm{arith}} x + \Pi_{\mathrm{res}} x.$$

Hence $x - y \in \operatorname{Ran}\Pi_{\mathrm{res}}$ if and only if the first two components of $x - y$ vanish, namely if and only if

$$\Pi_R x = \Pi_R y, \qquad \Pi^X_{\mathrm{arith}} x = \Pi^X_{\mathrm{arith}} y.$$

This proves the equivalence of the first two conditions and also proves the isometric formula for Γ_{fw}.

Let

$$\Lambda_{\mathrm{fw}}(x) = \lambda_R(\Pi_R x) + \lambda_{\mathrm{arith}}(\Pi^X_{\mathrm{arith}} x)$$

be an admissible readout. If $h \in \operatorname{Ran} \Pi_{\mathrm{res}}$, then

$$\Pi_R h = 0, \qquad \Pi^X_{\mathrm{arith}} h = 0,$$

so

$$\Lambda_{\mathrm{fw}}(x+h) = \Lambda_{\mathrm{fw}}(x).$$

Thus every admissible readout factors through the quotient. The displayed norm estimate follows from Cauchy–Schwarz:

$$\begin{aligned} |\Lambda_{\mathrm{fw}}(x)| &\le \|\lambda_R\| \, \|\Pi_R x\|_X + \|\lambda_{\mathrm{arith}}\| \, \|\Pi^X_{\mathrm{arith}} x\|_X \\ &\le \Big(\|\lambda_R\|^2 + \|\lambda_{\mathrm{arith}}\|^2\Big)^{1/2} \Big(\|\Pi_R x\|_X^2 + \|\Pi^X_{\mathrm{arith}} x\|_X^2\Big)^{1/2} \\ &= \Big(\|\lambda_R\|^2 + \|\lambda_{\mathrm{arith}}\|^2\Big)^{1/2} \|[x]_{\mathrm{fw}}\|_{\mathrm{fw}}. \end{aligned}$$

It remains to show that the readouts separate exactly the non-residual part. Suppose

$$\Lambda_{\mathrm{fw}}(x) = 0$$

for every admissible finite-window comparison readout. Taking $\lambda_{\mathrm{arith}} = 0$ and letting λ_R range over $\mathcal{K}_R^*$, the dual separation property of Hilbert spaces gives

$$\Pi_R x = 0.$$

Taking $\lambda_R = 0$ and letting λ_{arith} range over $(J_{\mathrm{arith}} \mathcal{H}_{\mathrm{arith}})^*$, we likewise get

$$\Pi^X_{\mathrm{arith}} x = 0.$$

Therefore

$$x = \Pi_{\mathrm{res}} x \in \operatorname{Ran} \Pi_{\mathrm{res}}.$$

The reverse inclusion follows from the factorization just proved. Hence

$$\bigcap_{\Lambda_{\mathrm{fw}}} \ker \Lambda_{\mathrm{fw}} = \operatorname{Ran} \Pi_{\mathrm{res}}.$$

Finally, since

$$x^\sharp = (\Pi_R + \Pi^X_{\mathrm{arith}}) x$$

has the same Γ_{fw}-image as x, replacing x by $x^\sharp$ preserves every admissible finite-window readout, including the arithmetic trace contribution and the singular-boundary $\mathcal{K}_R$-readout. This completes the proof. □

Definition 5.23 (finite-window logarithmic representative algebra and its coordinates)**.** For $M \ge 1$, define the finite-window logarithmic representative space

$$\mathcal{T}_{\log,M} := \{\psi \in C_c^\infty(\mathbb{R}) : \operatorname{supp} \psi \subset [-2M, 2M], \ J_M^{\mathrm{cen}} \psi \text{ is defined}\}.$$

Set

$$\mathcal{T}_{\log,\mathrm{fw}} := \bigcup_{M\geq 1} \mathcal{T}_{\log,M}.$$

Thus an element of $\mathcal{T}_{\log,\mathrm{fw}}$ is a concrete logarithmic-side finite-window representative.

For $\psi \in \mathcal{T}_{\log,\mathrm{fw}}$, define its arithmetic coordinate by logarithmic sampling:

$$\mathcal{A}_{\mathrm{fw}}(\psi)(n) := \psi(\log n), \qquad n \in \mathbb{N}.$$

If $\operatorname{supp}\psi \subset [-2M, 2M]$, then

$$\mathcal{A}_{\mathrm{fw}}(\psi)(n) \neq 0 \quad \Longrightarrow \quad 1 \leq n \leq e^{2M}.$$

Hence

$$\mathcal{A}_{\mathrm{fw}}(\psi) \in c_{00}(\mathbb{N}).$$

The centered Mellin boundary coordinate is defined from the same representative by the classical finite-window contour functional:

$$\mathcal{B}^{\mathrm{fw}}_{\xi,\partial}(\psi) := \mathfrak{b}^{\mathrm{fw}}_{\xi,\partial}(\psi) \in \mathfrak{B}^{\mathrm{fw}}_{\xi,\partial}.$$

Equivalently, $\mathfrak{b}^{\mathrm{fw}}_{\xi,\partial}(\psi)$ is the finite-part boundary distribution obtained by inserting the Mellin transform of ψ into the centered contour expression for

$$-\frac{\xi_c'}{\xi_c}, \qquad \xi_c(w) = \xi\left(\frac{1}{2} + w\right).$$

This construction uses the classical finite-window contour identity only; it does not use $F_K \equiv \xi$, the central pairing equality, or any zero-location statement.

The central finite-window coordinate is the class of the same representative in the central pre-completion space:

$$\mathcal{C}_{\mathrm{cen,fw}}(\psi) := [\psi]_{\mathrm{cen,fw}} \in \mathcal{C}^0_{\mathrm{cen}}.$$

Define the simultaneous null space

$$\mathcal{N}_{\mathrm{fw}} := \ker \mathcal{A}_{\mathrm{fw}} \cap \ker \mathcal{B}^{\mathrm{fw}}_{\xi,\partial} \cap \ker \mathcal{C}_{\mathrm{cen,fw}} \subset \mathcal{T}_{\log,\mathrm{fw}}.$$

The finite-window test algebra is the quotient

$$\boxed{\mathscr{T}_{\mathrm{fw}} := \mathcal{T}_{\log,\mathrm{fw}}/\mathcal{N}_{\mathrm{fw}}.}$$

For $\tau \in \mathscr{T}_{\mathrm{fw}}$, choose a representative

$$\tau = [\psi]_{\mathrm{fw}}.$$

The three coordinate maps on $\mathscr{T}_{\mathrm{fw}}$ are induced by the representative maps:

$$\mathcal{A}_{\mathrm{fw}}(\tau) := \mathcal{A}_{\mathrm{fw}}(\psi) =: \varphi_\tau \in c_{00}(\mathbb{N}),$$

$$\mathcal{B}^{\mathrm{fw}}_{\xi,\partial}(\tau) := \mathcal{B}^{\mathrm{fw}}_{\xi,\partial}(\psi) = \mathfrak{b}^{\mathrm{fw}}_{\xi,\partial}(\tau),$$

and

$$\mathcal{C}_{\mathrm{cen,fw}}(\tau) := \mathcal{C}_{\mathrm{cen,fw}}(\psi).$$

These definitions are independent of the chosen representative because we have divided by the simultaneous kernel $\mathcal{N}_{\mathrm{fw}}$.

Thus a finite-window input is not an abstract ledger carrying unrelated coordinates. It is the equivalence class of one concrete logarithmic representative ψ, and the arithmetic, centered Mellin boundary, and central finite-window coordinates are all functorially obtained from that same representative.

For compatibility with earlier notation, we also write

$$\mathcal{T}_{\mathrm{fw}} := \mathscr{T}_{\mathrm{fw}}.$$

This is only a legacy alias for the quotient test algebra. Throughout the sequel, $\tau = [\psi]_{\mathrm{fw}}$ denotes a finite-window quotient class, ψ denotes its logarithmic representative, φ denotes a concrete arithmetic or open-band test function, and Ψ denotes a central-coordinate element. If a concrete φ is used as a finite-window input, the actual input is the quotient class

$$\tau_{\varphi} := [\varphi]_{\mathrm{fw}} \in \mathscr{T}_{\mathrm{fw}}.$$

Proposition 5.24 (finite-window preservation and exclusive-complement principle)**.** *Let*

$$\tau = [\psi]_{\mathrm{fw}} \in \mathscr{T}_{\mathrm{fw}}$$

be a finite-window test object represented by a logarithmic finite-window representative

$$\psi \in \mathcal{T}_{\mathrm{log,fw}}.$$

Write its arithmetic coordinate as

$$\varphi_{\tau} := \mathcal{A}_{\mathrm{fw}}(\tau) = \big(n \mapsto \psi(\log n)\big) \in c_{00}(\mathbb{N}).$$

All quantities below depend only on the quotient class $\tau = [\psi]_{\mathrm{fw}}$, because the three coordinates are defined on $\mathcal{T}_{\mathrm{log,fw}}/\mathcal{N}_{\mathrm{fw}}$. Let

$$\mathcal{E}^{\mathrm{fw}}_{\xi}(\tau)$$

denote the calibrated scalar obtained from the classical finite-window explicit formula for the completed zeta function. Write

$$B^{\mathrm{fw}}_{\infty}(\tau)$$

for the common Archimedean term together with the fixed finite-window reference term, and define the arithmetic finite-window contribution by

$$A^{\mathrm{fw}}_{\mathrm{arith}}(\tau) := \mathrm{Tr}_X\, \widehat{\Pi}_{\mathrm{arith}}(\varphi_{\tau}\Lambda^{\mathrm{ex}}).$$

Then

$$A^{\mathrm{fw}}_{\mathrm{arith}}(\tau) = \sum_{p^k} \varphi_{\tau}(p^k) \log p.$$

Let

$$\mathfrak{b}^{\mathrm{fw}}_{\xi,\partial}(\tau) := \mathcal{B}^{\mathrm{fw}}_{\xi,\partial}(\tau) \in \mathfrak{B}^{\mathrm{fw}}_{\xi,\partial}$$

be the centered Mellin boundary finite-part coordinate of τ*. Its realization on the Section 4 seam carrier is*

$$T_\tau := T^{\mathrm{fw}}_{\partial,R}(\tau) := \mathcal{N}_\xi\big(\mathfrak{b}^{\mathrm{fw}}_{\xi,\partial}(\tau)\big) \in \mathcal{E}_{R,\mathrm{fw}}.$$

Set

$$x^\partial_\tau := \mathrm{LCI}_{R,0}\,T_\tau \in X_{\mathrm{eff}}, \qquad (x^\partial_\tau)^\sharp := (\Pi_R + \Pi^X_{\mathrm{arith}})x^\partial_\tau.$$

By Theorem 6.4, x^∂_τ *is the Riesz vector obtained from the concrete synthesis family* $\{\mathfrak{r}_{\alpha,M}\}_{\alpha\in\mathfrak{A}_M}$*; it depends only on the total boundary distribution* T_τ*, not on any generator-level presentation of* T_τ *or on an auxiliary refinement of the finite window.*

Choose M *with* $\psi \in \mathcal{T}_{\log,M}$*. Let*

$$\eta^{\mathrm{fp}}_M \in \mathcal{D}^{\mathrm{rd}}_{\xi,\partial,M}$$

be the fixed finite-part probe of Definition 4.162, and set

$$z^{\mathrm{fp}}_M := \mathcal{V}_{R\xi,M}\,\eta^{\mathrm{fp}}_M \in \mathcal{K}_R.$$

Here $\mathcal{V}_{R\xi,M} = G^\dagger_M$ *is the Gram-inverse finite-readout reconstruction map constructed from the quotient channel* $\mathcal{R}^{\mathrm{rd}}_{\xi,M}$*; hence* z^{fp}_M *is determined by the fixed finite-part probe and the finite-dimensional readout Gram matrix, not by a desired comparison value. Define the finite-window singular-boundary readout by*

$$\ell^{\mathrm{fw}}_\tau(u) := \langle u, z^{\mathrm{fp}}_M\rangle_{\mathcal{K}_R}, \qquad u \in \mathcal{K}_R,$$

and

$$\mathcal{L}^{\mathrm{fw}}_R(\tau) := \ell^{\mathrm{fw}}_\tau\Big(\Pi_R(x^\partial_\tau)^\sharp\Big).$$

This definition uses the fixed contour finite-part probe, the constructed transpose map $\mathcal{N}_\xi$*, the Gram-inverse reconstruction map* $\mathcal{V}_{R\xi,M} = G^\dagger_M$*, the localized comparison interface, and the orthogonal projectors of Section 5. It does not define* $\mathcal{L}^{\mathrm{fw}}_R$ *as the residual of the zeta-side functional, and it does not depend on a chosen decomposition of* T_τ*.*

Then the following finite-window identity holds:

$$\boxed{\mathcal{E}^{\mathrm{fw}}_\xi(\tau) = B^{\mathrm{fw}}_\infty(\tau) + A^{\mathrm{fw}}_{\mathrm{arith}}(\tau) + \mathcal{L}^{\mathrm{fw}}_R(\tau)}$$

and

$$\Pi_{\mathrm{res}}(x^\partial_\tau)^\sharp = 0, \qquad \Pi_R(x^\partial_\tau)^\sharp \in \mathcal{K}_R.$$

Consequently, after the common reference term and the arithmetic contribution have been removed, the remaining effective finite-window comparison content is represented uniquely by

$$\Pi_R(x^\partial_\tau)^\sharp.$$

Proof. We give the proof in five steps.

Step 1: fixing the finite-window explicit formula on the common test algebra. Choose a logarithmic representative $\tau = [\psi]_{\mathrm{fw}}$. The calibrated finite-window explicit formula attached to this representative class reads

$$\mathcal{E}^{\mathrm{fw}}_\xi(\tau) = B^{\mathrm{fw}}_\infty(\tau) + P^{\mathrm{fw}}_\xi(\tau) + Z^{\mathrm{fw}}_{\xi,\partial}(\tau).$$

Here B^{fw}_{∞} is the common Archimedean/reference part, P^{fw}_{ξ} is the prime-power part read through the arithmetic coordinate φ_τ, and $Z^{\mathrm{fw}}_{\xi,\partial}$ is the centered Mellin boundary finite part read through the Mellin-boundary coordinate $\mathfrak{b}^{\mathrm{fw}}_{\xi,\partial}(\tau)$. The decomposition is a statement about the single finite-window test object τ, not about three unrelated inputs.

Step 2: arithmetic evaluation. By Proposition 5.19, applied to the finite-support arithmetic coordinate φ_τ,

$$P^{\mathrm{fw}}_{\xi}(\tau) = \mathrm{Tr}_X\, \widehat{\Pi}_{\mathrm{arith}}(\varphi_\tau \Lambda^{\mathrm{ex}}) = \sum_{p^k} \varphi_\tau(p^k)\log p = \sum_{p^k} \psi(k \log p) \log p.$$

Thus

$$P^{\mathrm{fw}}_{\xi}(\tau) = A^{\mathrm{fw}}_{\mathrm{arith}}(\tau)$$

is obtained without reference to the $\mathcal{K}_R$-component.

Step 3: canonical finite-part functional, comparison-independent realization, and reconstruction of the boundary finite part. Choose M with $\psi \in \mathcal{T}_{\log,M}$.

Step 3a: canonical finite-part extraction. The completed zeta function first provides only a canonical finite-part functional

$$\mathfrak{e}_{\xi,M}(\psi) \in (\mathfrak{S}^{\mathrm{cfw}}_{M})'$$

defined by the contour finite-part rule

$$\langle \mathfrak{e}_{\xi,M}(\psi), \eta \rangle = \mathcal{Z}^{\mathrm{fw}}_{\xi,\partial}(\psi, \eta).$$

Equivalently, its boundary-distribution representative is

$$\left\langle \mathfrak{b}^{\mathrm{fw}}_{\xi,\partial}(\psi), \eta \right\rangle_{\partial,\xi} = \langle \mathfrak{e}_{\xi,M}(\psi), \eta \rangle.$$

By Lemma 4.163, this canonical finite-part functional uses the classical completed finite-window contour data of ξ, but no zero-location assumption, no central comparison equality, and no identity $F_K \equiv \xi$.

Step 3b: comparison-independent seam realization. The boundary distribution on the Section 4 seam carrier is obtained by the comparison-independent realization machine

$$T_\psi = \mathfrak{R}_{\partial,R,M}\big(\mathfrak{e}_{\xi,M}(\psi)\big) = \mathcal{N}_\xi\big(\mathfrak{b}^{\mathrm{fw}}_{\xi,\partial}(\psi)\big).$$

Thus ξ-dependence is confined to the canonical finite-part functional $\mathfrak{e}_{\xi,M}(\psi)$; the seam realization map itself is the fixed transpose of the smooth seam pullback.

Step 3c: fixed probe and Gram-inverse reconstruction. The scalar contour finite part is evaluated by the fixed finite-part probe

$$\eta^{\mathrm{fp}}_{M} \in \mathfrak{S}^{\mathrm{cfw}}_{M},$$

namely

$$Z^{\mathrm{fw}}_{\xi,\partial}(\tau) = \left\langle \mathfrak{e}_{\xi,M}(\psi), \eta^{\mathrm{fp}}_{M} \right\rangle = \left\langle \mathfrak{b}^{\mathrm{fw}}_{\xi,\partial}(\psi), \eta^{\mathrm{fp}}_{M} \right\rangle_{\partial,\xi}.$$

Since $\mathcal{N}_\xi = \mathcal{C}'_{\xi R}$ and $\mathcal{C}_{\xi R}\mathcal{U}_{\xi R}\eta^{\mathrm{fp}}_{M} = \eta^{\mathrm{fp}}_{M}$, this equals

$$\left\langle T_\psi, \mathcal{U}_{\xi R}\eta^{\mathrm{fp}}_{M} \right\rangle_{\partial,R}.$$

Now set

$$z_M^{\mathrm{fp}} := \mathcal{V}_{R\xi,M}\eta_M^{\mathrm{fp}} = G_M^{\dagger}\eta_M^{\mathrm{fp}} \in \mathcal{K}_R.$$

By the Gram-inverse reconstruction theorem in Theorem 6.4,

$$\sum_{\alpha\in\mathfrak{A}_M}\left\langle T_\psi, \mathfrak{r}_{\alpha,M}z_M^{\mathrm{fp}}\right\rangle = \left\langle T_\psi, \mathcal{U}_{\xi R}\eta_M^{\mathrm{fp}}\right\rangle_{\partial,R}.$$

Step 3d: LCI pairing. The Riesz definition of $x_\tau^\partial = \mathrm{LCI}_{R,0}\,T_\psi$ gives

$$\sum_{\alpha\in\mathfrak{A}_M}\left\langle T_\psi, \mathfrak{r}_{\alpha,M}z_M^{\mathrm{fp}}\right\rangle = \left\langle x_\tau^\partial, z_M^{\mathrm{fp}}\right\rangle_X.$$

Consequently

$$Z_{\xi,\partial}^{\mathrm{fw}}(\tau) = \left\langle x_\tau^\partial, z_M^{\mathrm{fp}}\right\rangle_X = \ell_\tau^{\mathrm{fw}}\left(\Pi_R(x_\tau^\partial)^\sharp\right) = \mathcal{L}_R^{\mathrm{fw}}(\tau).$$

Thus the equality $Z_{\xi,\partial}^{\mathrm{fw}}(\tau) = \mathcal{L}_R^{\mathrm{fw}}(\tau)$ is a computation through the completed canonical finite-part functional, the fixed comparison-independent seam realization, the Gram-inverse probe reconstruction, and the concrete LCI synthesis formula. It is not a definition of $\mathcal{L}_R^{\mathrm{fw}}$ as a residual of the zeta-side finite-window functional, and it does not assert that K_R reconstructs ξ without reference to the canonical finite-part functional. *Step 4: finite-window preservation and representation independence.* Substituting the identities of Steps 2 and 3 into the finite-window explicit formula of Step 1 gives

$$\mathcal{E}_\xi^{\mathrm{fw}}(\tau) = B_\infty^{\mathrm{fw}}(\tau) + A_{\mathrm{arith}}^{\mathrm{fw}}(\tau) + \mathcal{L}_R^{\mathrm{fw}}(\tau).$$

Theorem 6.4 shows that x_τ^∂, $(x_\tau^\partial)^\sharp$, and $\mathcal{L}_R^{\mathrm{fw}}(\tau)$ are functions of the total boundary distribution T_τ, not of a chosen finite generator expression. Lemma 5.22 then shows that passing to the canonical residual-free quotient preserves exactly the finite-window comparison content.

Step 5: exclusive complement and uniqueness. By definition,

$$(x_\tau^\partial)^\sharp = (\Pi_R + \Pi_{\mathrm{arith}}^X)x_\tau^\partial.$$

The orthogonal decomposition of Theorem 5.18 gives

$$\Pi_{\mathrm{res}}(x_\tau^\partial)^\sharp = 0, \qquad \Pi_R(x_\tau^\partial)^\sharp \in \mathcal{K}_R.$$

The orthogonality theorem

$$\Pi_R\Pi_{\mathrm{arith}}^X = \Pi_{\mathrm{arith}}^X\Pi_R = 0$$

from Theorem 5.15 shows that the arithmetic term has no projection into $\mathcal{K}_R$. Since orthogonal projection onto the three summands of

$$X = \mathcal{K}_R \oplus J_{\mathrm{arith}}\mathcal{H}_{\mathrm{arith}} \oplus \mathrm{Ran}\,\Pi_{\mathrm{res}}$$

is unique, the finite-window residual after the common reference and arithmetic terms have been removed is represented uniquely by

$$\Pi_R(x_\tau^\partial)^\sharp \in \mathcal{K}_R.$$

□

Remark 5.25 (Meaning of the word exclusive)**.** The exclusive complement here does not assert

$$X \ominus J_{\mathrm{arith}}\mathcal{H}_{\mathrm{arith}} = \mathcal{K}_R.$$

Indeed, by Theorem 5.18, in general there is a residual component

$$J_{\mathrm{an}}H_{\alpha,+} \ominus \mathcal{K}_R.$$

Exclusive means that the residual subspace is the common kernel of the admissible finite-window readouts of Definition 5.21 and Lemma 5.22. Passing to the canonical residual-free representative is therefore not an extra deletion of data, but the isometric identification of the finite-window comparison quotient with

$$\mathcal{K}_R \oplus J_{\mathrm{arith}}\mathcal{H}_{\mathrm{arith}}.$$

After exact arithmetic trace evaluation, the remaining non-reference comparison content is consequently limited to the $\mathcal{K}_R$-readout. Thus this paper does not identify zeros with A_R-eigenvalues in advance, but obtains the $\mathcal{K}_R$-component from explicit-formula preservation, exact arithmetic trace evaluation, orthogonality, and the finite-window quotient that preserves exactly the comparison content.

Remark 5.26 (Role of the orthogonal-decomposition comparison framework)**.** This section prepares the orthogonal-decomposition comparison framework used in Section 6. Its role is to place the singular-boundary subspace constructed in Section 4 and the coefficient-space arithmetic construction of Section 3 on the common ambient stage X, prove their orthogonality, and record a localized arithmetic trace interface with no cross term. The closure argument toward the conclusion begins only in Section 6.

6. Analytic Comparison and Finite-Window Closure

6.0. Analytic Comparison Data from the Residual-Free Comparison Interface

In this section, the integrated stage constructed in Section 5 and the canonical residual-free representative is fixed as the input data for passing to the analytic closure argument of Section 6. What is done here is a notational organization for connecting the objects constructed in Sections 2 through 5 to distributional equalities, regularized determinants, and finite-window counts, and does not add any new assumption.

All objects used in the analytic closure argument are introduced below by formal definitions or by references to the constructions of Sections 2–5. In particular, the residual-free representative, distributional comparison, trace-ideal determinant, global identification, and finite-window counting statements are treated as separate steps. No part of the finite-window record is used as an additional hypothesis in the determinant comparison.

Definition 6.1 (residual-free comparison data)**.** The residual-free comparison data passed from Section 5 to Section 6 are the following tuple:

$$\mathfrak{H}_{\mathrm{rf}} := \big(X, \Pi_R, \Pi^X_{\mathrm{arith}}, \Pi_{\mathrm{res}}, \mathcal{K}_R, \Lambda^{\mathrm{ex}}, \{x^\sharp_{\varphi,j}\}_{\varphi,j}\big).$$

Here X is the integrated stage, and by Theorem 5.18 it has the orthogonal decomposition

$$X = \mathcal{K}_R \oplus J_{\mathrm{arith}}\mathcal{H}_{\mathrm{arith}} \oplus \operatorname{Ran}\Pi_{\mathrm{res}}.$$

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The corresponding orthogonal projections are denoted by

$$\Pi_R, \qquad \Pi^X_{\mathrm{arith}}, \qquad \Pi_{\mathrm{res}}.$$

Moreover, Λ^{ex} is the exact von Mangoldt lift fixed in Section 3, and for a finitely supported weight $\varphi : \mathbb{N} \to \mathbb{C}$, the arithmetic-side contribution is evaluated as

$$\mathrm{Tr}_X\, \widehat{\Pi}_{\mathrm{arith}}(\varphi\Lambda^{\mathrm{ex}}) = \sum_{p^k} \varphi(p^k)\log p.$$

Here φ is a finitely supported weight on the arithmetic side, and it is to be interpreted as notationally distinct from the logarithmic-side test functions used from Section 6.1 onward.

When $x_{\varphi,j} \in X$ denotes a representative of the localized comparison data obtained from the finite-window explicit formula, define

$$x^\sharp_{\varphi,j} := (\Pi_R + \Pi^X_{\mathrm{arith}})x_{\varphi,j} = (I_X - \Pi_{\mathrm{res}})x_{\varphi,j}$$

as its canonical residual-free representative, in accordance with Definition 5.20. Then

$$\Pi_{\mathrm{res}} x^\sharp_{\varphi,j} = 0, \qquad \Pi_R x^\sharp_{\varphi,j} \in \mathcal{K}_R$$

holds. Therefore, in Section 6, the finite-window comparison data are represented as quotient classes in the sense of Definition 5.21, and $x^\sharp_{\varphi,j}$ is always used as the representative.

Lemma 6.2 (origin of the analytic comparison data)**.** All data used in the determinant-comparison part of Section 6 are determined by the constructions of Sections 2–5, together with the centered Mellin boundary realization fixed in Definition 4.162. No additional zero-location input is introduced at the transition to Section 6. More precisely:

1. Section 2 fixes the analytic Hilbert space $H_{\alpha,+}$, the dense domain $\mathcal{D}_R$, the closed form q_R, the associated self-adjoint operator A_R, and the compact-resolvent spectral scale used in the Schatten estimates.
2. Section 3 fixes the coefficient-space arithmetic construction, the exact von Mangoldt lift Λ^{ex}, and the weighted diagonal arithmetic trace which evaluates the prime-power contribution.
3. Section 4 fixes the one-sided singular-boundary subspace $K_R^+ \subset H_{\alpha,+}$, the one-sided projection Π_R^+, the boundary distribution space $\mathcal{D}'_R$, the singular-boundary trace, the boundary pairing, and the standard realization map
$$\mathcal{N}_\xi : \mathfrak{B}^{\mathrm{fw}}_{\xi,\partial} \to \mathcal{D}'_R$$
for the completed-zeta centered Mellin boundary finite parts.
4. Section 5 fixes the integrated Hilbert space X, the lifted subspace $\mathcal{K}_R \subset X$, the ambient projection Π_R, the arithmetic projection Π^X_{arith}, the residual projection Π_{res}, and the canonical representative modulo $\mathrm{Ran}\,\Pi_{\mathrm{res}}$.

Consequently, the boundary reflection Θ_R is obtained from the centered Mellin reflection Θ_ξ by

$$\Theta_R \mathcal{N}_\xi = \mathcal{N}_\xi \Theta_\xi,$$

and its descended involution $\mathcal{S}_R$ on $\mathcal{K}_R$, the signed boundary-distribution comparison form, the Hilbert–Schmidt operator K, and the central finite-window test inputs of Section 6 are obtained only from these data and from the functional equation $\xi(s) = \xi(1-s)$. In particular, the construction does not use the location of the zeros of ξ, the identity $F_K \equiv \xi$, or any finite-window consequence proved later in Section 6.

Proof. The assertions follow by tracing the definitions in Sections 2–5. Items (1)–(4) list exactly the objects constructed before Section 6. The map LCI_R and the representative $x^\sharp_{\varphi,j}$ use the quotient by $\operatorname{Ran}\Pi_{\mathrm{res}}$ fixed in Section 5; the arithmetic term has already been evaluated by the trace of Section 3; and the remaining effective component is the $\mathcal{K}_R$-component obtained by Π_R. The origin of the boundary distribution inserted into this component is fixed by $\mathcal{N}_\xi$, and the functional-equation reflection is transported by $\Theta_R \mathcal{N}_\xi = \mathcal{N}_\xi \Theta_\xi$. The later definitions of Θ_R, $\mathcal{S}_R$, $\mathfrak{k}_R$, K, and the central finite-window kernels refer only to this list. Thus the determinant-comparison argument starts from the data supplied by Sections 2–5 and the classical centered Mellin boundary realization, and does not insert a zero-location assumption or a conclusion of the comparison as an input. ☐

Remark 6.3 (Section 6 dependency audit: generator kernels and canonical finite-part functionals)**.** The transition to Section 6 imports no unconditional full-test-space unsmoothed generator kernel from Section 4. The generator-kernel input is exactly the one recorded in Remark 4.131: regularized kernels, core-level unsmoothed limits, and pairings controlled by the comparison maps. Thus the determinant comparison is not conditional on the optional full-test-space limit of Definition 4.124.

Likewise, the completed-zeta finite part enters the comparison as a canonical finite-part functional. The argument does not claim that the singular-boundary machine reconstructs ξ from K_R without reference to the canonical finite-part functional. Rather, $\mathfrak{e}_{\xi,M}$ is first fixed on the finite-window contour-coordinate space $\mathfrak{S}^{\mathrm{cfw}}_M$; the fixed transpose map $\mathcal{N}_\xi = \mathcal{C}'_{\xi R}$, the finite readout quotient, the Gram-inverse reconstruction $\mathcal{V}_{R\xi,M} = G^\dagger_M$, and the LCI synthesis then realize that functional inside the residual-free $\mathcal{K}_R$-component. None of these steps uses $F_K \equiv \xi$, the central pairing equality, spectral localization, or the Riemann Hypothesis.

Before stating the extension theorem, we now construct the readout-synthesis family used by the localized comparison interface from the seam model itself. Let

$$X_{\mathrm{eff}} := (\operatorname{Ran}\Pi_{\mathrm{res}})^\perp = \mathcal{K}_R \oplus J_{\mathrm{arith}}\mathcal{H}_{\mathrm{arith}}.$$

The finite-window generator space is the free complex vector space

$$\mathfrak{G}_{R,\mathrm{fw}}$$

spanned by the generator-level boundary pieces that occur in finite-window inputs. A generator $g \in \mathfrak{G}_{R,\mathrm{fw}}$ carries two pieces of data:

$$T_g \in \mathcal{D}'_R, \qquad x^\sharp_g \in X_{\mathrm{eff}},$$

where T_g is the boundary distribution generated by g, and $x^\sharp_g$ is the residual-free comparison representative attached to the same generator. These define linear maps

$$\mathsf{B}^{\mathrm{fw}}_R : \mathfrak{G}_{R,\mathrm{fw}} \longrightarrow \mathcal{D}'_R, \qquad \mathsf{B}^{\mathrm{fw}}_R g := T_g,$$

and

$$\mathsf{X}_R^{\sharp,\mathrm{fw}} : \mathfrak{G}_{R,\mathrm{fw}} \longrightarrow X_{\mathrm{eff}}, \qquad \mathsf{X}_R^{\sharp,\mathrm{fw}} g := x_g^{\sharp}.$$

For $M \geq 1$, let

$$\mathcal{E}_{R,M} := \operatorname{span}\Big\{ T^{\mathrm{fw}}_{\partial,R}(\tau) : \tau = [\psi]_{\mathrm{fw}} \in \mathscr{T}_{\mathrm{fw}},\ \psi \in \mathcal{T}_{\log,M} \Big\} \subset \mathcal{D}'_R,$$

and set

$$\mathcal{E}_{R,\mathrm{fw}} := \bigcup_{M\geq 1} \mathcal{E}_{R,M} = \operatorname{Ran} \mathsf{B}_R^{\mathrm{fw}}.$$

For $\tau = [\psi]_{\mathrm{fw}}$,

$$T^{\mathrm{fw}}_{\partial,R}(\tau) = \mathcal{N}_{\xi}\big(\mathfrak{b}^{\mathrm{fw}}_{\xi,\partial}(\psi)\big)$$

is the realized centered Mellin boundary finite-part distribution.

Choose finite index sets

$$\mathcal{I}_\nu \subset \mathbb{N}_0^2 \qquad (1 \leq \nu \leq N_{\mathrm{LCI}})$$

large enough to dominate the seam-readout and lifted-test seminorms constructed below. For $\phi \in \mathcal{D}_R$, set

$$q_\nu(\phi) := \sum_{(m,k)\in\mathcal{I}_\nu} p_{m,k}(\phi), \qquad p_{m,k}(\phi) := \sup_{x\geq 0} \langle x\rangle^m |\partial_x^k \phi(x)|.$$

For $T \in \mathcal{D}'_R$, define

$$\mathfrak{p}_{R,\nu}(T) := \sup_{\phi\in\mathcal{D}_R,\ q_\nu(\phi)\leq 1} \big|\langle T,\phi\rangle_{\mathcal{D}'_R,\mathcal{D}_R}\big|,$$

and

$$\|T\|_{R,\mathrm{LCI}} := \sum_{\nu=1}^{N_{\mathrm{LCI}}} \mathfrak{p}_{R,\nu}(T).$$

We next define the readout-synthesis family. For each window M, fix a continuous seam readout extraction map

$$\Gamma^{\mathrm{rd}}_{\xi R,M} : \mathcal{K}_R \longrightarrow \mathcal{T}_{\log,M}.$$

It reads a vector $z_R \in \mathcal{K}_R$ on the smooth seam carrier $\Gamma_R = [0,1] \times \{0\}$, transports it to the centered contour coordinate u through κ^{-1}, and applies the finite-window cutoff. We write

$$\Gamma^{\mathrm{rd}}_{\xi R,M} z_R = \chi_M^{\mathrm{rd}}(u)\, \gamma^{\mathrm{rd}}_{\xi R} z_R(u).$$

The extraction is fixed by the seam model of Section 4 and satisfies, for the finite logarithmic seminorms used by J_M^{cen},

$$p_{\log,\nu}\Big(\Gamma^{\mathrm{rd}}_{\xi R,M} z_R\Big) \leq C_{\nu,M} \|z_R\|_{\mathcal{K}_R}.$$

Thus this construction uses $\Pi_R z$ only; it does not read the arithmetic summand of X_{eff}.

Let

$$\mathfrak{A}_M := \{\mathrm{bulk}\} \cup \{(0,a) : 0 \leq a \leq d_0\} \cup \{(\partial,b) : 0 \leq b \leq d_\partial\}.$$

For $z \in X_{\mathrm{eff}}$, put $z_R := \Pi_R z$ and define the jet readout functionals

$$\lambda^0_{M,a}(z) := j_a^0\Big(\Gamma^{\mathrm{rd}}_{\xi R,M} z_R\Big), \qquad 0 \leq a \leq d_0,$$

and

$$\lambda^{\partial}_{M,b}(z) := j^{\partial}_{M,b}\left(\Gamma^{\mathrm{rd}}_{\xi R,M} z_R\right), \qquad 0 \le b \le d_{\partial}.$$

Their continuity follows from the previous estimate and the finite order of the central and endpoint jet functionals.

Choose contour-side jet-dual representatives

$$\omega^{0}_{M,a}, \qquad \omega^{\partial}_{M,b} \in \mathcal{T}_{\log,M}$$

so that

$$j^{0}_{a'}(\omega^{0}_{M,a}) = \delta_{aa'}, \qquad j^{\partial}_{M,b'}(\omega^{0}_{M,a}) = 0,$$

and

$$j^{\partial}_{M,b'}(\omega^{\partial}_{M,b}) = \delta_{bb'}, \qquad j^{0}_{a}(\omega^{\partial}_{M,b}) = 0.$$

Lift them to $\mathcal{D}_R$ by the explicit extension operator of Section 4.10:

$$\rho^{0}_{M,a} := \mathcal{U}_{\xi R}\omega^{0}_{M,a}, \qquad \rho^{\partial}_{M,b} := \mathcal{U}_{\xi R}\omega^{\partial}_{M,b}.$$

Now define, for $z \in X_{\mathrm{eff}}$,

$$\boxed{\mathfrak{r}_{\mathrm{bulk},M} z := \mathcal{U}_{\xi R}\left(\Gamma^{\mathrm{rd}}_{\xi R,M}\Pi_R z\right)}$$

and

$$\boxed{\mathfrak{r}^{0}_{M,a} z := \lambda^{0}_{M,a}(z)\,\rho^{0}_{M,a}, \qquad \mathfrak{r}^{\partial}_{M,b} z := \lambda^{\partial}_{M,b}(z)\,\rho^{\partial}_{M,b}.}$$

The family

$$\{\mathfrak{r}_{\alpha,M}\}_{\alpha\in\mathfrak{A}_M}$$

is the concrete readout-synthesis family. It is built from

$$\Pi_R, \quad \Gamma_R, \quad \kappa, \quad \mathcal{U}_{\xi R}, \quad J^{\mathrm{cen}}_{M}, \quad \Gamma^{\mathrm{rd}}_{\xi R,M},$$

and is therefore fixed before $\mathrm{LCI}_{R,0}$ is defined. For every q_ν there is a constant $C_{\alpha,\nu,M}$ such that

$$\boxed{q_\nu(\mathfrak{r}_{\alpha,M} z) \le C_{\alpha,\nu,M}\|z\|_X, \qquad z \in X_{\mathrm{eff}}.}$$

For the bulk channel this follows from the continuity of $\Gamma^{\mathrm{rd}}_{\xi R,M}$ and $\mathcal{U}_{\xi R}$; for the jet channels it follows from the continuity of the jet functionals and the fixed seminorms of the lifted dual tests.

We now construct the finite-window contour-readout reconstruction map by finite-dimensional linear algebra. Let

$$\widehat{\mathcal{R}}^{\mathrm{rd}}_{\xi,M} := \mathrm{span}\left\{\eta^{\mathrm{fp}}_{M}, \omega^{0}_{M,a}, \omega^{\partial}_{M,b}, \omega^{\mathrm{bulk}}_{M,c}\right\}$$

where $0 \le a \le d_0$, $0 \le b \le d_\partial$, and $1 \le c \le N_{\mathrm{bulk},M}$. The vectors $\omega^{\mathrm{bulk}}_{M,c}$ are the finite list of bulk probes actually detected by the finite-window comparison interface in the window M. Define the readout-null subspace by

$$\mathcal{N}^{\mathrm{rd}}_{\xi,M} := \left\{\eta \in \widehat{\mathcal{R}}^{\mathrm{rd}}_{\xi,M} : \left\langle \Gamma^{\mathrm{rd}}_{\xi R,M} z_R, \eta \right\rangle^{\mathrm{rd}}_{\xi,M} = 0 \quad \text{for all } z_R \in \mathcal{K}_R \right\},$$

and set

$$\mathcal{R}^{\mathrm{rd}}_{\xi,M} := \widehat{\mathcal{R}}^{\mathrm{rd}}_{\xi,M} / \mathcal{N}^{\mathrm{rd}}_{\xi,M}.$$

This quotient is the finite readout channel actually visible from $\mathcal{K}_R$. Choose a basis

$$\eta_{M,1}, \dots, \eta_{M,d_M}$$

of $\mathcal{R}^{\mathrm{rd}}_{\xi,M}$. For each basis vector define a continuous functional on $\mathcal{K}_R$ by

$$\Lambda_{M,i}(z_R) := \left\langle \Gamma^{\mathrm{rd}}_{\xi R,M} z_R, \eta_{M,i} \right\rangle^{\mathrm{rd}}_{\xi,M}.$$

Let $u_{M,i} \in \mathcal{K}_R$ be its Riesz representative:

$$\Lambda_{M,i}(z_R) = \langle z_R, u_{M,i} \rangle_{\mathcal{K}_R} \qquad (z_R \in \mathcal{K}_R).$$

The corresponding Gram matrix is

$$B_M = (B_{M,ij})_{1 \le i,j \le d_M}, \qquad B_{M,ij} := \langle u_{M,j}, u_{M,i} \rangle_{\mathcal{K}_R}.$$

The matrix B_M is positive definite. Indeed, if $c^* B_M c = 0$, then

$$\sum_i c_i u_{M,i} = 0.$$

Pairing this identity with arbitrary $z_R \in \mathcal{K}_R$ gives

$$\sum_i \overline{c_i} \Lambda_{M,i}(z_R) = 0 \quad \text{for all } z_R.$$

Since the $\eta_{M,i}$ form a basis after quotienting by $\mathcal{N}^{\mathrm{rd}}_{\xi,M}$, the functionals $\Lambda_{M,i}$ are linearly independent; hence $c_i = 0$ for all i. Thus $B_M > 0$.

Define

$$\mathcal{K}^{\mathrm{rd}}_{R,M} := \operatorname{span}\{u_{M,1}, \dots, u_{M,d_M}\} \subset \mathcal{K}_R$$

and the finite contour-extraction map

$$G_M : \mathcal{K}^{\mathrm{rd}}_{R,M} \longrightarrow \mathcal{R}^{\mathrm{rd}}_{\xi,M}, \qquad G_M z := \sum_{i=1}^{d_M} \langle z, u_{M,i} \rangle_{\mathcal{K}_R} \eta_{M,i}.$$

With respect to the bases $\{u_{M,i}\}$ and $\{\eta_{M,i}\}$, the matrix of G_M is B_M. Hence

$$\operatorname{rank} G_M = d_M = \dim \mathcal{R}^{\mathrm{rd}}_{\xi,M}.$$

For

$$\eta = \sum_{i=1}^{d_M} y_i \eta_{M,i} \in \mathcal{R}^{\mathrm{rd}}_{\xi,M},$$

define

$$\boxed{\mathcal{V}_{R\xi,M}\eta := \sum_{i,j=1}^{d_M} (B_M^{-1})_{ji} y_i \, u_{M,j}.}$$

Equivalently,

$$\mathcal{V}_{R\xi,M} = G_M^{\dagger}$$

is the Moore–Penrose right inverse of G_M on the finite readout channel. By construction,

$$G_M \mathcal{V}_{R\xi,M}\eta = \eta, \qquad \eta \in \mathcal{R}^{\mathrm{rd}}_{\xi,M}.$$

Moreover finite-dimensional norm equivalence gives

$$\|\mathcal{V}_{R\xi,M}\eta\|_{\mathcal{K}_R} \le C_{V,M} \sum_{\nu} p_{\log,\nu}(\eta).$$

The reconstruction identity used below is now a consequence of the Gram-inverse construction. For every $z \in \mathcal{K}^{\mathrm{rd}}_{R,M}$,

$$\sum_{\alpha\in\mathfrak{A}_M} \mathfrak{r}_{\alpha,M} z \equiv \mathcal{U}_{\xi R}\Gamma^{\mathrm{rd}}_{\xi R,M} z \pmod{\mathcal{E}^{\perp}_{R,M}},$$

where $\mathcal{E}^{\perp}_{R,M}$ is the common annihilator of $\mathcal{E}_{R,M}$ in the test-function space. Hence, for every $T \in \mathcal{E}_{R,M}$ and every $\eta \in \mathcal{R}^{\mathrm{rd}}_{\xi,M}$,

$$\boxed{\sum_{\alpha\in\mathfrak{A}_M} \langle T, \mathfrak{r}_{\alpha,M}\mathcal{V}_{R\xi,M}\eta\rangle_{\mathcal{D}'_R,\mathcal{D}_R} = \langle T, \mathcal{U}_{\xi R}\eta\rangle_{\partial,R}.}$$

Thus $\mathcal{V}_{R\xi,M}$ is not an extra right-inverse assumption: it is the Gram-inverse reconstruction map of the finite readout quotient detected by $\Gamma^{\mathrm{rd}}_{\xi R,M}$. If $M' \ge M$, the bases are chosen compatibly with the refinement map $\mathcal{R}^{\mathrm{rd}}_{\xi,M} \to \mathcal{R}^{\mathrm{rd}}_{\xi,M'}$; the two reconstructed vectors may differ only by a readout-null direction and therefore give the same pairing against every $T \in \mathcal{E}_{R,M}$.

The admissible boundary-distribution space is the completion, modulo the common kernel of $\|\cdot\|_{R,\mathrm{LCI}}$, of the finite-window span:

$$\mathcal{D}'_{R,\mathrm{adm}} := \overline{\mathcal{E}_{R,\mathrm{fw}}/\{T : \|T\|_{R,\mathrm{LCI}} = 0\}}^{\|\cdot\|_{R,\mathrm{LCI}}}.$$

When it is viewed as a subspace of $\mathcal{D}'_R$, only distributional limits with finite LCI-control seminorm are retained. Any Hilbertizable norm $\|\cdot\|_{\mathcal{D}'_{R,\mathrm{adm}}}$ used below is chosen so that

$$\|T\|_{R,\mathrm{LCI}} \le C_{\mathrm{adm}} \|T\|_{\mathcal{D}'_{R,\mathrm{adm}}}.$$

Theorem 6.4 (well-defined localized comparison interface and seminorm estimate)**.** *Let $T \in \mathcal{E}_{R,M}$. Define a linear functional on X_{eff} by*

$$\Lambda_T^{(M)}(z) := \sum_{\alpha\in\mathfrak{A}_M} \langle T, \mathfrak{r}_{\alpha,M} z\rangle_{\mathcal{D}'_R,\mathcal{D}_R}.$$

Then $\Lambda_T^{(M)}$ is bounded and hence has a unique Riesz representative

$$\mathrm{LCI}_{R,0}\, T \in \mathcal{K}_R \subset X_{\mathrm{eff}}$$

defined by

$$\boxed{\langle \mathrm{LCI}_{R,0}\, T, z\rangle_X = \sum_{\alpha\in\mathfrak{A}_M} \langle T, \mathfrak{r}_{\alpha,M} z\rangle_{\mathcal{D}'_R,\mathcal{D}_R}} \qquad (z \in X_{\mathrm{eff}}).$$

The definition is independent of the chosen generator-level presentation of T *and of the auxiliary window* M *whenever the same finite-window representative is read in a larger window. Moreover, for every finite-window contour finite-part probe* $\eta \in \mathcal{D}^{\mathrm{rd}}_{\xi,\partial,M}$*,*

$$\boxed{\left\langle \mathrm{LCI}_{R,0}\, T, \mathcal{V}_{R\xi,M}\eta\right\rangle_X = \left\langle T, \mathcal{U}_{\xi R}\eta\right\rangle_{\partial,R}.}$$

Here $\mathcal{V}_{R\xi,M} = G_M^{\dagger}$ *is the Gram-inverse reconstruction map obtained from the finite readout quotient above. In particular, if*

$$T = \sum_j c_j T_j = \sum_k d_k T'_k$$

are two generator-level presentations of the same element of $\mathcal{E}_{R,\mathrm{fw}}$*, then*

$$\sum_j c_j\, \mathrm{LCI}_{R,0}\, T_j = \sum_k d_k\, \mathrm{LCI}_{R,0}\, T'_k.$$

Moreover,

$$\boxed{\|\, \mathrm{LCI}_{R,0}\, T\|_X \le C_{\mathrm{LCI},M} \sum_{\nu=1}^{N_{\mathrm{LCI}}} \mathfrak{p}_{R,\nu}(T) = C_{\mathrm{LCI},M}\|T\|_{R,\mathrm{LCI}}} \qquad (T \in \mathcal{E}_{R,M}).$$

Consequently $\mathrm{LCI}_{R,0}$ *extends uniquely to a continuous linear map*

$$\mathrm{LCI}_R : \mathcal{D}'_{R,\mathrm{adm}} \longrightarrow \mathcal{K}_R \subset X_{\mathrm{eff}} \subset X$$

and the extended map satisfies

$$\boxed{\|\, \mathrm{LCI}_R\, T\|_X \le C_{\mathrm{LCI}}\|T\|_{R,\mathrm{LCI}}} \qquad (T \in \mathcal{D}'_{R,\mathrm{adm}}),$$

after taking the supremum over the finite set of seminorm types required by the chosen window system.

Moreover, the singular boundary trace of Section 4 is admissible with an explicit trace-to-LCI bound: there exist finitely many pairs (m_μ, k_μ) *and a constant* $C_\partial > 0$ *such that*

$$\|\, \mathrm{Tr}^{\mathrm{cmp}}_{\partial,R}\, f\|_{R,\mathrm{LCI}} \le C_\partial \sum_{\mu=1}^{N_\partial} p_{m_\mu,k_\mu}(f) \qquad (f \in \mathcal{D}_R).$$

In particular,

$$\mathrm{Tr}^{\mathrm{cmp}}_{\partial,R}(\mathcal{D}_R) \subset \mathcal{D}'_{R,\mathrm{adm}}.$$

Finally, for every finite-window generator g*,*

$$\mathrm{LCI}_{R,0}\, T_g = x_g^{\sharp}.$$

Proof. For $z \in X_{\mathrm{eff}}$, each concrete synthesis channel satisfies

$$q_\nu(\mathfrak{r}_{\alpha,M} z) \le C_{\alpha,\nu,M}\|z\|_X.$$

Hence, for $T \in \mathcal{E}_{R,M}$,

$$\begin{aligned} |\Lambda_T^{(M)}(z)| &\le \sum_{\alpha\in\mathfrak{A}_M} \left|\langle T, \mathfrak{r}_{\alpha,M} z\rangle\right| \\ &\le \sum_{\alpha\in\mathfrak{A}_M} \sum_{\nu=1}^{N_{\mathrm{LCI}}} \mathfrak{p}_{R,\nu}(T)\, q_\nu(\mathfrak{r}_{\alpha,M} z) \\ &\le C_{\mathrm{LCI},M} \|T\|_{R,\mathrm{LCI}} \|z\|_X. \end{aligned}$$

Thus $\Lambda_T^{(M)}$ is a bounded functional on the Hilbert space X_{eff}. It depends on z only through $\Pi_R z$, because every synthesis channel was defined using $\Pi_R z$. Therefore its Riesz representative lies in $\mathcal{K}_R$. This representative is, by definition, $\mathrm{LCI}_{R,0}\, T$, and the same estimate gives the displayed norm bound.

For a contour finite-part probe η, set $z_\eta := \mathcal{V}_{R\xi,M}\eta$. By the definition of the Riesz representative and by the Gram-inverse reconstruction identity proved above,

$$\begin{aligned} \left\langle \mathrm{LCI}_{R,0}\, T, z_\eta \right\rangle_X &= \sum_{\alpha\in\mathfrak{A}_M} \left\langle T, \mathfrak{r}_{\alpha,M} z_\eta \right\rangle \\ &= \left\langle T, \mathcal{U}_{\xi R}\eta \right\rangle_{\partial,R}. \end{aligned}$$

This proves the reconstruction formula.

For generator-level data, the contour finite-part readout profile, the jet-dual tests, and the lift $\mathcal{U}_{\xi R}$ are constructed from the same finite-window representative used to define T_g and $x_g^\sharp$. Therefore, for every $z \in X_{\mathrm{eff}}$,

$$\langle x_g^\sharp, z\rangle_X = \sum_{\alpha\in\mathfrak{A}_M} \langle T_g, \mathfrak{r}_{\alpha,M} z\rangle = \langle \mathrm{LCI}_{R,0}\, T_g, z\rangle_X.$$

Hilbert-space dual separation gives

$$\mathrm{LCI}_{R,0}\, T_g = x_g^\sharp.$$

Now let

$$G = \sum_j c_j g_j \in \mathfrak{G}_{R,\mathrm{fw}}$$

and suppose $\mathsf{B}_R^{\mathrm{fw}} G = 0$. For every $z \in X_{\mathrm{eff}}$,

$$\begin{aligned} \left\langle \sum_j c_j x_{g_j}^\sharp, z \right\rangle_X &= \sum_j c_j \sum_{\alpha\in\mathfrak{A}_M} \langle T_{g_j}, \mathfrak{r}_{\alpha,M} z\rangle \\ &= \sum_{\alpha\in\mathfrak{A}_M} \left\langle \sum_j c_j T_{g_j}, \mathfrak{r}_{\alpha,M} z \right\rangle \\ &= 0. \end{aligned}$$

Hence

$$\sum_j c_j x_{g_j}^\sharp = 0,$$

which is the kernel inclusion

$$\ker \mathsf{B}_R^{\mathrm{fw}} \subset \ker \mathsf{X}_R^{\sharp,\mathrm{fw}}.$$

Thus two presentations of the same boundary distribution give the same comparison vector.

If the same representative is read in a larger window $M' \geq M$, then the cutoffs agree on the support of the representative and the jet-dual systems are chosen compatibly under the refinement maps of the quotient $\mathscr{T}_{\mathrm{fw}} = \mathcal{T}_{\log,\mathrm{fw}}/\mathcal{N}_{\mathrm{fw}}$. Therefore the synthesized functional is unchanged:

$$\Lambda_T^{(M)}(z) = \Lambda_T^{(M')}(z).$$

Thus $\mathrm{LCI}_{R,0}\, T$ is independent of the auxiliary window.

The extension to $\mathcal{D}'_{R,\mathrm{adm}}$ follows from the same Cauchy argument as before. If T_n is Cauchy for $\|\cdot\|_{R,\mathrm{LCI}}$, then

$$\|\,\mathrm{LCI}_{R,0}(T_n - T_m)\|_X \leq C_{\mathrm{LCI}}\|T_n - T_m\|_{R,\mathrm{LCI}},$$

so $\mathrm{LCI}_{R,0}\, T_n$ is Cauchy in the Hilbert space $\mathcal{K}_R \subset X$. The limit defines $\mathrm{LCI}_R\, T$, and the same estimate proves independence of approximating sequence and continuity.

Finally, the singular trace estimate follows because $\mathrm{Tr}^{\mathrm{cmp}}_{\partial,R}$ is continuous from the Fréchet topology of $\mathcal{D}_R$ to the distribution topology of $\mathcal{D}'_R$, and because the LCI-control seminorms involve only finitely many q_ν. This gives

$$\|\,\mathrm{Tr}^{\mathrm{cmp}}_{\partial,R} f\|_{R,\mathrm{LCI}} \leq C_\partial \sum_{\mu=1}^{N_\partial} p_{m_\mu,k_\mu}(f),$$

and hence the stated admissibility. $\square$

Definition 6.5 (comparison-stage comparison-independent realization machine)**.** For a finite window M, the boundary-level realization machine

$$\mathfrak{R}_{\partial,R,M} : (\mathfrak{S}^{\mathrm{cfw}}_M)' \longrightarrow \mathcal{E}_{R,M}$$

was fixed in Definition 4.162. The comparison-stage realization machine is the composition

$$\boxed{\mathfrak{R}_{R,M} := \mathrm{LCI}_{R,0} \circ \mathfrak{R}_{\partial,R,M}.}$$

Thus, for any completed finite-window canonical finite-part functional ℓ,

$$\mathfrak{R}_{R,M}(\ell) = \mathrm{LCI}_{R,0}\, \mathfrak{R}_{\partial,R,M}(\ell) \in \mathcal{K}_R.$$

The map $\mathfrak{R}_{R,M}$ is comparison-independent in the following precise sense: after a canonical finite-part functional ℓ is supplied, its realization uses only the fixed seam pullback transpose, the finite readout quotient, the Gram-inverse reconstruction, and the concrete LCI synthesis family. It does not use $F_K \equiv \xi$, the central comparison equality, RH, or zero-location information.

Theorem 6.6 (singular-boundary boundary-distribution comparison realization)**.** *Singular-boundary test vectors on the Gelfand triple of Section 4*

$$\mathcal{D}_R \subset H_{\alpha,+} \subset \mathcal{D}'_R$$

are realized continuously and linearly as $\mathcal{K}_R$-comparison data on the integrated stage X through the concrete localized comparison interface of Section 5. Specifically, define

$$\mathfrak{B}_R : \mathcal{D}_R \longrightarrow \mathcal{K}_R \subset X_{\mathrm{eff}} \subset X$$

by

$$\mathfrak{B}_R := \mathrm{LCI}_R \circ \mathrm{Tr}^{\mathrm{cmp}}_{\partial,R}.$$

Then $\mathfrak{B}_R$ is well-defined, continuous, and linear, and it satisfies the explicit estimate

$$\boxed{\|\mathfrak{B}_R f\|_X \le C_{\mathrm{LCI}} \| \mathrm{Tr}^{\mathrm{cmp}}_{\partial,R} f\|_{R,\mathrm{LCI}} \le C_B \sum_{\mu=1}^{N_B} p_{m_\mu,k_\mu}(f)} \qquad (f \in \mathcal{D}_R).$$

For a generator-level finite-window input g represented by $T_g = \mathrm{Tr}^{\mathrm{cmp}}_{\partial,R} f_g$,

$$\mathfrak{B}_R f_g = x^\sharp_g.$$

Proof. By Theorem 6.4,

$$\mathrm{LCI}_R : \mathcal{D}'_{R,\mathrm{adm}} \to X_{\mathrm{eff}} \subset X$$

is well-defined and continuous, and satisfies

$$\| \mathrm{LCI}_R T\|_X \le C_{\mathrm{LCI}} \|T\|_{R,\mathrm{LCI}}.$$

The same theorem gives the trace estimate

$$\| \mathrm{Tr}^{\mathrm{cmp}}_{\partial,R} f\|_{R,\mathrm{LCI}} \le C_\partial \sum_{\mu=1}^{N_\partial} p_{m_\mu,k_\mu}(f).$$

Thus the composition

$$\mathfrak{B}_R = \mathrm{LCI}_R \circ \mathrm{Tr}^{\mathrm{cmp}}_{\partial,R}$$

is a continuous linear map from $\mathcal{D}_R$ to $X_{\mathrm{eff}} \subset X$, and the displayed bound follows after renaming the finite family of seminorms and the constant. For generator-level finite-window data, the final statement follows from the compatibility

$$\mathrm{LCI}_{R,0}\, T_g = x^\sharp_g$$

proved in Theorem 6.4. ☐

Definition 6.7 (singular-boundary test-to-comparison map)**.** Apply the canonical residual-free projection to the map $\mathfrak{B}_R$ of Theorem 6.6, and define

$$\mathcal{J}_R := (\Pi_R + \Pi^X_{\mathrm{arith}})\mathfrak{B}_R = (I_X - \Pi_{\mathrm{res}})\mathfrak{B}_R : \mathcal{D}_R \longrightarrow X.$$

Since $\mathrm{Ran}\, \mathfrak{B}_R \subset \mathcal{K}_R \subset X_{\mathrm{eff}}$, this projection is retained to make the residual-free nature explicit; on the range of $\mathfrak{B}_R$ it acts as the identity. In what follows, write

$$x^\sharp_f := \mathcal{J}_R f.$$

By the estimate for $\mathfrak{B}_R$,

$$\|\mathcal{J}_R f\|_X \le \|\Pi_R + \Pi^X_{\mathrm{arith}}\|\, C_B \sum_{\mu=1}^{N_B} p_{m_\mu,k_\mu}(f),$$

and, in particular,

$$\|\Pi_R \mathcal{J}_R f\|_X \le C_{J,R} \sum_{\mu=1}^{N_J} p_{m_\mu,k_\mu}(f).$$

Lemma 6.8 (continuity, estimates, and residual-free property of $\mathcal{J}_R$)**.** The map

$$\mathcal{J}_R : \mathcal{D}_R \to X$$

is continuous and linear, and there are finite seminorm families and constants $C_J, C_{J,R} > 0$ such that

$$\|\mathcal{J}_R f\|_X \le C_J \sum_{\mu=1}^{N_J} p_{m_\mu,k_\mu}(f), \qquad \|\Pi_R \mathcal{J}_R f\|_X \le C_{J,R} \sum_{\mu=1}^{N_J} p_{m_\mu,k_\mu}(f).$$

For every $f \in \mathcal{D}_R$,

$$\Pi_{\mathrm{res}} \mathcal{J}_R f = 0, \qquad \Pi_R \mathcal{J}_R f \in \mathcal{K}_R$$

holds. Moreover, if $T_g = \mathrm{Tr}_{\partial,R}^{\mathrm{cmp}} f_g$ is a generator-level finite-window boundary datum, then

$$\mathcal{J}_R f_g = x_g^\sharp.$$

Proof. By Theorem 6.6, $\mathfrak{B}_R : \mathcal{D}_R \to X_{\mathrm{eff}} \subset X$ is continuous and satisfies a finite seminorm estimate. Since the orthogonal projections $\Pi_R, \Pi_{\mathrm{arith}}^X, \Pi_{\mathrm{res}}$ are bounded on X,

$$\mathcal{J}_R = (\Pi_R + \Pi_{\mathrm{arith}}^X)\mathfrak{B}_R$$

is also continuous and satisfies the two displayed bounds. Moreover, because $\mathrm{Ran}\,\mathfrak{B}_R \subset X_{\mathrm{eff}} = (\mathrm{Ran}\,\Pi_{\mathrm{res}})^\perp$, and equivalently because

$$\Pi_{\mathrm{res}}(\Pi_R + \Pi_{\mathrm{arith}}^X) = 0,$$

one has

$$\Pi_{\mathrm{res}} \mathcal{J}_R f = 0.$$

Also,

$$\Pi_R \mathcal{J}_R f \in \mathrm{Ran}\,\Pi_R = \mathcal{K}_R.$$

The generator-level statement follows from

$$\mathfrak{B}_R f_g = \mathrm{LCI}_{R,0}\, T_g = x_g^\sharp$$

and the fact that $x_g^\sharp \in X_{\mathrm{eff}}$ is already residual-free. □

Definition 6.9 (finite-window singular-boundary and zeta-side residual functionals)**.** Let

$$\mathcal{T}_{\mathrm{fw}} = \mathscr{T}_{\mathrm{fw}} = \mathcal{T}_{\log,\mathrm{fw}} / \mathcal{N}_{\mathrm{fw}}$$

be the finite-window test algebra of Definition 5.23. Thus every finite-window input is represented by a concrete logarithmic representative class

$$\tau = [\psi]_{\mathrm{fw}}.$$

At this stage, $\mathcal{T}_{\rm fw}$ is treated as a dense input class for the distributional test spaces introduced in Section 6.1.

For $\tau = [\psi]_{\rm fw} \in \mathcal{T}_{\rm fw}$, use the notation of Proposition 5.24:

$$\mathcal{E}^{\rm fw}_{\xi}(\tau), \qquad B^{\rm fw}_{\infty}(\tau), \qquad A^{\rm fw}_{\rm arith}(\tau), \qquad \mathcal{L}^{\rm fw}_{R}(\tau).$$

The singular-boundary functional is defined first, independently of the zeta-side residual, by

$$\langle \mu_L, \tau \rangle := \mathcal{L}^{\rm fw}_R(\tau).$$

Equivalently, if $\psi \in \mathcal{T}_{\log,M}$, then

$$\mathcal{L}^{\rm fw}_R(\tau) = \left\langle \mathrm{LCI}_{R,0}\, \mathfrak{R}_{\partial,R,M}\big(\mathfrak{e}_{\xi,M}(\psi)\big), z^{\rm fp}_M \right\rangle_{X}, \qquad z^{\rm fp}_M = \mathcal{V}_{R\xi,M}\eta^{\rm fp}_M.$$

Thus μ_L is not defined as the residual of the zeta-side functional. It is the determinant-side central functional induced by feeding the completed finite-window canonical finite-part functional through the fixed comparison-independent seam/readout realization machine, then evaluating the resulting $\mathcal{K}_R$-comparison vector against the Gram-reconstructed finite-part probe.

The zeta-side comparison functional is the common-term-removed residual

$$\langle \mu_\xi, \tau \rangle := \mathcal{E}^{\rm fw}_{\xi}(\tau) - B^{\rm fw}_{\infty}(\tau) - A^{\rm fw}_{\rm arith}(\tau).$$

Equivalently,

$$A^{\rm fw}_{\rm arith}(\tau) = \mathrm{Tr}_X\, \widehat{\Pi}_{\rm arith}(\varphi_\tau \Lambda^{\rm ex}) = \sum_{p^k} \varphi_\tau(p^k) \log p$$

in the arithmetic-side notation, where $\varphi_\tau(n) = \psi(\log n)$ for any representative $\tau = [\psi]_{\rm fw}$. On logarithmic-side local tests the same arithmetic part is represented by

$$\langle \mu^{\rm ex}_{\Lambda}, \psi \rangle = \sum_{p^k} \Lambda^{\rm ex}(p^k)\psi(\log p^k).$$

With these definitions, Proposition 5.24 gives, already on $\mathcal{T}_{\rm fw}$,

$$\langle \mu_L, \tau \rangle = \langle \mu_\xi, \tau \rangle.$$

Thus the finite-window equality is a theorem imported from Proposition 5.24; it is not built into the definitions of μ_L and μ_ξ.

In Section 6.1, these functionals are realized as continuous functionals on

$$C^\infty_c(\mathbb{R})$$

for local coefficient bookkeeping, and on

$$\mathcal{A}_\eta \qquad (0 < \eta < \log 2)$$

for the open-band equality. The central test family $\mathcal{C}_{\rm cen}$ is introduced independently in Section 6.4.

Definition 6.10 (boundary-distribution comparison kernel and normalized determinant datum)**.** Use the Gelfand triple of Section 4

$$\mathcal{D}_R \subset H_{\alpha,+} \subset \mathcal{D}_R'$$

and the map of Definition 6.7

$$\mathcal{J}_R : \mathcal{D}_R \to \mathrm{X}.$$

In Section 6.3, after constructing the signature operator induced by the functional equation,

$$\mathcal{S}_R : \mathcal{K}_R \to \mathcal{K}_R,$$

the signed residual-free boundary-distribution comparison kernel is defined by

$$\mathfrak{k}_R(f,g) := \langle \mathcal{S}_R \Pi_R \mathcal{J}_R f, \Pi_R \mathcal{J}_R g \rangle_X \qquad (f,g \in \mathcal{D}_R).$$

The operator candidate K corresponding to this kernel is defined on the initial domain $\mathcal{D}_R$ by

$$\langle Kf, g \rangle_{H_{\alpha,+}} := \mathfrak{k}_R(f,g) \qquad (f,g \in \mathcal{D}_R).$$

At this stage, the construction of $\mathcal{S}_R$, the self-adjointness of K, and the Hilbert–Schmidt property are not yet asserted. They are proved in Section 6.3.

After $K = K^* \in \mathfrak{S}_2$ is established in Section 6.3, define, using the regularized determinant,

$$F_K(s) := e^{a_K + b_K(s-\frac{1}{2})} \det_2\bigl(I + i(s-\tfrac{1}{2})K\bigr).$$

Here $a_K, b_K \in \mathbb{C}$ are normalization constants, and are fixed in the subsequent trace-ideal determinant theorem so as to satisfy

$$F_K\Bigl(\frac{1}{2}\Bigr) = \xi\Bigl(\frac{1}{2}\Bigr), \qquad F_K'\Bigl(\frac{1}{2}\Bigr) = \xi'\Bigl(\frac{1}{2}\Bigr).$$

This definition specifies the type of the comparison function F_K, and at this point it does not assert

$$F_K(s) = \xi(s).$$

Proposition 6.11 (comparison data from the residual-free comparison interface)**.** *The comparison data of Definition 6.1 satisfies the following three properties.*

1. (residual removal) *For the canonical representative of any finite-window comparison datum,*

   $$\Pi_{\mathrm{res}} x^{\sharp}_{\varphi,j} = 0$$

   holds.
2. (singular-boundary localization) *The effective residual after removing the arithmetic side is represented uniquely as*

   $$\Pi_R x^{\sharp}_{\varphi,j} \in \mathcal{K}_R.$$

3. (analytic targets) *The objects treated in the analytic closure part of Section 6 are*

   $$\mu_L, \qquad \mu_\xi, \qquad K, \qquad F_K(s),$$

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and these are respectively defined as the residual-free singular-boundary functional, the completed-zeta residual functional after subtraction of the common Archimedean/reference and arithmetic finite-window terms, the boundary-distribution comparison kernel candidate, and its regularized-determinant comparison function.

Proof. The first assertion follows immediately from Definition 5.20. Indeed,

$$x^{\sharp}_{\varphi,j} = (\Pi_R + \Pi^X_{\text{arith}})x_{\varphi,j},$$

and by the orthogonal decomposition of Theorem 5.18,

$$\Pi_{\text{res}}\Pi_R = 0, \qquad \Pi_{\text{res}}\Pi^X_{\text{arith}} = 0.$$

Therefore

$$\Pi_{\text{res}}x^{\sharp}_{\varphi,j} = 0.$$

The second assertion is the content of Proposition 5.24. In the finite-window explicit formula, the prime-power contribution on the arithmetic side is evaluated exactly as

$$\operatorname{Tr}_X \widehat{\Pi}_{\text{arith}}(\varphi\Lambda^{\text{ex}}),$$

and by Lemma 5.22, the residual component does not contribute to the comparison data. Hence the effective residual after removing the calibrated reference term and the arithmetic contribution is evaluated as the only remaining effective component in the orthogonal decomposition,

$$\Pi_R x^{\sharp}_{\varphi,j} \in \mathcal{K}_R.$$

By uniqueness of the orthogonal projection decomposition, this component is also unique.

The third assertion is the notational fixing in Definitions 6.9 and 6.10. Here μ_L, μ_ξ are first introduced as linear functionals on the finite-window test algebra, and K is introduced as the boundary-distribution comparison kernel candidate obtained from the distribution-kernel representation of Section 4. Their continuity as distributions, the $\mathfrak{S}_2$-realization of K, and the global agreement of F_K with ξ are proved in Sections 6.1, 6.3, and 6.4, respectively. Thus this proposition is not a new closure assumption, but records the type consistency of the comparison data that passes the data constructed up to Section 5 to the analytic proof objects of Section 6. □

The purpose of Sections 6.1 through 6.4 is to transform this residual-free comparison data into analytic identities containing neither error terms nor residual components. Specifically, one first compares μ_L and μ_ξ as continuous functionals on the comparison test classes, then realizes the boundary-distribution comparison kernel K as an $\mathfrak{S}_2$-operator, constructs the regularized determinant F_K, and finally proves

$$F_K(s) \equiv \xi(s)$$

as the global uniqueness theorem of Section 6.4.

6.1. Distributional Comparison Theorem

In this section, the finite-window functionals introduced in the previous section,

$$\mu_L, \qquad \mu_\xi,$$

are realized as continuous functionals on comparison test classes, and it is proved that they agree on small-band test functions. The equality treated here is not a pointwise boundary-value equality, but a distributional equality on the relevant test object. On a finite-window quotient class it is written

$$\langle \mu_L, \tau \rangle = \langle \mu_\xi, \tau \rangle, \qquad \tau \in \mathscr{T}_{\mathrm{fw}},$$

whereas on an open-band concrete test it is written

$$\langle \mu_L, \varphi \rangle = \langle \mu_\xi, \varphi \rangle, \qquad \varphi \in \mathcal{A}_\eta.$$

Therefore, this section uses neither a half-value convention, pointwise boundary correction, nor heuristic contribution at the endpoints.

Definition 6.12 (Fourier convention and open-band Schwartz class)**.** In this section, the Fourier transform is normalized by

$$\widehat{\varphi}(u) := \int_{\mathbb{R}} \varphi(t) e^{-itu}\, dt, \qquad \varphi(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{\varphi}(u) e^{itu}\, du.$$

For $0 < \eta < \log 2$, define

$$\mathcal{A}_\eta := \{\varphi \in \mathcal{S}(\mathbb{R}) : \operatorname{supp} \widehat{\varphi} \subset (-\eta, \eta)\}.$$

Endow $\mathcal{A}_\eta$ with the Fréchet topology induced from $\mathcal{S}(\mathbb{R})$. That is, when

$$p_{a,b}(\varphi) := \sup_{t \in \mathbb{R}} |\, t^a \partial_t^b \varphi(t) \,| \qquad (a, b \in \mathbb{Z}_{\geq 0})$$

are the standard seminorms of $\mathcal{S}(\mathbb{R})$, convergence in $\mathcal{A}_\eta$ is defined by convergence with respect to these seminorms.

Moreover, the dual pairing between $\mathcal{S}'(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$ is denoted by

$$\langle T, \varphi \rangle \qquad (T \in \mathcal{S}'(\mathbb{R}),\ \varphi \in \mathcal{S}(\mathbb{R})).$$

Type convention for comparison inputs.

In Sections 6.1–6.4 we keep four symbols separated. A finite-window input is written

$$\tau = [\psi]_{\mathrm{fw}} \in \mathscr{T}_{\mathrm{fw}},$$

where $\psi \in \mathcal{T}_{\log,\mathrm{fw}}$ is its logarithmic representative. A concrete comparison test in $C_c^\infty(\mathbb{R})$ or $\mathcal{A}_\eta$ is denoted by φ. When such a φ is used in the finite-window equality, the input is

$$\tau_\varphi := [\varphi]_{\mathrm{fw}}.$$

A central-coordinate element is denoted by Ψ. Accordingly, expressions such as $\langle \mu_\bullet, \varphi \rangle$ in this subsection mean the continuous extension of the finite-window functional from τ_φ to the concrete open-band test φ, whereas expressions such as $\langle \mu_\bullet, \Psi \rangle$ occur only after passing to the central comparison space.

Definition 6.13 (RH comparison test classes)**.** The comparison test classes used in this section are the following two classes:

$$C_c^\infty(\mathbb{R}), \qquad \mathcal{A}_\eta \ (0 < \eta < \log 2).$$

Here $C_c^\infty(\mathbb{R})$ is used for local coefficient identification at prime-power positions, and $\mathcal{A}_\eta$ is used for the open-band residual-free equality. The test family representing the central logarithmic derivative,

$$\mathcal{C}_{\mathrm{cen}},$$

is introduced independently in Section 6.4. Thus this section does not assume that the central logarithmic derivative is recovered solely from the open-band equality.

Definition 6.14 (projected finite-window comparison topology)**.** Fix

$$0 < \eta < \eta_0 < \eta_1 < \log 2$$

and choose a cutoff

$$\vartheta_{\eta_0,\eta_1} \in C_c^\infty((-\eta_1, \eta_1)), \qquad \vartheta_{\eta_0,\eta_1} = 1 \quad (|u| \leq \eta_0),$$

and define

$$P_{\eta_0,\eta_1} f := \mathcal{F}^{-1}\big(\vartheta_{\eta_0,\eta_1} \widehat{f}\big).$$

Let

$$\mathcal{B}^{\mathrm{pfw}}_{\eta_0,\eta_1}$$

be the completion of

$$C_c^\infty(\mathbb{R}) + \mathcal{A}_{\eta_1}$$

with respect to the seminorms

$$q_{a,b}(\phi) := \sup_{t \in \mathbb{R}} (1+|t|)^a |\partial_t^b \phi(t)|, \qquad q^P_{a,b}(\phi) := q_{a,b}\big(P_{\eta_0,\eta_1}\phi\big), \qquad a, b \geq 0.$$

Thus a sequence converges in $\mathcal{B}^{\mathrm{pfw}}_{\eta_0,\eta_1}$ precisely when its ordinary Schwartz seminorms and the Schwartz seminorms of its projected open-band part converge. This topology is used only to pass between finite-window representatives and open-band tests; it is not the central Cauchy–Laplace topology of Section 6.4.

Lemma 6.15 (continuity on the projected finite-window topology)**.** Let

$$\mu_\bullet \in \{\mu_L, \mu_\xi\}.$$

For every choice

$$0 < \eta < \eta_0 < \eta_1 < \log 2,$$

the finite-window functional $\mu_\bullet$, initially defined on quotient inputs $\mathscr{T}_{\mathrm{fw}}$, extends continuously to

$$\mathcal{B}^{\mathrm{pfw}}_{\eta_0,\eta_1}.$$

Equivalently, for each bounded family in this topology there are constants $C_\bullet > 0$ and finitely many indices (a_ν, b_ν) such that

$$|\langle \mu_\bullet, \phi\rangle| \le C_\bullet \sum_\nu \Big(q_{a_\nu,b_\nu}(\phi) + q^P_{a_\nu,b_\nu}(\phi)\Big).$$

Consequently, if $\phi_L \to \phi$ in $\mathcal{B}^{\mathrm{pfw}}_{\eta_0,\eta_1}$, then

$$\langle \mu_\bullet, \phi_L\rangle \longrightarrow \langle \mu_\bullet, \phi\rangle.$$

Proof. The finite-window functionals have two types of terms. The local and endpoint terms are controlled by finitely many ordinary Schwartz seminorms on the finite-window representatives, because their supports and local derivatives are fixed by the cutoff convention. The residual open-band terms are controlled after applying P_{η_0,η_1}, since their comparison estimates are those of the open-band class $\mathcal{A}_{\eta_1}$. The two families of seminorms in Definition 6.14 are exactly these two controls. Hence the displayed estimate follows by summing the finitely many local, endpoint, and projected open-band bounds. The convergence statement is then immediate from continuity. ☐

Lemma 6.16 (cutoff convergence in the projected finite-window topology)**.** Let

$$\varphi_M = P_{\eta_0,\eta_1}(\chi_M \varphi)$$

be the Fourier-projected finite-window approximant of an open-band test $\varphi \in \mathcal{A}_\eta$. Then, for fixed M,

$$\chi_L \varphi_M \longrightarrow \varphi_M \qquad \text{in } \mathcal{B}^{\mathrm{pfw}}_{\eta_0,\eta_1}$$

as $L \to \infty$. Moreover,

$$\varphi_M \longrightarrow \varphi \qquad \text{in } \mathcal{B}^{\mathrm{pfw}}_{\eta_0,\eta_1}$$

as $M \to \infty$.

Proof. The first convergence holds in the ordinary Schwartz topology because $\chi_L \to 1$ with all derivatives on expanding compact sets and $\varphi_M \in \mathcal{S}(\mathbb{R})$. Since P_{η_0,η_1} is a continuous Fourier multiplier on $\mathcal{S}(\mathbb{R})$,

$$P_{\eta_0,\eta_1}(\chi_L \varphi_M) \longrightarrow P_{\eta_0,\eta_1}\varphi_M = \varphi_M$$

in the same topology. These are exactly the two families of seminorms defining $\mathcal{B}^{\mathrm{pfw}}_{\eta_0,\eta_1}$.

For the second convergence, $\chi_M\varphi \to \varphi$ in the Schwartz topology, and applying the same continuous projection gives

$$P_{\eta_0,\eta_1}(\chi_M \varphi) \to P_{\eta_0,\eta_1}\varphi = \varphi.$$

The projected seminorms converge by applying P_{η_0,η_1} once more. Thus convergence holds in the projected finite-window topology. ☐

Lemma 6.17 (Fourier-projected finite-window exhaustion of the open band)**.** Let

$$0 < \eta < \eta_0 < \eta_1 < \log 2.$$

Choose

$$\vartheta_{\eta_0,\eta_1} \in C_c^\infty((-\eta_1, \eta_1))$$

such that

$$\vartheta_{\eta_0,\eta_1}(u) = 1 \qquad (|u| \le \eta_0).$$

Define the Fourier projection

$$P_{\eta_0,\eta_1} f := \mathcal{F}^{-1}\big(\vartheta_{\eta_0,\eta_1} \widehat{f}\big).$$

Let χ_M be the finite-window cutoff in the logarithmic variable. For every

$$\varphi \in \mathcal{A}_\eta$$

set

$$\varphi_M := P_{\eta_0,\eta_1}(\chi_M \varphi).$$

Then

$$\varphi_M \in \mathcal{A}_{\eta_1}, \qquad \varphi_M \longrightarrow \varphi \quad \text{in the Fréchet topology of } \mathcal{A}_{\eta_1}.$$

Moreover, for each fixed M,

$$\chi_L \varphi_M \in \mathcal{T}_{\log,\mathrm{fw}} \qquad (L \ge 1),$$

and

$$\chi_L \varphi_M \longrightarrow \varphi_M \qquad \text{in } \mathcal{B}^{\mathrm{pfw}}_{\eta_0,\eta_1}.$$

Consequently, by Lemma 6.15, for

$$\mu_\bullet \in \{\mu_L, \mu_\xi\},$$

one has

$$\langle \mu_\bullet, \varphi \rangle = \lim_{M\to\infty} \lim_{L\to\infty} \langle \mu_\bullet, [\chi_L \varphi_M]_{\mathrm{fw}} \rangle.$$

Thus the passage from finite-window quotient inputs to the open-band class is made through Fourier projection and not through the false assertion that compactly supported logarithmic tests are themselves strictly band-limited.

Proof. Since $\varphi \in \mathcal{A}_\eta$, its Fourier transform is supported in $(-\eta, \eta) \subset (-\eta_0, \eta_0)$. Hence

$$P_{\eta_0,\eta_1} \varphi = \varphi.$$

As $M \to \infty$, $\chi_M \varphi \to \varphi$ in the Schwartz topology. The multiplier P_{η_0,η_1} is continuous on $\mathcal{S}(\mathbb{R})$, and its image has Fourier support in $(-\eta_1, \eta_1)$. Therefore

$$\varphi_M = P_{\eta_0,\eta_1}(\chi_M \varphi) \longrightarrow \varphi$$

in the Fréchet topology of $\mathcal{A}_{\eta_1}$.

For fixed M, φ_M need not be compactly supported. Multiplying by χ_L gives a genuine finite-window logarithmic representative, but this operation generally destroys strict band limitation. We therefore do not regard $\chi_L \varphi_M$ as an element converging in $\mathcal{A}_{\eta_1}$. Instead, the cutoff convergence is taken in $\mathcal{B}^{\mathrm{pfw}}_{\eta_0,\eta_1}$: the ordinary Schwartz seminorms converge because $\chi_L \to 1$ with all derivatives on the fixed Schwartz function φ_M, and the projected seminorms converge because P_{η_0,η_1} is continuous on $\mathcal{S}(\mathbb{R})$. Lemma 6.15 then gives the displayed iterated limit. $\square$

Remark 6.18 (open-band convention)**.** Throughout this section, assume

$$0 < \eta < \log 2.$$

The boundary band

$$\eta = \log 2$$

is treated separately from the open-band distributional equality, because the Fourier support touches the first prime-power position. The stability of the boundary band is deferred to the endpoint stability theorem of Section 6.2.

Definition 6.19 (completed von Mangoldt distribution on the logarithmic side)**.** Let Λ^{ex} be the exact von Mangoldt lift fixed in Section 3. The corresponding arithmetic distribution on the logarithmic side is denoted by

$$\mu_\Lambda^{\mathrm{ex}} \in (C_c^\infty(\mathbb{R}))'.$$

Namely, for $\psi \in C_c^\infty(\mathbb{R})$, define

$$\langle \mu_\Lambda^{\mathrm{ex}}, \psi \rangle := \sum_{p^k} \Lambda^{\mathrm{ex}}(p^k)\psi(\log p^k).$$

This sum is finite because $\operatorname{supp} \psi$ is compact. Hence $\mu_\Lambda^{\mathrm{ex}}$ is a distribution on local logarithmic tests. In this paper, this arithmetic singular part is not used as a tempered distribution on the whole of $\mathcal{S}(\mathbb{R})$. The open-band equality is defined on $\mathcal{A}_\eta$, and the central logarithmic transform is defined separately on $\mathcal{C}_{\mathrm{cen}}$.

Lemma 6.20 (coefficient support of the total zeta ledger and von Mangoldt lift)**.** Let

$$\mu_\xi^{\mathrm{tot}}$$

denote the total finite-window completed-zeta ledger before the common Archimedean/reference term and the arithmetic term are subtracted. In the notation of Definition 6.9,

$$\langle \mu_\xi^{\mathrm{tot}}, \varphi \rangle := \mathcal{E}_\xi^{\mathrm{fw}}(\varphi).$$

Then

$$\mu_\xi^{\mathrm{tot}} = \mu_{\xi,\mathrm{sm}} + \mu_\Lambda^{\mathrm{ex}} + \mu_\xi.$$

Here $\mu_{\xi,\mathrm{sm}}$ is the smooth distribution representing the Archimedean term and the calibrated reference term, $\mu_\Lambda^{\mathrm{ex}}$ is the arithmetic distribution of Definition 6.19, and μ_ξ is the common-term-removed zeta-side residual of Definition 6.9.

The arithmetic singular support of the total ledger is therefore contained in

$$\{\log p^k : \ p \text{ prime}, \ k \geq 1\},$$

and the principal delta coefficient of the arithmetic part at $\log p^k$ is

$$\Lambda^{\mathrm{ex}}(p^k).$$

Equivalently, if $\chi \in C_c^\infty(\mathbb{R})$ has support in a sufficiently small neighborhood of $\log p^k$ and contains no other prime-power positions, then

$$\langle \mu_\xi^{\mathrm{tot}}, \chi \rangle - \langle \mu_{\xi,\mathrm{sm}}, \chi \rangle - \langle \mu_\xi, \chi \rangle = \Lambda^{\mathrm{ex}}(p^k)\chi(\log p^k).$$

Proof. The total finite-window ledger is, by definition,

$$\mathcal{E}_{\xi}^{\mathrm{fw}} = B_{\infty}^{\mathrm{fw}} + A_{\mathrm{arith}}^{\mathrm{fw}} + \mu_{\xi}.$$

The first term is represented on the logarithmic side by the smooth distribution $\mu_{\xi,\mathrm{sm}}$. The arithmetic term is represented by the exact von Mangoldt distribution

$$\mu_{\Lambda}^{\mathrm{ex}},$$

whose coefficient at the prime-power position $\log p^k$ is, by Definition 6.19, $\Lambda^{\mathrm{ex}}(p^k)$. This gives

$$\mu_{\xi}^{\mathrm{tot}} = \mu_{\xi,\mathrm{sm}} + \mu_{\Lambda}^{\mathrm{ex}} + \mu_{\xi}.$$

Since the smooth term has no singular support and the residual μ_{ξ} is used after the common arithmetic term has been removed, the arithmetic singular support of the total ledger is precisely the support carried by $\mu_{\Lambda}^{\mathrm{ex}}$. Testing with a bump function χ isolating a single prime-power position removes all other delta components and yields the displayed identity. □

Theorem 6.21 (local arithmetic coefficient identification)**.** *The arithmetic singular part of the total completed-zeta finite-window ledger*

$$\mu_{\xi}^{\mathrm{tot}}$$

is given at prime-power positions by the exact von Mangoldt lift. Namely, the arithmetic singular support is contained in

$$\{\log p^k : \ p \text{ prime}, \ k \geq 1\},$$

and the principal delta coefficient at each $\log p^k$ *is*

$$\Lambda^{\mathrm{ex}}(p^k).$$

Proof. This is a theorem-level summary of Lemma 6.20. The assertion concerns the total zeta ledger before common-term subtraction. The residual functional μ_{ξ} used in the comparison is obtained only after the smooth Archimedean/reference term and the arithmetic von Mangoldt term have been removed. □

Lemma 6.22 (continuous realization of the zeta-side residual functional on the comparison tests)**.** The zeta-side residual functional of Definition 6.9,

$$\tau = [\psi]_{\mathrm{fw}} \longmapsto \langle \mu_{\xi}, \tau \rangle = \mathcal{E}_{\xi}^{\mathrm{fw}}(\tau) - B_{\infty}^{\mathrm{fw}}(\tau) - A_{\mathrm{arith}}^{\mathrm{fw}}(\tau) \qquad (\tau \in \mathscr{T}_{\mathrm{fw}}),$$

is realized, through its concrete representative ψ, as a continuous linear functional on the following two types of test classes:

$$C_c^{\infty}(\mathbb{R}), \qquad \mathcal{A}_{\eta} \ (0 < \eta < \log 2).$$

In particular, μ_{ξ} is well-defined on the test spaces required for the residual comparison.

Proof. On $C_c^{\infty}(\mathbb{R})$, the total completed-zeta ledger μ_{ξ}^{tot} acts as the ordinary local explicit-formula distribution. The smooth Archimedean/reference part is continuous on compact supports, and the arithmetic part $\mu_{\Lambda}^{\mathrm{ex}}$ is a finite sum on compactly supported tests. Therefore their difference,

$$\mu_{\xi} = \mu_{\xi}^{\mathrm{tot}} - \mu_{\xi,\mathrm{sm}} - \mu_{\Lambda}^{\mathrm{ex}},$$

is continuous on $C_c^\infty(\mathbb{R})$.

On the open-band class $\mathcal{A}_\eta$, the total finite-window explicit-formula functional, the common Archimedean/reference term, and the arithmetic trace term are all evaluated by continuous extension from the open-band coordinates of finite-window quotient classes. The compact Fourier support in a fixed open band controls these three components by finitely many seminorms of $\mathcal{A}_\eta$. Thus their difference μ_ξ is also continuous on $\mathcal{A}_\eta$. ☐

Lemma 6.23 (continuous realization of the residual-free singular-boundary functional on the comparison tests)**.** The residual-free singular-boundary functional of the previous section,

$$\tau = [\psi]_{\mathrm{fw}} \longmapsto \langle \mu_L, \tau \rangle = \mathcal{L}_R^{\mathrm{fw}}(\tau) \qquad (\tau \in \mathscr{T}_{\mathrm{fw}}),$$

is realized, through its concrete representative ψ, as a continuous linear functional on the following two types of test classes:

$$C_c^\infty(\mathbb{R}), \qquad \mathcal{A}_\eta \ (0 < \eta < \log 2).$$

More precisely, for each fixed open band $0 < \eta < \log 2$, there are finitely many defining seminorms $q_{\eta,\ell}$ on $\mathcal{A}_\eta$ and a constant $C_\eta > 0$ such that

$$|\langle \mu_L, \varphi \rangle| \le C_\eta \sum_{\ell=1}^{M_\eta} q_{\eta,\ell}(\varphi) \qquad (\varphi \in \mathcal{A}_\eta).$$

An analogous finite seminorm estimate holds on $C_c^\infty(\mathbb{R})$ on each fixed compact support set.

Proof. For a finite-window test φ, let

$$T_{\partial,R}^{\mathrm{fw}}(\varphi) = \mathcal{N}_\xi\big(\mathfrak{b}_{\xi,\partial}^{\mathrm{fw}}(\varphi)\big)$$

be its realized boundary finite-part distribution. By Theorem 6.4,

$$\| \mathrm{LCI}_R \, T_{\partial,R}^{\mathrm{fw}}(\varphi) \|_X \le C_{\mathrm{LCI}} \sum_{\nu=1}^{N_{\mathrm{LCI}}} \mathfrak{p}_{R,\nu}\big(T_{\partial,R}^{\mathrm{fw}}(\varphi)\big).$$

The natural realization map $\mathcal{N}_\xi$ of Section 4.10 is continuous with respect to the finite-window centered Mellin boundary seminorms. Hence, for finitely many seminorms $\mathfrak{q}_{\xi,\mu}$ on the centered Mellin boundary finite-part space,

$$\mathfrak{p}_{R,\nu}\Big(T_{\partial,R}^{\mathrm{fw}}(\varphi)\Big) \le C_\nu \sum_{\mu=1}^{M_\nu} \mathfrak{q}_{\xi,\mu}\Big(\mathfrak{b}_{\xi,\partial}^{\mathrm{fw}}(\varphi)\Big).$$

For $\varphi \in \mathcal{A}_\eta$, the compact Fourier support in the fixed band and the finite-window contour construction control the latter seminorms by finitely many defining seminorms $q_{\eta,\ell}$ of $\mathcal{A}_\eta$:

$$\mathfrak{q}_{\xi,\mu}\Big(\mathfrak{b}_{\xi,\partial}^{\mathrm{fw}}(\varphi)\Big) \le C_{\mu,\eta} \sum_{\ell=1}^{M_\eta} q_{\eta,\ell}(\varphi).$$

Combining the preceding estimates gives

$$\|\Pi_R x_\varphi^\sharp\|_X \le C_\eta \sum_{\ell=1}^{M_\eta} q_{\eta,\ell}(\varphi).$$

The singular-boundary readout defining $\mathcal{L}_R$ is a bounded readout on the $\mathcal{K}_R$-component, so

$$|\langle \mu_L, \varphi\rangle| = |\mathcal{L}_R(\varphi)| \le C\|\Pi_R x_\varphi^\sharp\|_X \le C_\eta \sum_{\ell=1}^{M_\eta} q_{\eta,\ell}(\varphi).$$

This proves continuity on $\mathcal{A}_\eta$. The proof on $C_c^\infty(\mathbb{R})$ is identical, with the usual compact-support seminorms replacing the open-band seminorms. The central logarithmic test family is handled later in Section 6.4 after the central comparison topology has been defined. ☐

Lemma 6.24 (continuity of the open-band restriction map)**.** μ_L and μ_ξ are restricted as continuous linear functionals on

$$\mathcal{A}_\eta.$$

More generally, for any comparison functional T realized continuously on $\mathcal{A}_\eta$,

$$T|_{\mathcal{A}_\eta} : \mathcal{A}_\eta \to \mathbb{C}$$

is continuous.

Proof. The Fréchet topology of Definition 6.12 is placed on $\mathcal{A}_\eta$. Lemma 6.22 and Lemma 6.23 state that μ_ξ and μ_L act continuously in this topology. The assertion follows. ☐

Lemma 6.25 (open-band residual-free comparison)**.** Let $0 < \eta < \log 2$. For every $\varphi \in \mathcal{A}_\eta$, the canonical residual-free comparison data satisfy

$$\langle \mu_L, \varphi\rangle - \langle \mu_\xi, \varphi\rangle = 0.$$

Proof. First let $\psi \in \mathcal{T}_{\mathrm{log,fw}}$ be a genuine finite-window logarithmic representative and put

$$\tau_\psi = [\psi]_{\mathrm{fw}} \in \mathscr{T}_{\mathrm{fw}}.$$

By Proposition 5.24,

$$\mathcal{E}_\xi^{\mathrm{fw}}(\tau_\psi) = B_\infty^{\mathrm{fw}}(\tau_\psi) + A_{\mathrm{arith}}^{\mathrm{fw}}(\tau_\psi) + \mathcal{L}_R^{\mathrm{fw}}(\tau_\psi).$$

Definition 6.9 gives

$$\langle \mu_L, \tau_\psi\rangle = \mathcal{L}_R^{\mathrm{fw}}(\tau_\psi)$$

and

$$\langle \mu_\xi, \tau_\psi\rangle = \mathcal{E}_\xi^{\mathrm{fw}}(\tau_\psi) - B_\infty^{\mathrm{fw}}(\tau_\psi) - A_{\mathrm{arith}}^{\mathrm{fw}}(\tau_\psi).$$

Substitution yields

$$\langle \mu_L, \tau_\psi\rangle = \langle \mu_\xi, \tau_\psi\rangle.$$

Thus the equality on finite-window quotient inputs is a direct consequence of Proposition 5.24, not a definition of μ_L as the residual part of μ_ξ.

Now take an arbitrary

$$\varphi \in \mathcal{A}_\eta.$$

Choose

$$0 < \eta < \eta_0 < \eta_1 < \log 2$$

and construct

$$\varphi_M = P_{\eta_0,\eta_1}(\chi_M \varphi)$$

as in Lemma 6.17. For each fixed M and L,

$$[\chi_L \varphi_M]_{\mathrm{fw}} \in \mathscr{T}_{\mathrm{fw}},$$

so the finite-window equality just proved gives

$$\langle \mu_L, [\chi_L \varphi_M]_{\mathrm{fw}} \rangle = \langle \mu_\xi, [\chi_L \varphi_M]_{\mathrm{fw}} \rangle.$$

Letting $L \to \infty$ and using the continuity of the comparison functionals gives

$$\langle \mu_L, \varphi_M \rangle = \langle \mu_\xi, \varphi_M \rangle.$$

Finally,

$$\varphi_M \to \varphi \qquad \text{in } \mathcal{A}_{\eta_1}$$

by Lemma 6.17, and Lemma 6.22, Lemma 6.23, and Lemma 6.24 give continuity on $\mathcal{A}_{\eta_1}$. Hence

$$\langle \mu_L, \varphi \rangle = \lim_{M\to\infty} \langle \mu_L, \varphi_M \rangle = \lim_{M\to\infty} \langle \mu_\xi, \varphi_M \rangle = \langle \mu_\xi, \varphi \rangle.$$

The compact-support step and the band-limited step are therefore separated by the Fourier projection P_{η_0,η_1}. ☐

Theorem 6.26 (distributional comparison theorem on the open band)**.** *Let* $0 < \eta < \log 2$*. Then* μ_L *and* μ_ξ *are realized as continuous linear functionals on*

$$\mathcal{A}_\eta,$$

and for every

$$\varphi \in \mathcal{A}_\eta$$

one has

$$\langle \mu_L, \varphi \rangle = \langle \mu_\xi, \varphi \rangle.$$

Proof. The continuous realizations of μ_ξ and μ_L on $\mathcal{A}_\eta$ were shown in Lemma 6.22 and Lemma 6.23. Their restrictions agree by Lemma 6.25. ☐

Theorem 6.27 (open-band residual-free equality)**.** *For* $0 < \eta < \log 2$*,*

$$\langle \mu_L, \varphi \rangle = \langle \mu_\xi, \varphi \rangle \qquad (\varphi \in \mathcal{A}_\eta)$$

holds.

Proof. This is the equality part of Theorem 6.26. The assertion here is the residual-free equality on open-band test functions whose Fourier support is contained in $(-\eta, \eta)$, and is used separately from the local coefficient identification of Theorem 6.21. ☐

Remark 6.28 (boundary band is not used in the open-band theorem)**.** Theorem 6.26 is the open-band statement for

$$0 < \eta < \log 2.$$

At the boundary value

$$\eta = \log 2,$$

the Fourier support may touch the first prime-power position, and therefore the proof of this section is not applied as it stands. The boundary band and finite-window endpoint stability are treated independently in Section 6.2.

6.2. Endpoint Stability Theorem

In this section, when connecting the open-band distributional equality obtained in Section 6.1 to the argument principle for finite windows, it is shown that band cutoff and endpoint regularization do not change the integer-valued zero count. What is needed here is not merely an estimate saying that the endpoint error is small, but stability saying that the integer value obtained by the argument principle is invariant. Thus this section is restricted to finite windows whose boundary does not pass through zeros, and endpoint contributions are handled by homotopy invariance.

Definition 6.29 (admissible finite window)**.** Let G be a holomorphic function on $\Omega \subset \mathbb{C}$. Let $0 < T_0 < T$, and let $R_\eta(T_0, T) \Subset \Omega$ be a bounded closed rectangle. Its positively oriented boundary is denoted by

$$\partial R_\eta(T_0, T).$$

This finite window $R_\eta(T_0, T)$ is said to be G-admissible if

$$G(s) \neq 0 \qquad (s \in \partial R_\eta(T_0, T))$$

holds. Then, by compactness,

$$m_R(G) := \inf_{s \in \partial R_\eta(T_0,T)} |G(s)| > 0.$$

In what follows, when $R_\eta(T_0, T)$ is clear, it is simply denoted by R.

Definition 6.30 (argument-principle count)**.** Let R be a G-admissible finite window. Define

$$N_{\mathrm{arg}}(G; R) := \frac{1}{2\pi i} \int_{\partial R} \frac{G'(s)}{G(s)}\, ds.$$

Since G is holomorphic in a neighborhood of R, the argument principle gives

$$N_{\mathrm{arg}}(G; R) = \sum_{\rho \in R^\circ,\, G(\rho)=0} m_G(\rho) \in \mathbb{Z}_{\geq 0}.$$

In particular, when $G = \xi$, write

$$N_{\mathrm{arg}}(R) := N_{\mathrm{arg}}(\xi; R).$$

Definition 6.31 (admissible analytic cutoff near a window)**.** Let $R = R_\eta(T_0, T)$ be a G-admissible finite window. Let $\{G_\Lambda\}_{\Lambda \geq 1}$ be a family of holomorphic functions on an open neighborhood $U_R \Subset \Omega$ of R. This family is said to be an admissible analytic cutoff with respect to R if, for some integer $m \geq 3$,

$$G_\Lambda \longrightarrow G \qquad \text{in } H^m(U_R) \quad (\Lambda \to \infty)$$

holds.

Then define the argument count after cutoff by G_Λ as

$$N_{\text{arg}}^{\text{cutoff}}(G_\Lambda; R) := \frac{1}{2\pi i} \int_{\partial R} \frac{G'_\Lambda(s)}{G_\Lambda(s)}\, ds,$$

provided that the right-hand side is defined, namely that G_Λ has no zero on ∂R.

Definition 6.32 (endpoint cutoff term)**.** Let R be a G-admissible finite window, and let $\{G_\Lambda\}_{\Lambda \geq 1}$ be an admissible analytic cutoff with respect to R. When G_Λ has no zero on ∂R, define the endpoint cutoff term by

$$E_{\Lambda,\partial R}(G) := \frac{1}{2\pi i} \int_{\partial R} \left(\frac{G'_\Lambda(s)}{G_\Lambda(s)} - \frac{G'(s)}{G(s)} \right) ds.$$

Namely,

$$E_{\Lambda,\partial R}(G) = N_{\text{arg}}^{\text{cutoff}}(G_\Lambda; R) - N_{\text{arg}}(G; R).$$

Lemma 6.33 (Sobolev trace control on the boundary)**.** Let U_R be a bounded Lipschitz neighborhood of R, and let $m \geq 3$. If

$$G_\Lambda \to G \qquad \text{in } H^m(U_R),$$

then

$$G_\Lambda|_{\partial R} \to G|_{\partial R} \quad \text{in } C^1(\partial R).$$

In particular,

$$\sup_{s \in \partial R} |G_\Lambda(s) - G(s)| \longrightarrow 0.$$

Proof. By the Sobolev trace theorem for Lipschitz boundaries, the restriction map

$$H^m(U_R) \longrightarrow H^{m-\frac{1}{2}}(\partial R)$$

is continuous. Here ∂R is a one-dimensional piecewise smooth compact curve, and since $m \geq 3$,

$$m - \frac{1}{2} > \frac{3}{2}.$$

Therefore, by the one-dimensional Sobolev embedding,

$$H^{m-\frac{1}{2}}(\partial R) \hookrightarrow C^1(\partial R)$$

is continuous. Hence

$$G_\Lambda \to G \quad \text{in } H^m(U_R)$$

implies

$$G_\Lambda|_{\partial R} \to G|_{\partial R} \quad \text{in } C^1(\partial R).$$

□

Lemma 6.34 (boundary non-vanishing stability)**.** Let R be a G-admissible finite window, and let $\{G_\Lambda\}$ be an admissible analytic cutoff with respect to R. Then, for all sufficiently large Λ,

$$G_\Lambda(s) \neq 0 \qquad (s \in \partial R)$$

holds. Moreover, for every $u \in [0, 1]$,

$$G_{\Lambda,u}(s) := (1-u)G(s) + uG_\Lambda(s)$$

satisfies

$$G_{\Lambda,u}(s) \neq 0 \qquad (s \in \partial R).$$

Proof. By Definition 6.29,

$$m_R(G) = \inf_{s \in \partial R} |G(s)| > 0.$$

By Lemma 6.33,

$$\sup_{s \in \partial R} |G_\Lambda(s) - G(s)| \longrightarrow 0.$$

Therefore, for sufficiently large Λ,

$$\sup_{s \in \partial R} |G_\Lambda(s) - G(s)| < \frac{1}{2} m_R(G)$$

holds.

For such Λ, for $s \in \partial R$ and $u \in [0, 1]$,

$$|G_{\Lambda,u}(s)| = |G(s) + u(G_\Lambda(s) - G(s))| \geq |G(s)| - |G_\Lambda(s) - G(s)| > \frac{1}{2} m_R(G) > 0.$$

The assertion follows. □

Lemma 6.35 (homotopy invariance of the argument count)**.** Let R be a G-admissible finite window, and let $\{G_\Lambda\}$ be an admissible analytic cutoff with respect to R. For sufficiently large Λ,

$$N_{\mathrm{arg}}^{\mathrm{cutoff}}(G_\Lambda; R) = N_{\mathrm{arg}}(G; R)$$

holds. Equivalently,

$$E_{\Lambda,\partial R}(G) = 0.$$

Proof. By Lemma 6.34, for sufficiently large Λ,

$$G_{\Lambda,u} = (1-u)G + uG_\Lambda \qquad (0 \leq u \leq 1)$$

has no zero on ∂R for every u. Therefore

$$u \longmapsto \frac{1}{2\pi i} \int_{\partial R} \frac{\partial_s G_{\Lambda,u}(s)}{G_{\Lambda,u}(s)}\, ds$$

is continuous with respect to $u \in [0,1]$. On the other hand, for each u, $G_{\Lambda,u}$ is holomorphic in a neighborhood of R, and has no zero on ∂R, so by the argument principle this quantity is an integer. A continuous integer-valued function is constant on an interval. Therefore the values at $u = 0$ and $u = 1$ are equal, and

$$N_{\arg}(G; R) = N_{\arg}^{\text{cutoff}}(G_\Lambda; R).$$

By Definition 6.32,

$$E_{\Lambda,\partial R}(G) = 0$$

also follows. □

Theorem 6.36 (endpoint stability theorem)**.** *Let $R = R_\eta(T_0, T)$ be a G-admissible finite window, and let $\{G_\Lambda\}_{\Lambda \geq 1}$ be an admissible analytic cutoff with respect to R. Then, for sufficiently large Λ,*

$$N_{\arg}^{\text{cutoff}}(G_\Lambda; R) = N_{\arg}(G; R).$$

In particular, when $G = \xi$,

$$N_{\arg}^{\text{cutoff}}(R_\eta(T_0, T)) = N_{\arg}(R_\eta(T_0, T)).$$

Proof. Applying Lemma 6.35 to G and $\{G_\Lambda\}$ gives, for sufficiently large Λ,

$$N_{\arg}^{\text{cutoff}}(G_\Lambda; R) = N_{\arg}(G; R).$$

In the case $G = \xi$, using the notation

$$N_{\arg}(R) = N_{\arg}(\xi; R)$$

gives the same conclusion. □

Corollary 6.37 (integer stability of endpoint cutoff term)**.** *Under the assumptions of Theorem 6.36, for sufficiently large Λ,*

$$E_{\Lambda,\partial R}(G) = 0.$$

Therefore endpoint cutoff term does not change the zero count given by the argument principle inside the finite window.

Proof. By Definition 6.32,

$$E_{\Lambda,\partial R}(G) = N_{\arg}^{\text{cutoff}}(G_\Lambda; R) - N_{\arg}(G; R).$$

By Theorem 6.36, the right-hand side is zero for sufficiently large Λ. □

Remark 6.38 (no zero-counting conclusion in this section)**.** The conclusion of this section is the stability that band cutoff and endpoint regularization do not change the integer-valued count of the argument principle in an admissible finite window. This section does not assert either that the zeros lie on the critical line or that no off-line zero exists. These counting consequences are treated in the subsequent arguments on the finite-window bridge and the defect staircase.

6.3. Trace-Ideal Determinant Theorem

In this section, using the compact-resolvent construction and the distribution-kernel representation of Section 4, we realize the boundary-distribution comparison kernel candidate K introduced in Section 6.0 as a Hilbert–Schmidt operator. After that, we introduce the finite-rank cutoff

$$K_N = P_N K P_N$$

and prove the local uniform convergence and coefficient transport

$$\det_2(I + zK_N) \longrightarrow \det_2(I + zK).$$

The conclusion of this section is the construction of the comparison function $F_K(s)$ and the stability of its Taylor coefficients, and here we do not yet assert

$$F_K(s) = \xi(s).$$

Definition 6.39 (reference spectral resolution)**.** For the trace-ideal estimates in this subsection, write

$$A_R := A_R^{\mathrm{ref}},$$

where

$$A_R^{\mathrm{ref}} = \mathcal{W}_R^{-1}(I - \Delta_{\Sigma,N})\mathcal{W}_R$$

is the Sobolev seam-chart regularizer constructed in Section 4.8 from

$$\mathcal{W}_R = \mathcal{E}_{\Sigma,R}\mathcal{E}_\alpha^{-1}U_\alpha.$$

Thus A_R is not the singular-boundary transport operator L_R^{res}; it is the positive compact-resolvent regularizer used only for trace smoothing and Schatten estimates.

The transported Sobolev scale is

$$H_R^s = \mathcal{W}_R^{-1}H_N^s(\Sigma_R),$$

and

$$\mathcal{D}_R^\infty = \bigcap_{q\geq 0} \mathrm{Dom}(I + A_R)^{q/2} = \mathcal{W}_R^{-1}C_N^\infty(\Sigma_R).$$

The effective dimension, elliptic order, and Weyl exponent are fixed concretely by

$$\boxed{d_{\mathrm{eff}} = 2, \qquad m_R = 2, \qquad \nu_R := \frac{m_R}{d_{\mathrm{eff}}} = 1.}$$

The Hilbert–Schmidt threshold is

$$a_{\mathrm{HS}} := \frac{d_{\mathrm{eff}}}{2m_R} = \frac{1}{2}.$$

The net trace-decay exponent and the actual comparison-trace smoothing exponent fixed in Section 4.8 are

$$\boxed{a_{\mathrm{tr}} > a_{\mathrm{HS}}, \qquad a_{\mathrm{cmp}} := a_{\mathrm{tr}} + \frac{s_R}{m_R}.}$$

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In the trace-ideal estimates below,

$$\mathrm{Tr}^{\mathrm{cmp}}_{\partial,R} = \mathrm{Tr}^{\mathrm{raw}}_{\partial,R}(I + A_R)^{-a_{\mathrm{cmp}}} = \mathrm{Tr}^{[s_R]}_{\partial,R}(I + A_R)^{-a_{\mathrm{tr}}}$$

is the trace used to define the comparison map. The raw restriction $\mathrm{Tr}^{\mathrm{raw}}_{\partial,R}$ is used only to define this smoothed comparison trace.

Choose an orthonormal basis

$$\{e_n\}_{n\geq 1}$$

of $H_{\alpha,+}$ consisting of eigenfunctions of A_R,

$$A_R e_n = \lambda_n e_n, \qquad 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots .$$

The eigenvalues satisfy

$$\lambda_n \geq c_{\mathrm{W}} n^{\nu_R} \qquad (n \geq n_{\mathrm{W}})$$

for constants $c_{\mathrm{W}} > 0$ and $n_{\mathrm{W}} \geq 1$. Moreover,

$$\mathrm{span}\{e_n : n \geq 1\} \subset \mathcal{D}_R^{\infty} \subset \mathcal{D}_R,$$

so the matrix coefficients $\mathfrak{t}_R(e_m, e_n)$ are defined on the original boundary-test domain. For each $N \geq 1$, define

$$P_N f := \sum_{n=1}^{N} \langle f, e_n \rangle_{H_{\alpha,+}} e_n.$$

Then

$$P_N \to I \qquad \text{strongly on } H_{\alpha,+}.$$

Definition 6.40 (functional-equation boundary reflection)**.** Let

$$\Theta_\xi : \mathfrak{B}^{\mathrm{fw}}_{\xi,\partial} \longrightarrow \mathfrak{B}^{\mathrm{fw}}_{\xi,\partial}$$

be the centered Mellin boundary reflection induced by

$$u \longmapsto -u, \qquad w = s - \frac{1}{2}.$$

On the smooth seam carrier $\Gamma_R = [0,1] \times \{0\}$, define the geometric seam reflection by

$$\theta_R(x, 0) := (1 - x, 0).$$

This is equivalently

$$\theta_R = \kappa \circ \Theta_\xi \circ \kappa^{-1}, \qquad \theta_R \circ \kappa = \kappa \circ \Theta_\xi.$$

Let

$$\Theta_R : \mathcal{D}'_R \longrightarrow \mathcal{D}'_R$$

be the distributional pushforward induced by θ_R, extended by continuity in the boundary-distribution topology fixed in Section 4. Then, on realized finite-window boundary distributions,

$$\Theta_R \mathcal{N}_\xi = \mathcal{N}_\xi \Theta_\xi$$

holds as a consequence of the preceding commutative identity and of $\mathcal{N}_\xi = \mathcal{C}'_{\xi R}$. Thus Θ_R is the smooth-seam realization of the functional equation

$$\xi\left(\frac{1}{2}+w\right) = \xi\left(\frac{1}{2}-w\right).$$

It is defined from the seam reflection and the transpose realization map, not from the location of the zeros of ξ.

Lemma 6.41 (basic properties of the boundary reflection)**.** The boundary reflection Θ_R is a bounded involution on $\mathcal{D}'_R$. Moreover,

$$\Theta_R^2 = I_{\mathcal{D}'_R},$$

and it preserves the boundary pairing used in the construction of the singular-boundary component. Equivalently, for admissible boundary distributions u, v for which the boundary pairing is defined,

$$\langle \Theta_R u, \Theta_R v\rangle_{\partial,R} = \langle u, v\rangle_{\partial,R}.$$

Proof. The seam reflection satisfies

$$\theta_R^2 = I_{\Gamma_R},$$

because $(x,0) \mapsto (1-x,0) \mapsto (x,0)$. Taking distributional pushforwards gives

$$\Theta_R^2 = I$$

on finite-window seam distributions, and continuity extends the identity to $\mathcal{D}'_R$.

The equality

$$\theta_R \circ \kappa = \kappa \circ \Theta_\xi$$

implies

$$\Theta_R \mathcal{N}_\xi = \mathcal{N}_\xi \Theta_\xi$$

because $\mathcal{N}_\xi = \mathcal{C}'_{\xi R}$. The centered Mellin boundary pairing is invariant under $u \mapsto -u$, and $\nu_R = \mathcal{L}^1_{[0,1]} \otimes \delta_0$ is invariant under $x \mapsto 1-x$. Therefore, for realized finite-window boundary distributions,

$$\left\langle \Theta_R \mathcal{N}_\xi b, \Theta_R \mathcal{N}_\xi c\right\rangle_{\partial,R} = \left\langle \mathcal{N}_\xi b, \mathcal{N}_\xi c\right\rangle_{\partial,R}.$$

Density of the realized finite-window boundary distributions and continuity of the boundary trace topology extend the identity to all admissible boundary distributions. Boundedness follows from the same seminorm-preserving estimate on the dense seam-distribution class and the continuous extension. ☐

Lemma 6.42 (topological realization of the boundary reflection)**.** Let

$$\mathcal{E}_{R,\mathrm{fw}} \subset \mathcal{D}'_R$$

be the finite-window boundary-distribution subspace used to construct the admissible boundary-distribution closure. Then $\mathcal{E}_{R,\mathrm{fw}}$ is dense in the boundary-distribution topology of $\mathcal{D}'_R$, and the reflection Θ_R is bounded with respect to the defining seminorms of that topology. In particular, if $u_n \to u$ in $\mathcal{D}'_R$, then

$$\Theta_R u_n \to \Theta_R u \qquad \text{in } \mathcal{D}'_R,$$

and the boundary pairing identities verified on $\mathcal{E}_{R,\mathrm{fw}}$ extend uniquely to all admissible limits.

Proof. In Section 4, $\mathcal{D}'_R$ is obtained as the distributional completion generated by finite-window boundary distributions subject to the boundary trace estimates and support constraints. Thus $\mathcal{E}_{R,\mathrm{fw}}$ is dense by definition of this completion. On the seam, Θ_R is the smooth map $x \mapsto 1 - x$, so derivatives and finite-order trace seminorms are transformed by the chain rule with uniformly bounded coefficients on each finite window. Consequently, for every defining seminorm q of the boundary-distribution topology there are a defining seminorm q' and a constant $C_q > 0$ such that

$$q(\Theta_R u) \leq C_q q'(u) \qquad (u \in \mathcal{E}_{R,\mathrm{fw}}).$$

The estimate extends by density and gives a bounded operator on $\mathcal{D}'_R$. The boundary pairing is continuous with respect to these seminorms, so the pairing preservation established on finite-window boundary distributions extends to admissible limits. □

Lemma 6.43 (compatibility with admissible boundary data and residual-free comparison)**.** The boundary reflection preserves the admissible boundary-distribution subspace:

$$\Theta_R \mathcal{D}'_{R,\mathrm{adm}} = \mathcal{D}'_{R,\mathrm{adm}}.$$

Moreover, it is compatible with the residual-free comparison interface in the following sense. If $y_1, y_2 \in \mathcal{D}'_{R,\mathrm{adm}}$ have the same $\mathcal{K}_R$-projected comparison component,

$$\Pi_R \,\mathrm{LCI}_R(y_1) = \Pi_R \,\mathrm{LCI}_R(y_2),$$

then

$$\Pi_R \,\mathrm{LCI}_R(\Theta_R y_1) = \Pi_R \,\mathrm{LCI}_R(\Theta_R y_2).$$

Consequently, the rule

$$\Pi_R \,\mathrm{LCI}_R(y) \longmapsto \Pi_R \,\mathrm{LCI}_R(\Theta_R y)$$

is well-defined on the $\mathcal{K}_R$-projected comparison range.

Proof. On the dense subspace of boundary distributions obtained from finite-window inputs, $\mathcal{E}_{R,\mathrm{fw}} \subset \mathcal{D}'_R$, Θ_R is the smooth seam reflection $(x, 0) \mapsto (1 - x, 0)$ transported from the centered Mellin reflection by κ. The finite-window comparison identity is invariant under this reflection, and the canonical residual-free representative of Section 5 is defined modulo $\mathrm{Ran}\,\Pi_{\mathrm{res}}$, which is orthogonal to the $\mathcal{K}_R$-component. Therefore equality of the $\mathcal{K}_R$-projected components is preserved by applying Θ_R.

In Theorem 6.4, $\mathcal{D}'_{R,\mathrm{adm}}$ was defined as the closed subspace satisfying the uniform boundary estimate in the closure of finite-window boundary inputs. Since Θ_R is bounded in the boundary-distribution topology by Lemma 6.42 and preserves the boundary pairing by Lemma 6.41, it preserves this closure and the same uniform estimate. Hence

$$\Theta_R \mathcal{D}'_{R,\mathrm{adm}} = \mathcal{D}'_{R,\mathrm{adm}}.$$

The well-definedness of the displayed rule follows from the first part of the statement and the residual-free quotient compatibility just proved. □

Lemma 6.44 (finite synthesis form of the LCI readout)**.** On a finite comparison window M, suppose the original localized comparison interface is given in the z-dependent readout form

$$\langle \mathrm{LCI}_{R,0}\, T, z\rangle_X = \sum_{\alpha\in A_M} \langle T, \mathfrak{r}_{\alpha,M}(z)\rangle_{\partial,R},$$

where

$$\mathfrak{r}_{\alpha,M} : X_{\mathrm{eff},M} \longrightarrow \mathcal{D}_R$$

is a finite-rank continuous probe operator. Then there are finitely many

$$\rho_{\alpha\beta,M} \in \mathcal{D}_R, \qquad u_{\alpha\beta,M} \in X_{\mathrm{eff},M} \subset X,$$

such that

$$\mathfrak{r}_{\alpha,M}(z) = \sum_\beta \langle z, u_{\alpha\beta,M}\rangle_X\, \rho_{\alpha\beta,M}$$

with the conjugation convention adjusted to the convention for the Hilbert inner product. Consequently,

$$\mathrm{LCI}_{R,M}(T) = \sum_{\alpha,\beta} \big\langle T, \rho_{\alpha\beta,M}\big\rangle_{\partial,R} u_{\alpha\beta,M}$$

is a finite synthesis form of the same finite-window LCI readout.

Proof. Because each $\mathfrak{r}_{\alpha,M}$ is finite-rank, its range has a finite basis $\rho_{\alpha\beta,M}$. Hence

$$\mathfrak{r}_{\alpha,M}(z) = \sum_\beta c_{\alpha\beta,M}(z)\rho_{\alpha\beta,M}$$

for continuous linear functionals $c_{\alpha\beta,M}$ on the finite-dimensional Hilbert space $X_{\mathrm{eff},M}$. By the finite-dimensional Riesz representation theorem, there are $u_{\alpha\beta,M} \in X_{\mathrm{eff},M}$ with

$$c_{\alpha\beta,M}(z) = \langle z, u_{\alpha\beta,M}\rangle_X$$

up to the fixed sesquilinear convention. Substitution gives

$$\begin{aligned} \langle \mathrm{LCI}_{R,0}\, T, z\rangle_X &= \sum_{\alpha,\beta} \langle z, u_{\alpha\beta,M}\rangle_X \langle T, \rho_{\alpha\beta,M}\rangle_{\partial,R} \\ &= \left\langle \sum_{\alpha,\beta} \langle T, \rho_{\alpha\beta,M}\rangle_{\partial,R} u_{\alpha\beta,M}, z \right\rangle_X . \end{aligned}$$

Since this holds for every z in the finite comparison range, the displayed finite synthesis formula follows. Thus the finite synthesis form is not a new definition of LCI; it is the Riesz decomposition of the earlier z-dependent readout formula. ☐

Lemma 6.45 (explicit Riesz probe from the LCI synthesis family)**.** On each finite comparison window M, write the localized comparison interface in the finite synthesis form

$$\mathrm{LCI}_{R,M}(T) = \sum_{\alpha\in A_M} \langle T, \rho_{\alpha,M}\rangle_{\partial,R}\, u_{\alpha,M}, \qquad \rho_{\alpha,M} \in \mathcal{D}_R, \quad u_{\alpha,M} \in X.$$

Here A_M is finite, and the vectors $u_{\alpha,M}$ span the finite-window comparison range

$$X_M^{\mathrm{cmp}} := \operatorname{span}\{u_{\alpha,M} : \alpha \in A_M\}.$$

For $z \in X_M^{\mathrm{cmp}}$, define

$$\mathfrak{r}_R^{(M)}(z) := \sum_{\alpha \in A_M} c_{\alpha,M}(z)\rho_{\alpha,M} \in \mathcal{D}_R,$$

where $c_{\alpha,M}(z)$ is the scalar dictated by the sesquilinear convention so that

$$\langle T, c_{\alpha,M}(z)\rho_{\alpha,M}\rangle_{\partial,R} = \langle T, \rho_{\alpha,M}\rangle_{\partial,R}\langle u_{\alpha,M}, z\rangle_X$$

for all T. With the convention that the X-inner product is linear in the first variable, this is simply $c_{\alpha,M}(z) = \langle u_{\alpha,M}, z\rangle_X$. Then

$$\boxed{\langle \mathrm{LCI}_{R,M}(T), z\rangle_X = \langle T, \mathfrak{r}_R^{(M)}(z)\rangle_{\partial,R}.}$$

The map $z \mapsto \mathfrak{r}_R^{(M)}(z)$ is continuous because it is finite-rank.

If $M' \geq M$ is a larger window and z is represented in both comparison ranges, then

$$\mathfrak{r}_R^{(M')}(z) - \mathfrak{r}_R^{(M)}(z)$$

annihilates all finite-window boundary distributions already visible in window M. Hence the Riesz probe is intrinsically an element of the quotient

$$\mathcal{D}_R/\mathcal{N}_R^{\mathrm{ann}}, \qquad \mathcal{N}_R^{\mathrm{ann}} := \{\rho \in \mathcal{D}_R : \langle T, \rho\rangle_{\partial,R} = 0 \text{ for all admissible finite-window } T\}.$$

After fixing the representative section already used in the finite-window readout construction, we denote this quotient class by

$$\mathfrak{r}_R(z).$$

All pairings below are independent of the chosen representative.

Definition 6.46 (Riesz probe associated with the localized comparison interface)**.** Let

$$\mathcal{K}_R^{\mathrm{cmp}} := \operatorname{span}\{\Pi_R \,\mathrm{LCI}_R(y) : y \in \mathcal{D}'_{R,\mathrm{adm}}\} \subset \mathcal{K}_R.$$

For $z \in \mathcal{K}_R^{\mathrm{cmp}}$, the Riesz probe

$$\mathfrak{r}_R(z) \in \mathcal{D}_R/\mathcal{N}_R^{\mathrm{ann}}$$

is the quotient class constructed in Lemma 6.45. Choosing the fixed finite-window representative section, it is characterized by

$$\boxed{\langle \mathrm{LCI}_R(y), z\rangle_X = \langle y, \mathfrak{r}_R(z)\rangle_{\partial,R} \qquad (y \in \mathcal{D}'_{R,\mathrm{adm}}).}$$

This identity is therefore not an additional duality assumption: it is obtained from the concrete finite-window LCI synthesis formula and then extended by continuity.

For

$$x_j = \Pi_R \,\mathrm{LCI}_R(y_j),$$

define the boundary comparison form by

$$\mathcal{B}_R(y_1, y_2) := \langle y_1, \mathfrak{r}_R(x_2) \rangle_{\partial,R}.$$

Then

$$\mathcal{B}_R(y_1, y_2) = \langle \Pi_R \operatorname{LCI}_R(y_1), \Pi_R \operatorname{LCI}_R(y_2) \rangle_X.$$

Thus the equality between the boundary comparison form and the Hilbert X-inner product is a consequence of the finite-window Riesz construction of LCI_R, not a normalization imposed after the fact.

Lemma 6.47 (kernel invariance of the comparison quotient)**.** Let

$$\mathcal{N}_R^{\mathrm{cmp}} := \{ y \in \mathcal{D}'_{R,\mathrm{adm}} : \Pi_R \operatorname{LCI}_R(y) = 0 \}.$$

Then

$$\Theta_R \mathcal{N}_R^{\mathrm{cmp}} = \mathcal{N}_R^{\mathrm{cmp}}.$$

Consequently, the rule

$$\Pi_R \operatorname{LCI}_R(y) \longmapsto \Pi_R \operatorname{LCI}_R(\Theta_R y)$$

is well-defined on the quotient comparison range.

Proof. If $y \in \mathcal{N}_R^{\mathrm{cmp}}$, then by Definition 6.46,

$$\| \Pi_R \operatorname{LCI}_R(y) \|_X^2 = \mathcal{B}_R(y, y) = 0.$$

The boundary reflection preserves the centered boundary pairing by Lemma 6.41, and the Riesz probe is transported contragrediently by the test-side pullback associated with Θ_R. Hence

$$\mathcal{B}_R(\Theta_R y, \Theta_R y) = \mathcal{B}_R(y, y) = 0.$$

Using the Riesz identity again gives

$$\| \Pi_R \operatorname{LCI}_R(\Theta_R y) \|_X^2 = 0,$$

so $\Theta_R y \in \mathcal{N}_R^{\mathrm{cmp}}$. Applying the same argument to $\Theta_R^{-1} = \Theta_R$ gives equality of the two kernels. ☐

Lemma 6.48 (Riesz inner-product transport for the descended reflection)**.** For

$$y_1, y_2 \in \mathcal{D}'_{R,\mathrm{adm}}, \qquad x_j := \Pi_R \operatorname{LCI}_R(y_j) \in \mathcal{K}_R,$$

the Riesz transport identity is

$$\boxed{\langle x_1, x_2 \rangle_X = \mathcal{B}_R(y_1, y_2) = \langle y_1, \mathfrak{r}_R(x_2) \rangle_{\partial,R}.}$$

Moreover, on the dense comparison range

$$\mathcal{K}_R^{\mathrm{cmp}} = \operatorname{span}\{ \Pi_R \operatorname{LCI}_R(y) : y \in \mathcal{D}'_{R,\mathrm{adm}} \},$$

the descended reflection satisfies

$$\langle \mathcal{S}_R x_1, \mathcal{S}_R x_2 \rangle_X = \langle x_1, x_2 \rangle_X$$

and

$$\langle \mathcal{S}_R x_1, x_2 \rangle_X = \langle x_1, \mathcal{S}_R x_2 \rangle_X.$$

Thus the self-adjointness of the descended reflection is obtained by transporting the boundary pairing through the Riesz representation of LCI_R.

Proof. The first displayed identity is the defining Riesz identity of Definition 6.46, with $z = x_2 = \Pi_R \, \mathrm{LCI}_R(y_2)$. Hence

$$\langle \Pi_R \, \mathrm{LCI}_R(y_1), \Pi_R \, \mathrm{LCI}_R(y_2) \rangle_X = \langle y_1, \mathfrak{r}_R(x_2) \rangle_{\partial,R} = \mathcal{B}_R(y_1, y_2).$$

On finite-window boundary representatives, Θ_R is the seam realization of the centered Mellin reflection $w \mapsto -w$. The functional equation

$$\xi_c(w) = \xi_c(-w)$$

preserves the centered boundary pairing. The test-side probe is transported by the pullback dual to this reflection; equivalently,

$$\mathfrak{r}_R(\mathcal{S}_R x) = \Theta_R^\sharp \mathfrak{r}_R(x)$$

on $\mathcal{K}_R^{\mathrm{cmp}}$, where $\Theta_R^\sharp$ denotes the test-probe pullback. Consequently,

$$\begin{aligned}\langle \mathcal{S}_R x_1, \mathcal{S}_R x_2 \rangle_X &= \mathcal{B}_R(\Theta_R y_1, \Theta_R y_2) \\ &= \mathcal{B}_R(y_1, y_2) \\ &= \langle x_1, x_2 \rangle_X,\end{aligned}$$

and similarly

$$\begin{aligned}\langle \mathcal{S}_R x_1, x_2 \rangle_X &= \mathcal{B}_R(\Theta_R y_1, y_2) \\ &= \mathcal{B}_R(y_1, \Theta_R y_2) \\ &= \langle x_1, \mathcal{S}_R x_2 \rangle_X.\end{aligned}$$

The kernel invariance in Lemma 6.47 ensures that the displayed formulas are independent of the chosen admissible representatives y_j. Since Θ_R, LCI_R, and the boundary pairing are continuous on the admissible closure, the identities pass from finite-window representatives to all $y_j \in \mathcal{D}'_{R,\mathrm{adm}}$. □

Proposition 6.49 (descent to the singular-boundary component). *The boundary reflection Θ_R descends to a bounded self-adjoint involution*

$$\mathcal{S}_R : \mathcal{K}_R \longrightarrow \mathcal{K}_R$$

on the residual-free $\mathcal{K}_R$-component. It satisfies

$$\mathcal{S}_R^* = \mathcal{S}_R, \qquad \mathcal{S}_R^2 = I_{\mathcal{K}_R}, \qquad \|\mathcal{S}_R\| = 1.$$

For every $f \in \mathcal{D}_R$,

$$\mathcal{S}_R \Pi_R \mathcal{J}_R f = \Pi_R \, \mathrm{LCI}_R\big(\Theta_R \, \mathrm{Tr}_{\partial,R}^{\mathrm{cmp}} f\big).$$

Proof. By Lemma 6.43 and Lemma 6.47, the rule

$$\Pi_R \, \mathrm{LCI}_R(y) \longmapsto \Pi_R \, \mathrm{LCI}_R(\Theta_R y)$$

is well-defined on the projected comparison range. This range is dense in the component $\mathcal{K}_R = \operatorname{Ran}\Pi_R$ generated by the residual-free comparison interface.

By Lemma 6.48, this rule is isometric and symmetric on the dense comparison range:

$$\langle \mathcal{S}_R x_1, \mathcal{S}_R x_2 \rangle_X = \langle x_1, x_2 \rangle_X, \qquad \langle \mathcal{S}_R x_1, x_2 \rangle_X = \langle x_1, \mathcal{S}_R x_2 \rangle_X.$$

It therefore extends uniquely to a bounded isometry

$$\mathcal{S}_R : \mathcal{K}_R \to \mathcal{K}_R.$$

The identity $\Theta_R^2 = I$ gives $\mathcal{S}_R^2 = I_{\mathcal{K}_R}$. Hence

$$\mathcal{S}_R^{-1} = \mathcal{S}_R.$$

Since an isometry satisfies $\mathcal{S}_R^{-1} = \mathcal{S}_R^*$, we obtain

$$\mathcal{S}_R^* = \mathcal{S}_R.$$

Finally, taking

$$y = \operatorname{Tr}_{\partial,R}^{\mathrm{cmp}} f$$

and using

$$\Pi_R \mathcal{J}_R f = \Pi_R \operatorname{LCI}_R(\operatorname{Tr}_{\partial,R}^{\mathrm{cmp}} f)$$

gives the displayed formula. □

Definition 6.50 ($\mathcal{K}_R$-signature operator)**.** In what follows, the self-adjoint involution

$$\mathcal{S}_R : \mathcal{K}_R \to \mathcal{K}_R$$

constructed in Proposition 6.49 is called the $\mathcal{K}_R$-signature operator. The operator $\mathcal{S}_R$ is not an operator for making the form positive-definite. It is only the self-adjoint involution induced by the functional-equation reflection after passage to the residual-free $\mathcal{K}_R$-component.

Definition 6.51 (signed residual-free $\mathcal{K}_R$-quadratic form)**.** For $f \in \mathcal{D}_R$, define

$$Q_R^{\mathrm{sgn}}(f) := \langle \mathcal{S}_R \Pi_R \mathcal{J}_R f,\, \Pi_R \mathcal{J}_R f \rangle_X.$$

This is not a positive quadratic form, but a Hermitian quadratic form representing the oriented component on the singular-boundary side.

Definition 6.52 (signed residual-free boundary-distribution comparison kernel)**.** Define the signed residual-free boundary-distribution comparison kernel by

$$\mathfrak{k}_R(f,g) := \langle \mathcal{S}_R \Pi_R \mathcal{J}_R f,\, \Pi_R \mathcal{J}_R g \rangle_X \qquad (f, g \in \mathcal{D}_R).$$

Equivalently, it is the sesquilinear form obtained as the polarization of Q_R^{sgn}. In what follows, the boundary-distribution comparison kernel candidate of Section 6.0 is evaluated as this signed kernel.

Lemma 6.53 (Hermitian symmetry of the signed boundary-distribution comparison kernel)**.**

$$\mathfrak{k}_R(f,g) = \overline{\mathfrak{k}_R(g,f)} \qquad (f,g \in \mathcal{D}_R)$$

holds. Moreover,

$$\mathfrak{k}_R(f,f) = Q_R^{\mathrm{sgn}}(f) \in \mathbb{R}.$$

In general, $\mathfrak{k}_R(f,f) \geq 0$ is not assumed.

Proof. By Definition 6.52,

$$\mathfrak{k}_R(f,g) = \langle \mathcal{S}_R \Pi_R \mathcal{J}_R f,\, \Pi_R \mathcal{J}_R g \rangle_X.$$

The only structural property used here is the self-adjointness

$$\mathcal{S}_R^* = \mathcal{S}_R$$

obtained in Proposition 6.49. Therefore

$$\langle \mathcal{S}_R \Pi_R \mathcal{J}_R f,\, \Pi_R \mathcal{J}_R g \rangle_X = \overline{\langle \mathcal{S}_R \Pi_R \mathcal{J}_R g,\, \Pi_R \mathcal{J}_R f \rangle_X}.$$

This gives

$$\mathfrak{k}_R(f,g) = \overline{\mathfrak{k}_R(g,f)}.$$

Taking $f = g$, the value is real. No positivity of Q_R^{sgn} is used or asserted. ☐

Remark 6.54 (non-circularity of the construction of K)**.** The construction of Θ_R, $\mathcal{S}_R$, the signed boundary-distribution comparison kernel $\mathfrak{k}_R$, and the operator K uses only the functional equation

$$\xi(s) = \xi(1-s),$$

the boundary distribution framework of Sections 4–5, and the orthogonal projection structure of X. It does not use any information about the location of the zeros of ξ. In particular, no positivity, Herglotz property, or spectral-localization statement equivalent to the Riemann Hypothesis is assumed in the definition of $\mathcal{S}_R$, $\mathfrak{k}_R$, or K.

Theorem 6.55 (quantitative Sobolev eigenvalue growth of the reference operator)**.** *For the smooth-seam Sobolev reference operator $A_R = A_R^{\mathrm{ref}}$ fixed in Section 4.8 and Definition 6.39,*

$$A_R e_n = \lambda_n e_n, \qquad 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots,$$

one has

$$\boxed{\lambda_n \geq c_W n^{\nu_R} = c_W n^{m_R/d_{\mathrm{eff}}} = c_W n} \qquad (n \geq n_W).$$

Consequently,

$$a_{\mathrm{tr}} > \frac{1}{2\nu_R} = \frac{d_{\mathrm{eff}}}{2m_R}$$

implies

$$\boxed{\sum_{n\geq 1} (1+\lambda_n)^{-2a_{\mathrm{tr}}} < \infty.}$$

Proof. By Section 4.8,

$$A_R^{\mathrm{ref}} = \mathcal{W}_R^{-1}(I - \Delta_{\Sigma,N})\mathcal{W}_R$$

on the smooth compact carrier $\Sigma_R = [0,1]^2$. The Neumann elliptic regularizer $I - \Delta_{\Sigma,N}$ has effective dimension $d_{\mathrm{eff}} = 2$ and order $m_R = 2$, hence the usual Weyl lower bound gives

$$\lambda_n \geq c_{\mathrm{W}} n^{m_R/d_{\mathrm{eff}}} = c_{\mathrm{W}} n^{\nu_R} \qquad (n \geq n_{\mathrm{W}}).$$

Unitary conjugation by $\mathcal{W}_R$ preserves the spectrum. For $n \geq n_{\mathrm{W}}$,

$$(1+\lambda_n)^{-2a_{\mathrm{tr}}} \leq C(1+n^{\nu_R})^{-2a_{\mathrm{tr}}} \leq C' n^{-2a_{\mathrm{tr}}\nu_R}.$$

Since $a_{\mathrm{tr}} > 1/(2\nu_R)$, the exponent satisfies $2a_{\mathrm{tr}}\nu_R > 1$, and the displayed series converges. The finitely many terms $n < n_{\mathrm{W}}$ do not affect convergence. ☐

Remark 6.56 (Schatten scale inherited from the smooth-seam Sobolev model)**.** Theorem 6.55 is the only spectral-growth input used in the Hilbert–Schmidt construction below. The operator $A_R = A_R^{\mathrm{ref}}$ is the Sobolev regularizer attached to the smooth seam carrier, not a zero-detecting operator. It is fixed before F_K is identified with ξ, and it contains no information about the location of the zeros of ξ. The trace-ideal argument uses only

$$d_{\mathrm{eff}} = 2, \qquad m_R = 2, \qquad \nu_R = 1, \qquad \lambda_n \geq c_{\mathrm{W}} n^{\nu_R},$$

together with the smoothed comparison trace and the LCI_R-control estimate proved earlier.

Theorem 6.57 (fixed trace-smoothing exponent and Hilbert–Schmidt summability)**.** *With*

$$a_{\mathrm{HS}} := \frac{1}{2\nu_R} = \frac{d_{\mathrm{eff}}}{2m_R},$$

the exponent a_{tr} fixed in Section 4.8 and Definition 6.39 satisfies

$$a_{\mathrm{tr}} > a_{\mathrm{HS}}.$$

Consequently,

$$\boxed{(I + A_R)^{-a_{\mathrm{tr}}} \in \mathfrak{S}_2(H_{\alpha,+})}$$

and equivalently

$$\boxed{\sum_{n\geq 1}(1+\lambda_n)^{-2a_{\mathrm{tr}}} < \infty.}$$

Proof. This is exactly the summability conclusion of Theorem 6.55. For a positive self-adjoint operator with eigenbasis $\{e_n\}$,

$$\|(I + A_R)^{-a_{\mathrm{tr}}}\|_{\mathfrak{S}_2}^2 = \sum_{n\geq 1}(1+\lambda_n)^{-2a_{\mathrm{tr}}}.$$

Thus the summability is equivalent to Hilbert–Schmidt membership. Unlike the earlier threshold formulation, no floating "sufficiently large" exponent is used below. The single exponent a_{tr} controls the net Hilbert–Schmidt decay, while the actual comparison-trace smoothing order is $a_{\mathrm{cmp}} = a_{\mathrm{tr}} + s_R/m_R$. ☐

Lemma 6.58 (construction of the raw and smoothed singular-boundary comparison traces)**.** The raw seam trace and the smoothed comparison trace are the two distinct operators

$$\mathrm{Tr}^{\mathrm{raw}}_{\partial,R} : H^{s_R}_R \longrightarrow \mathcal{D}'_{R,\mathrm{adm}}$$

and

$$\mathrm{Tr}^{\mathrm{cmp}}_{\partial,R} : H_{\alpha,+} \longrightarrow \mathcal{D}'_{R,\mathrm{adm}}.$$

They are given by

$$\boxed{\mathrm{Tr}^{\mathrm{raw}}_{\partial,R} f = \iota_{\Gamma R}\gamma_\Gamma \mathcal{W}_R f} \qquad (f \in H^{s_R}_R)$$

and

$$\boxed{\mathrm{Tr}^{\mathrm{cmp}}_{\partial,R} = \mathrm{Tr}^{\mathrm{raw}}_{\partial,R}(I + A_R)^{-a_{\mathrm{cmp}}} = \mathrm{Tr}^{[s_R]}_{\partial,R}(I + A_R)^{-a_{\mathrm{tr}}},}$$

where

$$\mathrm{Tr}^{[s_R]}_{\partial,R} := \mathrm{Tr}^{\mathrm{raw}}_{\partial,R}(I + A_R)^{-s_R/m_R} = \iota_{\Gamma R}\gamma_\Gamma \mathcal{W}_R (I + A_R)^{-s_R/m_R}.$$

Moreover,

$$\mathrm{Tr}^{[s_R]}_{\partial,R} : H_{\alpha,+} \longrightarrow \mathcal{D}'_{R,\mathrm{adm}}$$

is bounded and satisfies

$$\boxed{\| \mathrm{Tr}^{[s_R]}_{\partial,R} h\|_{R,\mathrm{LCI}} \le C_\partial \|h\|_{H_{\alpha,+}}} \qquad (h \in H_{\alpha,+}).$$

Proof. By construction in Section 4.8, s_R was chosen so that the raw seam restriction followed by the admissible boundary embedding obeys

$$\|\iota_{\Gamma R}\gamma_\Gamma u\|_{R,\mathrm{LCI}} \le C_\Gamma \|u\|_{H^{s_R}_N(\Sigma_R)}.$$

Since

$$H^{s_R}_R = \mathcal{W}_R^{-1} H^{s_R}_N(\Sigma_R),$$

the raw trace is a bounded map from $H^{s_R}_R$ into the R, LCI-controlled admissible boundary-distribution space. The operator

$$\mathcal{W}_R (I + A_R)^{-s_R/m_R}$$

maps $H_{\alpha,+}$ boundedly into $H^{s_R}_N(\Sigma_R)$. Therefore

$$\mathrm{Tr}^{[s_R]}_{\partial,R} = \iota_{\Gamma R}\gamma_\Gamma \mathcal{W}_R (I + A_R)^{-s_R/m_R}$$

is bounded from $H_{\alpha,+}$ to $\mathcal{D}'_{R,\mathrm{adm}}$ with the displayed R, LCI-estimate. Multiplying on the right by the net decay factor

$$(I + A_R)^{-a_{\mathrm{tr}}}$$

gives

$$\mathrm{Tr}^{\mathrm{cmp}}_{\partial,R} = \mathrm{Tr}^{[s_R]}_{\partial,R}(I + A_R)^{-a_{\mathrm{tr}}}.$$

Since

$$a_{\mathrm{cmp}} = a_{\mathrm{tr}} + \frac{s_R}{m_R},$$

this is the same as

$$\mathrm{Tr}^{\mathrm{raw}}_{\partial,R}(I+A_R)^{-a_{\mathrm{cmp}}}.$$

Thus the decay used below is a property of the smoothed comparison trace, not of the raw restriction alone. □

Lemma 6.59 (boundary trace smoothing on the reference eigenbasis)**.** There exists $C_\partial > 0$ such that, for every $f \in \mathcal{D}_R$,

$$\boxed{\|\mathrm{Tr}^{\mathrm{cmp}}_{\partial,R} f\|_{R,\mathrm{LCI}} \le C_\partial \|(I+A_R)^{-a_{\mathrm{tr}}} f\|_{H_{\alpha,+}}.}$$

In particular, for the reference eigenbasis $A_R e_n = \lambda_n e_n$,

$$\boxed{\|\mathrm{Tr}^{\mathrm{cmp}}_{\partial,R} e_n\|_{R,\mathrm{LCI}} \le C_\partial (1+\lambda_n)^{-a_{\mathrm{tr}}}} \qquad (n \ge 1).$$

Consequently, for any Hilbertizable admissible norm dominating the LCI-control seminorm,

$$\|\mathrm{Tr}^{\mathrm{cmp}}_{\partial,R} e_n\|_{\mathcal{D}'_{R,\mathrm{adm}}} \le C'_\partial (1+\lambda_n)^{-a_{\mathrm{tr}}}.$$

Proof. By Lemma 6.58,

$$\mathrm{Tr}^{\mathrm{cmp}}_{\partial,R} = \mathrm{Tr}^{[s_R]}_{\partial,R}(I+A_R)^{-a_{\mathrm{tr}}},$$

and $\mathrm{Tr}^{[s_R]}_{\partial,R}$ is bounded into the R, LCI-controlled admissible boundary space. Therefore

$$\|\mathrm{Tr}^{\mathrm{cmp}}_{\partial,R} f\|_{R,\mathrm{LCI}} \le C_\partial \|(I+A_R)^{-a_{\mathrm{tr}}} f\|_{H_{\alpha,+}}.$$

For $f = e_n$,

$$(I+A_R)^{-a_{\mathrm{tr}}} e_n = (1+\lambda_n)^{-a_{\mathrm{tr}}} e_n,$$

which proves the eigenbasis estimate. The final inequality follows from the chosen admissible norm dominating $\|\cdot\|_{R,\mathrm{LCI}}$. □

Lemma 6.60 (boundedness of the comparison extension)**.** The continuously extended localized comparison interface

$$\mathrm{LCI}_R : \mathcal{D}'_{R,\mathrm{adm}} \to X$$

is bounded by the explicit LCI-control seminorm:

$$\boxed{\|\mathrm{LCI}_R\, y\|_X \le C_{\mathrm{LCI}} \|y\|_{R,\mathrm{LCI}} = C_{\mathrm{LCI}} \sum_{\nu=1}^{N_{\mathrm{LCI}}} \mathfrak{p}_{R,\nu}(y)} \qquad (y \in \mathcal{D}'_{R,\mathrm{adm}}).$$

In particular,

$$\|\mathrm{LCI}_R\, y\|_X \le C'_{\mathrm{LCI}} \|y\|_{\mathcal{D}'_{R,\mathrm{adm}}}.$$

Proof. The displayed inequality is exactly the extension estimate in Theorem 6.4. The second inequality follows from the defining domination of the Hilbertizable admissible norm over the LCI-control seminorm. Thus this boundedness is a consequence of the explicit finite family of boundary dual seminorms, not a continuity assertion inserted after the fact. □

Theorem 6.61 (smoothing of the singular-boundary comparison map)**.** *There exists $C_J > 0$ such that, for every $f \in \mathcal{D}_R$,*

$$\boxed{\|\Pi_R \mathcal{J}_R f\|_X \le C_J \| \operatorname{Tr}^{\mathrm{cmp}}_{\partial,R} f\|_{R,\mathrm{LCI}}}$$

and hence

$$\boxed{\|\Pi_R \mathcal{J}_R f\|_X \le C_J' \|(I + A_R)^{-a_{\mathrm{tr}}} f\|_{H_{\alpha,+}}.}$$

In particular,

$$\boxed{\|\Pi_R \mathcal{J}_R e_n\|_X \le C_J' (1 + \lambda_n)^{-a_{\mathrm{tr}}}} \qquad (n \ge 1).$$

Proof. By definition,

$$\mathcal{J}_R = (\Pi_R + \Pi^X_{\mathrm{arith}})\mathfrak{B}_R = (\Pi_R + \Pi^X_{\mathrm{arith}})\, \mathrm{LCI}_R \operatorname{Tr}^{\mathrm{cmp}}_{\partial,R}.$$

Applying Π_R and using $\Pi_R \Pi^X_{\mathrm{arith}} = 0$, we obtain

$$\Pi_R \mathcal{J}_R = \Pi_R \, \mathrm{LCI}_R \operatorname{Tr}^{\mathrm{cmp}}_{\partial,R}.$$

Therefore, by Lemma 6.60,

$$\|\Pi_R \mathcal{J}_R f\|_X \le \|\Pi_R\| \, C_{\mathrm{LCI}} \| \operatorname{Tr}^{\mathrm{cmp}}_{\partial,R} f\|_{R,\mathrm{LCI}}.$$

This proves the first estimate. The second follows by inserting Lemma 6.59. For $f = e_n$, the eigenbasis identity

$$\|(I + A_R)^{-a_{\mathrm{tr}}} e_n\| = (1 + \lambda_n)^{-a_{\mathrm{tr}}}$$

gives the stated decay estimate. ☐

Theorem 6.62 (Schatten estimate for the signed boundary-distribution comparison kernel)**.** *Set*

$$u_n := \Pi_R \mathcal{J}_R e_n \in \mathcal{K}_R.$$

Then

$$\boxed{(u_n)_{n \ge 1} \in \ell^2(\mathcal{K}_R)}$$

and

$$\boxed{\sum_{m,n \ge 1} |\mathfrak{k}_R(e_m, e_n)|^2 < \infty.}$$

More explicitly,

$$\mathfrak{k}_R(e_m, e_n) = \langle \mathcal{S}_R u_m, u_n \rangle_X,$$

and

$$\sum_{m,n \ge 1} |\mathfrak{k}_R(e_m, e_n)|^2 \le \|\mathcal{S}_R\|^2 \left(\sum_{n \ge 1} \|u_n\|_X^2 \right)^2.$$

Proof. By Theorem 6.61,

$$\|u_n\|_X = \|\Pi_R \mathcal{J}_R e_n\|_X \le C_J' (1 + \lambda_n)^{-a_{\mathrm{tr}}}.$$

By Theorem 6.57,

$$\sum_{n \ge 1} (1 + \lambda_n)^{-2a_{\mathrm{tr}}} < \infty.$$

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Thus

$$\sum_{n\geq 1} \|u_n\|_X^2 < \infty,$$

so $(u_n) \in \ell^2(\mathcal{K}_R)$.

By definition of the signed boundary-distribution comparison kernel,

$$\mathfrak{k}_R(e_m, e_n) = \langle \mathcal{S}_R u_m, u_n \rangle_X.$$

Since $\mathcal{S}_R$ is bounded,

$$|\mathfrak{k}_R(e_m, e_n)| \leq \|\mathcal{S}_R\| \, \|u_m\|_X \|u_n\|_X.$$

Squaring and summing gives

$$\begin{aligned}\sum_{m,n\geq 1} |\mathfrak{k}_R(e_m, e_n)|^2 &\leq \|\mathcal{S}_R\|^2 \sum_{m,n\geq 1} \|u_m\|_X^2 \|u_n\|_X^2 \\ &= \|\mathcal{S}_R\|^2 \left(\sum_{n\geq 1} \|u_n\|_X^2 \right)^2 < \infty.\end{aligned}$$

□

Proposition 6.63 (Hilbert–Schmidt realization of the boundary-distribution comparison kernel)**.** *The boundary-distribution comparison kernel candidate K defined in Section 6.0 closes uniquely as a Hilbert–Schmidt operator on $H_{\alpha,+}$. Namely,*

$$K \in \mathfrak{S}_2(H_{\alpha,+}).$$

Moreover,

$$\|K\|_{\mathfrak{S}_2}^2 = \sum_{m,n\geq 1} |\langle K e_n, e_m \rangle_{H_{\alpha,+}}|^2 < \infty.$$

The assertion $K \in \mathfrak{S}_2$ is independent of the reference orthonormal basis used to prove it.

Proof. By Definition 6.52, the sesquilinear kernel on the initial domain $\mathcal{D}_R \subset H_{\alpha,+}$,

$$\mathfrak{k}_R(f, g) = \langle K f, g \rangle_{H_{\alpha,+}} \qquad (f, g \in \mathcal{D}_R),$$

is determined as the polarization of the residual-free $\mathcal{K}_R$-quadratic evaluation. Definition 6.39 ensures that

$$e_n \in \mathcal{D}_R^\infty \subset \mathcal{D}_R,$$

so the matrix entries $\mathfrak{k}_R(e_n, e_m)$ are defined on the initial domain. By Theorem 6.62,

$$\sum_{m,n\geq 1} |\mathfrak{k}_R(e_n, e_m)|^2 < \infty.$$

Define K_0 on finite linear combinations of the basis vectors by

$$K_0 e_n := \sum_{m\geq 1} \mathfrak{k}_R(e_n, e_m) e_m.$$

Then

$$\sum_{n\geq 1} \|K_0 e_n\|^2_{H_{\alpha,+}} = \sum_{m,n\geq 1} |\mathfrak{k}_R(e_n, e_m)|^2 < \infty.$$

Thus K_0 extends uniquely to a Hilbert–Schmidt operator on $H_{\alpha,+}$. Writing this extension as K, we obtain

$$K \in \mathfrak{S}_2(H_{\alpha,+}).$$

The Hilbert–Schmidt class is an operator ideal defined independently of a basis; the displayed basis computation only verifies membership. □

Lemma 6.64 (Hermitian symmetry of the residual-free boundary-distribution comparison kernel)**.** The boundary-distribution comparison kernel of Section 6.0,

$$\mathfrak{k}_R(f, g) \qquad (f, g \in \mathcal{D}_R),$$

is Hermitian. That is,

$$\mathfrak{k}_R(f, g) = \overline{\mathfrak{k}_R(g, f)} \qquad (f, g \in \mathcal{D}_R).$$

Proof. This is the content of Lemma 6.53. Namely, since the boundary-distribution comparison kernel of Section 6.0 is evaluated as the polarized kernel of Definition 6.52, it inherits the Hermitian symmetry coming from the inner product of the Hilbert space X. □

Theorem 6.65 (self-adjoint Hilbert–Schmidt realization)**.** *The boundary-distribution comparison kernel candidate K of Section 6.0 is realized as a self-adjoint Hilbert–Schmidt operator satisfying*

$$K = K^* \qquad \text{on } H_{\alpha,+}.$$

Proof. By Proposition 6.63, K closes uniquely as a Hilbert–Schmidt operator on $H_{\alpha,+}$. The symmetry of the initial sesquilinear form is exactly the Hermitian symmetry of Lemma 6.64. That lemma, in turn, uses only the self-adjointness

$$\mathcal{S}_R^* = \mathcal{S}_R$$

of the $\mathcal{K}_R$-involution and the Hilbert-space inner product on X. Thus, for $f, g \in \mathcal{D}_R$,

$$\langle Kf, g\rangle_{H_{\alpha,+}} = \mathfrak{k}_R(f, g) = \overline{\mathfrak{k}_R(g, f)} = \langle f, Kg\rangle_{H_{\alpha,+}}.$$

No positivity of Q_R^{sgn}, Herglotz property, or zero-localization assertion is used in this step. Since $\mathcal{D}_R$ is dense in $H_{\alpha,+}$, and K is bounded, the symmetry extends continuously to all of $H_{\alpha,+}$. Therefore

$$K = K^*$$

holds. □

Definition 6.66 (finite-rank compressions and central matrix readout)**.** Let P_N be the finite-rank projection of Definition 6.39, and set

$$E_N := P_N H_{\alpha,+}.$$

Let

$$\mathrm{tr}_N$$

denote the ordinary finite-dimensional trace on $\mathrm{End}(E_N)$. Define the finite-rank cutoff of the boundary-distribution comparison kernel K by

$$K_N := P_N K P_N.$$

When K_N is used inside tr_N, it is viewed as the self-adjoint endomorphism

$$K_N : E_N \to E_N,$$

and outside E_N it is extended by zero. Thus K_N is finite rank and

$$K_N \in \mathfrak{S}_1(H_{\alpha,+}) \cap \mathfrak{S}_2(H_{\alpha,+}).$$

For w in the central disk on which $I_{E_N} + iwK_N$ is invertible, define

$$C_N(w) := (I_{E_N} + iwK_N)^{-1}$$

and the finite-dimensional boundary central matrix

$$\mathcal{H}_N(w) := i\big(C_N(w) - I_{E_N}\big)K_N.$$

Equivalently, since $C_N(w)$ commutes with K_N,

$$\boxed{\mathcal{H}_N(w) = w\, C_N(w) K_N^2.}$$

The finite-rank boundary central functional is the trace of this matrix:

$$\boxed{\langle \mu_{L,N}, \Psi_w^{\mathrm{cen}} \rangle := \mathrm{tr}_N\, \mathcal{H}_N(w).}$$

This definition uses only the compressed boundary-distribution comparison matrix K_N; it is not a determinant-side definition.

Lemma 6.67 (Hilbert–Schmidt convergence of finite-rank compressions).

$$\|K_N - K\|_{\mathfrak{S}_2} \longrightarrow 0 \qquad (N \to \infty).$$

Proof. $P_N \to I$ converges strongly on $H_{\alpha,+}$. For a Hilbert–Schmidt operator K, a bounded strongly convergent sequence of operators P_N satisfies

$$\|(I - P_N)K\|_{\mathfrak{S}_2} \to 0, \qquad \|K(I - P_N)\|_{\mathfrak{S}_2} \to 0.$$

Indeed, using the orthonormal basis $\{e_n\}$,

$$\|(I - P_N)K\|_{\mathfrak{S}_2}^2 = \sum_{j \geq 1} \|(I - P_N)Ke_j\|_{H_{\alpha,+}}^2.$$

For each j, $(I - P_N)Ke_j \to 0$, and moreover

$$\|(I - P_N)Ke_j\|^2 \leq \|Ke_j\|^2$$

and

$$\sum_j \|Ke_j\|^2 = \|K\|_{\mathfrak{S}_2}^2 < \infty.$$

Therefore the first convergence follows by dominated convergence. The second convergence is identical.

Now

$$K - P_N K P_N = (I - P_N)K + P_N K(I - P_N).$$

Thus, by the triangle inequality,

$$\|K - K_N\|_{\mathfrak{S}_2} \le \|(I - P_N)K\|_{\mathfrak{S}_2} + \|P_N K(I - P_N)\|_{\mathfrak{S}_2}.$$

The right-hand side converges to zero as $N \to \infty$. Hence

$$\|K_N - K\|_{\mathfrak{S}_2} \to 0.$$

□

Definition 6.68 (regularized Fredholm determinant)**.** Let $A \in \mathfrak{S}_2(H_{\alpha,+})$. Define the regularized Fredholm determinant by

$$\det_2(I + A) := \det\big((I + A)e^{-A}\big).$$

The standard properties of the regularized Fredholm determinant, trace ideals, and $\det_2$ follow [4,5]. The right-hand side is defined as an ordinary Fredholm determinant because

$$(I + A)e^{-A} - I \in \mathfrak{S}_1.$$

Equivalently, if the eigenvalue sequence of A, counted with algebraic multiplicities, is denoted by $\{\lambda_j(A)\}$, then

$$\det_2(I + A) = \prod_j (1 + \lambda_j(A))e^{-\lambda_j(A)}.$$

This product converges under the $\mathfrak{S}_2$-condition.

In particular, for $z \in \mathbb{C}$, write

$$D_K(z) := \det_2(I + zK).$$

Remark 6.69 (first trace renormalization)**.** In $\det_2$, the first trace term is removed by normalization. Indeed, in the range $|z|\|K\| < 1$,

$$\log D_K(z) = \sum_{m=2}^{\infty} \frac{(-1)^{m+1}}{m} z^m \operatorname{Tr}(K^m).$$

Therefore

$$\frac{d}{dz} \log D_K(z)\bigg|_{z=0} = 0.$$

For this reason, in the comparison with the completed zeta function, the constant term and the linear term must be normalized separately by an exponential factor

$$e^{a+bs}.$$

Lemma 6.70 (continuity of $\det_2$ in Hilbert–Schmidt norm)**.** For every $R > 0$,

$$\sup_{|z|\le R} |\det_2(I + zK_N) - \det_2(I + zK)| \longrightarrow 0.$$

Namely,

$$D_{K_N}(z) := \det_2(I + zK_N)$$

converges to

$$D_K(z) = \det_2(I + zK)$$

locally uniformly on compact sets in the z-plane.

Proof. For Hilbert–Schmidt operators A, B, the regularized determinant satisfies the following Lipschitz-type estimate. There exists a universal constant $C > 0$ such that

$$|\det_2(I + A) - \det_2(I + B)| \le \|A - B\|_{\mathfrak{S}_2} \exp\Big(C\big(1 + \|A\|_{\mathfrak{S}_2} + \|B\|_{\mathfrak{S}_2}\big)^2\Big).$$

Set $A = zK_N$ and $B = zK$. By Lemma 6.67,

$$\|K_N - K\|_{\mathfrak{S}_2} \to 0.$$

Moreover,

$$\|K_N\|_{\mathfrak{S}_2} \le \|K\|_{\mathfrak{S}_2}.$$

Therefore, for $|z| \le R$,

$$\|zK_N - zK\|_{\mathfrak{S}_2} \le R\|K_N - K\|_{\mathfrak{S}_2} \to 0,$$

and the exponential factor is uniformly bounded with respect to N and z. Hence

$$\sup_{|z|\le R} |D_{K_N}(z) - D_K(z)| \to 0.$$

□

Lemma 6.71 (trace-power convergence)**.** For each $m \ge 2$,

$$K_N^m \longrightarrow K^m \qquad \text{in } \mathfrak{S}_1.$$

Therefore

$$\mathrm{Tr}(K_N^m) \longrightarrow \mathrm{Tr}(K^m).$$

Proof. $K_N \to K$ in $\mathfrak{S}_2$, and in particular also in operator norm:

$$\|K_N - K\| \le \|K_N - K\|_{\mathfrak{S}_2} \to 0.$$

Moreover, $\sup_N \|K_N\| < \infty$.

First, for $m = 2$,

$$K_N^2 - K^2 = (K_N - K)K_N + K(K_N - K).$$

By the Schatten Hölder inequality,

$$\|(K_N - K)K_N\|_{\mathfrak{S}_1} \leq \|K_N - K\|_{\mathfrak{S}_2}\|K_N\|_{\mathfrak{S}_2},$$

and

$$\|K(K_N - K)\|_{\mathfrak{S}_1} \leq \|K\|_{\mathfrak{S}_2}\|K_N - K\|_{\mathfrak{S}_2}.$$

Thus $K_N^2 \to K^2$ in $\mathfrak{S}_1$.

For $m \geq 3$,

$$K_N^m - K^m = \sum_{\ell=0}^{m-1} K_N^{\ell}(K_N - K)K^{m-1-\ell}.$$

In each term, $K_N - K$ converges in $\mathfrak{S}_2$; taking one of the remaining factors as an $\mathfrak{S}_2$-factor and estimating the other factors as bounded operators, the product converges to zero in $\mathfrak{S}_1$. That is, there exists a constant $C_m > 0$ such that

$$\|K_N^m - K^m\|_{\mathfrak{S}_1} \leq C_m\|K_N - K\|_{\mathfrak{S}_2} \to 0.$$

Hence $K_N^m \to K^m$ in $\mathfrak{S}_1$, and continuity of the trace gives

$$\mathrm{Tr}(K_N^m) \to \mathrm{Tr}(K^m).$$

□

Lemma 6.72 (non-vanishing at the central point)**.** One has

$$\xi\left(\frac{1}{2}\right) \neq 0.$$

Proof. For $0 < s < 1$, the Dirichlet eta function is represented by the convergent alternating series

$$\eta_{\mathrm{Dir}}(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}.$$

At $s = \frac{1}{2}$, the terms $n^{-1/2}$ are positive, decrease monotonically to zero, and

$$1 - \frac{1}{\sqrt{2}} > 0.$$

The alternating-series estimate therefore gives

$$\eta_{\mathrm{Dir}}\left(\frac{1}{2}\right) > 0.$$

Since

$$\eta_{\mathrm{Dir}}(s) = (1 - 2^{1-s})\zeta(s)$$

on $0 < s < 1$ by the usual analytic continuation of the eta function, and

$$1 - 2^{1/2} \neq 0,$$

we obtain

$$\zeta\Big(\frac{1}{2}\Big) \neq 0.$$

The remaining factors in

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$$

are nonzero at $s = \frac{1}{2}$, and hence

$$\xi\Big(\frac{1}{2}\Big) \neq 0.$$

□

Definition 6.73 (normalized determinant comparison function)**.** Set

$$D_K(z) := \det_2(I + zK).$$

Define the comparison function by

$$F_K(s) := e^{a_K + b_K(s-\frac{1}{2})} D_K\Big(i(s-\frac{1}{2})\Big).$$

The normalization constants $a_K, b_K \in \mathbb{C}$ are fixed by

$$F_K\Big(\frac{1}{2}\Big) = \xi\Big(\frac{1}{2}\Big), \qquad F_K'\Big(\frac{1}{2}\Big) = \xi'\Big(\frac{1}{2}\Big).$$

Namely, since

$$D_K(0) = 1, \qquad D_K'(0) = 0,$$

take

$$a_K := \log \xi\Big(\frac{1}{2}\Big), \qquad b_K := \frac{\xi'(\frac{1}{2})}{\xi(\frac{1}{2})}.$$

Here we use Lemma 6.72. The branch of the logarithm is used only to fix the value at this single point, and the definition of F_K is independent of the branch by

$$e^{a_K} = \xi\Big(\frac{1}{2}\Big).$$

Remark 6.74 (normalization does not encode zero locations)**.** The constants a_K and b_K fix only the value and the first derivative, equivalently the first logarithmic derivative, at the central point. They do not prescribe any zero of F_K, and they do not contain any information about the location of the zeros of ξ. The zero set of F_K is determined only by the regularized determinant factor

$$\det_2\big(I + i(s-\tfrac{1}{2})K\big),$$

where K was constructed independently of zero-location information as explained in Remark 6.54.

Lemma 6.75 (entireness of the normalized determinant comparison function)**.** $F_K(s)$ is an entire function of $s \in \mathbb{C}$. Moreover, for each N,

$$F_{K_N}(s) := e^{a_K + b_K(s-\frac{1}{2})} \det_2\big(I + i(s-\tfrac{1}{2})K_N\big)$$

is also entire, and

$$F_{K_N} \to F_K$$

locally uniformly on compact sets.

Proof. The map $z \mapsto \det_2(I + zK)$ is an entire function. Therefore

$$s \mapsto D_K\Big(i(s - \frac{1}{2})\Big)$$

is also an entire function. The exponential factor

$$e^{a_K + b_K(s - \frac{1}{2})}$$

is also entire, and hence F_K is entire.

For the same reason, F_{K_N} is also entire. By Lemma 6.70,

$$\det_2(I + zK_N) \to \det_2(I + zK)$$

locally uniformly on compact sets in z. The map

$$s \mapsto z = i(s - \tfrac{1}{2})$$

sends compact sets to compact sets, and therefore

$$F_{K_N} \to F_K$$

also locally uniformly on compact sets in s. □

Theorem 6.76 (coefficient transport for the regularized determinant). *In a sufficiently small neighborhood of the origin, take the branch of*

$$\log D_K(z)$$

satisfying

$$\log D_K(0) = 0.$$

Then, for each $m \geq 2$,

$$\frac{d^m}{dz^m} \log D_{K_N}(z)\Big|_{z=0} \longrightarrow \frac{d^m}{dz^m} \log D_K(z)\Big|_{z=0}.$$

Moreover,

$$\frac{d^m}{dz^m} \log D_K(z)\Big|_{z=0} = (-1)^{m+1}(m-1)!\,\mathrm{Tr}(K^m) \qquad (m \geq 2).$$

Likewise,

$$\frac{d}{dz} \log D_K(z)\Big|_{z=0} = 0.$$

Proof. Since $D_K(0) = 1$, there exists $r > 0$ such that

$$D_K(z) \neq 0 \qquad (|z| < r).$$

By Lemma 6.70, $D_{K_N} \to D_K$ uniformly on $\overline{B(0,r)}$. Therefore, for sufficiently large N,

$$D_{K_N}(z) \neq 0 \qquad (|z| < r),$$

and

$$\log D_{K_N}(z) \to \log D_K(z)$$

uniformly on every $\overline{B(0,r')}$, $0 < r' < r$. By Cauchy's integral formula, the derivatives also converge in each order $m \geq 0$. Namely,

$$\frac{d^m}{dz^m} \log D_{K_N}(z)\Big|_{z=0} \to \frac{d^m}{dz^m} \log D_K(z)\Big|_{z=0}.$$

On the other hand, for $|z|\|K\| < 1$,

$$\log D_K(z) = \sum_{m=2}^{\infty} \frac{(-1)^{m+1}}{m} z^m \operatorname{Tr}(K^m).$$

Hence, for $m \geq 2$,

$$\frac{d^m}{dz^m} \log D_K(z)\Big|_{z=0} = m!\frac{(-1)^{m+1}}{m} \operatorname{Tr}(K^m) = (-1)^{m+1}(m-1)! \operatorname{Tr}(K^m).$$

Moreover, since there is no linear term,

$$\frac{d}{dz} \log D_K(z)\Big|_{z=0} = 0.$$

□

Corollary 6.77 (coefficient transport for F_K). *Set $w = s - \frac{1}{2}$. In a neighborhood of the origin, take the branch of*

$$\log F_K\left(\frac{1}{2} + w\right)$$

satisfying

$$\log F_K\left(\frac{1}{2}\right) = \log \xi\left(\frac{1}{2}\right).$$

Then

$$\log F_K\left(\frac{1}{2} + w\right) = a_K + b_K w + \log D_K(iw),$$

and for $m \geq 2$,

$$\frac{d^m}{dw^m} \log F_K\left(\frac{1}{2} + w\right)\Big|_{w=0} = i^m(-1)^{m+1}(m-1)! \operatorname{Tr}(K^m).$$

Furthermore, the Taylor coefficients of

$$\log F_{K_N}\left(\frac{1}{2} + w\right) \longrightarrow \log F_K\left(\frac{1}{2} + w\right)$$

converge in each order.

Proof. By Definition 6.73,

$$F_K\left(\frac{1}{2}+w\right)=e^{a_K+b_K w}D_K(iw).$$

Therefore

$$\log F_K\left(\frac{1}{2}+w\right)=a_K+b_K w+\log D_K(iw).$$

Applying Theorem 6.76 with $z=iw$, for $m\geq 2$ one obtains

$$\frac{d^m}{dw^m}\log D_K(iw)\bigg|_{w=0}=i^m\frac{d^m}{dz^m}\log D_K(z)\bigg|_{z=0}=i^m(-1)^{m+1}(m-1)!\,\mathrm{Tr}(K^m).$$

The linear coefficient is fixed by b_K, and the constant term is fixed by a_K. The coefficient convergence from finite-rank cutoffs also follows from Theorem 6.76. □

Proposition 6.78 (compatibility with the distributional comparison coefficients)**.** *The open-band coefficient comparison obtained from the distributional comparison theorem of Section 6.1 transports continuously to the trace-ideal coefficients*

$$\mathrm{Tr}(K^m)\qquad(m\geq 2)$$

constructed in this section. Namely, setting the coefficient sequence obtained by finite-rank cutoffs as

$$c_m(K_N):=\frac{(-1)^{m+1}}{m}\,\mathrm{Tr}(K_N^m)\qquad(m\geq 2),$$

one has

$$c_m(K_N)\to c_m(K):=\frac{(-1)^{m+1}}{m}\,\mathrm{Tr}(K^m),$$

and this limiting coefficient is compatible with the calibrated coefficient comparison on the open band obtained from

$$\langle\mu_L,\varphi\rangle=\langle\mu_\xi,\varphi\rangle\qquad(\varphi\in\mathcal{A}_\eta,\ 0<\eta<\log 2)$$

in Section 6.1.

Proof. The convergence $c_m(K_N)\to c_m(K)$ follows immediately from Lemma 6.71. In Section 6.1, it was shown that μ_L and μ_ξ agree as continuous linear functionals on $\mathcal{A}_\eta$. On the other hand, the $\mathcal{K}_R$-component of μ_L is represented by the kernel of K in Section 6.0, and in the finite-rank approximation it is represented as finite trace coefficients by K_N. Therefore the open-band comparison quantity represented by finite-rank coefficients transports to

$$c_m(K)$$

as

$$N\to\infty.$$

This transport is due to $\mathfrak{S}_2$-convergence and trace-power convergence, and does not use any additional endpoint convention or pointwise boundary value. □

Theorem 6.79 (trace-ideal determinant theorem)**.** *The boundary-distribution comparison kernel candidate K of Section 6.0 is realized as*

$$K=K^*,\qquad K\in\mathfrak{S}_2(H_{\alpha,+}).$$

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Moreover, the finite-rank cutoffs

$$K_N = P_N K P_N$$

satisfy

$$\|K_N - K\|_{\mathfrak{S}_2} \to 0,$$

and

$$\det{}_2(I + zK_N) \longrightarrow \det{}_2(I + zK)$$

locally uniformly on compact sets in the z-plane.

The normalized comparison function

$$F_K(s) = e^{a_K + b_K(s-\frac{1}{2})} \det{}_2\big(I + i(s - \tfrac{1}{2})K\big)$$

is an entire function and is normalized so as to satisfy

$$F_K\Big(\frac{1}{2}\Big) = \xi\Big(\frac{1}{2}\Big), \qquad F_K'\Big(\frac{1}{2}\Big) = \xi'\Big(\frac{1}{2}\Big).$$

Moreover, its Taylor coefficients are transported degree by degree from the finite-rank cutoffs.

Proof. The fact that $K \in \mathfrak{S}_2$ was shown in Proposition 6.63. The self-adjointness $K = K^*$ follows from Theorem 6.65. The Hilbert–Schmidt convergence of the finite-rank cutoffs follows from Lemma 6.67. The local uniform convergence of $\det_2$ follows from Lemma 6.70. The definition and normalization of F_K are given by Definition 6.73. The entireness of F_K follows from Lemma 6.75. Finally, the degree-by-degree transport of Taylor coefficients was shown in Theorem 6.76 and Corollary 6.77. ☐

Remark 6.80 (no global identification in this section)**.** In this section, F_K was constructed by the regularized determinant, and it was proved that its coefficients can be transported from finite-rank approximations. However, this section does not yet conclude

$$F_K(s) = \xi(s).$$

This global agreement is proved in the global uniqueness theorem of Section 6.4.

6.4. Global Uniqueness Theorem

In this section, we globally identify the regularized-determinant comparison function constructed in Section 6.3,

$$F_K(s) = e^{a_K + b_K(s-\frac{1}{2})} \det{}_2\big(I + i(s - \tfrac{1}{2})K\big),$$

with the completed zeta function $\xi(s)$. The uniqueness principle used in this section is only the identity theorem of complex analysis. That is, we use only the fact that if two entire functions agree on a nonempty open set, then they agree on the whole plane. Carlson-type theorems, the Phragmén–Lindelöf principle, or other growth-type uniqueness theorems are not used in the identity proof of this section.

Lemma 6.81 (common holomorphic domain)**.** Both F_K and ξ are entire functions on $\mathbb{C}$.

Proof. That F_K is entire was shown in Lemma 6.75. On the other hand,

$$\xi(s) := \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$$

is the completed zeta function; the simple pole of $\zeta(s)$ at $s=1$ is removed by the factor $s-1$, and the poles of $\Gamma(s/2)$ at the negative even integers are cancelled by the trivial zeros of $\zeta(s)$. Therefore $\xi(s)$ is an entire function. ☐

Lemma 6.82 (growth of the determinant comparison function)**.** For every $s \in \mathbb{C}$, there exists a constant $C_K > 0$ such that

$$|F_K(s)| \le C_K \exp\Big(|b_K|\,|s-\tfrac{1}{2}| + \frac{1}{2}\,\|K\|^2_{\mathfrak{S}_2}\,|s-\tfrac{1}{2}|^2\Big).$$

Consequently, F_K is an entire function of order at most 2.

Proof. For a Hilbert–Schmidt operator $A \in \mathfrak{S}_2$, the regularized determinant satisfies

$$|\det_2(I+A)| \le \exp\Big(\frac{1}{2}\|A\|^2_{\mathfrak{S}_2}\Big).$$

Taking $A = i(s-\frac{1}{2})K$, one has

$$\|A\|_{\mathfrak{S}_2} = |s-\tfrac{1}{2}|\,\|K\|_{\mathfrak{S}_2}.$$

Therefore

$$\Big|\det_2\big(I + i(s-\tfrac{1}{2})K\big)\Big| \le \exp\Big(\frac{1}{2}|s-\tfrac{1}{2}|^2\|K\|^2_{\mathfrak{S}_2}\Big).$$

Furthermore,

$$\Big|e^{a_K + b_K(s-\frac{1}{2})}\Big| \le e^{\mathrm{Re}\,a_K}\exp\Big(|b_K|\,|s-\tfrac{1}{2}|\Big).$$

Thus, taking $C_K = e^{\mathrm{Re}\,a_K}$, the asserted estimate follows. This estimate gives

$$\log\log|F_K(s)| = O(\log|s|)$$

in the form of order at most 2. ☐

Lemma 6.83 (growth of the completed zeta function)**.** The completed zeta function $\xi(s)$ is an entire function of order 1. In particular, for every $\varepsilon > 0$, there exist constants $C_\varepsilon, A_\varepsilon > 0$ such that

$$|\xi(s)| \le C_\varepsilon \exp\Big(A_\varepsilon |s|^{1+\varepsilon}\Big) \qquad (s \in \mathbb{C})$$

holds.

Proof. Use the representation

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s).$$

The growth of $\Gamma(s/2)$ in vertical strips is controlled exponentially by Stirling's formula. Moreover, $\zeta(s)$, as a meromorphic function, has growth of order at most 1, and its only pole at $s=1$ is removed by the factor $s-1$. By the functional equation

$$\xi(s) = \xi(1-s),$$

the estimates in the left and right half-planes are transferred to each other. Therefore ξ is an entire function of order 1, and the stated $(1+\varepsilon)$-type estimate follows. $\square$

Corollary 6.84 (growth of the difference)**.**

$$H_K(s) := F_K(s) - \xi(s)$$

is an entire function of order at most 2.

Proof. By Lemma 6.81, H_K is entire. By Lemma 6.82, F_K has order at most 2, and by Lemma 6.83, ξ has order 1. Therefore the difference H_K has order at most 2. $\square$

Definition 6.85 (logarithmic germs at the central point)**.** Set $w = s - \frac{1}{2}$. By the normalization of Section 6.3 and Lemma 6.72,

$$F_K\left(\frac{1}{2}\right) = \xi\left(\frac{1}{2}\right) \neq 0.$$

Hence there exists $r_0 > 0$ such that

$$F_K\left(\frac{1}{2} + w\right) \neq 0, \qquad \xi\left(\frac{1}{2} + w\right) \neq 0 \qquad (|w| < r_0).$$

On this disk, define

$$\mathcal{L}_K(w) := \log F_K\left(\frac{1}{2} + w\right), \qquad \mathcal{L}_\xi(w) := \log \xi\left(\frac{1}{2} + w\right)$$

by the branches satisfying

$$\mathcal{L}_K(0) = \mathcal{L}_\xi(0) = \log \xi\left(\frac{1}{2}\right).$$

For each $m \geq 0$, write the central logarithmic coefficients as

$$c_m(F_K) := \frac{1}{m!} \frac{d^m}{dw^m} \mathcal{L}_K(w)\bigg|_{w=0}, \qquad c_m(\xi) := \frac{1}{m!} \frac{d^m}{dw^m} \mathcal{L}_\xi(w)\bigg|_{w=0}.$$

Definition 6.86 (central Cauchy–Laplace kernel)**.** Take $r_0 > 0$ in Definition 6.85 smaller if necessary. For $|w| < r_0$, define

$$h_w(u) := \frac{e^{wu} - 1}{u}, \qquad h_w(0) := w.$$

The value at $u = 0$ fills the removable singularity, and the resulting kernel is holomorphic in w. Let

$$\mathcal{E}_{\mathrm{cen}}$$

denote the vector space spanned by finite linear combinations of the kernels h_w and their w-derivatives.

Definition 6.87 (raw finite-window central kernels)**.** Fix an even function $\chi \in C_c^\infty(\mathbb{R})$ such that

$$\chi(u) = 1 \quad (|u| \leq 1), \qquad \chi(u) = 0 \quad (|u| \geq 2), \qquad 0 \leq \chi \leq 1,$$

and set $\chi_M(u) := \chi(u/M)$ for $M \geq 1$. For $|w| < r_0$, define the raw finite-window kernel

$$h^{\mathrm{fw}}_{w,M}(u) := \chi_M(u) h_w(u),$$

and let

$$\widetilde{\Upsilon}^{\mathrm{fw}}_{w,M} := [h^{\mathrm{fw}}_{w,M}]_{\mathrm{fw}}$$

denote the corresponding finite-window test input before the central finite-part subtraction. The cutoff function χ is fixed once and for all throughout the central comparison argument.

Definition 6.88 (central finite-jet map)**.** The local singular orders in the completed finite-window explicit formula determine two non-negative integers

$$d_0, \qquad d_\partial.$$

These integers are fixed once and for all. They do not depend on the cutoff scale M, the central parameter w, or the values of the pairings with μ_L and μ_ξ.

For a finite-window kernel ϕ, write the central jet coordinates as

$$j^0_a(\phi) := \partial^a_u \phi(0), \qquad 0 \le a \le d_0,$$

and define

$$J^0_{d_0}(\phi) := \big(j^0_a(\phi)\big)_{0\le a\le d_0}.$$

For each $M \ge 1$, write the endpoint jet coordinates on the cutoff transition annulus as

$$j^\partial_{M,b}(\phi) := \mathcal{E}^\partial_{M,b}(\phi), \qquad 0 \le b \le d_\partial.$$

The endpoint functional is fixed by the cutoff scheme and is supported in

$$A^\partial_M := \{u \in \mathbb{R} : \ M \le |u| \le 2M\}.$$

The endpoint singular transition is encoded, before either pairing is evaluated, by the finite-order differential normal form

$$\mathcal{D}^\partial_M := \sum_{j=0}^{d_\partial} a^{\mathrm{loc}}_{M,j}(u)\, \partial^j_u \qquad \text{on } A^\partial_M.$$

The coefficients $a^{\mathrm{loc}}_{M,j}$ are fixed by the completed finite-window local normal form and by the chosen cutoff convention. They carry no side label $\bullet$; the Archimedean/reference, arithmetic-trace, and singular-boundary endpoint contributions will be reduced to this single operator. More explicitly, after the rescaling $\chi_M(u) = \chi(u/M)$, it may be written in the form

$$\mathcal{E}^\partial_{M,b}(\phi) = \sum_{\ell\le b} \int_{M\le|u|\le 2M} e_{M,b,\ell}(u)\, \partial^\ell_u \phi(u)\, du,$$

where the coefficient functions $e_{M,b,\ell}$ are fixed finite linear combinations of derivatives of χ_M. Thus the endpoint jet is also a finite, pre-pairing object.

The central finite-jet map is

$$J^{\mathrm{cen}}_M : \mathcal{C}^0_{\mathrm{cen}} \longrightarrow \mathcal{J}^{\mathrm{cen}}_M := J^0_{d_0} \oplus J^\partial_{M,d_\partial},$$

given on representatives by

$$J^{\mathrm{cen}}_M(\phi) := \Big(\big(j^0_a(\phi)\big)^{d_0}_{a=0}, \big(j^\partial_{M,b}(\phi)\big)^{d_\partial}_{b=0}\Big).$$

For the central Cauchy–Laplace kernel

$$h_w(u) = \frac{e^{wu} - 1}{u}$$

one has the convergent expansion

$$h_w(u) = \sum_{a=0}^{\infty} \frac{w^{a+1}}{(a+1)!}\, u^a.$$

Consequently,

$$\partial_u^a h_w(0) = \frac{w^{a+1}}{a+1}, \qquad a \geq 0.$$

Since $\chi_M = 1$ near $u = 0$,

$$\partial_u^a h^{\mathrm{fw}}_{w,M}(0) = \partial_u^a h_w(0) = \frac{w^{a+1}}{a+1}$$

for every fixed a. On the endpoint annulus,

$$\partial_u^\ell h^{\mathrm{fw}}_{w,M}(u) = \sum_{r=0}^{\ell} \binom{\ell}{r} M^{-r} \chi^{(r)}(u/M)\, \partial_u^{\ell-r} h_w(u),$$

so every endpoint jet of $h^{\mathrm{fw}}_{w,M}$ is an explicit finite combination of cutoff-transition derivatives and derivatives of h_w.

The universal endpoint transition coefficients are defined by the cutoff commutator

$$\mathcal{D}^{\partial}_M(\chi_M \phi) - \chi_M \mathcal{D}^{\partial}_M \phi = \sum_{\ell=0}^{d_\partial} q^{\partial}_{M,\ell}(u)\, \partial_u^\ell \phi(u) \qquad \text{on } A^{\partial}_M.$$

By the Leibniz rule,

$$q^{\partial}_{M,\ell}(u) := \sum_{j=\ell+1}^{d_\partial} \binom{j}{\ell} a^{\mathrm{loc}}_{M,j}(u)\, \partial_u^{j-\ell} \chi_M(u) = \sum_{j=\ell+1}^{d_\partial} \binom{j}{\ell} M^{-(j-\ell)} a^{\mathrm{loc}}_{M,j}(u)\, \chi^{(j-\ell)}(u/M),$$

with the convention that an empty sum is zero. These functions $q^{\partial}_{M,\ell}$ are universal for the three sides and replace any side-dependent symbolic coefficient functions.

Definition 6.89 (central local normal form)**.** In a fixed neighborhood $|u| < \epsilon_0$ of the central point, the central local singular principal part of the Archimedean/reference, arithmetic-trace, and singular-boundary sides is represented by one finite-order local differential normal form

$$\mathcal{D}^0_M := \sum_{j=0}^{d_0} a^0_{M,j}(u)\, \partial_u^j.$$

The coefficients $a^0_{M,j}$ are smooth near $u = 0$ and are fixed by the completed finite-window local convention and the central jet convention. They carry no side label. Since the finite-window cutoff satisfies $\chi_M = 1$ near $u = 0$, no cutoff-derivative commutator occurs in the central local term.

Definition 6.90 (central principal-part jet functional)**.** Let

$$J^0_{d_\Gamma} \phi := \left(\partial_u^q \phi(0)\right)_{0 \leq q \leq d_\Gamma}$$

be the finite central jet used by the Hadamard finite-part convention. The central finite-part extraction rule is the fixed linear functional

$$\Gamma^0_M : J^0_{d_\Gamma} \longrightarrow \mathbb{C}, \qquad \Gamma^0_M(v) = \sum_{q=0}^{d_\Gamma} \gamma^0_{M,q} v_q.$$

The coefficients $\gamma^0_{M,q}$ are part of the central finite-part normalization and are fixed before the three pairings are evaluated. They carry no side label

$$\bullet \in \{\infty, \mathrm{arith}, R\}.$$

Lemma 6.91 (explicit central jet coefficients)**.** Let

$$\mathcal{D}^0_M = \sum_{j=0}^{d_D} a^0_{M,j}(u)\,\partial^j_u$$

be the common central local normal form of Definition 6.89. After enlarging the central jet order d_0 if necessary, assume $d_0 \geq d_D + d_\Gamma$. For each side

$$\bullet \in \{\infty, \mathrm{arith}, R\},$$

the central principal part is

$$\mathrm{PP}^0_{M,\bullet}(\phi) = \Gamma^0_M\Big(J^0_{d_\Gamma}(\mathcal{D}^0_M\phi)\Big) = \sum_{a=0}^{d_0} c^0_{M,a}\,\partial^a_u\phi(0),$$

where the coefficients are the side-independent finite sums

$$\boxed{c^0_{M,a} = \sum_{j=0}^{d_D}\sum_{q=0}^{d_\Gamma} \mathbf{1}_{\{0\leq a-j\leq q\}} \binom{q}{a-j} \gamma^0_{M,q}\,\partial^{q-a+j}_u a^0_{M,j}(u)\Big|_{u=0}.}$$

Equivalently, if

$$\theta^0_{M,a}, \qquad \partial^b_u\theta^0_{M,a}(0) = \delta_{ab},$$

is the J^0_M-dual central test basis, then

$$c^0_{\infty,M,a} = c^0_{\mathrm{arith},M,a} = c^0_{R,M,a} = \mathrm{PP}^0_{M,\bullet}(\theta^0_{M,a}) = c^0_{M,a}.$$

For the central Cauchy–Laplace kernel

$$h_w(u) = \frac{e^{wu}-1}{u} = \sum_{a=0}^{\infty} \frac{w^{a+1}}{(a+1)!}u^a,$$

the common central principal part is

$$C^0_{\infty,M}(w) = C^0_{\mathrm{arith},M}(w) = C^0_{R,M}(w) = \sum_{a=0}^{d_0} c^0_{M,a}\frac{w^{a+1}}{a+1}.$$

Proof. For each q,

$$\partial_u^q\big(a^0_{M,j}(u)\partial_u^j\phi(u)\big)\big|_{u=0} = \sum_{\ell=0}^{q}\binom{q}{\ell}\,\partial_u^{q-\ell}a^0_{M,j}(u)\Big|_{u=0}\partial_u^{j+\ell}\phi(0).$$

Putting $a = j+\ell$ gives the displayed formula for $c^0_{M,a}$. Since Γ^0_M and $\mathcal{D}^0_M$ have no side label, the resulting coefficient ledger is common to the Archimedean/reference, arithmetic-trace, and singular-boundary sides.

On the arithmetic side, there is no additional central prime-power singularity:

$$\operatorname{supp}\mu^{\mathrm{ex}}_\Lambda \subset \{\log p^k : p^k \geq 2\} \subset [\log 2, \infty),$$

whereas the central band is chosen inside $|u| < \eta_1 < \log 2$. On the singular-boundary side, seam realization and finite readout reconstruction preserve the central contour jet,

$$J^0_M(\mathcal{N}_\xi b) = J^0_M(b)$$

on the finite readout quotient channel. Hence all three sides apply the same functional

$$\Gamma^0_M \circ J^0_{d_\Gamma} \circ \mathcal{D}^0_M.$$

Finally,

$$\partial_u^a h_w(0) = \frac{w^{a+1}}{a+1}$$

follows from the Taylor expansion of h_w. Substituting this jet into the common coefficient ledger gives the displayed equality of the three central polynomials. □

Definition 6.92 (principal-part space, jet-dual basis, and counterterm operator)**.** For every $M \geq 1$, let

$$\mathcal{P}^{\mathrm{cen}}_M$$

be the finite-dimensional vector space spanned by fixed local-principal-part basis elements

$$p^0_{M,a}, \qquad p^\partial_{M,b} \qquad (0 \leq a \leq d_0,\ 0 \leq b \leq d_\partial).$$

The basis elements may depend on M, but only through the fixed rescaling

$$\chi_M(u) = \chi(u/M)$$

and the associated finite-window localization. They are fixed before either central pairing is evaluated.

There is a canonical local-principal-part embedding

$$\iota^{\mathrm{cen}}_M : \mathcal{P}^{\mathrm{cen}}_M \hookrightarrow \mathcal{C}^0_{\mathrm{cen}}$$

which regards a local principal part as the corresponding finite-window reference test input. The basis is chosen to be J^{cen}_M-dual:

$$J^{\mathrm{cen}}_M\big(\iota^{\mathrm{cen}}_M p^0_{M,a}\big) = e^0_a, \qquad J^{\mathrm{cen}}_M\big(\iota^{\mathrm{cen}}_M p^\partial_{M,b}\big) = e^\partial_b,$$

where e^0_a and e^∂_b are the standard coordinate vectors in the central and endpoint jet components. In particular, the cross-coordinates vanish.

For coefficient extraction in Lemma 6.95, set

$$\theta^0_{M,a} := \iota^{\mathrm{cen}}_M p^0_{M,a}, \qquad \theta^\partial_{M,b} := \iota^{\mathrm{cen}}_M p^\partial_{M,b}.$$

These are not additional data. They are the embedded J^{cen}_M-dual test basis elements. Thus

$$j^0_{a'}(\theta^0_{M,a}) = \delta_{a,a'}, \qquad j^\partial_{M,b'}(\theta^0_{M,a}) = 0,$$

and

$$j^0_a(\theta^\partial_{M,b}) = 0, \qquad j^\partial_{M,b'}(\theta^\partial_{M,b}) = \delta_{b,b'}.$$

Define the universal principal-part map

$$P^{\mathrm{cen}}_M : \mathcal{J}^{\mathrm{cen}}_M \longrightarrow \mathcal{P}^{\mathrm{cen}}_M$$

by

$$P^{\mathrm{cen}}_M\big((\alpha_a)_{a=0}^{d_0}, (\beta_b)_{b=0}^{d_\partial}\big) := \sum_{a=0}^{d_0} \alpha_a\, p^0_{M,a} + \sum_{b=0}^{d_\partial} \beta_b\, p^\partial_{M,b}.$$

The associated finite-rank counterterm operator on the pre-completion finite-window space is

$$Q^{\mathrm{cen}}_M := \iota^{\mathrm{cen}}_M P^{\mathrm{cen}}_M J^{\mathrm{cen}}_M : \mathcal{C}^0_{\mathrm{cen}} \longrightarrow \mathcal{C}^0_{\mathrm{cen}}.$$

The jet-duality gives

$$J^{\mathrm{cen}}_M Q^{\mathrm{cen}}_M = J^{\mathrm{cen}}_M, \qquad (Q^{\mathrm{cen}}_M)^2 = Q^{\mathrm{cen}}_M,$$

and therefore

$$J^{\mathrm{cen}}_M(I - Q^{\mathrm{cen}}_M) = 0.$$

Thus the central counterterm associated with h_w and the window M is the principal-part vector

$$\mathrm{CT}^{\mathrm{cen}}_M(h_w) := P^{\mathrm{cen}}_M J^{\mathrm{cen}}_M(h^{\mathrm{fw}}_{w,M}) \in \mathcal{P}^{\mathrm{cen}}_M,$$

and its embedded finite-window reference element is

$$Q^{\mathrm{cen}}_M(h^{\mathrm{fw}}_{w,M}) = \iota^{\mathrm{cen}}_M\, \mathrm{CT}^{\mathrm{cen}}_M(h_w) \in \mathcal{C}^0_{\mathrm{cen}}.$$

No coefficient in Q^{cen}_M is chosen from the values of

$$\langle \mu_L, \cdot \rangle, \qquad \langle \mu_\xi, \cdot \rangle.$$

Definition 6.93 (finite-window central cutoff inputs)**.** For $M \ge 1$ and $|w| < r_0$, the regularized finite-window central test input is

$$\Psi^{\mathrm{fw}}_{w,M} := (I - Q^{\mathrm{cen}}_M)[h^{\mathrm{fw}}_{w,M}]_{\mathrm{fw}} \in \mathcal{C}^0_{\mathrm{cen}}.$$

Equivalently, using the embedded principal-part element,

$$\Psi^{\mathrm{fw}}_{w,M} = [h^{\mathrm{fw}}_{w,M}]_{\mathrm{fw}} - \iota^{\mathrm{cen}}_M P^{\mathrm{cen}}_M J^{\mathrm{cen}}_M(h^{\mathrm{fw}}_{w,M}) = [h^{\mathrm{fw}}_{w,M}]_{\mathrm{fw}} - \iota^{\mathrm{cen}}_M \mathrm{CT}^{\mathrm{cen}}_M(h_w).$$

The logarithmic-side representative of this regularized input is

$$\psi^{\mathrm{reg}}_{w,M} := (I - Q^{\mathrm{cen}}_M)h^{\mathrm{fw}}_{w,M}.$$

By the projection identity above,

$$J^{\mathrm{cen}}_M(\psi^{\mathrm{reg}}_{w,M}) = 0.$$

The subtraction is an algebraic subtraction in the pre-completion finite-window test-input space $\mathcal{C}^0_{\mathrm{cen}}$. It is not performed after applying either central pairing.

The data

$$\chi_M, \quad h_w, \quad J^{\mathrm{cen}}_M, \quad P^{\mathrm{cen}}_M, \quad \iota^{\mathrm{cen}}_M$$

fix the finite-rank operator Q^{cen}_M and hence $\Psi^{\mathrm{fw}}_{w,M}$ before the values of

$$\langle \mu_L, \Psi^{\mathrm{fw}}_{w,M} \rangle, \qquad \langle \mu_\xi, \Psi^{\mathrm{fw}}_{w,M} \rangle$$

are evaluated. Thus Q^{cen}_M is a pre-pairing finite-jet operation and cannot encode the desired equality of the two pairings.

Definition 6.94 (central finite-window representatives as quotient finite-window inputs)**.** For $M \geq 1$ and $|w| < r_0$, let

$$\psi^{\mathrm{reg}}_{w,M}$$

denote the logarithmic-side smooth representative of the regularized central finite-window input

$$\Psi^{\mathrm{fw}}_{w,M}.$$

Equivalently,

$$\psi^{\mathrm{reg}}_{w,M} := h^{\mathrm{fw}}_{w,M} - Q^{\mathrm{cen}}_M h^{\mathrm{fw}}_{w,M},$$

where the second term means the smooth finite-window representative of the embedded principal-part reference element. By Lemma 6.95,

$$J^{\mathrm{cen}}_M \psi^{\mathrm{reg}}_{w,M} = 0$$

and

$$\mathrm{PP}^{\mathrm{cen}}_{M,\bullet}(\psi^{\mathrm{reg}}_{w,M}) = 0, \qquad \bullet \in \{\infty, \mathrm{arith}, R\}.$$

Thus $\psi^{\mathrm{reg}}_{w,M}$ is a finite-window representative whose common local principal part has already been removed on all three sides. Since both $h^{\mathrm{fw}}_{w,M}$ and the finite-jet reference element have finite-window support, there is a fixed constant C_χ depending only on the cutoff convention such that

$$\mathrm{supp}\, \psi^{\mathrm{reg}}_{w,M} \subset [-C_\chi M, C_\chi M].$$

With the present normalization of the cutoff, one may take $C_\chi = 2$. Hence

$$\psi^{\mathrm{reg}}_{w,M} \in \mathcal{T}_{\log,\mathrm{fw}}.$$

Define the central-to-finite-window representative map

$$\mathfrak{T}_M^{\mathrm{cen}\to\mathrm{fw}}$$

by

$$\boxed{\tau_{w,M} := \mathfrak{T}_M^{\mathrm{cen}\to\mathrm{fw}}(\Psi_{w,M}^{\mathrm{fw}}) := [\psi_{w,M}^{\mathrm{reg}}]_{\mathrm{fw}} \in \mathscr{T}_{\mathrm{fw}}.}$$

By Definition 5.23, the three coordinates of $\tau_{w,M}$ are induced from the same representative $\psi_{w,M}^{\mathrm{reg}}$:

$$\mathcal{C}_{\mathrm{cen,fw}}(\tau_{w,M}) = [\psi_{w,M}^{\mathrm{reg}}]_{\mathrm{cen,fw}} = \Psi_{w,M}^{\mathrm{fw}},$$

$$\mathcal{A}_{\mathrm{fw}}(\tau_{w,M}) = \varphi_{w,M}, \qquad \varphi_{w,M}(n) := \psi_{w,M}^{\mathrm{reg}}(\log n),$$

and

$$\mathcal{B}_{\xi,\partial}^{\mathrm{fw}}(\tau_{w,M}) = \mathfrak{b}_{\xi,\partial}^{\mathrm{fw}}(\psi_{w,M}^{\mathrm{reg}}).$$

Thus the boundary coordinate is not chosen independently; it is obtained by inserting the same logarithmic representative $\psi_{w,M}^{\mathrm{reg}}$ into the centered Mellin finite-window contour identity.

In particular,

$$\varphi_{w,M}(n) \neq 0 \quad\Longrightarrow\quad 1 \le n \le e^{2M},$$

and hence

$$\varphi_{w,M} \in c_{00}(\mathbb{N}).$$

For any central pairing symbol, we use the convention

$$\langle \mu_\bullet, \Psi_{w,M}^{\mathrm{fw}} \rangle := \langle \mu_\bullet, \tau_{w,M} \rangle, \qquad \mu_\bullet \in \{\mu_L, \mu_\xi\}.$$

Thus $\Psi_{w,M}^{\mathrm{fw}}$ denotes the central coordinate of the quotient finite-window test object $\tau_{w,M} = [\psi_{w,M}^{\mathrm{reg}}]_{\mathrm{fw}}$, while the actual input to Proposition 5.24 is $\tau_{w,M}$.

Lemma 6.95 (explicit endpoint transition coefficients and principal-part cancellation)**.** For every $M \ge 1$, each side

$$\bullet \in \{\infty, \mathrm{arith}, R\}$$

has central and endpoint local-principal-part functionals

$$\mathrm{PP}_{M,\bullet}^{0}, \qquad \mathrm{PP}_{M,\bullet}^{\partial}, \qquad \mathrm{PP}_{M,\bullet}^{\mathrm{cen}} = \mathrm{PP}_{M,\bullet}^{0} + \mathrm{PP}_{M,\bullet}^{\partial}.$$

The central coefficients are extracted from the embedded central jet-dual test basis by

$$c_{\bullet,a}^{0} := \mathrm{PP}_{M,\bullet}^{0}(\theta_{M,a}^{0}), \qquad 0 \le a \le d_0.$$

For the central Cauchy–Laplace kernel, define

$$\mathrm{C}_{\bullet}^{0}(w) := \mathrm{PP}_{M,\bullet}^{0}(h_{w,M}^{\mathrm{fw}}).$$

Since $\chi_M = 1$ near $u = 0$,

$$C^0_\bullet(w) = \sum_{a=0}^{d_0} c^0_{\bullet,a} \frac{w^{a+1}}{a+1}, \qquad c^0_{\bullet,a} = \frac{1}{a!}\,\partial_w^{a+1} C^0_\bullet(w)\Big|_{w=0}.$$

Regular Taylor terms that are not singular principal parts are kept in the regular Archimedean/reference term and are not placed in Q^{cen}_M.

For the endpoint part, let

$$A^\partial_M = \{u \in \mathbb{R} : \ M \le |u| \le 2M\}$$

and let

$$\mathcal{D}^\partial_M = \sum_{j=0}^{d_\partial} a^{\mathrm{loc}}_{M,j}(u)\partial_u^j$$

be the universal endpoint local differential normal form of Definition 6.88. The cutoff commutator has the explicit Leibniz expansion

$$\mathcal{D}^\partial_M(\chi_M\phi) - \chi_M\mathcal{D}^\partial_M\phi = \sum_{\ell=0}^{d_\partial} q^\partial_{M,\ell}(u)\partial_u^\ell\phi(u),$$

where

$$q^\partial_{M,\ell}(u) = \sum_{j=\ell+1}^{d_\partial} \binom{j}{\ell} M^{-(j-\ell)} a^{\mathrm{loc}}_{M,j}(u)\chi^{(j-\ell)}(u/M).$$

Hence the endpoint coefficients are computed by the side-independent formula

$$c^\partial_{M,b} := \sum_{\ell=0}^{d_\partial} \int_{A^\partial_M} q^\partial_{M,\ell}(u)\,\partial_u^\ell \theta^\partial_{M,b}(u)\,du, \qquad 0 \le b \le d_\partial.$$

Equivalently,

$$c^\partial_{\infty,M,b} = c^\partial_{\mathrm{arith},M,b} = c^\partial_{R,M,b} = c^\partial_{M,b}.$$

The central coefficients also agree:

$$c^0_{\infty,a} = c^0_{\mathrm{arith},a} = c^0_{R,a} =: c^0_a.$$

Consequently, for every finite-window representative ϕ,

$$\mathrm{PP}^{\mathrm{cen}}_{M,\bullet}(\phi) = \ell^{\mathrm{loc}}_M\big(J^{\mathrm{cen}}_M\phi\big), \qquad \bullet \in \{\infty, \mathrm{arith}, R\},$$

where the single local coefficient ledger is

$$\ell^{\mathrm{loc}}_M\big((\alpha_a),(\beta_b)\big) := \sum_{a=0}^{d_0} c^0_a\alpha_a + \sum_{b=0}^{d_\partial} c^\partial_{M,b}\beta_b.$$

In particular,

$$\mathrm{PP}^{\mathrm{cen}}_{M,\bullet}\big((I - Q^{\mathrm{cen}}_M)\phi\big) = 0, \qquad \bullet \in \{\infty, \mathrm{arith}, R\}.$$

Thus, for

$$\psi^{\mathrm{reg}}_{w,M} = (I - Q^{\mathrm{cen}}_M)h^{\mathrm{fw}}_{w,M},$$

one has

$$J^{\mathrm{cen}}_M(\psi^{\mathrm{reg}}_{w,M}) = 0, \qquad \mathrm{PP}^{\mathrm{cen}}_{M,\bullet}(\psi^{\mathrm{reg}}_{w,M}) = 0 \quad (\bullet \in \{\infty, \mathrm{arith}, R\}).$$

Proof. The central extraction follows from Lemma 6.91 and from the J^{cen}_M-dual basis. The expansion

$$h_w(u) = \sum_{a=0}^{\infty} \frac{w^{a+1}}{(a+1)!}u^a$$

gives

$$\partial_u^a h_w(0) = \frac{w^{a+1}}{a+1}.$$

Since $\chi_M = 1$ near $u = 0$, the central jet of $h^{\mathrm{fw}}_{w,M}$ is the same. Differentiating

$$C^0_\bullet(w) = \sum_{a=0}^{d_0} c^0_{\bullet,a} \frac{w^{a+1}}{a+1}$$

$a + 1$ times at $w = 0$ gives

$$\partial_w^{a+1} C^0_\bullet(w)\Big|_{w=0} = a!\, c^0_{\bullet,a}.$$

Only the singular central principal part is entered into the counterterm; regular Taylor terms remain in the regular Archimedean/reference contribution.

For the endpoint part, all endpoint contributions are supported in

$$A^\partial_M = \{M \le |u| \le 2M\}.$$

By Definition 6.88, the endpoint local normal form is the finite-order operator

$$\mathcal{D}^\partial_M = \sum_{j=0}^{d_\partial} a^{\mathrm{loc}}_{M,j}(u)\partial_u^j.$$

Applying the Leibniz rule to $\mathcal{D}^\partial_M(\chi_M \phi)$ gives

$$\mathcal{D}^\partial_M(\chi_M\phi) = \chi_M \mathcal{D}^\partial_M \phi + \sum_{\ell=0}^{d_\partial} \left(\sum_{j=\ell+1}^{d_\partial} \binom{j}{\ell} a^{\mathrm{loc}}_{M,j}(u)\partial_u^{j-\ell}\chi_M(u) \right) \partial_u^\ell \phi(u).$$

Since $\partial_u^{j-\ell}\chi_M(u) = M^{-(j-\ell)}\chi^{(j-\ell)}(u/M)$, this proves the explicit formula for

$$q^\partial_{M,\ell}(u).$$

Evaluating the endpoint principal part on the endpoint-dual basis element

$$\theta^\partial_{M,b}$$

therefore gives

$$c^{\partial}_{M,b} = \sum_{\ell=0}^{d_\partial} \int_{A^{\partial}_M} q^{\partial}_{M,\ell}(u)\, \partial_u^{\ell} \theta^{\partial}_{M,b}(u)\, du.$$

We now verify the three sides. On the Archimedean/reference side, the completed local factors and the finite-window endpoint convention reduce the endpoint singular term to the cutoff commutator of the same operator $\mathcal{D}^{\partial}_M$. Thus

$$\mathrm{PP}^{\partial}_{M,\infty}(\phi) = \sum_{\ell=0}^{d_\partial} \int_{A^{\partial}_M} q^{\partial}_{M,\ell}(u)\, \partial_u^{\ell} \phi(u)\, du.$$

Regular terms from the gamma factor, the $s(s-1)$-factor, and the $\pi^{-s/2}$-normalization that are not singular principal parts remain in B^{fw}_{∞}.

On the arithmetic-trace side, the finite-window quotient construction gives

$$\varphi_\tau(n) = \psi(\log n).$$

The prime-power sampling

$$\sum_{p^k} \psi(k \log p) \log p$$

is the regular arithmetic contribution. It does not create an independent endpoint singular normal form. The endpoint principal part comes only from the same cutoff transition convention; hence

$$\mathrm{PP}^{\partial}_{M,\mathrm{arith}}(\phi) = \sum_{\ell=0}^{d_\partial} \int_{A^{\partial}_M} q^{\partial}_{M,\ell}(u)\, \partial_u^{\ell} \phi(u)\, du.$$

On the singular-boundary side, the contour finite jet is transported by

$$\mathcal{N}_\xi = \mathcal{C}'_{\xi R}, \qquad \mathcal{C}_{\xi R} \mathcal{U}_{\xi R} = I,$$

and is read through the LCI synthesis family and the Gram-inverse reconstruction

$$\{\mathfrak{r}_{\alpha,M}\}_{\alpha \in \mathfrak{A}_M}, \qquad \mathcal{V}_{R\xi,M} = G_M^{\dagger}.$$

These maps preserve the finite endpoint coordinates on the readout quotient channel. Consequently, the singular-boundary endpoint principal part is again the same cutoff-transition functional:

$$\mathrm{PP}^{\partial}_{M,R}(\phi) = \sum_{\ell=0}^{d_\partial} \int_{A^{\partial}_M} q^{\partial}_{M,\ell}(u)\, \partial_u^{\ell} \phi(u)\, du.$$

This proves the equality of the endpoint coefficients across the three sides. The equality of the central coefficients is obtained similarly by testing the central principal part on the J^{cen}_M-dual basis $\theta^0_{M,a}$; the arithmetic prime-power sampling has no additional singular support at $u = 0$, and the singular-boundary transfer preserves the contour central jet. Hence the three central extractions yield the common c^0_a.

Thus all three local principal parts factor through the same ledger

$$\ell^{\mathrm{loc}}_M \circ J^{\mathrm{cen}}_M.$$

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Finally, Definition 6.92 gives

$$J_M^{\mathrm{cen}} Q_M^{\mathrm{cen}} = J_M^{\mathrm{cen}}, \qquad J_M^{\mathrm{cen}}(I - Q_M^{\mathrm{cen}}) = 0.$$

Therefore

$$\mathrm{PP}^{\mathrm{cen}}_{M,\bullet}((I - Q_M^{\mathrm{cen}})\phi) = \ell_M^{\mathrm{loc}}(J_M^{\mathrm{cen}}(I - Q_M^{\mathrm{cen}})\phi) = 0$$

for all three sides. Taking $\phi = h^{\mathrm{fw}}_{w,M}$ gives the asserted statement for $\psi^{\mathrm{reg}}_{w,M}$.

The calculation uses only the fixed cutoff χ_M, the central kernel h_w, the finite jet maps, the seam realization maps, and the LCI readout synthesis. It does not use a central pairing equality, $F_K \equiv \xi$, zero localization, or the Riemann Hypothesis. ☐

Lemma 6.96 (seminorm control of the finite-jet counterterm)**.** Let $B \Subset \{|w| < r_0\}$. For every defining seminorm

$$p_{B,N,m,\sigma}$$

of $\mathcal{C}_{\mathrm{cen}}$, there exist finitely many defining seminorms

$$p_{B,N_\nu,m_\nu,\sigma_\nu}$$

and a constant $C_B > 0$, independent of M, such that, for every finite-window family Φ used in the central comparison,

$$p_{B,N,m,\sigma}(Q_M^{\mathrm{cen}}\Phi) = p_{B,N,m,\sigma}(\iota_M^{\mathrm{cen}} P_M^{\mathrm{cen}} J_M^{\mathrm{cen}}(\Phi)) \le C_B \sum_\nu p_{B,N_\nu,m_\nu,\sigma_\nu}(\Phi).$$

In particular, the finite-jet counterterm is a continuous finite-rank operator with respect to the central comparison seminorms on the finite-window families appearing in the proof.

Proof. The map J_M^{cen} consists of finitely many evaluations of u-derivatives at $u = 0$ and finitely many endpoint functionals on $M \le |u| \le 2M$. Each such functional is controlled by finitely many of the seminorms defining the central comparison topology, after increasing N, m, σ if necessary. The map P_M^{cen} is finite rank, and the embedding ι_M^{cen} inserts the resulting finite local principal part into the fixed finite-window test-input model. Since the orders d_0, d_∂ are uniform in M, only finitely many seminorm types are needed. The dependence on M is only through the fixed rescaling $\chi_M(u) = \chi(u/M)$, which is already controlled by the finite-window seminorms. This gives the stated estimate for Q_M^{cen}. ☐

Remark 6.97 (the counterterm does not encode the comparison equality)**.** The finite-jet operator

$$Q_M^{\mathrm{cen}} = \iota_M^{\mathrm{cen}} P_M^{\mathrm{cen}} J_M^{\mathrm{cen}}$$

and the counterterm

$$\mathrm{CT}_M^{\mathrm{cen}}(h_w)$$

are fixed before the two central pairings are evaluated. They depend only on the fixed cutoff χ_M, the central kernel h_w, the finite-jet map J_M^{cen}, and the universal principal-part map P_M^{cen}. They do not depend on the values of

$$\langle \mu_L, \Psi^{\mathrm{fw}}_{w,M} \rangle, \qquad \langle \mu_\xi, \Psi^{\mathrm{fw}}_{w,M} \rangle,$$

and therefore cannot encode the central residual-free equality. The operator Q_M^{cen} is only a pre-pairing algebraic removal of the common local principal part.

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Equivalently, Lemma 6.95 writes the three local principal parts as

$$\mathrm{PP}^{\mathrm{cen}}_{M,\bullet} = \ell^{\mathrm{loc}}_M \circ J^{\mathrm{cen}}_M, \qquad \bullet \in \{\infty, \mathrm{arith}, R\},$$

with a single coefficient ledger ℓ^{loc}_M. The counterterm kills this common ledger because

$$J^{\mathrm{cen}}_M (I - Q^{\mathrm{cen}}_M) = 0,$$

not because either central pairing has been evaluated.

Definition 6.98 (central comparison topology)**.** The central comparison topology is the locally convex topology generated as follows. For every compact set $B \Subset \{|w| < r_0\}$, every integer $m \geq 0$, every integer $N \geq 0$, and every

$$\sigma > \sup_{w \in B} |\operatorname{Re} w|,$$

define, on kernel representatives $\Phi = \{\phi_w(u)\}_{w \in B}$,

$$p_{B,N,m,\sigma}(\Phi) := \max_{0 \leq j \leq m} \max_{0 \leq \ell \leq N} \sup_{\substack{w \in B \\ u \in \mathbb{R}}} e^{-\sigma |u|} (1 + |u|)^N \left| \partial_w^j \partial_u^\ell \phi_w(u) \right|.$$

Let $\mathcal{C}^0_{\mathrm{cen}}$ be the vector space spanned by the finite-window central test inputs

$$\Psi^{\mathrm{fw}}_{w,M}$$

and their finite w-derivatives. For each M, the finite-dimensional counterterm space

$$\mathcal{P}^{\mathrm{cen}}_M$$

is regarded as a subspace of $\mathcal{C}^0_{\mathrm{cen}}$ via the canonical local-principal-part embedding

$$\iota^{\mathrm{cen}}_M : \mathcal{P}^{\mathrm{cen}}_M \hookrightarrow \mathcal{C}^0_{\mathrm{cen}}.$$

Thus the expression

$$[h^{\mathrm{fw}}_{w,M}]_{\mathrm{fw}} - \mathrm{CT}^{\mathrm{cen}}_M(h_w)$$

is formed in the algebraic space $\mathcal{C}^0_{\mathrm{cen}}$, not after applying either central pairing. The space $\mathcal{C}_{\mathrm{cen}}$ is the locally convex completion of $\mathcal{C}^0_{\mathrm{cen}}$, modulo zero seminorms, with respect to the seminorms above. Central finite-part limits such as Ψ^{cen}_w are not inserted as additional generators of the topology. They denote elements of the completion only after the corresponding finite-window family has been proved Cauchy in these seminorms; for the family used here this is exactly Lemma 6.104.

The seminorms defining $\mathcal{C}_{\mathrm{cen}}$ are fixed before the functionals

$$\mu_L, \qquad \mu_\xi$$

are applied. In particular, the topology depends only on the finite-window cutoff structure, the central kernel family, and its w- and u-derivatives; it does not depend on the values of

$$\langle \mu_L, \Phi \rangle \quad \text{or} \quad \langle \mu_\xi, \Phi \rangle.$$

Definition 6.99 (central Cauchy–Laplace comparison subspace)**.** Let $0 < r < r_0$ be the central radius used below. The central Cauchy–Laplace comparison subspace is

$$\mathcal{C}_{\mathrm{CL}}(r) := \overline{\operatorname{span}\left\{\partial_w^j \Psi_{w,M}^{\mathrm{fw}} : |w| < r,\ M \geq 1,\ j \geq 0\right\}}^{\mathcal{C}_{\mathrm{cen}}}.$$

For a compact set $B \Subset \{|w| < r\}$ and an integer $m \geq 0$, set

$$\mathcal{C}_{\mathrm{CL}}(B, m) := \overline{\operatorname{span}\left\{\partial_w^j \Psi_{w,M}^{\mathrm{fw}} : w \in B,\ M \geq 1,\ 0 \leq j \leq m\right\}}^{\mathcal{C}_{\mathrm{cen}}}.$$

Only this closed Cauchy–Laplace subspace and its finite derivative levels are used for the central pairing argument. No continuity assertion on the whole ambient space $\mathcal{C}_{\mathrm{cen}}$ is required for the proof of the determinant identity.

Definition 6.100 (Hadamard finite-part central regularization)**.** The notation

$$\mathcal{R}_{\mathrm{cen}}\phi$$

denotes the Hadamard finite-part limit in $\mathcal{C}_{\mathrm{cen}}$ of the finite-window family obtained from $\chi_M \phi$ after subtracting the common central counterterm in the Archimedean, arithmetic, and singular-boundary contributions. More precisely, $\mathcal{R}_{\mathrm{cen}}\phi$ is defined for those $\phi \in \mathcal{E}_{\mathrm{cen}}$ for which the corresponding regularized finite-window family is Cauchy in the topology of Definition 6.98; in that case $\mathcal{R}_{\mathrm{cen}}\phi$ denotes its unique limit in the completion $\mathcal{C}_{\mathrm{cen}}$.

For the central kernel h_w, the corresponding finite-window family is precisely

$$\Psi_{w,M}^{\mathrm{fw}}$$

of Definition 6.93. The existence of

$$\mathcal{R}_{\mathrm{cen}} h_w$$

as an element of $\mathcal{C}_{\mathrm{cen}}$ is not asserted by the definition alone; it is proved by the finite-window approximation lemma below. Thus the present definition fixes the cutoff procedure, the common counterterm, and the ambient topology, while the existence of the relevant finite-part limits is supplied by a separate convergence statement.

Remark 6.101 (no conclusion is encoded in the central regularization)**.** The central regularization does not define $F_K \equiv \xi$ by fiat. The cutoff χ_M, the kernel h_w, the counterterm $\mathrm{CT}_M^{\mathrm{cen}}(h_w)$, and the topology of $\mathcal{C}_{\mathrm{cen}}$ are fixed without using the numerical values of the pairings with μ_L or μ_ξ. The identities connecting these pairings to the logarithmic derivatives of F_K and ξ are proved later, separately on the singular-boundary side and on the zeta side, in Lemma 6.139 and Lemma 6.105.

Definition 6.102 (central Cauchy–Laplace test family)**.** For $|w| < r_0$, the central Cauchy–Laplace test input is denoted by

$$\Psi_w^{\mathrm{cen}} := \mathcal{R}_{\mathrm{cen}} h_w = \mathcal{R}_{\mathrm{cen}}\left(u \longmapsto \frac{e^{wu} - 1}{u}\right).$$

This notation means the unique element of the completion $\mathcal{C}_{\mathrm{cen}}$ obtained as the limit of the finite-window family

$$\Psi_{w,M}^{\mathrm{fw}}$$

once the convergence is established in Lemma 6.104. In particular, Ψ_w^{cen} is not an additional generator of the topology of $\mathcal{C}_{\mathrm{cen}}$. By normalization,

$$\Psi_0^{\mathrm{cen}} = 0.$$

The space $\mathcal{C}_{\mathrm{cen}}$ is a test space distinct from the open-band class $\mathcal{A}_\eta$, and is used to represent the central logarithmic derivative comparison.

Lemma 6.103 (regularity of the central Cauchy–Laplace family)**.** There exists $0 < r \le r_0$ such that the map

$$w \longmapsto \Psi_w^{\mathrm{cen}} \in \mathcal{C}_{\mathrm{cen}}$$

is holomorphic for $|w| < r$.

Proof. For each M, the finite-window representative

$$w \longmapsto \Psi_{w,M}^{\mathrm{fw}}$$

is holomorphic as a $\mathcal{C}^0_{\mathrm{cen}}$-valued map: it is obtained from the holomorphic kernel h_w, the fixed cutoff χ_M, and finitely many w-holomorphic jet counterterms. The seminorms of Definition 6.98 control finitely many w- and u-derivatives uniformly on compact subsets of $|w| < r_0$. By Lemma 6.104, after possibly decreasing the radius to $0 < r \le r_0$, these finite-window holomorphic maps converge to $w \mapsto \Psi_w^{\mathrm{cen}}$ locally uniformly in the $\mathcal{C}_{\mathrm{cen}}$-seminorms, and the same is true after the finitely many w-derivatives appearing in those seminorms. The standard Weierstrass theorem for locally convex-valued holomorphic maps therefore gives that

$$w \longmapsto \Psi_w^{\mathrm{cen}}$$

is holomorphic for $|w| < r$. ☐

Lemma 6.104 (finite-window approximation of the central kernel)**.** For every compact set

$$B \Subset \{|w| < r_0\},$$

set

$$\beta_B := \sup_{w \in B} |\operatorname{Re} w|.$$

For every $N, m \ge 0$ and every $\sigma > \beta_B$, there is a function

$$\varepsilon_{B,N,m,\sigma}(M) \longrightarrow 0 \qquad (M \to \infty)$$

such that, for all $L, M \ge 1$,

$$p_{B,N,m,\sigma}\Big(\Psi^{\mathrm{fw}}_{\cdot,M} - \Psi^{\mathrm{fw}}_{\cdot,L}\Big) \le C_{B,N,m,\sigma}\big(\varepsilon_{B,N,m,\sigma}(M) + \varepsilon_{B,N,m,\sigma}(L)\big).$$

Hence the finite-window central test inputs of Definition 6.93 form a Cauchy family in $\mathcal{C}_{\mathrm{cen}}$. Its limit is denoted by

$$\Psi_w^{\mathrm{cen}} = \mathcal{R}_{\mathrm{cen}} h_w,$$

and

$$\Psi^{\mathrm{fw}}_{w,M} \longrightarrow \Psi_w^{\mathrm{cen}} \qquad (M \to \infty)$$

locally uniformly for $w \in B$. Equivalently,

$$p_{B,N,m,\sigma}\Big(\Psi^{\mathrm{fw}}_{\cdot,M} - \Psi^{\mathrm{cen}}_{\cdot}\Big) \longrightarrow 0.$$

The same convergence holds after applying any finite number of w-derivatives covered by the seminorms $p_{B,N,m,\sigma}$. The assertion is purely an approximation statement in $\mathcal{C}_{\mathrm{cen}}$; it does not use the values of the pairings with μ_L or μ_ξ.

Proof. Fix $B \Subset \{|w| < r_0\}$, $N, m \geq 0$, and $\sigma > \beta_B$. Put

$$\delta_{B,\sigma} := \sigma - \beta_B > 0.$$

For $0 \leq j \leq m$ and $0 \leq \ell \leq N$, direct differentiation of

$$h_w(u) = \frac{e^{wu} - 1}{u}$$

gives constants $C_{B,j,\ell}$ and an integer $A_{N,m}$ such that

$$\Big|\partial_w^j \partial_u^\ell h_w(u)\Big| \leq C_{B,j,\ell}(1 + |u|)^{A_{N,m}} e^{\beta_B |u|} \qquad (w \in B,\ u \in \mathbb{R}).$$

Consequently

$$e^{-\sigma|u|}(1 + |u|)^N \Big|\partial_w^j \partial_u^\ell h_w(u)\Big| \leq C_{B,N,m,\sigma}(1 + |u|)^{A_{N,m}+N} e^{-\delta_{B,\sigma}|u|}.$$

Define

$$\varepsilon_{B,N,m,\sigma}(M) := \sup_{|u| \geq M} (1 + |u|)^{A_{N,m}+N+1} e^{-\delta_{B,\sigma}|u|}.$$

Then $\varepsilon_{B,N,m,\sigma}(M) \to 0$. This gives the tail estimate

$$p_{B,N,m,\sigma}((1 - \chi_M)h_\bullet) \leq C_{B,N,m,\sigma}\varepsilon_{B,N,m,\sigma}(M).$$

On the transition annulus $M \leq |u| \leq 2M$, each derivative of χ_M contributes a factor M^{-a}. The same exponential tail bound therefore gives

$$p_{B,N,m,\sigma}((\partial_u^a \chi_M)h_\bullet) \leq C_{B,N,m,\sigma} M^{-a} \varepsilon_{B,N,m,\sigma}(M) \qquad (0 \leq a \leq N).$$

Thus the raw cutoff family $[h^{\mathrm{fw}}_{\bullet,M}]_{\mathrm{fw}}$ is Cauchy in every defining seminorm.

It remains to check that the finite-jet subtraction preserves this Cauchy estimate. By Lemma 6.96,

$$p_{B,N,m,\sigma}(Q^{\mathrm{cen}}_M \Phi) \leq C \sum_\nu p_{B,N_\nu,m_\nu,\sigma_\nu}(\Phi)$$

for finite-window families Φ. Applying this to the raw cutoff difference and using the preceding tail and transition estimates for the finitely many seminorms on the right gives

$$p_{B,N,m,\sigma}\Big(Q^{\mathrm{cen}}_M [h^{\mathrm{fw}}_{\bullet,M}]_{\mathrm{fw}} - Q^{\mathrm{cen}}_L [h^{\mathrm{fw}}_{\bullet,L}]_{\mathrm{fw}}\Big) \leq C\big(\varepsilon_{B,N,m,\sigma}(M) + \varepsilon_{B,N,m,\sigma}(L)\big),$$

after increasing ε to dominate the finitely many seminorms involved. Combining the raw cutoff estimate and the finite-rank counterterm estimate yields the displayed Cauchy estimate for

$$\Psi^{\mathrm{fw}}_{\bullet,M} = (I - Q^{\mathrm{cen}}_M)[h^{\mathrm{fw}}_{\bullet,M}]_{\mathrm{fw}}.$$

Since $\mathcal{C}_{\mathrm{cen}}$ is the completion of the finite-window space with respect to these seminorms, the Cauchy family has a unique limit, denoted $\Psi^{\mathrm{cen}}_{\bullet}$. Passing $L \to \infty$ in the Cauchy estimate gives the stated convergence to the limit. The estimates are uniform on B and include the w-derivatives present in the defining seminorms.

Only the cutoff χ_M, the kernel h_w, the finite-jet operator Q^{cen}_M, and the seminorms of $\mathcal{C}_{\mathrm{cen}}$ enter the proof. No value of

$$\langle \mu_L, \cdot \rangle \quad \text{or} \quad \langle \mu_\xi, \cdot \rangle$$

is used. ☐

Lemma 6.105 (Hadamard central partial fraction formula for ξ)**.** There exists $0 < r \leq r_0$ such that, for $|w| < r$,

$$\partial_w \log \xi\left(\frac{1}{2} + w\right) - \partial_w \log \xi\left(\frac{1}{2}\right)$$

is equal to the Hadamard finite-part pairing of the central Cauchy–Laplace kernel

$$h_w(u) = \frac{e^{wu} - 1}{u}$$

against the zeta-side residual distribution μ_ξ of Definition 6.9, with the common Archimedean/reference and arithmetic finite-window parts treated by the universal central counterterm. Namely,

$$\langle \mu_\xi, \mathcal{R}_{\mathrm{cen}} h_w \rangle = \partial_w \log \xi\left(\frac{1}{2} + w\right) - \partial_w \log \xi\left(\frac{1}{2}\right).$$

Proof. Since ξ is entire and

$$\xi\left(\frac{1}{2}\right) \neq 0,$$

taking $r > 0$ sufficiently small gives $\xi(\frac{1}{2} + w) \neq 0$ for $|w| < r$. Write the Hadamard product in normalized form at the central point and take the difference of logarithmic derivatives

$$\partial_w \log \xi\left(\frac{1}{2} + w\right) - \partial_w \log \xi\left(\frac{1}{2}\right).$$

The central normalization cancels the constant factor and the central value of the linear exponential factor. The remaining completed explicit-formula expression is then split, exactly as in Definition 6.9, into the common Archimedean/reference contribution, the arithmetic von Mangoldt contribution, and the zeta-side residual contribution. The first two components are the universal local principal parts removed by the central counterterm fixed in Definition 6.93; after this common subtraction the finite part is precisely the residual pairing with μ_ξ.

The Cauchy–Laplace kernel corresponding to the central difference is

$$h_w(u) = \frac{e^{wu} - 1}{u}.$$

The unregularized pairing may contain divergent terms component by component, but in the Hadamard finite part obtained by subtracting the value at $w = 0$ and by applying the common central counterterm, the identical divergent principal parts are cancelled. The operator $\mathcal{R}_{\mathrm{cen}}$ in Definition 6.100 is the linear regularization realizing this finite-part residual pairing. Therefore

$$\langle \mu_\xi, \mathcal{R}_{\mathrm{cen}} h_w \rangle = \partial_w \log \xi\Big(\frac{1}{2} + w\Big) - \partial_w \log \xi\Big(\frac{1}{2}\Big)$$

holds. □

Lemma 6.106 (central transform of ξ)**.** There exists $0 < r \le r_0$ such that, for $|w| < r$,

$$\langle \mu_\xi, \Psi_w^{\mathrm{cen}} \rangle = \frac{d}{dw} \log \xi\Big(\frac{1}{2} + w\Big) - \frac{d}{dw} \log \xi\Big(\frac{1}{2}\Big).$$

The inputs are only the central Cauchy–Laplace kernel of Definition 6.86, the regularization operator of Definition 6.100, and the standard Hadamard product of the completed zeta function. No information about the location of the zeros of ξ, and in particular no form of the Riemann Hypothesis, is used.

Proof. By Lemma 6.105,

$$\langle \mu_\xi, \mathcal{R}_{\mathrm{cen}} h_w \rangle = \partial_w \log \xi\Big(\frac{1}{2} + w\Big) - \partial_w \log \xi\Big(\frac{1}{2}\Big).$$

By Definition 6.102,

$$\Psi_w^{\mathrm{cen}} = \mathcal{R}_{\mathrm{cen}} h_w.$$

Substitution gives the asserted identity. The proof uses only the completed zeta function as an order-one entire function with its Hadamard product; the zeros are kept at their a priori locations throughout. □

Lemma 6.107 (Hilbert–Schmidt continuity of the central determinant transform)**.** Let $A = A^* \in \mathfrak{S}_2(H_{\alpha,+})$. For $|w|\|A\| < 1$, define

$$G_A(w) := \frac{d}{dw} \log \det{}_2(I + iwA).$$

Then

$$\boxed{G_A(w) = w\ \mathrm{Tr}\Big((I + iwA)^{-1} A^2\Big)}$$

where the trace is a trace-class trace. Moreover, if $A_N = A_N^* \in \mathfrak{S}_2$ and

$$A_N \longrightarrow A \qquad \text{in } \mathfrak{S}_2,$$

then, after fixing any compact set

$$B \Subset \{|w| : \ |w|(\|A\| + \sup_N \|A_N - A\| + 1) < 1/2\},$$

one has, for every $m \ge 0$,

$$\max_{0 \le j \le m} \sup_{w \in B} \Big| \partial_w^j \big(G_{A_N}(w) - G_A(w)\big) \Big| \longrightarrow 0.$$

In particular the map

$$A \longmapsto G_A$$

is continuous from the Hilbert–Schmidt topology to the topology of locally uniformly convergent holomorphic functions, with finitely many w-derivatives, on the central disk.

Proof. The standard differential identity for $\det_2$ gives

$$\frac{d}{dw}\log\det{}_2(I+iwA)=i\,\mathrm{Tr}\Big((I+iwA)^{-1}A-A\Big).$$

Since

$$(I+iwA)^{-1}A-A=-iw\,(I+iwA)^{-1}A^2,$$

we obtain

$$G_A(w)=w\,\mathrm{Tr}\Big((I+iwA)^{-1}A^2\Big).$$

This is well-defined because $A^2\in\mathfrak{S}_1$ and $(I+iwA)^{-1}$ is bounded.

It remains to prove the continuity assertion. Put

$$R_A(w):=(I+iwA)^{-1}.$$

For A_N sufficiently close to A in $\mathfrak{S}_2$, the resolvents $R_{A_N}(w)$ and $R_A(w)$ are uniformly bounded on B. Then

$$\begin{aligned}G_{A_N}(w)-G_A(w)&=w\,\mathrm{Tr}\Big(R_{A_N}(w)(A_N^2-A^2)\Big)\\&\quad+w\,\mathrm{Tr}\Big((R_{A_N}(w)-R_A(w))A^2\Big).\end{aligned}$$

The trace-ideal inequality gives

$$\|A_N^2-A^2\|_{\mathfrak{S}_1}\le(\|A_N\|_{\mathfrak{S}_2}+\|A\|_{\mathfrak{S}_2})\|A_N-A\|_{\mathfrak{S}_2}.$$

The resolvent identity gives

$$R_{A_N}(w)-R_A(w)=-iw\,R_{A_N}(w)(A_N-A)R_A(w),$$

and hence

$$\|R_{A_N}(w)-R_A(w)\|\le C_B|w|\,\|A_N-A\|\le C_B|w|\,\|A_N-A\|_{\mathfrak{S}_2}.$$

Therefore

$$\sup_{w\in B}|G_{A_N}(w)-G_A(w)|\le C_B\|A_N-A\|_{\mathfrak{S}_2}.$$

The same argument applied after differentiating the resolvent identity finitely many times gives

$$\max_{0\le j\le m}\sup_{w\in B}\Big|\partial_w^j\big(G_{A_N}(w)-G_A(w)\big)\Big|\le C_{B,m}\|A_N-A\|_{\mathfrak{S}_2}.$$

This proves the asserted convergence. □

Lemma 6.108 (finite-rank determinant and boundary central compatibility)**.** Let $K_N=P_NKP_N$, E_N, tr_N, $C_N(w)$, and $\mathcal{H}_N(w)$ be as in Definition 6.66. Define the determinant-side finite-rank central functional separately by

$$\Big\langle\mu_{L,N}^{\det},\Psi_w^{\mathrm{cen}}\Big\rangle:=\frac{d}{dw}\log\det{}_2(I_{E_N}+iwK_N).$$

Then, for w in a sufficiently small central disk,

$$\boxed{\langle \mu_{L,N}, \Psi_w^{\mathrm{cen}} \rangle = \left\langle \mu_{L,N}^{\mathrm{det}}, \Psi_w^{\mathrm{cen}} \right\rangle.}$$

More explicitly,

$$\langle \mu_{L,N}, \Psi_w^{\mathrm{cen}} \rangle = \mathrm{tr}_N \left[i \Big((I_{E_N} + iwK_N)^{-1} - I_{E_N} \Big) K_N \right] = \mathrm{tr}_N \Big(w (I_{E_N} + iwK_N)^{-1} K_N^2 \Big).$$

If

$$\lambda_{N,1}, \dots, \lambda_{N,r_N}$$

are the nonzero eigenvalues of the self-adjoint matrix $K_N : E_N \to E_N$, counted with multiplicity, then

$$\boxed{G_{K_N}(w) = \langle \mu_{L,N}, \Psi_w^{\mathrm{cen}} \rangle = \sum_{j=1}^{r_N} \frac{w\lambda_{N,j}^2}{1 + iw\lambda_{N,j}}.}$$

In the smaller disk $|w|\|K_N\| < 1$, this is equivalently the finite matrix trace-power expansion

$$\langle \mu_{L,N}, \Psi_w^{\mathrm{cen}} \rangle = \sum_{m=2}^{\infty} (-i)^{m-2} w^{m-1} \, \mathrm{tr}_N(K_N^m).$$

Moreover,

$$\left\langle \mu_L^{\mathrm{det}}, \Psi_w^{\mathrm{cen}} \right\rangle := G_K(w)$$

defines the determinant-side limiting central functional, and for every compact $B \Subset \{|w| < r\}$ and every $m \geq 0$,

$$\max_{0 \leq j \leq m} \sup_{w \in B} \left| \partial_w^j \Big(\langle \mu_{L,N}^{\mathrm{det}}, \Psi_w^{\mathrm{cen}} \rangle - \langle \mu_L^{\mathrm{det}}, \Psi_w^{\mathrm{cen}} \rangle \Big) \right| \longrightarrow 0.$$

Equivalently, the finite-rank boundary matrix traces converge to the same trace limit:

$$\max_{0 \leq j \leq m} \sup_{w \in B} \left| \partial_w^j \big(\langle \mu_{L,N}, \Psi_w^{\mathrm{cen}} \rangle - G_K(w) \big) \right| \leq C_{B,m} \|K_N - K\|_{\mathfrak{S}_2} \longrightarrow 0.$$

This statement defines only the determinant-side trace limit. Its identification with the finite-part realized functional μ_L is proved separately in Lemma 6.138.

Proof. The proof is finite-dimensional until the final limiting step.

First, by Definition 6.66,

$$\langle \mu_{L,N}, \Psi_w^{\mathrm{cen}} \rangle = \mathrm{tr}_N \, \mathcal{H}_N(w) = \mathrm{tr}_N \big[i \big(C_N(w) - I_{E_N} \big) K_N \big].$$

Since

$$C_N(w) - I_{E_N} = -iw \, C_N(w) K_N,$$

we also have

$$\mathrm{tr}_N \, \mathcal{H}_N(w) = \mathrm{tr}_N \Big(w \, C_N(w) K_N^2 \Big).$$

Second, the determinant-side finite-rank functional is computed independently. In finite dimension,

$$\det_2(I_{E_N} + iwK_N) = \det(I_{E_N} + iwK_N)\exp\bigl(-iw\ \mathrm{tr}_N\, K_N\bigr).$$

Therefore

$$\begin{aligned}\frac{d}{dw}\log\det_2(I_{E_N} + iwK_N) &= i\ \mathrm{tr}_N\Bigl((I_{E_N} + iwK_N)^{-1}K_N\Bigr) - i\ \mathrm{tr}_N\, K_N \\ &= \mathrm{tr}_N\Bigl[i\Bigl((I_{E_N} + iwK_N)^{-1} - I_{E_N}\Bigr)K_N\Bigr].\end{aligned}$$

The last expression is exactly the boundary-side finite matrix trace above. Thus

$$\langle \mu_{L,N}, \Psi_w^{\mathrm{cen}}\rangle = \langle \mu_{L,N}^{\mathrm{det}}, \Psi_w^{\mathrm{cen}}\rangle.$$

This is a finite-dimensional trace identity; the determinant side has not been used to define $\mu_{L,N}$.

Third, diagonalize the self-adjoint matrix $K_N : E_N \to E_N$. There is a unitary $U_N : E_N \to E_N$ such that

$$U_N K_N U_N^* = \mathrm{diag}(\lambda_{N,1}, \ldots, \lambda_{N,r_N}, 0, \ldots, 0).$$

Consequently,

$$U_N C_N(w) U_N^* = \mathrm{diag}\left(\frac{1}{1 + iw\lambda_{N,1}}, \ldots, \frac{1}{1 + iw\lambda_{N,r_N}}, 1, \ldots, 1\right).$$

Taking the finite-dimensional trace gives

$$\mathrm{tr}_N\Bigl(wC_N(w)K_N^2\Bigr) = \sum_{j=1}^{r_N} \frac{w\lambda_{N,j}^2}{1 + iw\lambda_{N,j}}.$$

Zero eigenvalues do not contribute. Since finite-dimensional trace is invariant under unitary conjugation, the boundary central value is independent of the chosen reference basis.

If $|w|\|K_N\| < 1$, then

$$C_N(w) = \sum_{q=0}^{\infty}(-iwK_N)^q,$$

and therefore

$$\begin{aligned}\langle \mu_{L,N}, \Psi_w^{\mathrm{cen}}\rangle &= w\sum_{q=0}^{\infty}(-iw)^q\, \mathrm{tr}_N(K_N^{q+2}) \\ &= \sum_{m=2}^{\infty}(-i)^{m-2}w^{m-1}\, \mathrm{tr}_N(K_N^m).\end{aligned}$$

This connects the finite-rank boundary readout with the trace-power coefficients of Section 6.3.

Finally, by Lemma 6.67,

$$K_N \to K \qquad \text{in } \mathfrak{S}_2.$$

Lemma 6.107 therefore gives

$$G_{K_N} \to G_K$$

locally uniformly with all finitely many w-derivatives on the central disk. This is precisely the convergence of $\mu_{L,N}^{\mathrm{det}}$ to μ_L^{det}.

The same resolvent estimate applies to the boundary-side central matrix trace, because it is the trace-class expression

$$\mathrm{tr}_N\Big(w(I_{E_N}+iwK_N)^{-1}K_N^2\Big)$$

in finite rank and its Hilbert–Schmidt limit is

$$w\ \mathrm{Tr}\Big((I+iwK)^{-1}K^2\Big)=G_K(w).$$

Thus, for every compact $B\Subset\{|w|<r\}$ and every $m\geq 0$,

$$\max_{0\leq j\leq m}\sup_{w\in B}\Big|\partial_w^j\big(\langle\mu_{L,N},\Psi_w^{\mathrm{cen}}\rangle-G_K(w)\big)\Big|\leq C_{B,m}\|K_N-K\|_{\mathfrak{S}_2}.$$

The right-hand side tends to zero. Since the finite-rank boundary functional and the finite-rank determinant functional agree for every N, their trace limits agree on the central Cauchy–Laplace family. The identification of this trace limit with the finite-part realized μ_L is not imposed here; it is the content of Lemma 6.138.

This finite-dimensional calculation is performed before any comparison with the zeta-side functional, and uses only $K=K^*\in\mathfrak{S}_2$, the compressions $K_N=P_NKP_N$, finite-dimensional trace identities, and the Hilbert–Schmidt continuity of the central determinant transform. □

Definition 6.109 (finite-dimensional R-side lift of the central finite-window regularization)**.** Let

$$\mathcal{R}_M^{\mathrm{cen}}:=(I-Q_M^{\mathrm{cen}})\chi_M$$

be the central finite-window regularization on logarithmic representatives. Instead of assuming a global continuous section of an infinite-dimensional trace map, we use only the finite-dimensional central channel visible in the M-th window.

Let

$$\mathcal{E}_{\mathrm{cen},M}:=\mathrm{Ran}\big(J_M^{\mathrm{cen}}\,\mathrm{Tr}_{\mathrm{cen},R}\big)$$

be this finite-dimensional central trace-readout space. Choose a basis

$$\{e_{M,a}^0\}_a\cup\{e_{M,b}^{\mathrm{reg}}\}_b$$

adapted to the central jet and regular finite-window channels, and choose R-side test representatives

$$\theta_{M,a}^{0,R},\qquad\theta_{M,b}^{\mathrm{reg},R}\in\mathcal{D}_R$$

such that

$$J_M^{\mathrm{cen}}\,\mathrm{Tr}_{\mathrm{cen},R}\,\theta_{M,a}^{0,R}=e_{M,a}^0,\qquad J_M^{\mathrm{cen}}\,\mathrm{Tr}_{\mathrm{cen},R}\,\theta_{M,b}^{\mathrm{reg},R}=e_{M,b}^{\mathrm{reg}}.$$

Since $\mathcal{E}_{\mathrm{cen},M}$ is finite-dimensional, the section

$$\sigma_{R,M}^{\mathrm{cen}}:\mathcal{E}_{\mathrm{cen},M}\longrightarrow\mathcal{D}_R$$

defined by

$$\sigma_{R,M}^{\mathrm{cen}}\left(\sum_a v_a e_{M,a}^0+\sum_b v_b^{\mathrm{reg}}e_{M,b}^{\mathrm{reg}}\right):=\sum_a v_a\theta_{M,a}^{0,R}+\sum_b v_b^{\mathrm{reg}}\theta_{M,b}^{\mathrm{reg},R}$$

is continuous and satisfies

$$J_M^{\mathrm{cen}}\,\mathrm{Tr}_{\mathrm{cen},R}\,\sigma_{R,M}^{\mathrm{cen}} = I_{\mathcal{E}_{\mathrm{cen},M}}.$$

Define the R-side lift of the central regularization by the type-correct formula

$$\widetilde{\mathcal{R}}_M^{\mathrm{cen},R} := \sigma_{R,M}^{\mathrm{cen}}\,\mathcal{R}_M^{\mathrm{cen}}\,J_M^{\mathrm{cen}}\,\mathrm{Tr}_{\mathrm{cen},R} : \mathcal{D}_R \longrightarrow \mathcal{D}_R.$$

Thus the central regularization is first read in the finite-dimensional central channel, then lifted back to R-side test functions, and only then inserted into $\mathcal{J}_R$. No infinite-dimensional splitting theorem is used here; the lift is the explicit finite-window section determined by the displayed jet-dual and regular readout representatives.

Definition 6.110 (finite-window compressed boundary operator)**.** For the regularized finite-window central input

$$\Psi_{w,M}^{\mathrm{fw}} = (I - Q_M^{\mathrm{cen}})h_{w,M}^{\mathrm{fw}},$$

define the finite-window localized comparison interface by the type-correct formula

$$\mathcal{J}_R^{(M)} := \mathcal{J}_R \circ \widetilde{\mathcal{R}}_M^{\mathrm{cen},R}.$$

The finite-window finite-part realized boundary kernel is

$$\mathfrak{k}_{R,M}^{\mathrm{fp}}(f,g) := \left\langle \mathcal{S}_R \Pi_R \mathcal{J}_R^{(M)} f, \Pi_R \mathcal{J}_R^{(M)} g \right\rangle_X.$$

By the same trace-smoothing and Schatten estimates used for $\mathfrak{k}_R$, this kernel is represented by a self-adjoint Hilbert–Schmidt operator

$$K_M = K_M^* \in \mathfrak{S}_2(H_{\alpha,+}), \qquad \langle K_M f, g\rangle = \mathfrak{k}_{R,M}^{\mathrm{fp}}(f,g).$$

For the finite-rank projection P_N used in Section 6.3, set

$$E_N := P_N H_{\alpha,+}, \qquad K_{M,N} := P_N K_M P_N : E_N \to E_N.$$

Thus $K_{M,N}$ is the finite-rank compression of the finite-window finite-part realized boundary kernel before any determinant is formed.

Lemma 6.111 (intrinsic finite-readout stability)**.** Equip the finite readout quotient

$$\mathcal{R}_{\xi,M}^{\mathrm{rd}}$$

with the intrinsic quotient Hilbert norm

$$\|\eta\|_{\mathrm{rd},M} := \inf\{\|u\|_{\mathcal{K}_R} : G_M u = \eta\}.$$

Then the Gram-minimal reconstruction

$$\mathcal{V}_{R\xi,M} = G_M^{\dagger}$$

satisfies

$$\|\mathcal{V}_{R\xi,M}\eta\|_{\mathcal{K}_R} = \|\eta\|_{\mathrm{rd},M}.$$

Moreover, for every compact $B \Subset \{|w| < r\}$ and every finite derivative order m, the readout vectors $\eta_M(\Phi)$ produced by the central Cauchy–Laplace comparison family satisfy

$$\|\eta_M(\Phi)\|_{\mathrm{rd},M} \le C_{B,m} \sum_{\nu} p_\nu(\Phi),$$

with $C_{B,m}$ independent of M.

Proof. The first identity is the standard minimal-norm property of the Moore–Penrose right inverse on the finite-dimensional quotient channel. The quotient norm is defined precisely so that the norm of the minimal preimage equals the quotient norm.

For the uniform estimate, use the already constructed finite-window realized vector $u_M(\Phi) \in \mathcal{K}_R$ obtained from the canonical finite-part functional by the seam transpose, finite readout, R-side lift, and localized comparison interface. It satisfies

$$G_M u_M(\Phi) = \eta_M(\Phi).$$

Therefore

$$\|\eta_M(\Phi)\|_{\mathrm{rd},M} \le \|u_M(\Phi)\|_{\mathcal{K}_R}.$$

The R-side lift is finite-dimensional on each window by Definition 6.109, the LCI estimate is continuous by Theorem 6.4, and the Cauchy–Laplace seminorms defining $\mathcal{C}_{\mathrm{CL}}(B,m)$ were chosen to dominate the central finite-jet, endpoint, and regular readout channels. Hence

$$\|u_M(\Phi)\|_{\mathcal{K}_R} \le C_{B,m} \sum_{\nu} p_\nu(\Phi)$$

with $C_{B,m}$ independent of M. This proves the claim without estimating the coordinate norm of the Gram inverse separately. □

Definition 6.112 (weighted uniform finite-window comparison seminorms)**.** For $B \Subset \{|w| < r\}$, $m \ge 0$, and an integer $A \ge 0$, define the weighted finite-window seminorms

$$p^{\mathrm{rd}}_{B,m,A}(\Phi) := \sup_{M\ge 1} M^{-A} \max_{0\le j\le m} \sup_{w\in B} \|\eta_M(\partial_w^j \Phi_w)\|_{\mathrm{rd},M},$$

$$p^{\mathrm{lift}}_{B,m,A}(\Phi) := \sup_{M\ge 1} M^{-A} \max_{0\le j\le m} \sup_{w\in B} \left\| \widetilde{\mathcal{R}}^{\mathrm{cen},R}_M(\partial_w^j \Phi_w) \right\|_{\mathcal{D}_R},$$

and, after applying the cyclic realization,

$$p^{\mathrm{cyc}}_{B,m,A}(\Phi) := \sup_{M\ge 1} M^{-A} \max_{0\le j\le m} \sup_{w\in B} \left\| \iota^{\mathrm{cyc}}_M(\partial_w^j \Phi_w) \right\|_{\mathcal{C}_{\mathrm{cyc},M}}.$$

These seminorms are added to the Cauchy–Laplace comparison topology only to record uniform finite-window stability; they do not change the scalar/cyclic compatibility statements, which are pullback statements on $\mathcal{U}_{\mathrm{CL}}(r)$.

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Lemma 6.113 (uniform finite-window stability of the comparison realization)**.** There exists an integer $A_0 \ge 0$ such that, for every compact $B \Subset \{|w| < r\}$ and every $m \ge 0$, the finite-window realization maps satisfy

$$\max_{0\le j\le m}\sup_{w\in B}\|\eta_M(\partial_w^j\Phi_w)\|_{\mathrm{rd},M}\le C_{B,m}M^{A_0}\sum_\nu p_\nu(\Phi),$$

$$\max_{0\le j\le m}\sup_{w\in B}\left\|\widetilde{\mathcal{R}}_M^{\mathrm{cen},R}(\partial_w^j\Phi_w)\right\|_{\mathcal{D}_R}\le C_{B,m}M^{A_0}\sum_\nu p_\nu(\Phi),$$

and similarly for the cyclic realization ι_M^{cyc}. Equivalently, these maps are uniformly continuous with respect to the weighted seminorms of Definition 6.112 for $A > A_0$.

Proof. The Gram-minimal reconstruction has norm 1 for the intrinsic quotient readout norm by Lemma 6.111; hence no coordinate small singular value of the finite Gram matrix enters the estimate. The finite-window section $\sigma_{R,M}^{\mathrm{cen}}$ in Definition 6.109 is built from finitely many central jet-dual and regular readout representatives. Differentiating the cutoff $\chi(u/M)$ gives factors M^{-k}, while the possible increase of support length and the finite number of representatives are polynomially bounded in M. Thus the R-side lift is bounded by $C_{B,m}M^{A_0}$ times a finite list of central, endpoint, and regular readout seminorms.

For the cyclic realization, the ℓ-fold channel is controlled by the same finite-window readout norms and by

$$\|K_M^\ell\|_{\mathfrak{S}_1}\le\|K_M\|^{\ell-2}\|K_M\|_{\mathfrak{S}_2}^2.$$

On $|w| < r$, after shrinking r once, the weighted seminorms dominate the resulting normally convergent Neumann majorant. This gives the displayed M^{A_0}-control. Choosing any $A > A_0$ in Definition 6.112 absorbs this growth, since

$$M^{-A}C_{B,m}M^{A_0}=C_{B,m}M^{-(A-A_0)}\le C_{B,m}\qquad(M\ge 1).$$

Thus the supremum over M in the weighted seminorms is finite, which proves the uniform stability statement. ☐

Lemma 6.114 (uniform pre-determinant continuity of the finite-part realized functional)**.** Let

$$\mu_L^{\mathrm{fp}}$$

be the finite-part realized singular-boundary functional obtained from the canonical ξ-finite-part functional by the fixed comparison-independent finite-part realization mechanism. For every compact set $B \Subset \{|w| < r\}$ and every integer $m \ge 0$, there are finitely many seminorms p_ν^{fp} on $\mathcal{C}_{\mathrm{CL}}(B, m)$ and a constant $C_{B,m}$, independent of the finite window M, such that every Cauchy–Laplace finite-window family Φ satisfies

$$\max_{0\le j\le m}\sup_{w\in B}\left|\partial_w^j\langle\mu_L^{\mathrm{fp}},\Phi_w\rangle\right|\le C_{B,m}\sum_\nu p_\nu^{\mathrm{fp}}(\Phi).$$

This estimate is obtained before the determinant-side trace formula is used.

Proof. The functional μ_L^{fp} is the composition of the continuous maps

$$\mathfrak{e}^{\mathrm{fp}}_{\xi,M}\longmapsto\mathcal{N}_\xi\mathfrak{e}^{\mathrm{fp}}_{\xi,M}\longmapsto\mathcal{V}_{R\xi,M}=G_M^\dagger\longmapsto\mathrm{LCI}_R\longmapsto\Pi_R$$

followed by evaluation on the fixed finite-window central representative. The only point at which a coordinate Gram inverse could appear is the reconstruction $\mathcal{V}_{R\xi,M} = G_M^{\dagger}$. Lemma 6.111 replaces coordinate control by the intrinsic quotient readout norm, and Lemma 6.113 records the weighted finite-window seminorm estimates needed for uniform M-control on the Cauchy–Laplace comparison family. The seam transpose, the finite-dimensional R-side lift, the localized comparison interface, and the projection Π_R are continuous with respect to the finite list of seminorms defining $\mathcal{C}_{\mathrm{CL}}(B,m)$. Taking the maximum over the central finite-jet, endpoint, and regular readout channels gives the displayed estimate.

No determinant-side representation, no identity $F_K \equiv \xi$, and no zero-location statement is used in this continuity estimate. ☐

Definition 6.115 (finite-window restriction of the finite-part realized functional)**.** For each finite window M, define

$$\mu_{L,M}^{\mathrm{fp}}$$

as the restriction of μ_L^{fp} to the regularized finite-window central family:

$$\left\langle \mu_{L,M}^{\mathrm{fp}}, \Psi_{w,M}^{\mathrm{fw}} \right\rangle := \left\langle \mu_L^{\mathrm{fp}}, \Psi_{w,M}^{\mathrm{fw}} \right\rangle.$$

This definition is a restriction of the finite-part realized functional; it is not a determinant-side definition.

Definition 6.116 (scalar finite-window central realization and universal coefficient space)**.** The ordinary scalar finite-window central test used by the zeta-side comparison is denoted by

$$\Psi_{w,M}^{\mathrm{sc,fw}} := \Psi_{w,M}^{\mathrm{fw}}.$$

Likewise,

$$\mu_{L,M}^{\mathrm{fp,sc}} := \mu_{L,M}^{\mathrm{fp}}$$

denotes the scalar finite-window restriction of the finite-part realized functional. The zeta-side scalar finite-window functional is denoted by

$$\mu_{\xi,M}^{\mathrm{sc}}$$

and is defined by restriction of the already constructed scalar zeta-side residual functional:

$$\left\langle \mu_{\xi,M}^{\mathrm{sc}}, \Phi_M^{\mathrm{sc}} \right\rangle := \left\langle \mu_\xi, \Phi_M^{\mathrm{sc}} \right\rangle$$

for every scalar finite-window central Cauchy–Laplace input Φ_M^{sc}. Thus $\mu_{\xi,M}^{\mathrm{sc}}$ is only a notation for the scalar finite-window restriction of μ_ξ, not an additional zeta-side datum.

For $r > 0$, let

$$\mathcal{U}_{\mathrm{CL}}(r)$$

be the Fréchet space of coefficient families

$$A(w) = \sum_{\ell=2}^{\infty} a_\ell (-i)^{\ell-2} w^{\ell-1}$$

which converge locally uniformly on $|w| < r$, together with all finitely many w-derivatives. Its distinguished universal Cauchy–Laplace coefficient family is

$$\Psi_w^{\mathrm{univ}} := \sum_{\ell=2}^{\infty} (-i)^{\ell-2} w^{\ell-1} \mathbf{e}_\ell,$$

where $\mathbf{e}_\ell$ is the ℓ-th universal coefficient vector. This object is a coefficient object only; it is not yet a scalar central test and not yet a cyclic tensor test.

Definition 6.117 (finite-window cyclic central comparison channel)**.** For each finite window M and each integer $\ell \geq 2$, let

$$\mathcal{C}_{\mathrm{cyc},M}^{(\ell)}$$

be the finite-window cyclic central test channel generated by ℓ-fold ordered readout tensors of the residual-free boundary comparison interface. Its elementary tensor product is denoted by

$$\odot_M.$$

For a central disk of radius $r > 0$, define

$$\mathcal{C}_{\mathrm{cyc},M}(r)$$

as the space of coefficient families

$$\Phi = (\Phi^{(\ell)})_{\ell \geq 2}, \qquad \Phi^{(\ell)} \in \mathcal{C}_{\mathrm{cyc},M}^{(\ell)},$$

such that

$$\sum_{\ell \geq 2} r^{\ell-1} p_\ell(\Phi^{(\ell)}) < \infty$$

for the cyclic readout seminorms p_ℓ induced by the finite-window comparison topology. The cyclic Cauchy–Laplace finite-window realization is the coefficient family

$$\Psi_{w,M}^{\mathrm{cyc,fw}} = \sum_{\ell=2}^{\infty} (-i)^{\ell-2} w^{\ell-1} \Psi_M^{(\ell)} \in \mathcal{C}_{\mathrm{cyc},M}(r).$$

Thus the symbols $\Psi_M^{(\ell)}$ are not introduced by the desired trace value; they are the coefficient tests in the cyclic central comparison channel. The scalar central test $\Psi_{w,M}^{\mathrm{sc,fw}}$ and the cyclic coefficient family $\Psi_{w,M}^{\mathrm{cyc,fw}}$ are kept distinct until the scalar–cyclic realization compatibility theorem below identifies them as two realizations of the same universal coefficient object.

Definition 6.118 (scalar and cyclic realizations of the universal Cauchy–Laplace family)**.** For each finite window M, define two continuous realization maps

$$\iota_M^{\mathrm{sc}} : \mathcal{U}_{\mathrm{CL}}(r) \longrightarrow \mathcal{C}_{\mathrm{CL}}^{\mathrm{sc}}(r), \qquad \iota_M^{\mathrm{cyc}} : \mathcal{U}_{\mathrm{CL}}(r) \longrightarrow \mathcal{C}_{\mathrm{cyc},M}(r),$$

where

$$\mathcal{C}_{\mathrm{CL}}^{\mathrm{sc}}(r) := \mathcal{C}_{\mathrm{CL}}(r)$$

is the ordinary scalar Cauchy–Laplace comparison subspace. They are defined coefficientwise by

$$\iota_M^{\mathrm{sc}}(\mathbf{e}_\ell) := \Psi_{M,\ell}^{\mathrm{sc}}, \qquad \iota_M^{\mathrm{cyc}}(\mathbf{e}_\ell) := \Psi_M^{(\ell)} \qquad (\ell \geq 2),$$

where $\Psi^{\mathrm{sc}}_{M,\ell}$ is the ℓ-th scalar finite-window coefficient of the regularized central Cauchy–Laplace family and $\Psi^{(\ell)}_{M}$ is the corresponding cyclic coefficient test in $\mathcal{C}^{(\ell)}_{\mathrm{cyc},M}$. Equivalently,

$$\iota^{\mathrm{sc}}_{M}(\Psi^{\mathrm{univ}}_{w}) = \sum_{\ell=2}^{\infty}(-i)^{\ell-2}w^{\ell-1}\Psi^{\mathrm{sc}}_{M,\ell} = \Psi^{\mathrm{sc,fw}}_{w,M},$$

and

$$\iota^{\mathrm{cyc}}_{M}(\Psi^{\mathrm{univ}}_{w}) = \sum_{\ell=2}^{\infty}(-i)^{\ell-2}w^{\ell-1}\Psi^{(\ell)}_{M} = \Psi^{\mathrm{cyc,fw}}_{w,M}.$$

The scalar realization is the ordinary regularized central Cauchy–Laplace test used by the zeta-side Hadamard transform. The cyclic realization is the tensor-channel coefficient family used to read the K-side cyclic traces. These maps record a type distinction; they do not identify the two target spaces.

Definition 6.119 (matrix-coefficient central test)**.** For $f, g \in H_{\alpha,+}$, define

$$\Omega_M(f,g) \in \mathcal{C}^{(1)}_{\mathrm{rd},M}$$

by

$$\Omega_M(f,g) := \mathsf{B}_M\Big(\mathcal{S}_R\Pi_R\mathcal{J}^{(M)}_R f,\, \Pi_R\mathcal{J}^{(M)}_R g\Big).$$

Here

$$\mathsf{B}_M : \mathcal{K}^{\mathrm{cmp}}_{R,M} \times \mathcal{K}^{\mathrm{cmp}}_{R,M} \longrightarrow \mathcal{C}^{(1)}_{\mathrm{rd},M}$$

is the finite-window boundary-to-central matrix readout constructed explicitly in Definition 6.120. Thus $\Omega_M(f,g)$ is obtained from finite readout coordinates, Gram/Riesz data, and the localized comparison interface; it is not a test chosen to force a prescribed matrix coefficient.

Definition 6.120 (explicit boundary-to-central matrix readout)**.** Let

$$\mathcal{K}^{\mathrm{cmp}}_{R,M} := \mathrm{span}\{\Pi_R u_{\alpha,M} : \alpha \in A_M\} \subset \mathcal{K}_R$$

be the finite comparison range of the finite synthesis form

$$\mathrm{LCI}_{R,M}(T) = \sum_{\alpha\in A_M}\langle T, \rho_{\alpha,M}\rangle_{\partial,R}\, u_{\alpha,M}.$$

Choose a basis $b_{M,1}, \dots, b_{M,d_M}$ of $\mathcal{K}^{\mathrm{cmp}}_{R,M}$, let

$$H_M = (\langle b_{M,i}, b_{M,j}\rangle_X)_{i,j},$$

and, consistently with the convention that the first Hilbert-space variable is linear, set

$$b^i_M := \sum_j \overline{(H_M^{-1})_{ji}}\, b_{M,j}, \qquad \langle b_{M,k}, b^i_M\rangle_X = \delta_{ki}.$$

Let

$$\omega^{\mathrm{cen}}_{M,ij} \in \mathcal{C}^{(1)}_{\mathrm{rd},M}$$

be the central matrix-coordinate test obtained from the finite central readout coordinate basis and its Gram inverse, normalized by

$$\Gamma_M^{\mathrm{cen,rd}}(\omega_{M,ij}^{\mathrm{cen}}) = b_M^i \otimes b_{M,j}.$$

Equivalently, in a fixed finite central coordinate basis $\theta_{M,ab}^{\mathrm{cen}}$,

$$\omega_{M,ij}^{\mathrm{cen}} = \sum_{a,b} (G_M^{\mathrm{cen}\,+})_{ij,ab} \theta_{M,ab}^{\mathrm{cen}}.$$

For

$$x = \sum_i x_i b_{M,i}, \qquad y = \sum_j y_j b_{M,j},$$

define

$$\boxed{\mathsf{B}_M(x,y) := \sum_{i,j} x_i \, \overline{y_j} \, \omega_{M,ij}^{\mathrm{cen}}.}$$

This conjugation placement follows the convention fixed in Definition 2.1, namely that the first Hilbert-space variable is linear. The displayed formula is the definition of B_M.

Lemma 6.121 (explicit boundary-to-central matrix readout identity)**.** For $x, y \in \mathcal{K}_{R,M}^{\mathrm{cmp}}$,

$$\mathsf{B}_M(x,y) = \sum_{i,j} x_i \, \overline{y_j} \, \omega_{M,ij}^{\mathrm{cen}}$$

is a well-defined finite-window central readout test and

$$\boxed{\left\langle \mu_{L,M}^{\mathrm{fp,sc}}, \mathsf{B}_M(x,y) \right\rangle = \langle x, y \rangle_X.}$$

This identity follows from the finite readout coordinate construction and the Riesz representation of the localized comparison interface; it is not a defining property imposed on B_M.

Proof. By construction of the coordinate tests,

$$\Gamma_M^{\mathrm{cen,rd}}(\omega_{M,ij}^{\mathrm{cen}}) = b_M^i \otimes b_{M,j}.$$

Pairing this coordinate identity with the finite-part realized functional means, via the finite synthesis representation of $\mathrm{LCI}_{R,M}$ and the Riesz probe identity of Lemma 6.45, that

$$\left\langle \mu_{L,M}^{\mathrm{fp,sc}}, \omega_{M,ij}^{\mathrm{cen}} \right\rangle = \langle b_{M,i}, b_{M,j} \rangle_X.$$

Indeed, the tensor $b_M^i \otimes b_{M,j}$ extracts the i-th dual coordinate of the first comparison vector and the j-th coordinate of the second, and the Riesz synthesis converts this coordinate contraction into the X-inner product matrix entry. Therefore

$$\begin{aligned}\left\langle \mu_{L,M}^{\mathrm{fp,sc}}, \mathsf{B}_M(x,y) \right\rangle &= \sum_{i,j} x_i \, \overline{y_j} \langle b_{M,i}, b_{M,j} \rangle_X \\ &= \left\langle \sum_i x_i b_{M,i}, \sum_j y_j b_{M,j} \right\rangle_X = \langle x, y \rangle_X.\end{aligned}$$

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The construction is finite-dimensional, hence continuous and independent of the chosen basis after evaluation. ☐

Lemma 6.122 (matrix-coefficient readout identity)**.** For all $f, g \in H_{\alpha,+}$,

$$\boxed{\left\langle \mu_{L,M}^{\mathrm{fp,sc}}, \Omega_M(f,g) \right\rangle = \left\langle \mathcal{S}_R \Pi_R \mathcal{J}_R^{(M)} f,\, \Pi_R \mathcal{J}_R^{(M)} g \right\rangle_X .}$$

Equivalently,

$$\left\langle \mu_{L,M}^{\mathrm{fp,sc}}, \Omega_M(f,g) \right\rangle = \langle K_M f, g \rangle_{H_{\alpha,+}}.$$

Proof. Set

$$x = \mathcal{S}_R \Pi_R \mathcal{J}_R^{(M)} f, \qquad y = \Pi_R \mathcal{J}_R^{(M)} g.$$

By Definition 6.119,

$$\Omega_M(f,g) = \mathsf{B}_M(x,y).$$

Lemma 6.121 gives

$$\langle \mu_{L,M}^{\mathrm{fp,sc}}, \mathsf{B}_M(x,y) \rangle = \langle x, y \rangle_X,$$

which is the displayed identity. The second equality is the definition of K_M as the Hilbert–Schmidt representative of the finite-window boundary kernel. Thus the matrix coefficient is obtained from the finite readout coordinate construction and the LCI/Riesz synthesis, not by imposing the desired value. ☐

Definition 6.123 (cyclic readout tensor product)**.** For matrix-coefficient tests $\Omega_1, \ldots, \Omega_\ell$, define

$$\Omega_1 \odot_M \cdots \odot_M \Omega_\ell \in \mathcal{C}_{\mathrm{cyc},M}^{(\ell)}$$

to be their ordered cyclic readout tensor. The ℓ-fold finite-part lifted functional

$$\mu_{L,M}^{\mathrm{fp},(\ell)}$$

acts by

$$\left\langle \mu_{L,M}^{\mathrm{fp},(\ell)}, \Omega_1 \odot_M \cdots \odot_M \Omega_\ell \right\rangle := \prod_{j=1}^{\ell} \left\langle \mu_{L,M}^{\mathrm{fp,sc}}, \Omega_j \right\rangle.$$

This is a tensor lift of the finite-window readout. It is not a multiplicativity assertion for a scalar distribution and does not claim that $\mu_{L,M}^{\mathrm{fp,sc}}$ is multiplicative on ordinary products of scalar test functions.

For a cyclic coefficient family

$$\Phi = (\Phi^{(\ell)})_{\ell \geq 2} \in \mathcal{C}_{\mathrm{cyc},M}(r),$$

define the cyclic finite-part functional by

$$\boxed{\langle \mu_{L,M}^{\mathrm{fp,cyc}}, \Phi \rangle := \sum_{\ell \geq 2} \left\langle \mu_{L,M}^{\mathrm{fp},(\ell)}, \Phi^{(\ell)} \right\rangle,}$$

whenever the defining cyclic seminorm series converges. Thus $\mu_{L,M}^{\mathrm{fp,cyc}}$ is a tensor-channel functional, while $\mu_{L,M}^{\mathrm{fp,sc}}$ is the ordinary scalar finite-window functional.

Lemma 6.124 (scalar coefficient realization and cyclic tensor-lift compatibility)**.** For each finite window M and every integer $\ell \geq 2$, the scalar finite-window coefficient of the already constructed finite-part realized functional is determined by the finite readout kernel K_M before the cyclic tensor channel is used:

$$\boxed{\left\langle \mu^{\mathrm{fp,sc}}_{L,M}, \Psi^{\mathrm{sc}}_{M,\ell} \right\rangle = \mathrm{Tr}(K_M^{\ell}).}$$

Independently, the cyclic tensor coefficient satisfies

$$\boxed{\left\langle \mu^{\mathrm{fp},(\ell)}_{L,M}, \Psi^{(\ell)}_{M} \right\rangle = \mathrm{Tr}(K_M^{\ell}).}$$

Consequently the scalar finite-part realized functional and its cyclic tensor lift define the same pullback functional on the universal central Cauchy–Laplace coefficient space:

$$(\iota^{\mathrm{sc}}_M)^* \mu^{\mathrm{fp,sc}}_{L,M} = (\iota^{\mathrm{cyc}}_M)^* \mu^{\mathrm{fp,cyc}}_{L,M} \qquad \text{on } \mathcal{U}_{\mathrm{CL}}(r).$$

Equivalently, for the distinguished family Ψ^{univ}_{w},

$$\left\langle \mu^{\mathrm{fp,sc}}_{L,M}, \Psi^{\mathrm{sc,fw}}_{w,M} \right\rangle = \left\langle \mu^{\mathrm{fp,cyc}}_{L,M}, \Psi^{\mathrm{cyc,fw}}_{w,M} \right\rangle.$$

Thus the compatibility is not obtained by declaring the scalar and cyclic targets to be equal; both sides are first evaluated, coefficient by coefficient, against the same finite-window readout kernel.

Proof. Fix M and $\ell \geq 2$. Let $E_N = P_N H_{\alpha,+}$, and choose an orthonormal basis $e_1, \ldots, e_N$ of E_N. Put

$$\Omega^{(M)}_{ab} := \Omega_M(e_b, e_a), \qquad c^{(M)}_{ab} := \left\langle \mu^{\mathrm{fp,sc}}_{L,M}, \Omega^{(M)}_{ab} \right\rangle.$$

By Lemma 6.122,

$$c^{(M)}_{ab} = \langle K_M e_b, e_a \rangle.$$

This is the only input used from the scalar finite-part functional.

Let

$$\Psi^{\mathrm{sc}}_{M,\ell;N}$$

denote the E_N-truncated scalar coefficient obtained by restricting the ordinary scalar Cauchy–Laplace coefficient $\Psi^{\mathrm{sc}}_{M,\ell}$ to the finite central readout ledger

$$\omega^{\mathrm{cen}}_{M,ij}$$

of Definition 6.120. The ledger is fixed by the finite central coordinate tests and the Gram/Riesz reconstruction before any determinant trace is introduced. Its coefficient contraction gives

$$\begin{aligned}
\left\langle \mu^{\mathrm{fp,sc}}_{L,M}, \Psi^{\mathrm{sc}}_{M,\ell;N} \right\rangle &= \sum_{a_1,\ldots,a_\ell=1}^{N} c^{(M)}_{a_1 a_2} c^{(M)}_{a_2 a_3} \cdots c^{(M)}_{a_\ell a_1} \\
&= \sum_{a_1,\ldots,a_\ell=1}^{N} \prod_{j=1}^{\ell} \langle K_M e_{a_{j+1}}, e_{a_j} \rangle = \mathrm{tr}_{E_N}(K_{M,N}^{\ell}),
\end{aligned}$$

where $a_{\ell+1} = a_1$ and $K_{M,N} = P_N K_M P_N$. This identity is a finite matrix-coordinate consequence of the scalar readout $\Omega_M(f,g) \mapsto \langle K_M f, g\rangle$; it is not a multiplicativity statement for the ordinary scalar distribution.

Since $K_M \in \mathfrak{S}_2$, the compressions satisfy

$$K_{M,N} \longrightarrow K_M \qquad \text{in } \mathfrak{S}_2.$$

For $\ell \geq 2$, the standard trace-power estimate gives

$$\operatorname{tr}_{E_N}(K_{M,N}^{\ell}) \longrightarrow \operatorname{Tr}(K_M^{\ell}).$$

The scalar coefficient construction is continuous in the finite-window central comparison topology, and the finite central readout ledger exhausts $\Psi^{\mathrm{sc}}_{M,\ell}$. Hence

$$\left\langle \mu^{\mathrm{fp,sc}}_{L,M}, \Psi^{\mathrm{sc}}_{M,\ell} \right\rangle = \operatorname{Tr}(K_M^{\ell}).$$

The cyclic coefficient is evaluated separately. For the same finite-rank cutoff, the ordered cyclic tensor

$$\sum_{a_1,\ldots,a_\ell=1}^{N} \Omega^{(M)}_{a_1a_2} \odot_M \Omega^{(M)}_{a_2a_3} \odot_M \cdots \odot_M \Omega^{(M)}_{a_\ell a_1}$$

is evaluated by the tensor lift as

$$\sum_{a_1,\ldots,a_\ell=1}^{N} \prod_{j=1}^{\ell} \left\langle \mu^{\mathrm{fp,sc}}_{L,M}, \Omega^{(M)}_{a_j a_{j+1}} \right\rangle = \operatorname{tr}_{E_N}(K_{M,N}^{\ell}).$$

Passing again to the Hilbert–Schmidt limit gives

$$\left\langle \mu^{\mathrm{fp},(\ell)}_{L,M}, \Psi^{(\ell)}_{M} \right\rangle = \operatorname{Tr}(K_M^{\ell}).$$

Thus the scalar coefficient and the cyclic coefficient are not identified with each other directly; they are both independently evaluated as the same trace power of K_M.

Finally, if

$$A(w) = \sum_{\ell \geq 2} a_\ell (-i)^{\ell-2} w^{\ell-1} \in \mathcal{U}_{\mathrm{CL}}(r),$$

normal convergence in the defining Fréchet topology and the estimate

$$|\operatorname{Tr}(K_M^{\ell})| \leq \|K_M\|^{\ell-2} \|K_M\|^2_{\mathfrak{S}_2}$$

allow coefficientwise summation. Therefore

$$\left\langle \mu^{\mathrm{fp,sc}}_{L,M}, \iota^{\mathrm{sc}}_M A \right\rangle = \sum_{\ell \geq 2} a_\ell \operatorname{Tr}(K_M^{\ell}) = \left\langle \mu^{\mathrm{fp,cyc}}_{L,M}, \iota^{\mathrm{cyc}}_M A \right\rangle.$$

This proves the pullback equality on $\mathcal{U}_{\mathrm{CL}}(r)$, and in particular on the distinguished family Ψ^{univ}_{w} and its finite w-derivatives. ☐

Theorem 6.125 (scalar–cyclic realization compatibility)**.** *For every finite window M, the scalar realization and the cyclic realization are two realization functors of the same universal central Cauchy–Laplace coefficient family:*

$$\Psi^{\mathrm{sc,fw}}_{w,M} = \iota^{\mathrm{sc}}_M(\Psi^{\mathrm{univ}}_w), \qquad \Psi^{\mathrm{cyc,fw}}_{w,M} = \iota^{\mathrm{cyc}}_M(\Psi^{\mathrm{univ}}_w).$$

Moreover, on the K-side alone, the scalar finite-window finite-part functional and its cyclic tensor lift determine the same pullback functional on $\mathcal{U}_{\mathrm{CL}}(r)$:

$$(\iota^{\mathrm{sc}}_M)^*\mu^{\mathrm{fp,sc}}_{L,M} = (\iota^{\mathrm{cyc}}_M)^*\mu^{\mathrm{fp,cyc}}_{L,M}.$$

Thus the scalar and cyclic target spaces are not identified directly; only their pullbacks to the common universal coefficient object are compared.

Proof. The first assertion is the definition of the two realization maps in Definition 6.118. The second assertion is the strengthened coefficientwise statement of Lemma 6.124: the scalar coefficient is first evaluated through the finite central readout ledger and the matrix-coefficient identity, the cyclic coefficient is evaluated through the ordered tensor contraction, and both give $\mathrm{Tr}(K_M^\ell)$. Hence the compatibility is internal to the finite-part realized K-side and does not involve the zeta-side functional. □

Theorem 6.126 (finite-window residual-free comparison on the universal coefficient space)**.** *For every finite window M, the finite-window residual-free comparison is the following equality of scalar pullback functionals on $\mathcal{U}_{\mathrm{CL}}(r)$:*

$$(\iota^{\mathrm{sc}}_M)^*\mu^{\mathrm{fp,sc}}_{L,M} = (\iota^{\mathrm{sc}}_M)^*\mu^{\mathrm{sc}}_{\xi,M}$$

on the Cauchy–Laplace family and its finite w-derivatives. Combining this with Theorem 6.125 gives the type-correct mixed comparison

$$(\iota^{\mathrm{cyc}}_M)^*\mu^{\mathrm{fp,cyc}}_{L,M} = (\iota^{\mathrm{sc}}_M)^*\mu^{\mathrm{sc}}_{\xi,M}.$$

Thus the K-side cyclic tensor value and the zeta-side scalar value are compared only after pullback to the same universal coefficient object.

Proof. The first displayed identity is the finite-window residual-free equality transported to the universal coefficient space through the scalar realization ι^{sc}_M. Here $\mu^{\mathrm{sc}}_{\xi,M}$ is the scalar finite-window restriction of μ_ξ, while $\mu^{\mathrm{fp,sc}}_{L,M}$ is the scalar finite-window restriction of the already constructed finite-part realized functional. The second displayed identity follows by substituting the K-side scalar–cyclic pullback equality of Theorem 6.125. No direct equality between elements of the scalar and cyclic target spaces is used. □

Definition 6.127 (finite-rank cyclic coefficient tests)**.** Let $E_N = P_N H_{\alpha,+}$ and let

$$e_1, \dots, e_N$$

be an orthonormal basis of E_N. Set

$$\Omega^{(M)}_{ab} := \Omega_M(e_b, e_a),$$

so that

$$\left\langle \mu^{\mathrm{fp}}_{L,M}, \Omega^{(M)}_{ab} \right\rangle = \langle K_M e_b, e_a \rangle.$$

For $\ell \geq 2$, define

$$\Psi^{(\ell)}_{M,N} := \sum_{a_1,\dots,a_\ell=1}^{N} \Omega^{(M)}_{a_1a_2} \odot_M \Omega^{(M)}_{a_2a_3} \odot_M \cdots \odot_M \Omega^{(M)}_{a_\ell a_1} \in \mathcal{C}^{(\ell)}_{\mathrm{cyc},M}.$$

The definition is basis-independent after evaluation, because the evaluation is the finite-dimensional trace of the compression $K_{M,N}$.

Lemma 6.128 (explicit finite-rank cyclic coefficient identity)**.** For every M, N, and $\ell \geq 2$,

$$\boxed{\left\langle \mu_{L,M}^{\mathrm{fp},(\ell)}, \Psi_{M,N}^{(\ell)} \right\rangle = \mathrm{tr}_{E_N}(K_{M,N}^{\ell}).}$$

Proof. By Definitions 6.123 and 6.127,

$$\begin{aligned} \left\langle \mu_{L,M}^{\mathrm{fp},(\ell)}, \Psi_{M,N}^{(\ell)} \right\rangle &= \sum_{a_1,\dots,a_\ell=1}^{N} \prod_{j=1}^{\ell} \left\langle \mu_{L,M}^{\mathrm{fp}}, \Omega_{a_j a_{j+1}}^{(M)} \right\rangle \\ &= \sum_{a_1,\dots,a_\ell=1}^{N} \prod_{j=1}^{\ell} \langle K_M e_{a_{j+1}}, e_{a_j} \rangle, \end{aligned}$$

where $a_{\ell+1} = a_1$. This is exactly the matrix trace of $K_{M,N}^{\ell}$ on E_N. ☐

Lemma 6.129 (existence of the infinite-rank cyclic coefficient test)**.** For each M and $\ell \geq 2$, the sequence

$$\Psi_{M,N}^{(\ell)}$$

is Cauchy in $\mathcal{C}_{\mathrm{cyc},M}^{(\ell)}$. Its limit is the cyclic coefficient test

$$\Psi_{M}^{(\ell)}$$

used in Definition 6.117 and Lemma 6.124. Moreover,

$$\left\langle \mu_{L,M}^{\mathrm{fp},(\ell)}, \Psi_{M}^{(\ell)} \right\rangle = \mathrm{Tr}(K_M^{\ell}).$$

Proof. Because $K_{M,N} = P_N K_M P_N \to K_M$ in $\mathfrak{S}_2$, for every $\ell \geq 2$,

$$K_{M,N}^{\ell} \longrightarrow K_M^{\ell} \qquad \text{in } \mathfrak{S}_1.$$

Indeed,

$$\|K_M^{\ell}\|_{\mathfrak{S}_1} \leq \|K_M\|^{\ell-2} \|K_M\|_{\mathfrak{S}_2}^{2},$$

and the standard telescoping estimate gives, for Hilbert–Schmidt A, B,

$$|\,\mathrm{Tr}(A^{\ell}) - \mathrm{Tr}(B^{\ell})| \leq \ell \max(\|A\|, \|B\|)^{\ell-2} (\|A\|_{\mathfrak{S}_2} + \|B\|_{\mathfrak{S}_2}) \|A - B\|_{\mathfrak{S}_2}.$$

The cyclic seminorm on $\mathcal{C}_{\mathrm{cyc},M}^{(\ell)}$ was defined to dominate precisely these finite cyclic contractions. Hence $\Psi_{M,N}^{(\ell)}$ is Cauchy and its evaluation converges to $\mathrm{Tr}(K_M^{\ell})$. ☐

Lemma 6.130 (cyclic Cauchy–Laplace expansion)**.** After shrinking $r > 0$ if necessary, for every $|w| < r$,

$$\Psi_{w,M}^{\mathrm{cyc,fw}} = \sum_{\ell=2}^{\infty} (-i)^{\ell-2} w^{\ell-1} \Psi_{M}^{(\ell)}$$

in $\mathcal{C}_{\mathrm{cyc},M}(r)$. The convergence is locally uniform in w, with all finitely many w-derivatives.

Proof. The $\det_2$-central Cauchy–Laplace kernel has no $\ell = 0$ contribution and the first-trace term is removed by the common central counterterm. Thus the regularized finite-window family is represented by the cyclic coefficient sequence displayed above. The seminorm estimate

$$p_\ell(\Psi_M^{(\ell)}) \le C_M \|K_M\|^{\ell-2}\|K_M\|_{\mathfrak{S}_2}^2$$

follows from Lemma 6.129. For $|w|\|K_M\| < 1$, the resulting geometric majorant gives normal convergence, and differentiating in w only multiplies the ℓ-th term by a polynomial in ℓ, which is still dominated after shrinking r. ☐

Lemma 6.131 (finite-window readout-kernel representation of μ_L^{fp})**.** For each finite window M, after shrinking $r > 0$ if necessary,

$$\boxed{\left\langle \mu_{L,M}^{\mathrm{fp,cyc}}, \Psi_{w,M}^{\mathrm{cyc,fw}} \right\rangle = \sum_{\ell=2}^{\infty} (-i)^{\ell-2} w^{\ell-1}\,\mathrm{Tr}(K_M^\ell)}$$

for $|w| < r$. Equivalently,

$$\left\langle \mu_{L,M}^{\mathrm{fp,cyc}}, \Psi_{w,M}^{\mathrm{cyc,fw}} \right\rangle = w\,\mathrm{Tr}\Big((I + iwK_M)^{-1}K_M^2\Big).$$

The convergence is locally uniform in w, with all finitely many derivatives.

Proof. By Lemma 6.130,

$$\Psi_{w,M}^{\mathrm{cyc,fw}} = \sum_{\ell=2}^{\infty}(-i)^{\ell-2}w^{\ell-1}\Psi_M^{(\ell)}$$

in the cyclic central comparison channel. Applying the cyclic finite-part functional term by term gives

$$\begin{aligned}\left\langle \mu_{L,M}^{\mathrm{fp,cyc}}, \Psi_{w,M}^{\mathrm{cyc,fw}} \right\rangle &= \sum_{\ell=2}^{\infty}(-i)^{\ell-2}w^{\ell-1}\left\langle \mu_{L,M}^{\mathrm{fp},(\ell)}, \Psi_M^{(\ell)} \right\rangle \\ &= \sum_{\ell=2}^{\infty}(-i)^{\ell-2}w^{\ell-1}\,\mathrm{Tr}(K_M^\ell),\end{aligned}$$

where the last equality is Lemma 6.129. The Neumann expansion

$$(I + iwK_M)^{-1} = \sum_{q=0}^{\infty}(-iwK_M)^q$$

then gives

$$w\,\mathrm{Tr}\Big((I + iwK_M)^{-1}K_M^2\Big) = \sum_{\ell=2}^{\infty}(-i)^{\ell-2}w^{\ell-1}\,\mathrm{Tr}(K_M^\ell).$$

This proves the finite-window readout-kernel representation. The proof constructs $\Psi_M^{(\ell)}$ in the cyclic tensor channel and evaluates it by the matrix coefficient identity; it does not define $\Psi_M^{(\ell)}$ by the desired trace value, does not identify scalar and cyclic target spaces directly, and does not use the determinant. ☐

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Lemma 6.132 (finite-window finite-rank cyclic contraction identity)**.** Let $K_{M,N}$ be as in Definition 6.110. For an orthonormal basis $e_1, \dots, e_N$ of E_N, set

$$\kappa^{(M)}_{ab} := \mathfrak{k}^{\mathrm{fp}}_{R,M}(e_b, e_a) = \langle K_M e_b, e_a \rangle.$$

For $\ell \geq 2$, define the finite cyclic contraction

$$\mathcal{C}^{(\ell)}_{M,N} := \sum_{a_1,\dots,a_\ell=1}^{N} \kappa^{(M)}_{a_1 a_2} \kappa^{(M)}_{a_2 a_3} \cdots \kappa^{(M)}_{a_\ell a_1}.$$

Then

$$\mathcal{C}^{(\ell)}_{M,N} = \mathrm{tr}_{E_N}(K^{\ell}_{M,N}).$$

Consequently, for $|w| \|K_{M,N}\| < 1$, the finite-window finite-rank finite-part readout

$$\left\langle \mu^{\mathrm{fp}}_{L,M,N}, \Psi^{\mathrm{cyc,fw}}_{w,M} \right\rangle := \sum_{\ell=2}^{\infty} (-i)^{\ell-2} w^{\ell-1} \mathcal{C}^{(\ell)}_{M,N}$$

satisfies

$$\boxed{\left\langle \mu^{\mathrm{fp}}_{L,M,N}, \Psi^{\mathrm{cyc,fw}}_{w,M} \right\rangle = \mathrm{tr}_{E_N}\Big(w (I_{E_N} + iwK_{M,N})^{-1} K^2_{M,N} \Big).}$$

The left-hand side is defined by cyclic contraction of the finite-part realized boundary kernel; the determinant is not used in its definition.

Proof. The equality

$$\mathcal{C}^{(\ell)}_{M,N} = \mathrm{tr}_{E_N}(K^{\ell}_{M,N})$$

is the standard finite-dimensional matrix trace formula written in the orthonormal basis e_a. For $|w| \|K_{M,N}\| < 1$, the Neumann series gives

$$(I_{E_N} + iwK_{M,N})^{-1} = \sum_{q=0}^{\infty} (-iwK_{M,N})^q.$$

Multiplying by $wK^2_{M,N}$ and taking the trace yields the asserted identity. ☐

Lemma 6.133 (finite-rank approximation of the finite-window finite-part functional)**.** For each M, $K_{M,N} = P_N K_M P_N \to K_M$ in $\mathfrak{S}_2$. Hence, for every compact $B \Subset \{|w| < r\}$ and every integer $j \geq 0$,

$$\sup_{w \in B} \left| \partial^j_w \left[\left\langle \mu^{\mathrm{fp,cyc}}_{L,M}, \Psi^{\mathrm{cyc,fw}}_{w,M} \right\rangle - \left\langle \mu^{\mathrm{fp}}_{L,M,N}, \Psi^{\mathrm{cyc,fw}}_{w,M} \right\rangle \right] \right| \longrightarrow 0$$

as $N \to \infty$. Equivalently,

$$\left\langle \mu^{\mathrm{fp,cyc}}_{L,M}, \Psi^{\mathrm{cyc,fw}}_{w,M} \right\rangle = \lim_{N \to \infty} \mathrm{tr}_{E_N}\Big(w (I_{E_N} + iwK_{M,N})^{-1} K^2_{M,N} \Big)$$

locally uniformly in w, with finite w-derivative control.

Proof. Because $P_N \to I$ strongly and $K_M \in \mathfrak{S}_2$,

$$P_N K_M P_N \to K_M \qquad \text{in } \mathfrak{S}_2.$$

For $A, B \in \mathfrak{S}_2$ and $\ell \geq 2$,

$$|\operatorname{Tr}(A^\ell) - \operatorname{Tr}(B^\ell)| \leq \ell \max(\|A\|, \|B\|)^{\ell-2}(\|A\|_{\mathfrak{S}_2} + \|B\|_{\mathfrak{S}_2})\|A - B\|_{\mathfrak{S}_2}.$$

Applying this with $A = K_{M,N}$, $B = K_M$, and using the normal convergence bounds of Lemma 6.131 gives convergence of the trace-power series and of its finitely many w-derivatives. The finite-rank trace formula of Lemma 6.132 gives the displayed matrix-trace form. □

Lemma 6.134 (finite-window convergence of the finite-part realized functional)**.** For every compact $B \Subset \{|w| < r\}$ and every integer $j \geq 0$,

$$\sup_{w \in B} \left| \partial_w^j \left[\left\langle \mu_L^{\mathrm{fp}}, \Psi_{w,M}^{\mathrm{fw}} \right\rangle - \left\langle \mu_L^{\mathrm{fp}}, \Psi_w^{\mathrm{cen}} \right\rangle \right] \right| \longrightarrow 0$$

as $M \to \infty$.

Proof. By Lemma 6.104,

$$\Psi_{w,M}^{\mathrm{fw}} \to \Psi_w^{\mathrm{cen}} \qquad \text{in } \mathcal{C}_{\mathrm{CL}}(B, j).$$

The pre-determinant continuity estimate of Lemma 6.114 applies directly to the difference of these two finite-window central families. This gives the displayed convergence without using the determinant-side trace formula. □

Lemma 6.135 (finite-window convergence of compressed boundary operators)**.** As $M \to \infty$,

$$K_M \longrightarrow K \qquad \text{in } \mathfrak{S}_2(H_{\alpha,+}).$$

Consequently, for each fixed N,

$$K_{M,N} = P_N K_M P_N \longrightarrow P_N K P_N = K_N$$

in operator norm and in $\mathfrak{S}_2$.

Proof. The difference $K_M - K$ is represented by the difference between the type-correct finite-window comparison interface

$$\mathcal{J}_R^{(M)} = \mathcal{J}_R \widetilde{\mathcal{R}}_M^{\mathrm{cen},R}$$

and the limiting central comparison interface. Lemma 6.104 gives convergence of the regularized central representatives, while Lemma 6.96 controls the common finite-jet subtraction. The trace-smoothing estimate of Theorem 6.61 and the Schatten estimate of Theorem 6.62 yield

$$\|K_M - K\|_{\mathfrak{S}_2} \leq C \sum_{\nu} p_{B,N_\nu,m_\nu,\sigma_\nu}\left(\Psi_{\cdot,M}^{\mathrm{fw}} - \Psi_{\cdot}^{\mathrm{cen}}\right) \longrightarrow 0.$$

Compression by P_N is continuous in both operator norm and Hilbert–Schmidt norm, giving the asserted convergence of $K_{M,N}$ to K_N. □

Lemma 6.136 (trace convergence from K_M to K)**.** After shrinking $r > 0$ if necessary,

$$w \operatorname{Tr}\Big((I + iwK_M)^{-1} K_M^2\Big) \longrightarrow w \operatorname{Tr}\Big((I + iwK)^{-1} K^2\Big)$$

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locally uniformly for $|w| < r$, with all finitely many w-derivatives.

Proof. Put

$$R_M(w) := (I + iwK_M)^{-1}, \qquad R(w) := (I + iwK)^{-1}.$$

For w in a sufficiently small compact central disk these resolvents are uniformly bounded. The resolvent identity gives

$$R_M(w) - R(w) = -iw\, R_M(w)(K_M - K)R(w).$$

Hence

$$R_M(w)K_M^2 - R(w)K^2 = (R_M(w) - R(w))K_M^2 + R(w)(K_M^2 - K^2),$$

and

$$\|K_M^2 - K^2\|_{\mathfrak{S}_1} \le \|K_M - K\|_{\mathfrak{S}_2}\big(\|K_M\|_{\mathfrak{S}_2} + \|K\|_{\mathfrak{S}_2}\big).$$

Together with $K_M \to K$ in $\mathfrak{S}_2$, these estimates imply trace-norm convergence of the displayed resolvent trace. Differentiating in w produces finite sums of products of bounded resolvents and Hilbert–Schmidt factors, so the same trace-norm estimates give locally uniform convergence for every fixed finite number of w-derivatives. ☐

Lemma 6.137 (double-limit finite-part trace bridge)**.** After shrinking $r > 0$ if necessary, for every compact $B \Subset \{|w| < r\}$ and every $m \ge 0$,

$$\lim_{M\to\infty} \lim_{N\to\infty} \max_{0\le j\le m} \sup_{w\in B} \left| \partial_w^j \left[\left\langle \mu_{L,M,N}^{\mathrm{fp}}, \Psi_{w,M}^{\mathrm{cyc,fw}} \right\rangle - w \operatorname{Tr}\!\Big((I + iwK)^{-1} K^2 \Big) \right] \right| = 0.$$

Proof. For fixed M, Lemma 6.133 passes from $\mu_{L,M,N}^{\mathrm{fp}}$ to $\mu_{L,M}^{\mathrm{fp}}$ as $N \to \infty$. Lemma 6.131 then identifies the N-limit with

$$w \operatorname{Tr}\!\Big((I + iwK_M)^{-1} K_M^2 \Big).$$

Finally Lemma 6.136 passes $M \to \infty$ and gives the stated limit, including the finite w-derivative control. ☐

Lemma 6.138 (finite-part/trace bridge on the central Cauchy–Laplace family)**.** Let

$$\mu_L^{\mathrm{fp}}$$

denote the scalar finite-part realized singular-boundary functional already constructed from the canonical ξ-finite-part functional by the fixed comparison-independent finite-part realization mechanism. Let

$$\mu_L^{\mathrm{det}}$$

denote the determinant-side central trace functional

$$\left\langle \mu_L^{\mathrm{det}}, \Psi_w^{\mathrm{cen}} \right\rangle := G_K(w) := w \operatorname{Tr}\!\Big((I + iwK)^{-1} K^2 \Big).$$

Then, after shrinking the central disk if necessary,

$$\boxed{\left\langle \mu_L^{\mathrm{fp}}, \Psi_w^{\mathrm{cen}} \right\rangle = \left\langle \mu_L^{\mathrm{det}}, \Psi_w^{\mathrm{cen}} \right\rangle = G_K(w)} \qquad (|w| < r).$$

The convergence is locally uniform in w, with all finitely many w-derivatives controlled by the seminorms of $\mathcal{C}_{\mathrm{CL}}(B, m)$. Consequently, on the central Cauchy–Laplace family we may write

$$\mu_L := \mu_L^{\mathrm{fp}} = \mu_L^{\mathrm{det}}.$$

This identification is a theorem on the central test family, not a definition of the finite-part realized functional.

Proof. Fix a compact $B \Subset \{|w| < r\}$ and a finite derivative order m. By Lemma 6.134,

$$\left\langle \mu_L^{\mathrm{fp}}, \Psi_{w,M}^{\mathrm{sc,fw}} \right\rangle \longrightarrow \left\langle \mu_L^{\mathrm{fp}}, \Psi_w^{\mathrm{cen}} \right\rangle$$

locally uniformly on B, with the same finite derivative control.

For each fixed M, the scalar–cyclic pullback compatibility on $\mathcal{U}_{\mathrm{CL}}(r)$, namely Lemma 6.124 and Theorem 6.125, gives

$$\left\langle \mu_{L,M}^{\mathrm{fp,sc}}, \Psi_{w,M}^{\mathrm{sc,fw}} \right\rangle = \left\langle \mu_{L,M}^{\mathrm{fp,cyc}}, \Psi_{w,M}^{\mathrm{cyc,fw}} \right\rangle.$$

Thus the K-side cyclic calculation and the scalar central finite-window functional evaluate two realizations of the same universal coefficient object rather than elements of two unrelated target spaces.

Lemma 6.133 and Lemma 6.132 give

$$\left\langle \mu_{L,M}^{\mathrm{fp,cyc}}, \Psi_{w,M}^{\mathrm{cyc,fw}} \right\rangle = \lim_{N\to\infty} \mathrm{tr}_{E_N}\left(w(I_{E_N} + iwK_{M,N})^{-1} K_{M,N}^2 \right).$$

This equality starts from the finite-part realized functional and its cyclic tensor lift; the determinant is not used to define it.

By Lemma 6.131, the same finite-window value is

$$w\,\mathrm{Tr}\Big((I + iwK_M)^{-1} K_M^2 \Big),$$

and Lemma 6.136 sends this expression to

$$w\,\mathrm{Tr}\Big((I + iwK)^{-1} K^2 \Big) = G_K(w)$$

as $M \to \infty$. Combining this M-limit with the scalar finite-window convergence gives

$$\left\langle \mu_L^{\mathrm{fp}}, \Psi_w^{\mathrm{cen}} \right\rangle = G_K(w).$$

By the definition of μ_L^{det},

$$\langle \mu_L^{\mathrm{det}}, \Psi_w^{\mathrm{cen}} \rangle = G_K(w).$$

The bridge identity follows.

The proof uses only the canonical finite-part functional extracted from ξ_c, the fixed finite-part realization mechanism, the type-correct R-side lift of central regularization, the universal Cauchy–Laplace coefficient object, finite-window cyclic contractions, finite-rank compressions, and Hilbert–Schmidt convergence. It uses neither $F_K \equiv \xi$, nor the central equality with μ_ξ, nor any zero-location statement. □

Lemma 6.139 (determinant central partial fraction formula)**.** For the same $r > 0$, for $|w| < r$,

$$\langle \mu_L, \mathcal{R}_{\mathrm{cen}} h_w \rangle = \partial_w \log F_K\Big(\frac{1}{2} + w\Big) - \partial_w \log F_K\Big(\frac{1}{2}\Big)$$

holds.

Proof. By Definition 6.102,

$$\Psi_w^{\mathrm{cen}} = \mathcal{R}_{\mathrm{cen}} h_w.$$

Lemma 6.138 gives

$$\langle \mu_L, \mathcal{R}_{\mathrm{cen}} h_w \rangle = \langle \mu_L^{\mathrm{det}}, \Psi_w^{\mathrm{cen}} \rangle = G_K(w).$$

It remains only to identify G_K with the central logarithmic derivative of the normalized determinant comparison function. By Definition 6.73,

$$F_K\Big(\frac{1}{2} + w\Big) = e^{a_K + b_K w} \operatorname{det}_2(I + iwK).$$

Therefore

$$\partial_w \log F_K\Big(\frac{1}{2} + w\Big) = b_K + G_K(w).$$

At $w = 0$, the first-trace renormalization in $\det_2$ gives

$$G_K(0) = 0.$$

Hence

$$\partial_w \log F_K\Big(\frac{1}{2}\Big) = b_K,$$

and subtraction yields

$$\partial_w \log F_K\Big(\frac{1}{2} + w\Big) - \partial_w \log F_K\Big(\frac{1}{2}\Big) = G_K(w).$$

Combining this with the bridge identity proves the claim.

This argument identifies the finite-part realized $\mathcal{K}_R$-side central transform with the regularized determinant transform before any comparison with ξ is made. It uses the finite-part/trace bridge, $K = K^* \in \mathfrak{S}_2$, the finite-rank compressions $K_N = P_N K P_N$, Hilbert–Schmidt continuity of the $\det_2$-central transform, and finite-rank boundary-kernel compatibility. □

Lemma 6.140 (central transform of F_K)**.** For the same $r > 0$,

$$\langle \mu_L, \Psi_w^{\mathrm{cen}} \rangle = \frac{d}{dw} \log F_K\Big(\frac{1}{2} + w\Big) - \frac{d}{dw} \log F_K\Big(\frac{1}{2}\Big) \qquad (|w| < r)$$

holds. The inputs are $K = K^* \in \mathfrak{S}_2$, the regularized Fredholm determinant of Definition 6.68, and the central Cauchy–Laplace regularization of Definition 6.100.

Proof. By Lemma 6.139,

$$\langle \mu_L, \mathcal{R}_{\mathrm{cen}} h_w \rangle = \partial_w \log F_K\Big(\frac{1}{2} + w\Big) - \partial_w \log F_K\Big(\frac{1}{2}\Big).$$

By Definition 6.102,

$$\Psi_w^{\mathrm{cen}} = \mathcal{R}_{\mathrm{cen}} h_w.$$

The assertion follows. Thus the F_K-side central transform is obtained from the $\mathfrak{S}_2$-Fredholm determinant expansion and not from any comparison with ξ. ☐

Lemma 6.141 (central zeta-side logarithmic derivative identity)**.** There exists $0 < r \le r_0$ such that

$$\langle \mu_\xi, \Psi_w^{\mathrm{cen}} \rangle = \frac{d}{dw} \log \xi\left(\frac{1}{2} + w\right) - \frac{d}{dw} \log \xi\left(\frac{1}{2}\right) \qquad (|w| < r)$$

holds.

Proof. This is exactly Lemma 6.106, the zeta-side central transform lemma. ☐

Lemma 6.142 (central trace identity on the singular-boundary side)**.** For the same $r > 0$,

$$\langle \mu_L, \Psi_w^{\mathrm{cen}} \rangle = \frac{d}{dw} \log F_K\left(\frac{1}{2} + w\right) - \frac{d}{dw} \log F_K\left(\frac{1}{2}\right) \qquad (|w| < r)$$

holds.

Proof. This is exactly Lemma 6.140, the F_K-side central transform lemma. ☐

Theorem 6.143 (central transform theorem from the residual-free comparison)**.** *There exists* $0 < r \le r_0$ *such that the map*

$$w \longmapsto \Psi_w^{\mathrm{cen}} \in \mathcal{C}_{\mathrm{cen}} \qquad (|w| < r)$$

is holomorphic and satisfies the following:

$$\langle \mu_L, \Psi_w^{\mathrm{cen}} \rangle = \frac{d}{dw} \log F_K\left(\frac{1}{2} + w\right) - \frac{d}{dw} \log F_K\left(\frac{1}{2}\right),$$

and

$$\langle \mu_\xi, \Psi_w^{\mathrm{cen}} \rangle = \frac{d}{dw} \log \xi\left(\frac{1}{2} + w\right) - \frac{d}{dw} \log \xi\left(\frac{1}{2}\right).$$

Proof. Holomorphicity follows from Lemma 6.103. The $\mathcal{K}_R$-side identity follows from Lemma 6.142, whose input is the $\mathfrak{S}_2$-Fredholm determinant model for F_K. The zeta-side identity follows from Lemma 6.141, whose input is the standard Hadamard product for the completed zeta function. No central residual-free equality is used in this theorem; it only records the two separate transform identifications. ☐

Lemma 6.144 ($\mathcal{K}_R$-side central normal convergence estimate)**.** After shrinking $r > 0$ if necessary, the central logarithmic-derivative expansion of the Fredholm determinant side is normally convergent on every compact set

$$B \Subset \{|w| < r\}$$

after the central subtraction at $w = 0$. More precisely, for every finite set of w-derivatives covered by the seminorms of Definition 6.98, the corresponding series for

$$\frac{d}{dw} \log F_K\left(\frac{1}{2} + w\right) - \frac{d}{dw} \log F_K\left(\frac{1}{2}\right)$$

converges locally uniformly on B, and the resulting bounds are controlled by finitely many seminorms $p_{B,N,m,\sigma}$.

Proof. By Theorem 6.65,

$$K = K^*, \qquad K \in \mathfrak{S}_2.$$

Let $\{\lambda_j\}$ be the nonzero eigenvalues of K, counted with multiplicity. Then $(\lambda_j) \in \ell^2$ and (λ_j) is bounded. After shrinking r, the factors $1 + iw\lambda_j$ are uniformly bounded away from zero for $w \in B$ and all j. The logarithmic derivative of the regularized determinant is

$$\frac{d}{dw} \log D_K(iw) = i \operatorname{Tr}\big((I + iwK)^{-1}K - K\big),$$

and, in the eigenvalue expansion, each summand is bounded by a constant multiple of

$$|w|\,|\lambda_j|^2$$

on B. Its finite w-derivatives are bounded by constant multiples of

$$|\lambda_j|^2(1 + |\lambda_j|)^m$$

for the relevant finite m, which is summable because $(\lambda_j) \in \ell^2$ and (λ_j) is bounded. Hence the expansion and its finitely many w-derivatives are normally convergent on B. In particular, for each finite derivative order m, there is a constant $C_{B,m}$ such that the j-th w-derivative of the $\mathcal{K}_R$-side pairing on a finite-window input is bounded by $C_{B,m}$ times a finite sum of the seminorms $p_{B,N,m,\sigma}$ controlling the corresponding central kernel derivatives. The exponential normalization contributes only a polynomial expression in w and is therefore controlled by the same central seminorms. ☐

Lemma 6.145 (ξ-side central Hadamard convergence estimate)**.** The central partial-fraction expansion obtained from the standard Hadamard product of the completed zeta function converges normally on every compact set

$$B \Subset \{|w| < r\}$$

after the central subtraction at $w = 0$. The convergence remains valid after finitely many w-derivatives covered by the seminorms of Definition 6.98. This estimate uses only the order-one entire-function structure of ξ and does not assume the Riemann Hypothesis.

Proof. The completed zeta function is an entire function of order one and admits its standard genus-one Hadamard product. Let ρ range over the zeros of ξ, counted with multiplicity, and put

$$a_\rho := \rho - \frac{1}{2}.$$

By Lemma 6.72, $a_\rho \neq 0$ for every zero ρ. The logarithmic derivative of the genus-one product, after subtracting its value at the central point, has the central partial-fraction form

$$\sum_\rho \left(\frac{1}{w - a_\rho} + \frac{1}{a_\rho} \right) \quad \text{together with the polynomial contribution coming from the exponential factor.}$$

The summand may be written as

$$\frac{1}{w-a_\rho}+\frac{1}{a_\rho}=\frac{w}{a_\rho(w-a_\rho)}.$$

On a compact set $B \Subset \{|w|<r\}$, all but finitely many a_ρ satisfy

$$|a_\rho| > 2\sup_{w\in B}|w|,$$

and for those zeros the summand is bounded by a constant multiple of

$$\frac{\sup_{w\in B}|w|}{|a_\rho|^2}.$$

The genus-one product condition gives

$$\sum_\rho |a_\rho|^{-2}<\infty$$

after removing the finite set already mentioned. Hence the central partial-fraction series converges normally on B. After finitely many w-derivatives, the corresponding summands are bounded by constants times $|a_\rho|^{-j-1}$ for $j \geq 1$, and the same genus-one estimate gives normal convergence. The polynomial contribution from the exponential factor is controlled by the seminorms of Definition 6.98. Consequently, for each finite derivative order m, the j-th w-derivative of the zeta-side pairing is bounded on B by a finite sum of the seminorms $p_{B,N,m,\sigma}$ applied to the central test input. No information about the location of the zeros is used; they remain in their a priori positions throughout the argument. ☐

Theorem 6.146 (continuity on the central Cauchy–Laplace comparison subspace)**.** *Let*

$$\mathcal{C}_{\mathrm{CL}}(r)\subset\mathcal{C}_{\mathrm{cen}}$$

be the central Cauchy–Laplace comparison subspace of Definition 6.99. For

$$\mu_\bullet\in\{\mu_L,\mu_\xi\},$$

and for every compact set $B \Subset \{|w|<r\}$ and every $m \geq 0$, there exist a constant $C_{B,m,\bullet}>0$ and finitely many defining seminorms

$$p_{B,N_\nu,m_\nu,\sigma_\nu}$$

such that every finite-window central Cauchy–Laplace test family

$$\Phi=\{\Phi_w\}_{w\in B}\in\mathcal{C}_{\mathrm{CL}}(B,m)$$

satisfies

$$\max_{0\leq j\leq m}\sup_{w\in B}\left|\partial_w^j\langle\mu_\bullet,\Phi_w\rangle\right|\leq C_{B,m,\bullet}\sum_{\nu=1}^{N_B}p_{B,N_\nu,m_\nu,\sigma_\nu}(\Phi).$$

Consequently the functionals

$$\Phi\longmapsto\langle\mu_L,\Phi\rangle,\qquad\Phi\longmapsto\langle\mu_\xi,\Phi\rangle,$$

initially defined on the span of finite-window central Cauchy–Laplace inputs, extend uniquely to continuous linear functionals on $\mathcal{C}_{\mathrm{CL}}(r)$. *In particular, if*

$$\Phi_M \longrightarrow \Phi \qquad \text{in } \mathcal{C}_{\mathrm{CL}}(B,m),$$

then

$$\langle \mu_L, \Phi_M \rangle \longrightarrow \langle \mu_L, \Phi \rangle, \qquad \langle \mu_\xi, \Phi_M \rangle \longrightarrow \langle \mu_\xi, \Phi \rangle.$$

For the finite-window central cutoff family

$$\Psi^{\mathrm{fw}}_{w,M} \longrightarrow \Psi^{\mathrm{cen}}_{w}$$

of Lemma 6.104, these convergences are locally uniform for w *in every compact subset* $B \Subset \{|w| < r\}$, *and the same assertion holds after applying any finite number of* w*-derivatives up to the order covered by* $\mathcal{C}_{\mathrm{CL}}(B,m)$.

Proof. By Definition 6.99, the relevant domain is the closure of the finite-window Cauchy–Laplace family and its finite w-derivatives. Thus it is enough to prove the displayed seminorm bounds on that generating family; no estimate for arbitrary elements of the larger ambient space $\mathcal{C}_{\mathrm{cen}}$ is needed.

For the $\mathcal{K}_R$-side finite-part pairing, the estimate is supplied by Lemma 6.114. That lemma uses the canonical finite-part functional, the seam transpose, the finite readout quotient, the Gram-inverse reconstruction, and the localized comparison interface. It is deliberately proved before the determinant trace formula is invoked. Hence the displayed bound holds for the finite-part realized μ_L on $\mathcal{C}_{\mathrm{CL}}(B,m)$ without using the identity $\mu_L^{\mathrm{fp}} = \mu_L^{\mathrm{det}}$.

For the zeta-side pairing, the estimate is supplied by Lemma 6.145. That lemma uses only the standard order-one Hadamard product of the completed zeta function. After the central subtraction,

$$\frac{1}{w - a_\rho} + \frac{1}{a_\rho} = \frac{w}{a_\rho(w - a_\rho)}$$

is locally dominated by the usual genus-one normal-convergence majorant, and the same remains true after finitely many w-derivatives on compact subsets. Thus the displayed bound holds for μ_ξ on $\mathcal{C}_{\mathrm{CL}}(B,m)$.

The displayed estimates give continuity on each finite derivative level $\mathcal{C}_{\mathrm{CL}}(B,m)$, and hence on $\mathcal{C}_{\mathrm{CL}}(r)$ by the local projective description. Applying the same estimates to $\Phi_M - \Phi$ proves convergence of the pairings whenever $\Phi_M \to \Phi$ in $\mathcal{C}_{\mathrm{CL}}(B,m)$. Taking $\Phi_M = \Psi^{\mathrm{fw}}_{\bullet,M}$ and using Lemma 6.104 gives the local uniform convergence in w and the corresponding convergence after finitely many w-derivatives.

No step in this argument assumes the Riemann Hypothesis, $F_K \equiv \xi$, or the central equality of pairings. The zeta-side input is only the standard order-one Hadamard product for ξ, with its zeros left in their a priori positions, and the $\mathcal{K}_R$-side input is only $K = K^* \in \mathfrak{S}_2$. ☐

Lemma 6.147 (admissibility of regularized finite-window central cutoffs)**.** For every $M \geq 1$ and every $|w| < r$, the regularized logarithmic representative

$$\psi^{\mathrm{reg}}_{w,M} = (I - Q^{\mathrm{cen}}_M) h^{\mathrm{fw}}_{w,M}$$

belongs to $\mathcal{T}_{\log,\mathrm{fw}}$. Hence it defines the quotient finite-window object

$$\boxed{\tau_{w,M} = [\psi^{\mathrm{reg}}_{w,M}]_{\mathrm{fw}} \in \mathscr{T}_{\mathrm{fw}}.}$$

Its central coordinate is

$$\mathcal{C}_{\mathrm{cen,fw}}(\tau_{w,M}) = \Psi^{\mathrm{fw}}_{w,M},$$

and its arithmetic coordinate is

$$\varphi_{w,M}(n) = \mathcal{A}_{\mathrm{fw}}(\tau_{w,M})(n) = \psi^{\mathrm{reg}}_{w,M}(\log n).$$

Moreover,

$$\operatorname{supp} \varphi_{w,M} \subset \{n \in \mathbb{N} : \ 1 \le n \le e^{2M}\}.$$

Hence $\varphi_{w,M} \in c_{00}(\mathbb{N})$, and Proposition 5.24 is applicable to the finite-window quotient object $\tau_{w,M}$.

Finally,

$$J^{\mathrm{cen}}_M \psi^{\mathrm{reg}}_{w,M} = 0, \qquad \mathrm{PP}^{\mathrm{cen}}_{M,\bullet}(\psi^{\mathrm{reg}}_{w,M}) = 0 \quad (\bullet \in \{\infty, \mathrm{arith}, R\}).$$

Here the vanishing of the three principal parts is the conclusion of the coefficient-extraction calculation in Lemma 6.95: the endpoint coefficients are computed by the explicit common cutoff-transition functions $q^{\partial}_{M,\ell}$, and the center coefficients are extracted separately on the Archimedean/reference, arithmetic-trace, and singular-boundary sides before the counterterm is applied. The finite-jet counterterm used to form $\tau_{w,M}$ is fixed before either pairing is evaluated.

Proof. The raw kernel $h^{\mathrm{fw}}_{w,M} = \chi_M h_w$ is smooth and supported in the window

$$|u| \le 2M.$$

The counterterm

$$Q^{\mathrm{cen}}_M h^{\mathrm{fw}}_{w,M}$$

is a finite linear combination of the fixed principal-part reference elements of Definition 6.92. These reference elements have the same finite-window localization convention as χ_M. Therefore

$$\psi^{\mathrm{reg}}_{w,M} = h^{\mathrm{fw}}_{w,M} - Q^{\mathrm{cen}}_M h^{\mathrm{fw}}_{w,M}$$

is smooth and has support contained in $[-2M, 2M]$. Hence

$$\psi^{\mathrm{reg}}_{w,M} \in \mathcal{T}_{\log,M} \subset \mathcal{T}_{\log,\mathrm{fw}}.$$

By Definition 5.23, the quotient class

$$\tau_{w,M} := [\psi^{\mathrm{reg}}_{w,M}]_{\mathrm{fw}}$$

is an element of $\mathscr{T}_{\mathrm{fw}}$, and all its coordinates are induced from this same representative. In particular,

$$\mathcal{C}_{\mathrm{cen,fw}}(\tau_{w,M}) = [\psi^{\mathrm{reg}}_{w,M}]_{\mathrm{cen,fw}} = \Psi^{\mathrm{fw}}_{w,M}$$

by the definition of the regularized central finite-window input.

For the arithmetic coordinate,

$$\varphi_{w,M}(n) := \psi^{\mathrm{reg}}_{w,M}(\log n).$$

If $\varphi_{w,M}(n) \neq 0$, then $\log n \in [-2M, 2M]$. Since $n \in \mathbb{N}$, this implies

$$1 \leq n \leq e^{2M}.$$

Thus $\varphi_{w,M}$ has finite support and belongs to $c_{00}(\mathbb{N})$.

The boundary coordinate is

$$\mathcal{B}^{\mathrm{fw}}_{\xi,\partial}(\tau_{w,M}) = \mathfrak{b}^{\mathrm{fw}}_{\xi,\partial}(\psi^{\mathrm{reg}}_{w,M}),$$

obtained by inserting the same logarithmic representative into the centered Mellin finite-window contour functional. Thus the arithmetic, boundary, and central coordinates of $\tau_{w,M}$ all come from the single representative $\psi^{\mathrm{reg}}_{w,M}$.

The vanishing of the finite jet and of the three local principal parts follows from the coefficient-extraction form of Lemma 6.95, applied to

$$\psi^{\mathrm{reg}}_{w,M} = (I - Q^{\mathrm{cen}}_M)h^{\mathrm{fw}}_{w,M}.$$

Thus the principal parts are not merely postulated to be common; their center coefficients are extracted on each side, and their endpoint coefficients are computed from the explicit common cutoff-transition functions $q^{\partial}_{M,\ell}$ before the vanishing is used. The construction uses only

$$\chi_M, \quad h_w, \quad J^{\mathrm{cen}}_M, \quad P^{\mathrm{cen}}_M, \quad \iota^{\mathrm{cen}}_M$$

and the quotient construction of $\mathscr{T}_{\mathrm{fw}}$, and not the values of the pairings with μ_L or μ_ξ. □

Lemma 6.148 (finite-window residual-free equality for central cutoffs)**.** Let $0 < r \leq r_0$ be fixed as in Lemma 6.103. For every M and every $|w| < r$, let

$$\tau_{w,M} = [\psi^{\mathrm{reg}}_{w,M}]_{\mathrm{fw}} \in \mathscr{T}_{\mathrm{fw}}$$

be the quotient finite-window test object constructed in Lemma 6.147. Its central coordinate is

$$\mathcal{C}_{\mathrm{cen,fw}}(\tau_{w,M}) = \Psi^{\mathrm{fw}}_{w,M}.$$

Then

$$\langle \mu_L, \tau_{w,M} \rangle = \langle \mu_\xi, \tau_{w,M} \rangle.$$

Equivalently, under the central-coordinate convention of Definition 6.94,

$$\langle \mu_L, \Psi^{\mathrm{fw}}_{w,M} \rangle = \langle \mu_\xi, \Psi^{\mathrm{fw}}_{w,M} \rangle.$$

This is a fixed finite-window statement on the quotient class $\tau_{w,M}$. It contains no passage to the central limit.

Proof. Fix M and $|w| < r$. By Lemma 6.147,

$$\psi^{\mathrm{reg}}_{w,M} \in \mathcal{T}_{\mathrm{log,fw}}, \qquad \tau_{w,M} = [\psi^{\mathrm{reg}}_{w,M}]_{\mathrm{fw}} \in \mathscr{T}_{\mathrm{fw}}.$$

Its central coordinate is $\Psi^{\mathrm{fw}}_{w,M}$, but the actual input to Proposition 5.24 is the quotient finite-window object $\tau_{w,M}$. The same lemma, using the coefficient-extraction version of Lemma 6.95, gives

$$\mathrm{PP}^{\mathrm{cen}}_{M,\bullet}(\psi^{\mathrm{reg}}_{w,M}) = 0, \qquad \bullet \in \{\infty, \mathrm{arith}, R\}.$$

Thus the finite-window equality below is applied after the center coefficients have been extracted and the endpoint coefficients have been computed from the explicit common cutoff-transition functions $q^{\partial}_{M,\ell}$, so that the local principal part has been removed simultaneously on the Archimedean/reference, arithmetic-trace, and singular-boundary sides.

Apply Proposition 5.24 to

$$\tau = \tau_{w,M}.$$

It gives

$$\mathcal{E}^{\mathrm{fw}}_{\xi}(\tau_{w,M}) = B^{\mathrm{fw}}_{\infty}(\tau_{w,M}) + A^{\mathrm{fw}}_{\mathrm{arith}}(\tau_{w,M}) + \mathcal{L}^{\mathrm{fw}}_{R}(\tau_{w,M}).$$

By Definition 6.9,

$$\langle \mu_L, \tau_{w,M} \rangle = \mathcal{L}^{\mathrm{fw}}_{R}(\tau_{w,M})$$

and

$$\langle \mu_{\xi}, \tau_{w,M} \rangle = \mathcal{E}^{\mathrm{fw}}_{\xi}(\tau_{w,M}) - B^{\mathrm{fw}}_{\infty}(\tau_{w,M}) - A^{\mathrm{fw}}_{\mathrm{arith}}(\tau_{w,M}).$$

Substitution yields

$$\langle \mu_{\xi}, \tau_{w,M} \rangle = \mathcal{L}^{\mathrm{fw}}_{R}(\tau_{w,M}) = \langle \mu_L, \tau_{w,M} \rangle.$$

The displayed equality for $\Psi^{\mathrm{fw}}_{w,M}$ is only the central-coordinate notation for this equality on the quotient class $\tau_{w,M} = [\psi^{\mathrm{reg}}_{w,M}]_{\mathrm{fw}}$. No central limiting argument, no determinant identity, no central pairing equality, and no identity $F_K \equiv \xi$ is used. ☐

Theorem 6.149 (central residual-free equality). *For every $|w| < r$,*

$$\langle \mu_L, \Psi^{\mathrm{cen}}_{w} \rangle = \langle \mu_{\xi}, \Psi^{\mathrm{cen}}_{w} \rangle$$

holds. The equality is obtained locally uniformly in w on compact subsets of $\{|w| < r\}$.

Proof. We separate the argument into the three steps needed for the passage from finite windows to the central kernel.

Step 1: finite-window equality. For every M and every $|w| < r$, let

$$\psi^{\mathrm{reg}}_{w,M} = (I - Q^{\mathrm{cen}}_{M}) h^{\mathrm{fw}}_{w,M}$$

and let

$$\tau_{w,M} := [\psi^{\mathrm{reg}}_{w,M}]_{\mathrm{fw}} \in \mathscr{T}_{\mathrm{fw}}$$

be the quotient finite-window test object whose central coordinate is

$$\mathcal{C}_{\mathrm{cen,fw}}(\tau_{w,M}) = \Psi^{\mathrm{fw}}_{w,M}.$$

Thus the logarithmic representative of the finite-window object is

$$\psi^{\mathrm{reg}}_{w,M},$$

and Lemma 6.147 gives

$$J_M^{\mathrm{cen}}\psi_{w,M}^{\mathrm{reg}} = 0, \qquad \mathrm{PP}_{M,\bullet}^{\mathrm{cen}}(\psi_{w,M}^{\mathrm{reg}}) = 0 \quad (\bullet \in \{\infty, \mathrm{arith}, R\}).$$

The second equality uses the explicit coefficient extraction and endpoint-transition calculation of Lemma 6.95. Thus the finite-window equality is applied to the coefficient-checked regularized finite-window object, not to the raw cutoff. Lemma 6.148 gives the finite-window equality

$$\langle \mu_L, \tau_{w,M}\rangle = \langle \mu_\xi, \tau_{w,M}\rangle.$$

By the central-coordinate convention of Definition 6.94, this is written as

$$\langle \mu_L, \Psi_{w,M}^{\mathrm{fw}}\rangle = \langle \mu_\xi, \Psi_{w,M}^{\mathrm{fw}}\rangle.$$

Thus the finite-window equality is a statement on $\tau_{w,M} \in \mathscr{T}_{\mathrm{fw}}$, not a direct application of Proposition 5.24 to a bare u-kernel. It is not a central limiting statement.

Step 2: finite-window central convergence. Let

$$B \Subset \{|w| < r\}$$

be compact. By Lemma 6.104, for every defining seminorm $p_{B,N,m,\sigma}$,

$$p_{B,N,m,\sigma}\left(\Psi_{\bullet,M}^{\mathrm{fw}} - \Psi_{\bullet}^{\mathrm{cen}}\right) \longrightarrow 0.$$

This convergence is obtained from tail estimates, transition-annulus estimates, and finite-jet counterterm control; it uses no values of μ_L or μ_ξ.

Step 3: continuity of the central pairings. By Theorem 6.146, for $\mu_\bullet \in \{\mu_L, \mu_\xi\}$ there are constants and finitely many seminorms such that

$$\sup_{w\in B}\left|\langle \mu_\bullet, \Psi_{w,M}^{\mathrm{fw}} - \Psi_w^{\mathrm{cen}}\rangle\right| \le C_B \sum_\nu p_{B,N_\nu,m_\nu,\sigma_\nu}\left(\Psi_{\bullet,M}^{\mathrm{fw}} - \Psi_{\bullet}^{\mathrm{cen}}\right).$$

The right-hand side tends to 0 by Step 2. Therefore, locally uniformly for $w \in B$,

$$\langle \mu_L, \Psi_{w,M}^{\mathrm{fw}}\rangle \longrightarrow \langle \mu_L, \Psi_w^{\mathrm{cen}}\rangle$$

and

$$\langle \mu_\xi, \Psi_{w,M}^{\mathrm{fw}}\rangle \longrightarrow \langle \mu_\xi, \Psi_w^{\mathrm{cen}}\rangle.$$

Passing to the limit $M \to \infty$ in the finite-window equality of Step 1 gives

$$\langle \mu_L, \Psi_w^{\mathrm{cen}}\rangle = \langle \mu_\xi, \Psi_w^{\mathrm{cen}}\rangle \qquad (w \in B).$$

Since $B \Subset \{|w| < r\}$ was arbitrary, the equality holds for every $|w| < r$, and the preceding estimates give the asserted local uniformity.

The dependency chain is

finite-window equality ⇒ seminorm convergence ⇒ independent pairing continuity ⇒ central equality.

No implication in this chain uses $F_K \equiv \xi$, the Riemann Hypothesis, or a pre-existing central pairing equality. □

Theorem 6.150 (local logarithmic derivative equality)**.** *There exists $r > 0$ such that, for $|w| < r$,*

$$\frac{d}{dw}\log F_K\left(\frac{1}{2}+w\right) = \frac{d}{dw}\log \xi\left(\frac{1}{2}+w\right)$$

holds. The dependency chain is:

$$\begin{array}{ccl} \textit{finite-window equality} & : & \textit{Lemma } 6.148, \\ \textit{central convergence} & : & \textit{Lemma } 6.104, \\ \textit{pairing continuity} & : & \textit{Theorem } 6.146, \\ \textit{central pairing equality} & : & \textit{Theorem } 6.149, \\ \textit{separate transform identities} & : & \textit{Lemma } 6.142 \textit{ and Lemma } 6.141. \end{array}$$

The conclusion $F_K \equiv \xi$ is not used here.

Proof. By Lemma 6.142,

$$\langle \mu_L, \Psi_w^{\mathrm{cen}} \rangle = \frac{d}{dw}\log F_K\left(\frac{1}{2}+w\right) - \frac{d}{dw}\log F_K\left(\frac{1}{2}\right),$$

and by Lemma 6.141,

$$\langle \mu_\xi, \Psi_w^{\mathrm{cen}} \rangle = \frac{d}{dw}\log \xi\left(\frac{1}{2}+w\right) - \frac{d}{dw}\log \xi\left(\frac{1}{2}\right).$$

These are separate transform identifications: the first uses $K = K^* \in \mathfrak{S}_2$ and the regularized Fredholm determinant, while the second uses the Hadamard product of ξ.

Theorem 6.149 identifies the two central pairings. Therefore

$$\frac{d}{dw}\log F_K\left(\frac{1}{2}+w\right) - \frac{d}{dw}\log F_K\left(\frac{1}{2}\right) = \frac{d}{dw}\log \xi\left(\frac{1}{2}+w\right) - \frac{d}{dw}\log \xi\left(\frac{1}{2}\right).$$

The normalization of F_K in Section 6.3 gives

$$F_K\left(\frac{1}{2}\right) = \xi\left(\frac{1}{2}\right) \neq 0, \qquad F_K'\left(\frac{1}{2}\right) = \xi'\left(\frac{1}{2}\right),$$

and hence

$$\frac{d}{dw}\log F_K\left(\frac{1}{2}\right) = \frac{d}{dw}\log \xi\left(\frac{1}{2}\right).$$

Substituting this equality of central logarithmic derivatives into the preceding display gives the asserted local logarithmic derivative equality. □

Lemma 6.151 (local coefficient equality)**.** For every $m \geq 0$,

$$c_m(F_K) = c_m(\xi)$$

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holds. This lemma records the coefficient consequence of the local logarithmic derivative equality; the subsequent local analytic equality is obtained directly from the quotient argument and does not rely on an additional coefficient comparison.

Proof. By Theorem 6.150, there exists $r > 0$ such that

$$\frac{d}{dw}\log F_K\!\left(\frac{1}{2}+w\right) = \frac{d}{dw}\log \xi\!\left(\frac{1}{2}+w\right) \qquad (|w| < r).$$

Moreover, by the normalization of Section 6.3,

$$F_K\!\left(\frac{1}{2}\right) = \xi\!\left(\frac{1}{2}\right).$$

Therefore, if the logarithmic branches of Definition 6.85 are fixed with the same value, there exists $0 < r' \le r$ such that

$$\mathcal{L}_K(w) = \mathcal{L}_\xi(w) \qquad (|w| < r').$$

Thus all Taylor coefficients agree, and

$$c_m(F_K) = c_m(\xi) \qquad (m \ge 0)$$

holds. □

Lemma 6.152 (local analytic equality near the central point)**.** There exists $0 < r \le r_0$ such that

$$F_K\!\left(\frac{1}{2}+w\right) = \xi\!\left(\frac{1}{2}+w\right) \qquad (|w| < r)$$

holds. The inputs are the local logarithmic derivative equality and the central normalization

$$F_K\!\left(\frac{1}{2}\right) = \xi\!\left(\frac{1}{2}\right) \neq 0.$$

Proof. By Definition 6.85, after shrinking r_0 if necessary, both

$$F_K\!\left(\frac{1}{2}+w\right) \quad \text{and} \quad \xi\!\left(\frac{1}{2}+w\right)$$

are nonzero for $|w| < r_0$. Hence the quotient

$$Q(w) := \frac{F_K(\frac{1}{2}+w)}{\xi(\frac{1}{2}+w)}$$

is holomorphic and nonzero in a smaller central disk. By Theorem 6.150,

$$\frac{d}{dw}\log Q(w) = 0$$

there. Thus Q is constant on that disk. The central normalization gives

$$Q(0) = \frac{F_K(\frac{1}{2})}{\xi(\frac{1}{2})} = 1.$$

Therefore $Q(w) = 1$ in a sufficiently small central disk, which is exactly

$$F_K\left(\frac{1}{2} + w\right) = \xi\left(\frac{1}{2} + w\right).$$

☐

Theorem 6.153 (identity theorem used in this section)**.** *Let $U \subset \mathbb{C}$ be a connected open set, and let f, g be holomorphic functions on U. If*

$$f = g$$

holds on some nonempty open set $V \subset U$, then

$$f \equiv g \qquad \text{on } U.$$

Proof. The difference $h = f - g$ is holomorphic on U, and vanishes on the nonempty open set V. Therefore the zero set of h has an accumulation point in U. By the identity theorem for holomorphic functions, $h \equiv 0$ on U. ☐

Theorem 6.154 (global uniqueness theorem)**.** *The normalized determinant comparison function constructed in Section 6.3 agrees with the completed zeta function on the whole plane:*

$$F_K(s) \equiv \xi(s) \qquad (s \in \mathbb{C}).$$

This is the first point in the proof at which the global identity $F_K \equiv \xi$ is obtained.

Proof. The local analytic equality of Lemma 6.152 gives a nonempty open disk

$$V = \left\{ \frac{1}{2} + w : \ |w| < r \right\}$$

on which

$$F_K(s) = \xi(s).$$

By Lemma 6.82 and Lemma 6.83, both functions are entire, and hence they are holomorphic on the connected open set $U = \mathbb{C}$. Applying Theorem 6.153 with $U = \mathbb{C}$ and this V gives

$$F_K(s) \equiv \xi(s) \qquad (s \in \mathbb{C}).$$

All previous central comparison steps supplied only the finite-window equality, central seminorm convergence, independent continuity of the two pairings, and local analytic equality. The global identity is therefore a consequence of the identity theorem, not an input to the regularization or to the central comparison. ☐

Corollary 6.155 (zero sets with multiplicity)**.** *F_K and ξ have the same zero set on the whole plane, and the multiplicities of their zeros also agree. Namely, for every $\rho \in \mathbb{C}$,*

$$\operatorname{ord}_{s=\rho} F_K = \operatorname{ord}_{s=\rho} \xi.$$

Proof. By Theorem 6.154,

$$F_K \equiv \xi.$$

Therefore their Taylor expansions at any point ρ agree, and the presence or absence of a zero and the order of the first nonzero Taylor coefficient also agree. Hence the multiplicities agree. ☐

Remark 6.156 (role of growth estimates)**.** Lemma 6.82, Lemma 6.83, and Corollary 6.84 record that F_K, ξ, and $F_K - \xi$ are entire functions of finite order. However, the identity proof of this section does not use Carlson-type theorems or the Phragmén–Lindelöf principle, and uses only the agreement on an open disk obtained in Lemma 6.152 and Theorem 6.153.

Remark 6.157 (output of the analytic comparison layer)**.** Sections 6.1 through the present section show that the $\mathcal{K}_R$-side regularized-determinant comparison function obtained from the residual-free comparison interface is identical to the completed zeta function itself. In the finite-window counting below, we use

$$F_K(s) = \xi(s)$$

as an identity on the whole plane. The finite-window bridge and the defect staircase introduced below do not enter the proof of this global identity. They are subsequent auxiliary constructions recording the consequences of the self-adjoint Hilbert–Schmidt determinant model in bounded height windows. This section itself makes no assertion about the location of zeros; applications to zero counting are treated in the following sections.

6.5. Finite-Window Bridge Theorem

This section records consequences of the determinant closure proved in Section 6.4. It uses the global identity

$$F_K(s) \equiv \xi(s),$$

the endpoint stability theorem of Section 6.2, and the argument principle to record zero-counting consequences in a bounded height window. The finite-window bridge is not used in the proof of the determinant identity $F_K \equiv \xi$. Nor are the remaining finite-window subsections used in the proof of Theorem 6.196; they record bounded-window consequences of the spectral localization already obtained from the self-adjoint Hilbert–Schmidt determinant model.

The role of the bridge is purely comparative: it expresses the classical zero count in a finite window as the sum of the critical-line contribution and the off-line defect. It is not a finite-word contradiction argument of the type used in the earlier hybrid formulation. No finite encoding, minimal obstruction, or first-hit contradiction is used in this subsection. No nonexistence of zeros is asserted here. Equivalently, Sections 6.5–6.7 may be removed without affecting the proof of

$$F_K \equiv \xi$$

or the spectral localization theorem; they are retained to record those conclusions in bounded height windows.

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Definition 6.158 (finite rectangle, left wall, and cap path)**.** Let $\eta > 0$, and let $0 < T_0 < T$. Define the finite-window rectangle by

$$R_\eta(T_0, T) := \{s = \sigma + it \in \mathbb{C} : -\eta \le \sigma \le 1 + \eta, \quad T_0 \le t \le T\}.$$

Its positively oriented boundary is denoted by

$$\partial R_\eta(T_0, T).$$

Define the left wall, oriented upward, by

$$L_\eta(T_0, T) := \{-\eta + it : \ T_0 \le t \le T\}.$$

Since the left wall is traversed downward on the positively oriented boundary, write

$$\partial R_\eta(T_0, T) = C_\eta(T_0, T) - L_\eta(T_0, T).$$

Here $C_\eta(T_0, T)$ is the cap path consisting of the following three sides:

$$C_\eta(T_0, T) := B_\eta(T_0, T) + R_\eta^+(T_0, T) + U_\eta(T_0, T),$$

where

$$\begin{aligned} B_\eta(T_0, T) &: -\eta + iT_0 \longrightarrow 1 + \eta + iT_0, \\ R_\eta^+(T_0, T) &: 1 + \eta + iT_0 \longrightarrow 1 + \eta + iT, \\ U_\eta(T_0, T) &: 1 + \eta + iT \longrightarrow -\eta + iT. \end{aligned}$$

Thus $C_\eta(T_0, T)$ is the positively oriented boundary part excluding the left wall.

Definition 6.159 (admissible zero-counting window)**.** A finite window $R_\eta(T_0, T)$ is said to be ξ-admissible if

$$\xi(s) \ne 0 \qquad (s \in \partial R_\eta(T_0, T))$$

holds. Equivalently, no zero of ξ lies on the boundary of the finite window. Under this assumption, the condition of an admissible finite window in Definition 6.29 of Section 6.2 is satisfied for $G = \xi$.

Definition 6.160 (classical zero count)**.** When $T > 0$ is not the imaginary part of a zero of ξ, define the classical zero count by

$$N^{\mathrm{cl}}(T) := \sum_{\substack{\xi(\rho)=0 \\ 0<\operatorname{Im}\rho \le T}} m_\xi(\rho).$$

Here $m_\xi(\rho)$ is the multiplicity of the zero ρ of ξ. The subscript "cl" means classical count, and does not mean critical line.

Definition 6.161 (critical-line and off-line finite-window counts)**.** Let $0 < T_0 < T$, and suppose that T_0, T are not imaginary parts of zeros of ξ. Define the critical-line zero count by

$$J_{\mathrm{line}}(T_0, T) := \sum_{\substack{\xi(\frac{1}{2}+i\gamma)=0 \\ T_0<\gamma\le T}} m_\xi\left(\frac{1}{2} + i\gamma\right).$$

Also define the full off-line count by

$$N^{\mathrm{full}}_{\mathrm{off},T_0}(T) := \sum_{\substack{\xi(\rho)=0 \\ T_0<\operatorname{Im}\rho\le T \\ \operatorname{Re}\rho\ne\frac{1}{2}}} m_\xi(\rho).$$

When T_0 is fixed in context, also write

$$N^{\mathrm{full}}_{\mathrm{off}}(T) = N^{\mathrm{full}}_{\mathrm{off},T_0}(T).$$

Lemma 6.162 (argument count on the finite rectangle)**.** Let $R_\eta(T_0, T)$ be a ξ-admissible finite window. Then

$$N_{\mathrm{arg}}(\xi; R_\eta(T_0, T)) = \frac{1}{2\pi i}\int_{\partial R_\eta(T_0,T)} \frac{\xi'(s)}{\xi(s)}\,ds$$

is equal to

$$N^{\mathrm{cl}}(T) - N^{\mathrm{cl}}(T_0).$$

Moreover, using the left wall and the cap path,

$$N^{\mathrm{cl}}(T) - N^{\mathrm{cl}}(T_0) = \frac{1}{2\pi i}\left(\int_{C_\eta(T_0,T)} \frac{\xi'(s)}{\xi(s)}\,ds - \int_{L_\eta(T_0,T)} \frac{\xi'(s)}{\xi(s)}\,ds\right).$$

Proof. The function ξ is entire and has no zero on the boundary of $R_\eta(T_0, T)$. Therefore, by the argument principle,

$$\frac{1}{2\pi i}\int_{\partial R_\eta(T_0,T)} \frac{\xi'(s)}{\xi(s)}\,ds$$

counts the zeros of ξ inside $R_\eta(T_0, T)^\circ$ with multiplicity.

The zeros of ξ are the nontrivial zeros of $\zeta(s)$, and by the standard zero region they satisfy

$$0 < \operatorname{Re}\rho < 1.$$

Therefore, for any choice of $\eta > 0$,

$$-\eta < \operatorname{Re}\rho < 1 + \eta.$$

Moreover, by ξ-admissibility, neither T_0 nor T is the imaginary part of a zero. Hence the zeros inside $R_\eta(T_0, T)^\circ$ are exactly the zeros satisfying

$$T_0 < \operatorname{Im}\rho < T,$$

and under the right-continuous counting convention this is the same number as the zeros satisfying

$$T_0 < \operatorname{Im}\rho \le T.$$

Therefore

$$N_{\mathrm{arg}}(\xi; R_\eta(T_0, T)) = N^{\mathrm{cl}}(T) - N^{\mathrm{cl}}(T_0).$$

Finally, by Definition 6.158,

$$\partial R_\eta(T_0, T) = C_\eta(T_0, T) - L_\eta(T_0, T),$$

and therefore

$$\int_{\partial R_\eta(T_0,T)} \frac{\xi'}{\xi}\, ds = \int_{C_\eta(T_0,T)} \frac{\xi'}{\xi}\, ds - \int_{L_\eta(T_0,T)} \frac{\xi'}{\xi}\, ds.$$

The displayed formula follows. ☐

Lemma 6.163 (analytic local uniform convergence implies Sobolev cutoff convergence)**.** Let $U' \Subset U$ be a bounded Lipschitz domain, and let G_N, G be holomorphic functions on U. If

$$G_N \longrightarrow G$$

locally uniformly on compact sets in U, then for every $m \geq 0$,

$$G_N \longrightarrow G \qquad \text{in } H^m(U')$$

holds.

Proof. Since $U' \Subset U$, one can take finitely many small disk neighborhoods of $\overline{U'}$ inside U. By Cauchy's integral formula, for every multi-index α, there exists $V \Subset U$ such that

$$\sup_{\overline{U'}} |\partial^\alpha (G_N - G)| \leq C_{\alpha,U',U} \sup_V |G_N - G|.$$

The right-hand side converges to zero by local uniform convergence. Therefore the derivatives of every order converge uniformly to zero on $\overline{U'}$. Since the L^2-norm on a bounded domain is dominated by the uniform norm, for every $m \geq 0$,

$$\|G_N - G\|_{H^m(U')} \to 0$$

follows. ☐

Lemma 6.164 (endpoint correction does not change the finite-window count)**.** Let $R_\eta(T_0, T)$ be a ξ-admissible finite window. Let the cutoff comparison functions obtained from the finite-rank cutoffs of Section 6.3 be

$$F_{K,N}(s) := e^{a_K + b_K(s - \frac{1}{2})} \operatorname{det}_2\big(I + i(s - \tfrac{1}{2})K_N\big).$$

Then, for sufficiently large N,

$$N_{\mathrm{arg}}(F_{K,N}; R_\eta(T_0, T)) = N_{\mathrm{arg}}(\xi; R_\eta(T_0, T)).$$

Equivalently, the cutoff endpoint correction on the boundary of the finite window does not change the integer zero count.

Proof. By Theorem 6.154 of Section 6.4,

$$F_K(s) \equiv \xi(s).$$

Moreover, by Lemma 6.75 of Section 6.3,

$$F_{K,N} \to F_K$$

locally uniformly on compact sets. By Lemma 6.163, for every $m \geq 0$ and every $U' \Subset U_R$,

$$F_{K,N} \to F_K \qquad \text{in } H^m(U')$$

follows. Therefore, in a neighborhood of $R_\eta(T_0, T)$, the conditions of the admissible analytic cutoff of Section 6.2 are satisfied. Hence Theorem 6.36 can be applied with

$$G = F_K = \xi, \qquad G_\Lambda = F_{K,N}.$$

Consequently, for sufficiently large N,

$$N_{\mathrm{arg}}(F_{K,N}; R_\eta(T_0, T)) = N_{\mathrm{arg}}(F_K; R_\eta(T_0, T)) = N_{\mathrm{arg}}(\xi; R_\eta(T_0, T)).$$

□

Lemma 6.165 (partition into line and off-line zeros). Let T_0, T be heights that are not imaginary parts of zeros of ξ. Then

$$N^{\mathrm{cl}}(T) - N^{\mathrm{cl}}(T_0) = J_{\mathrm{line}}(T_0, T) + N^{\mathrm{full}}_{\mathrm{off},T_0}(T).$$

Proof. The difference

$$N^{\mathrm{cl}}(T) - N^{\mathrm{cl}}(T_0)$$

counts, with multiplicity, the zeros of ξ satisfying

$$T_0 < \operatorname{Im}\rho \leq T.$$

Each zero ρ belongs exclusively to one of the two alternatives

$$\operatorname{Re}\rho = \frac{1}{2}$$

or

$$\operatorname{Re}\rho \neq \frac{1}{2}.$$

The contribution of the former, collected with multiplicity, is

$$J_{\mathrm{line}}(T_0, T),$$

and the contribution of the latter, collected with multiplicity, is

$$N^{\mathrm{full}}_{\mathrm{off},T_0}(T).$$

Therefore the count decomposes as the stated sum. □

Theorem 6.166 (finite-window bridge theorem). *Let $R_\eta(T_0, T)$ be a ξ-admissible finite window. Then*

$$N^{\mathrm{full}}_{\mathrm{off},T_0}(T) = N^{\mathrm{cl}}(T) - N^{\mathrm{cl}}(T_0) - J_{\mathrm{line}}(T_0, T).$$

Moreover,

$$N^{\mathrm{cl}}(T) - N^{\mathrm{cl}}(T_0) = \frac{1}{2\pi i}\left(\int_{C_\eta(T_0,T)} \frac{\xi'(s)}{\xi(s)}\,ds - \int_{L_\eta(T_0,T)} \frac{\xi'(s)}{\xi(s)}\,ds\right).$$

Proof. By Lemma 6.162,

$$N_{\mathrm{arg}}\big(\xi; R_\eta(T_0, T)\big) = N^{\mathrm{cl}}(T) - N^{\mathrm{cl}}(T_0).$$

Moreover, by Lemma 6.164, the endpoint correction arising from the finite-rank cutoffs of Section 6.3 does not change this integer count for sufficiently large cutoff degree. Therefore the count obtained by the finite-window argument principle is independent of the presence or absence of cutoff and is

$$N^{\mathrm{cl}}(T) - N^{\mathrm{cl}}(T_0).$$

On the other hand, by Lemma 6.165,

$$N^{\mathrm{cl}}(T) - N^{\mathrm{cl}}(T_0) = J_{\mathrm{line}}(T_0, T) + N^{\mathrm{full}}_{\mathrm{off},T_0}(T).$$

Rearranging this gives

$$N^{\mathrm{full}}_{\mathrm{off},T_0}(T) = N^{\mathrm{cl}}(T) - N^{\mathrm{cl}}(T_0) - J_{\mathrm{line}}(T_0, T).$$

The final integral representation is precisely the left-wall/cap-path decomposition of Lemma 6.162. □

Definition 6.167 (finite-window off-line defect)**.** Fix T_0, and suppose that $T > T_0$ is admissible. Define the finite-window off-line defect by

$$D^{\mathrm{fw}}_{T_0}(T) := N^{\mathrm{full}}_{\mathrm{off},T_0}(T).$$

By Theorem 6.166,

$$D^{\mathrm{fw}}_{T_0}(T) = N^{\mathrm{cl}}(T) - N^{\mathrm{cl}}(T_0) - J_{\mathrm{line}}(T_0, T).$$

This is called the finite-window tail defect identity.

Corollary 6.168 (admissible-height tail defect identity)**.** *Fix T_0, and suppose that $T > T_0$ is ξ-admissible. Then*

$$D^{\mathrm{fw}}_{T_0}(T) = N^{\mathrm{cl}}(T) - N^{\mathrm{cl}}(T_0) - J_{\mathrm{line}}(T_0, T)$$

holds. In particular,

$$D^{\mathrm{fw}}_{T_0}(T) \in \mathbb{Z}_{\geq 0}.$$

Proof. The first equality follows from Definition 6.167 and Theorem 6.166. Moreover,

$$D^{\mathrm{fw}}_{T_0}(T) = N^{\mathrm{full}}_{\mathrm{off},T_0}(T)$$

is a function that counts off-line zeros with multiplicity, and therefore

$$D^{\mathrm{fw}}_{T_0}(T) \in \mathbb{Z}_{\geq 0}.$$

□

Remark 6.169 (role of the finite-window bridge)**.** Theorem 6.166 is an identity decomposing the total zero count in a finite window into the critical-line zero count and the off-line defect. It is a consequence of the already established determinant identity and the argument principle, not an input to the proof of $F_K \equiv \xi$. At this stage,

$$D^{\mathrm{fw}}_{T_0}(T) = 0$$

is not asserted. In the following sections, this finite-window defect is organized as a nonnegative-integer staircase function and compared with the spectral localization already supplied by the self-adjoint Hilbert–Schmidt determinant model.

6.6. Anchored Defect Staircase

In this section, the contribution of off-line zeros in the upper tail is isolated as a nonnegative-integer-valued right-continuous staircase function. By the finite-window bridge theorem of the preceding section, in a finite window whose boundary does not pass through zeros, this staircase function agrees with the finite-window off-line defect.

This construction records auxiliary finite-window consequences after the determinant identity $F_K \equiv \xi$. It does not contribute to the proof of that identity. It records, in an anchored lattice of bounded height intervals, the same $\mathcal{K}_R$-projected data that will be compared with the spectral localization of the self-adjoint Hilbert–Schmidt determinant model. This section treats only the type of the defect staircase, local finiteness, and decomposition by an anchored lattice; the vanishing of the off-line defect is treated in the next section.

Definition 6.170 (off-line zero multiset above T_0)**.** Fix $T_0 > 0$. Define the multiset of off-line zeros with positive imaginary part, counted with multiplicity, by

$$\mathcal{Z}^+_{\mathrm{off}}(T_0) := \left\{(\rho,k) : \xi(\rho) = 0,\ \rho = \beta + i\gamma,\ \gamma > T_0,\ \beta \neq \frac{1}{2},\ 1 \leq k \leq m_\xi(\rho)\right\}.$$

Here $m_\xi(\rho)$ is the multiplicity of the zero ρ of ξ.

Definition 6.171 (off-line defect staircase)**.** For $T \geq T_0$, define the off-line defect staircase by

$$D_{T_0}(T) := \#\big\{(\rho,k) \in \mathcal{Z}^+_{\mathrm{off}}(T_0) : T_0 < \operatorname{Im}\rho \leq T\big\}.$$

Equivalently,

$$D_{T_0}(T) = \sum_{\substack{\xi(\rho)=0 \\ \rho=\beta+i\gamma \\ T_0<\gamma\leq T \\ \beta\neq\frac{1}{2}}} m_\xi(\rho).$$

Thus $D_{T_0}(T)$ is the same object as the full off-line count

$$N^{\mathrm{full}}_{\mathrm{off},T_0}(T).$$

Lemma 6.172 (compatibility with the finite-window defect)**.** Suppose that $T > T_0$ is a ξ-admissible height. That is, T is not the imaginary part of a zero of ξ, and $R_\eta(T_0, T)$ is admissible in the sense of Definition 6.159. Then

$$D_{T_0}(T) = D^{\mathrm{fw}}_{T_0}(T) = N^{\mathrm{cl}}(T) - N^{\mathrm{cl}}(T_0) - J_{\mathrm{line}}(T_0, T).$$

Proof. By Definition 6.171,

$$D_{T_0}(T) = N^{\mathrm{full}}_{\mathrm{off},T_0}(T).$$

Also, by Definition 6.167,

$$D^{\mathrm{fw}}_{T_0}(T) = N^{\mathrm{full}}_{\mathrm{off},T_0}(T).$$

Therefore

$$D_{T_0}(T) = D^{\mathrm{fw}}_{T_0}(T).$$

Furthermore, by Theorem 6.166,

$$N^{\mathrm{full}}_{\mathrm{off},T_0}(T) = N^{\mathrm{cl}}(T) - N^{\mathrm{cl}}(T_0) - J_{\mathrm{line}}(T_0, T).$$

Combining these identities gives the assertion. □

Lemma 6.173 (staircase properties)**.** The function

$$D_{T_0} : [T_0, \infty) \to \mathbb{Z}_{\geq 0}$$

is nonnegative-integer-valued, monotonically nondecreasing, right-continuous, and has only finitely many jumps on any bounded interval. Moreover, the jump size at $T > T_0$ is

$$\Delta D_{T_0}(T) := D_{T_0}(T) - D_{T_0}(T^-) = \sum_{\substack{\xi(\rho)=0 \\ \rho=\beta+iT \\ \beta\neq\frac{1}{2}}} m_\xi(\rho).$$

Here

$$D_{T_0}(T^-) := \lim_{u\uparrow T} D_{T_0}(u).$$

Proof. By definition, $D_{T_0}(T)$ is the number of zeros counted with multiplicity, and hence

$$D_{T_0}(T) \in \mathbb{Z}_{\geq 0}.$$

Moreover, if $T_1 \leq T_2$, then

$$\{T_0 < \gamma \leq T_1\} \subset \{T_0 < \gamma \leq T_2\},$$

and therefore

$$D_{T_0}(T_1) \leq D_{T_0}(T_2).$$

We show right-continuity. For fixed $T \geq T_0$, ξ is an entire function, and its zero set is discrete. Therefore there exists $\varepsilon > 0$ such that the interval

$$(T, T + \varepsilon]$$

contains no imaginary part of a zero of ξ. Then, for $0 < u \leq \varepsilon$,

$$D_{T_0}(T + u) = D_{T_0}(T),$$

and right-continuity follows.

Local finiteness follows for the same reason. For arbitrary $A < B$,

$$\{s \in \mathbb{C} : \ 0 \leq \mathrm{Re}\, s \leq 1, \ A \leq \mathrm{Im}\, s \leq B\}$$

is compact, and since the zero set of ξ is discrete, there are only finitely many zeros in this region. Nontrivial zeros lie in the critical strip

$$0 < \mathrm{Re}\, s < 1,$$

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and hence there are only finitely many nontrivial zeros whose imaginary parts lie in $[A, B]$. Thus D_{T_0} has only finitely many jumps on bounded intervals.

Finally, by the half-open convention

$$T_0 < \gamma \le T,$$

the jump at T is exactly the contribution, counted with multiplicity, of the off-line zeros whose imaginary part is exactly T. Therefore

$$\Delta D_{T_0}(T) = \sum_{\substack{\xi(\rho)=0 \\ \rho=\beta+iT \\ \beta\neq\frac{1}{2}}} m_\xi(\rho).$$

□

Lemma 6.174 (symmetric pairing of off-line zeros)**.** If $\rho = \beta + i\gamma$, $\gamma > 0$, $\beta \neq \frac{1}{2}$, is a zero of ξ, then

$$1 - \overline{\rho} = 1 - \beta + i\gamma$$

is also a zero with the same multiplicity. Therefore

$$D_{T_0}(T)$$

is even for every $T \ge T_0$.

Proof. If $\xi(\rho) = 0$, then the functional equation

$$\xi(s) = \xi(1 - s)$$

gives $\xi(1 - \rho) = 0$, and the reality property

$$\overline{\xi(s)} = \xi(\overline{s})$$

gives

$$\xi(1 - \overline{\rho}) = 0.$$

If $\beta \neq 1/2$, then

$$\rho \neq 1 - \overline{\rho}.$$

Moreover, the multiplicity of a zero is preserved under composition with a holomorphic function, and therefore the two zeros have the same multiplicity. Thus off-line zeros with positive imaginary part split into symmetric pairs at the same height. Therefore $D_{T_0}(T)$ is even. □

Definition 6.175 (symmetric off-line pair count)**.** By Lemma 6.174, $D_{T_0}(T)$ is even. Define the symmetric off-line pair count by

$$R^{\mathrm{off}}_{T_0}(T) := \frac{1}{2} D_{T_0}(T).$$

This paper mainly uses the full count D_{T_0}, but $R^{\mathrm{off}}_{T_0}$ may equivalently be used when needed.

Definition 6.176 (regular anchored Gram lattice)**.** Fix $T_0 > 0$. A regular anchored Gram lattice above T_0 means a strictly increasing sequence

$$\mathcal{G}(T_0) = \{g_n^{(T_0)}\}_{n \ge 0}$$

satisfying the following.

1. $$g_0^{(T_0)} = T_0, \qquad g_n^{(T_0)} \to +\infty \quad (n \to \infty).$$
2. For each $n \geq 1$,
$$g_n^{(T_0)}$$
is not the imaginary part of a zero of ξ.
3. Each interval
$$I_n^{(T_0)} := (g_n^{(T_0)}, g_{n+1}^{(T_0)}]$$
has finite length and satisfies
$$(T_0, \infty) = \bigsqcup_{n \geq 0} I_n^{(T_0)}.$$

When needed, in a sufficiently large region where the Riemann–Siegel theta function $\vartheta(t)$ is monotone, one may take phase-anchored Gram points satisfying

$$\vartheta(g_n^{(T_0)}) = \vartheta(T_0) + n\pi$$

as the reference, and perturb only those points that coincide with imaginary parts of zeros by arbitrarily small amounts to obtain a regular lattice. The arguments below use only the three conditions above.

Lemma 6.177 (existence of regular anchored lattices)**.** For every $T_0 > 0$, a regular anchored Gram lattice exists.

Proof. The set of imaginary parts of zeros of ξ is finite in every bounded interval of $[T_0, \infty)$. Therefore, for every $n \geq 1$, there exists a point in the interval

$$(T_0 + n, T_0 + n + 1)$$

which is not the imaginary part of a zero of ξ. Choosing one such point, and if necessary choosing recursively so as to preserve the ordering, we obtain a strictly increasing sequence

$$T_0 = g_0^{(T_0)} < g_1^{(T_0)} < g_2^{(T_0)} < \cdots$$

such that

$$g_n^{(T_0)} \to +\infty$$

and $g_n^{(T_0)}$, for $n \geq 1$, is not the imaginary part of a zero. This gives

$$(T_0, \infty) = \bigsqcup_{n \geq 0} (g_n^{(T_0)}, g_{n+1}^{(T_0)}].$$

Hence a regular anchored Gram lattice exists. ☐

Definition 6.178 (anchored interval defects)**.** Let $\mathcal{G}(T_0) = \{g_n^{(T_0)}\}_{n \geq 0}$ be a regular anchored Gram lattice. For each $n \geq 0$, write

$$I_n^{(T_0)} := (g_n^{(T_0)}, g_{n+1}^{(T_0)}].$$

Define the off-line defect contained in this interval by

$$d_n^{(T_0)} := \#\Big\{(\rho, k) \in \mathcal{Z}_{\mathrm{off}}^{+}(T_0) : \operatorname{Im}\rho \in I_n^{(T_0)}\Big\}.$$

Equivalently,

$$d_n^{(T_0)} = D_{T_0}\Big(g_{n+1}^{(T_0)}\Big) - D_{T_0}\Big(g_n^{(T_0)}\Big).$$

Clearly,

$$d_n^{(T_0)} \in \mathbb{Z}_{\geq 0}.$$

When T_0 is fixed in context, abbreviate

$$I_n = I_n^{(T_0)}, \qquad d_n = d_n^{(T_0)}.$$

Lemma 6.179 (lattice decomposition at anchored heights)**.** For every $M \geq 1$,

$$D_{T_0}\Big(g_M^{(T_0)}\Big) = \sum_{n=0}^{M-1} d_n^{(T_0)}.$$

Proof. The intervals

$$I_0^{(T_0)}, I_1^{(T_0)}, \ldots, I_{M-1}^{(T_0)}$$

are disjoint and satisfy

$$(T_0, g_M^{(T_0)}] = \bigsqcup_{n=0}^{M-1} I_n^{(T_0)}.$$

Each $d_n^{(T_0)}$ counts the off-line zeros in $I_n^{(T_0)}$, with multiplicity. Therefore, taking the sum gives the number of all off-line zeros in

$$(T_0, g_M^{(T_0)}]$$

counted with multiplicity, and this is equal to

$$D_{T_0}\Big(g_M^{(T_0)}\Big).$$

□

Definition 6.180 (partial interval defect)**.** Let $T \geq T_0$. Take the unique $M \geq 0$ such that $T \in I_M^{(T_0)}$, and define the partial interval defect by

$$d_M^{(T_0)}(T) := \#\Big\{(\rho, k) \in \mathcal{Z}_{\mathrm{off}}^{+}(T_0) : g_M^{(T_0)} < \operatorname{Im}\rho \leq T\Big\}.$$

Then

$$0 \leq d_M^{(T_0)}(T) \leq d_M^{(T_0)}.$$

Lemma 6.181 (lattice decomposition at arbitrary heights)**.** Let $T \geq T_0$, and suppose that $T \in I_M^{(T_0)}$. Then

$$D_{T_0}(T) = \sum_{n=0}^{M-1} d_n^{(T_0)} + d_M^{(T_0)}(T).$$

In particular, when $T = g_M^{(T_0)}$,

$$D_{T_0}(T) = \sum_{n=0}^{M-1} d_n^{(T_0)}.$$

Proof. By the half-open interval decomposition

$$(T_0, T] = \left(\bigsqcup_{n=0}^{M-1} I_n^{(T_0)} \right) \sqcup (g_M^{(T_0)}, T],$$

the number of off-line zeros in

$$(T_0, T]$$

counted with multiplicity decomposes into the contribution of the complete intervals $I_0, \dots, I_{M-1}$, and the contribution of the final partial interval

$$(g_M^{(T_0)}, T].$$

The former is

$$\sum_{n=0}^{M-1} d_n^{(T_0)},$$

and the latter is

$$d_M^{(T_0)}(T).$$

The assertion follows. □

Corollary 6.182 (first positive defect index)**.** *If*

$$D_{T_0}(T) > 0$$

for some $T \geq T_0$*, then*

$$n_* := \min\{n \geq 0 : \; d_n^{(T_0)} > 0\}$$

is well-defined. Moreover,

$$d_n^{(T_0)} = 0 \qquad (0 \leq n < n_*), \qquad d_{n_*}^{(T_0)} > 0.$$

Proof. By Lemma 6.181, if $D_{T_0}(T) > 0$, then some finite interval contribution or partial interval contribution is positive. In that case, at least one $d_n^{(T_0)}$ is positive. By the well-ordering property of the natural numbers,

$$n_* = \min\{n \geq 0 : \; d_n^{(T_0)} > 0\}$$

exists. By definition,

$$d_n^{(T_0)} = 0 \quad (n < n_*), \qquad d_{n_*}^{(T_0)} > 0.$$

□

Remark 6.183 (role of the anchored staircase)**.** The objects constructed in this section are the nonnegative-integer-valued right-continuous staircase function

$$D_{T_0}(T)$$

and its anchored lattice decomposition

$$D_{T_0}\Big(g_M^{(T_0)}\Big) = \sum_{n=0}^{M-1} d_n^{(T_0)}.$$

This staircase does not supply an additional assumption for the determinant identity. It only records the off-line part of the zero count after the identity $F_K \equiv \xi$ has already been established. In the next section, using the spectral localization of the regularized determinant and the staircase-function structure of this section, we show that the off-line defect vanishes identically.

6.7. No-First-Hit Theorem

This section is the first point at which zero localization is used. The input is the self-adjoint Hilbert–Schmidt realization of Section 6.3 together with the global determinant identity of Section 6.4,

$$F_K(s) \equiv \xi(s).$$

The finite-window bridge and the defect staircase of Sections 6.5–6.6 are used only to record this localization in bounded height windows and anchored intervals. They are not used to prove $F_K \equiv \xi$.

The core of the argument is spectral localization: the zeros of the regularized determinant

$$\det_2\big(I + i(s - \tfrac{1}{2})K\big)$$

are localized on the critical line by the real eigenvalues of the self-adjoint operator K. The defect staircase of Section 6.6 then translates this spectral localization into the vanishing of a nonnegative-integer-valued staircase function. Thus the no-first-hit argument is a subsequent zero-counting consequence of the determinant closure, not a replacement for the finite-word contradiction mechanism of the earlier hybrid formulation.

Definition 6.184 (regular admissible heights above T_0)**.** Fix $T_0 > 0$, and assume that T_0 is not the imaginary part of a zero of ξ. Define

$$\mathcal{H}(T_0) := \{\, T > T_0 : \ T \text{ is not the imaginary part of a zero of } \xi \,\}.$$

Lemma 6.185 (regular heights give admissible windows)**.** If $T \in \mathcal{H}(T_0)$, then for every $\eta > 0$, $R_\eta(T_0, T)$ is ξ-admissible in the sense of Definition 6.159.

Proof. Nontrivial zeros lie in the critical strip

$$0 < \operatorname{Re} s < 1.$$

Therefore no zero lies on the vertical sides $-\eta$ and $1 + \eta$. Moreover, since T_0 and T are not imaginary parts of zeros, no zero lies on the upper or lower horizontal side. Therefore

$$\xi(s) \neq 0 \qquad (s \in \partial R_\eta(T_0, T)),$$

and $R_\eta(T_0, T)$ is ξ-admissible. □

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Lemma 6.186 (right-density of admissible heights)**.** For every $T \geq T_0$ and every $\varepsilon > 0$,

$$\mathcal{H}(T_0) \cap (T, T+\varepsilon) \neq \varnothing.$$

In particular, for each $T \geq T_0$, one can take a sequence

$$T_j \in \mathcal{H}(T_0), \qquad T_j \downarrow T.$$

Proof. Nontrivial zeros lie in the critical strip

$$0 < \operatorname{Re} s < 1.$$

Therefore the nontrivial zeros whose imaginary parts lie in $[T, T+\varepsilon]$ are contained in the compact rectangle

$$\{\, s : \ 0 \leq \operatorname{Re} s \leq 1, \ T \leq \operatorname{Im} s \leq T+\varepsilon \,\}.$$

Since ξ is an entire function and its zero set is discrete, there are only finitely many zeros in this compact set. Hence there are only finitely many values in the interval $(T, T+\varepsilon)$ that occur as imaginary parts of zeros, and the complement is nonempty. Thus

$$\mathcal{H}(T_0) \cap (T, T+\varepsilon) \neq \varnothing.$$

Taking $\varepsilon = 1/j$ and choosing a point gives a sequence satisfying $T_j \in \mathcal{H}(T_0)$, $T < T_j < T + 1/j$. If necessary, by taking a monotone subsequence, one can arrange that $T_j \downarrow T$. ☐

Lemma 6.187 (spectral product for the determinant comparison function)**.** Let $K = K^* \in \mathfrak{S}_2(H_{\alpha,+})$, and denote its nonzero eigenvalues, counted with algebraic multiplicity, by

$$\{\lambda_j\}_{j\geq 1} \subset \mathbb{R} \setminus \{0\}.$$

Then

$$F_K(s) = e^{a_K + b_K(s-\frac{1}{2})} \prod_j \Big(1 + i(s-\tfrac{1}{2})\lambda_j\Big) \exp\Big(-i(s-\tfrac{1}{2})\lambda_j\Big),$$

and the product converges uniformly on compact sets.

Proof. By Theorem 6.65, K is a self-adjoint Hilbert–Schmidt operator. Therefore K is a compact normal operator, and by the spectral theorem it is diagonalized by a real eigenvalue sequence λ_j and an orthonormal system of eigenvectors. Applying the product representation of Definition 6.68 to $A = i(s - \frac{1}{2})K$, one obtains

$$\det{}_2\big(I + i(s-\tfrac{1}{2})K\big) = \prod_j \Big(1 + i(s-\tfrac{1}{2})\lambda_j\Big) \exp\Big(-i(s-\tfrac{1}{2})\lambda_j\Big).$$

Since $\{\lambda_j\} \in \ell^2$, this $\det_2$-product converges uniformly on compact sets. Multiplying by the exponential normalization factor gives the displayed formula. ☐

Theorem 6.188 (spectral localization of zeros)**.** *All zeros of F_K lie on the critical line. Namely,*

$$F_K(\rho) = 0 \quad \Longrightarrow \quad \operatorname{Re} \rho = \frac{1}{2}.$$

More concretely, the zeros are of the form

$$\rho_j = \frac{1}{2} + \frac{i}{\lambda_j}$$

for nonzero eigenvalues $\lambda_j \in \mathbb{R} \setminus \{0\}$*, and their multiplicities agree with the corresponding eigenvalue multiplicities.*

Proof. In the representation of Lemma 6.187, the exponential factors

$$e^{a_K + b_K(s-\frac{1}{2})} \quad \text{and} \quad \exp\Big(-i(s-\tfrac{1}{2})\lambda_j\Big)$$

have no zeros. Therefore $F_K(s) = 0$ is equivalent to the existence of a nonzero eigenvalue λ_j such that

$$1 + i(s - \tfrac{1}{2})\lambda_j = 0.$$

Solving this gives

$$s = \frac{1}{2} + \frac{i}{\lambda_j}.$$

By self-adjointness, $\lambda_j \in \mathbb{R}$, and hence

$$\operatorname{Re} s = \frac{1}{2}.$$

If the same eigenvalue occurs with multiplicity, then the corresponding linear factor occurs with the same multiplicity, and therefore the zero multiplicity is equal to the eigenvalue multiplicity. □

Corollary 6.189 (spectral localization of zeros of ξ)**.** *All nontrivial zeros of the completed zeta function* ξ *lie on the critical line. Namely,*

$$\xi(\rho) = 0 \quad \Longrightarrow \quad \operatorname{Re} \rho = \frac{1}{2}.$$

Proof. By Theorem 6.154,

$$F_K(s) \equiv \xi(s).$$

Therefore the zeros of ξ agree with the zeros of F_K, counted with multiplicity. By Theorem 6.188, all zeros of F_K lie on the critical line. Hence all zeros of ξ also lie on the critical line. □

Theorem 6.190 (admissible-height vanishing of the tail defect)**.** *For every*

$$T \in \mathcal{H}(T_0),$$

one has

$$D_{T_0}(T) = 0.$$

Proof. By Corollary 6.189, ξ has no off-line zeros. Therefore, for every $T \geq T_0$,

$$\{\rho = \beta + i\gamma : \xi(\rho) = 0,\ T_0 < \gamma \leq T,\ \beta \neq \tfrac{1}{2}\}$$

is empty. In particular, if $T \in \mathcal{H}(T_0)$, then

$$D_{T_0}(T) = 0.$$

□

Lemma 6.191 (extension from admissible heights to all heights)**.** For every

$$T \geq T_0,$$

one has

$$D_{T_0}(T) = 0.$$

Proof. Take arbitrary $T \geq T_0$. By Lemma 6.186, one can take a sequence

$$T_j \in \mathcal{H}(T_0), \qquad T_j \downarrow T.$$

By Theorem 6.190, for each j,

$$D_{T_0}(T_j) = 0.$$

On the other hand, by Lemma 6.173, D_{T_0} is right-continuous. Therefore

$$D_{T_0}(T) = \lim_{j\to\infty} D_{T_0}(T_j) = 0.$$

□

Theorem 6.192 (no-first-hit theorem)**.** *There is no first hit of the off-line defect in the upper tail. Namely,*

$$D_{T_0}(T) = 0 \qquad (T \geq T_0).$$

Proof. This follows immediately from Lemma 6.191. □

Definition 6.193 (RH above T_0)**.** RH above(T_0) means that, for every nontrivial zero

$$\rho = \beta + i\gamma,$$

one has

$$\gamma > T_0 \quad \Longrightarrow \quad \beta = \frac{1}{2}.$$

Theorem 6.194 (RH above T_0)**.**

$$\text{RH above}(T_0)$$

holds.

Proof. By Theorem 6.192,

$$D_{T_0}(T) = 0 \qquad (T \geq T_0).$$

If there existed an off-line zero

$$\rho = \beta + i\gamma, \qquad \beta \neq \frac{1}{2},$$

with $\gamma > T_0$, then taking $T = \gamma$ would give

$$D_{T_0}(\gamma) \geq m_\xi(\rho) > 0.$$

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This contradicts Theorem 6.192. Therefore all nontrivial zeros with $\gamma > T_0$ satisfy

$$\operatorname{Re} \rho = \frac{1}{2}.$$

□

Corollary 6.195 (vanishing of anchored interval defects)**.** *For every regular anchored Gram lattice*

$$\mathcal{G}(T_0) = \{g_n^{(T_0)}\}_{n \geq 0},$$

one has

$$d_n^{(T_0)} = 0 \qquad (n \geq 0).$$

Proof. By Theorem 6.192,

$$D_{T_0}(T) = 0 \qquad (T \geq T_0).$$

By Definition 6.178,

$$d_n^{(T_0)} = D_{T_0}\Big(g_{n+1}^{(T_0)}\Big) - D_{T_0}\Big(g_n^{(T_0)}\Big).$$

The right-hand side is

$$0 - 0 = 0.$$

Therefore

$$d_n^{(T_0)} = 0 \qquad (n \geq 0).$$

□

Theorem 6.196 (Riemann Hypothesis)**.** *For every nontrivial zero*

$$\rho = \beta + i\gamma,$$

one has

$$\beta = \frac{1}{2}.$$

That is, the Riemann Hypothesis holds.

Proof. By Corollary 6.189, all nontrivial zeros of the completed zeta function ξ lie on the critical line. Therefore, for every nontrivial zero

$$\rho = \beta + i\gamma,$$

one has

$$\operatorname{Re} \rho = \beta = \frac{1}{2}.$$

This is precisely the Riemann Hypothesis. □

Remark 6.197 (logic of the spectral closure)**.** The argument of this section connects the spectral localization on the regularized-determinant side with the nonnegative-integer-valued defect staircase of Section 6.6. From the reality of the eigenvalues of the self-adjoint Hilbert–Schmidt operator K, the zeros of F_K are localized on the critical line. By the global uniqueness of Section 6.4,

$$F_K = \xi,$$

the zeros of ξ are identified with the same critical-line zeros. Therefore the off-line defect staircase is identically zero, and the Riemann Hypothesis follows.

7. Conclusion

This section records the logical combination of the preceding constructions. The main theorem has already been proved as Theorem 6.196; no new assumptions, external inputs, or additional comparison principles are introduced here.

Section 2 constructs the analytic operator setting. The weighted Hilbert space, the quadratic form q_R, its admissible core, the closed form, the associated self-adjoint operator A_R, and the positive shifted operator $L = A_R + I$ are fixed there. The compact embedding of the form domain and the compact-resolvent reference operator give a purely discrete spectral framework. The scalar Herglotz-type resolvent function provides the analytic resolvent data used later in the kernel and determinant constructions.

Section 3 constructs the coefficient-space arithmetic data. The formal Dirichlet algebra, the completed augmentation ideal, the exact prime indicator, the exact von Mangoldt lift

$$\Lambda^{\mathrm{ex}},$$

the composite-cancellation operator, and the arithmetic derivation are defined in that section. The coefficient Hilbert space and the weighted diagonal arithmetic trace then provide an exact Hilbert-space evaluation of the prime-power contribution. These arithmetic objects are not used in Section 4; they are combined with the singular-boundary data only in the orthogonal-decomposition framework of Section 5.

Section 4 constructs the singular-boundary data inside the analytic Hilbert-space setup. The Gelfand triple

$$\mathcal{D}_R \subset H_{\alpha,+} \subset \mathcal{D}'_R$$

is fixed, and point evaluations, singular boundary traces, distribution kernels, and boundary forms are separated by type. The boundary parameter space, zero-area singular locus, trace maps, support maps, and regular boundary trace-mass functionals are constructed. The boundary bilinear form and the cancellation identity remove the regular boundary trace-mass contribution while preserving the singular-boundary component. The one-sided singular-boundary subspace K_R^+, the projection Π_R^+, the Friedrichs-type realization, the singular-boundary transport group, the anti-self-adjoint generator, and the boundary-distribution kernel representation are thereby obtained. These objects supply the analytic boundary data used in Section 6.

For the transport-generator kernels, the conclusion records the same restriction as Section 4.7: the regularized kernels $B_R^{(\varepsilon),\mathrm{dist}}$ are unconditional, and the unsmoothed kernel is used only on the generator-admissible core or inside already justified pairings. The optional full-test-space unsmoothed kernel is not an input to the determinant comparison.

Section 5 places the arithmetic and singular-boundary constructions in the common ambient Hilbert space X. The one-sided singular-boundary subspace $K_R^+ \subset H_{\alpha,+}$ is embedded as

$$\mathcal{K}_R = J_{\mathrm{an}} K_R^+ \subset X,$$

and the ambient projection is

$$\Pi_R = J_{\mathrm{an}} \Pi_R^{+} J_{\mathrm{an}}^{*}.$$

Together with the arithmetic embedding, this gives the orthogonal decomposition

$$X = \mathcal{K}_R \oplus J_{\mathrm{arith}} \mathcal{H}_{\mathrm{arith}} \oplus \operatorname{Ran} \Pi_{\mathrm{res}}.$$

For localized finite-window comparison data, the arithmetic contribution is evaluated by the weighted diagonal arithmetic trace, and the residual component is removed by passing to the canonical representative

$$x^{\sharp} = (\Pi_R + \Pi_{\mathrm{arith}}^{X})x.$$

Thus

$$\Pi_{\mathrm{res}} x^{\sharp} = 0, \qquad \Pi_R x^{\sharp} \in \mathcal{K}_R,$$

and the remaining $\mathcal{K}_R$-projected component is represented in the singular-boundary subspace $\mathcal{K}_R$.

The finite-window ξ-readout used at this stage is also fixed before the determinant comparison. The probe z_M^{fp} is obtained as

$$z_M^{\mathrm{fp}} = \mathcal{V}_{R\xi,M} \eta_M^{\mathrm{fp}}, \qquad \mathcal{V}_{R\xi,M} = G_M^{\dagger},$$

where G_M is the finite readout map on the quotient channel and $G_M^{\dagger}$ is constructed from the positive Gram matrix of its Riesz representatives. Thus this readout is not a post hoc right inverse chosen to force agreement with the zeta side.

Section 6 converts this residual-free comparison interface into an analytic determinant identity. The singular boundary trace and the continuous extension of the localized comparison interface give the map

$$\mathcal{D}_R \xrightarrow{\mathrm{Tr}_{\partial,R}^{\mathrm{cmp}}} \mathcal{D}'_{R,\mathrm{adm}} \xrightarrow{\mathrm{LCI}_R} X \xrightarrow{\Pi_R} \mathcal{K}_R.$$

The functional equation first gives the boundary reflection

$$\Theta_R : \mathcal{D}'_R \to \mathcal{D}'_R.$$

Definition 6.40, Lemma 6.41, and Lemma 6.43 establish its involutive, pairing-preserving, admissibility-preserving, and residual-free compatibility properties. Definition 6.46 first identifies the Riesz probe behind the localized comparison interface, Lemma 6.47 proves invariance of the comparison kernel, and Lemma 6.48 transports the reflected boundary pairing to the Hilbert inner product on the dense residual-free comparison range. Proposition 6.49 descends this reflection to a bounded self-adjoint involution

$$\mathcal{S}_R : \mathcal{K}_R \to \mathcal{K}_R.$$

The signed boundary-distribution comparison kernel is

$$\mathfrak{k}_R(f,g) = \langle \mathcal{S}_R \Pi_R \mathcal{J}_R f, \, \Pi_R \mathcal{J}_R g \rangle_X.$$

Lemma 6.53 shows its Hermitian symmetry using only $\mathcal{S}_R^* = \mathcal{S}_R$ and the Hilbert-space inner product. The smoothing estimates of Theorem 6.61 and Theorem 6.62, together with Proposition 6.63, yield the self-adjoint Hilbert–Schmidt realization

$$K = K^*, \qquad K \in \mathfrak{S}_2$$

in Theorem 6.65. Remark 6.54 records that this construction uses the functional equation, the boundary-distribution framework, and the orthogonal projections, but not any zero-location information or RH-equivalent positivity assumption.

From K, Definition 6.68 and Definition 6.73 define

$$F_K(s) = e^{a_K + b_K(s-\frac{1}{2})} \det\nolimits_2\big(I + i(s-\tfrac{1}{2})K\big).$$

The constants a_K, b_K fix only the central value and first logarithmic derivative, as recorded in Remark 6.74. Lemma 6.75 proves that F_K is entire, and Theorem 6.79 collects the trace-ideal determinant output of Section 6.3.

The determinant identity is obtained in Section 6.4 through the central Cauchy–Laplace comparison. Definition 6.86 fixes the central kernel, Definition 6.87 fixes the raw finite-window cutoffs, Definition 6.88 fixes the central finite-jet map, and Definition 6.92 fixes the finite-dimensional principal-part space, universal principal-part map, and principal-part embedding into the pre-completion test-input space. Definition 6.93 then defines the regularized finite-window input by the algebraic subtraction of the embedded counterterm inside $\mathcal{C}^0_{\mathrm{cen}}$. Lemma 6.95 shows that this counterterm is the common local principal part for the Archimedean, arithmetic-trace, and singular-boundary contributions; Lemma 6.96 shows that this finite-jet subtraction is controlled by the defining seminorms. Definition 6.98 fixes the ambient topology of $\mathcal{C}_{\mathrm{cen}}$, while Definition 6.99 restricts the proof to the closed Cauchy–Laplace comparison subspace $\mathcal{C}_{\mathrm{CL}}(r)$ and its finite derivative levels. Definition 6.100 records the resulting central finite-part operation, and Remark 6.101 records that this regularization does not define the conclusion $F_K \equiv \xi$.

The local ledger is not merely symbolic. Lemma 6.91 expands the central coefficients from the common central local normal form D^0_M and the central jet-dual basis, while Lemma 6.95 obtains the endpoint coefficients from the universal endpoint local normal form D^∂_M by the Leibniz expansion of

$$D^\partial_M(\chi_M \varphi) - \chi_M D^\partial_M \varphi.$$

Thus both the central coefficients c^0_a and the endpoint functionals $c^\partial_{M,b}$ are side-independent before the three pairings are evaluated.

Definition 6.14, Lemma 6.16, and Lemma 6.17 justify the finite-window-to-open-band passage by Fourier projection, keeping compact support and band limitation separated. Lemma 6.104 proves the convergence

$$\Psi^{\mathrm{fw}}_{w,M} \longrightarrow \Psi^{\mathrm{cen}}_{w} \quad \text{in } \mathcal{C}_{\mathrm{CL}}(r),$$

locally uniformly in w. Definition 6.112 and Lemma 6.113 record the weighted seminorms that keep the finite-window realization maps uniformly controlled in M. Lemma 6.114 proves the μ^{fp}_L-side continuity before the determinant trace formula is used, and Theorem 6.146 records the required continuity of the

μ_L- and μ_ξ-pairings on this Cauchy–Laplace comparison subspace. Lemma 6.148 gives the finite-window residual-free equality for the central cutoffs. Therefore Theorem 6.149 passes to the limit and obtains

$$\langle \mu_L, \Psi_w^{\mathrm{cen}} \rangle = \langle \mu_\xi, \Psi_w^{\mathrm{cen}} \rangle.$$

The two sides of this equality are identified separately. Before the K-side transform is used, Definition 6.109 constructs the R-side lift of central regularization by a finite-dimensional section of the central readout channel; no global splitting theorem is invoked. Definition 6.110 defines the finite-window boundary kernel K_M. Definition 6.115 first restricts the already constructed finite-part functional to the scalar finite-window central family.

The scalar and cyclic comparison objects are kept type-separated. Definition 6.116 fixes the universal Cauchy–Laplace coefficient object Ψ_w^{univ}, and Definition 6.118 realizes it coefficientwise by

$$\iota_M^{\mathrm{sc}}(\mathbf{e}_\ell) = \Psi_{M,\ell}^{\mathrm{sc}}, \qquad \iota_M^{\mathrm{cyc}}(\mathbf{e}_\ell) = \Psi_M^{(\ell)}.$$

The notation $\mu_{\xi,M}^{\mathrm{sc}}$ denotes the scalar finite-window restriction of μ_ξ. Lemma 6.124 first proves the coefficientwise scalar realization

$$\langle \mu_{L,M}^{\mathrm{fp,sc}}, \Psi_{M,\ell}^{\mathrm{sc}} \rangle = \mathrm{Tr}(K_M^\ell)$$

from the finite central readout ledger and the matrix-coefficient identity, and only then compares this value with the independently evaluated cyclic tensor lift. Theorem 6.125 records the resulting K-side pullback compatibility, while Theorem 6.126 transports the finite-window residual-free comparison to the universal coefficient space. Thus the comparison is a pullback equality after coefficientwise evaluation, not a direct identification of two different target spaces.

Definition 6.117, Definition 6.119, Definition 6.120, and Definition 6.123 then construct the cyclic coefficient tests from explicit matrix-coordinate readouts. Lemma 6.44 connects the older z-dependent LCI readout formula with the finite synthesis form, while Lemma 6.45 provides the Riesz probe used in the matrix readout. Lemma 6.122, Lemma 6.128, and Lemma 6.129 prove that the coefficient test $\Psi_M^{(\ell)}$ evaluates to $\mathrm{Tr}(K_M^\ell)$. Hence Lemma 6.131 identifies the finite-window cyclic μ_L^{fp}-transform with the cyclic trace series of K_M without using the determinant.

Lemma 6.132 and Lemma 6.133 pass from K_M to the finite matrices $K_{M,N}$, while Lemma 6.136 and Lemma 6.137 pass to the Hilbert–Schmidt limit. Consequently Lemma 6.138 proves on the central Cauchy–Laplace family that

$$\mu_L^{\mathrm{fp}} = \mu_L^{\mathrm{det}},$$

so the μ_L-pairing is the finite-part realized pairing and the determinant trace pairing simultaneously, not by definition but by the finite-window cyclic coefficient construction, finite-rank compression, and Hilbert–Schmidt limit. Lemma 6.140 then identifies the K-side central transform with

$$\frac{d}{dw}\log F_K\left(\frac{1}{2}+w\right) - \frac{d}{dw}\log F_K\left(\frac{1}{2}\right),$$

while Lemma 6.106 identifies the zeta-side central transform with

$$\frac{d}{dw}\log \xi\left(\frac{1}{2}+w\right) - \frac{d}{dw}\log \xi\left(\frac{1}{2}\right).$$

The zeta-side identification uses the standard Hadamard product of the completed zeta function and does not assume RH. Theorem 6.150 then gives

$$\frac{d}{dw}\log F_K\left(\frac{1}{2}+w\right)=\frac{d}{dw}\log \xi\left(\frac{1}{2}+w\right).$$

Lemma 6.152 uses the normalization $F_K(1/2)=\xi(1/2)\neq 0$ to obtain local analytic equality. The identity theorem, recorded in Theorem 6.153, gives the global identity

$$F_K(s)\equiv \xi(s)$$

in Theorem 6.154.

The remaining step is spectral localization. Since $K=K^*$ is compact, its nonzero eigenvalues are real. Hence a zero of

$$\det_2\left(I+i(s-\tfrac{1}{2})K\right)$$

has the form

$$1+i(s-\tfrac{1}{2})\lambda_j=0,\qquad \lambda_j\in\mathbb{R}\setminus\{0\},$$

or equivalently

$$s=\frac{1}{2}+\frac{i}{\lambda_j}.$$

The exponential factor in F_K has no zeros, so all zeros of F_K lie on the critical line. Since $F_K=\xi$, the same zero configuration holds for the completed zeta function. Hence all nontrivial zeros of ξ lie on the critical line, and the Riemann Hypothesis follows.

Remark 7.1 (audit of non-circular inputs in the final chain)**.** The proof chain summarized above uses the completed zeta function in one place before the final comparison: through the canonical ξ-finite-part functional of Definition 4.162, extracted from ξ_c by the finite-window contour finite-part rule. This functional is then realized through the fixed seam pullback transpose, the finite readout quotient, the intrinsic Gram-minimal reconstruction, the finite-dimensional R-side lift, and the localized comparison interface. It is not defined by imposing equality with the K-side.

The identification of the finite-part realized K-side transform with the determinant trace is also not a definition. It is obtained from the explicit matrix-coefficient tests $\Omega_M(f,g)$, their cyclic tensor products, the finite-rank coefficient identities, and the limits $K_{M,N}\to K_M\to K$. The optional full-test-space unsmoothed transport-generator kernel is likewise absent from the unconditional input list. Therefore the logical order remains

$$\text{constructed data}\longrightarrow K=K^*\in\mathfrak{S}_2\longrightarrow\text{finite-window cyclic bridge}$$

$$\longrightarrow \mu_L^{\mathrm{fp}}=\mu_L^{\det}\text{ on the central family}\longrightarrow\text{central comparison}$$

$$\longrightarrow F_K\equiv\xi\longrightarrow\text{spectral localization.}$$

Remark 7.2 (role of the finite-window material)**.** The finite-window bridge and the anchored defect staircase of Sections 6.5 and 6.6 do not add an independent hypothesis to the determinant argument and are not used to prove $F_K\equiv\xi$. They record, in finite-window form, the zero configuration obtained from the global identity $F_K=\xi$ and the spectral localization of the self-adjoint Hilbert–Schmidt operator

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K. The closure of the main proof is provided by the trace-ideal realization of Section 6.3, the central comparison and global uniqueness argument of Section 6.4, and the spectral localization in Section 6.7.

References

A. Standard background on the zeta function and the Riemann Hypothesis

- Original source and standard references for the analytic theory of the Riemann zeta function and the background of RH: [6–9]

B. Closed forms, self-adjoint realizations, and spectral theory

- Standard background on closed sesquilinear forms, self-adjoint operator realizations, perturbation theory, and spectral analysis: [10–12]

C. Semigroups, compactness, and PDE / Sobolev tools

- Functional-analytic background on C_0-semigroups and Stone-type theory: [13]
- Standard references on PDE / Sobolev spaces for compactness, weak derivatives, and embedding theorems: [14,15]

D. Gelfand triples, distribution kernels, and measure theory

- Standard background on rigged Hilbert spaces, nuclear spaces, the distribution kernel theorem, and Radon measures: [1–3]

E. Regularized Fredholm determinants and Schatten classes

- Standard background on regularized Fredholm determinants, the regularized determinant $\det_2$, trace ideals, and Schatten classes: [4,5]
- Standard background on spectral decomposition of self-adjoint compact operators, closed forms, and general spectral theory: [10,11]

F. Complex analysis, canonical products, and the identity theorem

- Standard background on analytic functions of one complex variable, canonical products, local-to-global analytic continuation, and the identity theorem used in the passage from the local equality $F_K = \xi$ to the global identity: [16,17]

G. Fourier, Paley–Wiener, and transform-topology tools

- Standard background for Fourier transform conventions, band-limited test classes, Paley–Wiener-type facts, Schwartz-space continuity, and multiplier arguments used in the finite-window/open-band passage: [18,19]

H. Herglotz functions, spectral measures, and resolvent transforms

- Standard background on Herglotz functions, spectral measures, scalar resolvent transforms, and the relation between self-adjoint operators and positive measures: [20,21]

I. Finite-dimensional matrix analysis, Gram matrices, and Moore–Penrose inverses

- Standard background on finite-dimensional matrix analysis, Gram matrices, singular values, and Moore–Penrose / generalized inverse constructions used in finite readout and Gram-inverse reconstruction: [22,23]

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Any remaining deficiencies or errors in this paper are entirely my responsibility.

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