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A Note on Odd Perfect Numbers

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Abstract: For over two millennia, the question of whether odd perfect numbers—positive integers whose proper divisors sum to the number itself—exist has captivated mathematicians, from Euclid's elegant construction of even perfect numbers via Mersenne primes to Euler's probing of their odd counterparts. This paper makes significant progress on this ancient conjecture by presenting a rigorous proof by contradiction that odd perfect numbers not divisible by 3 cannot exist. We define the abundancy index, $I(n) = \frac{\sigma(n)}{n}$, where $\sigma(n)$ is the divisor sum function, and leverage its properties alongside the p-adic order and radical of a number. Assuming the existence of an odd perfect number N not divisible by 3, with $I(N) = 2$, we apply a novel lemma to express $I(N)$ as a product over its prime factors. The proof leverages deep connections between analytic number theory (zeta function bounds) and multiplicative properties of divisors (abundancy indices), demonstrating the power of combining these tools to resolve classical conjectures. The assumption that an odd perfect number avoids divisibility by 3 yields a contradiction.

Keywords: odd perfect numbers; divisor sum function; abundancy index function; prime numbers

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1. Introduction

For centuries, mathematicians have been captivated by the enigmatic allure of perfect numbers, defined as positive integers whose proper divisors sum precisely to the number itself [1]. This fascination traces back to ancient Greece, where Euclid devised an elegant formula for generating even perfect numbers through Mersenne primes, numbers of the form $2^p - 1$ where p is prime [1]. His discovery not only provided a systematic way to construct such numbers, like 6, 28, and 496, but also sparked a profound question that has endured through the ages: could there exist odd perfect numbers, defying the pattern of their even counterparts? This tantalizing mystery, rooted in the simplicity of natural numbers, has fueled mathematical curiosity and inspired relentless exploration.

The quest for odd perfect numbers has been marked by both ingenuity and frustration, as the absence of a definitive example or proof has kept the problem alive for millennia. Early mathematicians, guided by intuition, leaned toward the conjecture that all perfect numbers might be even, yet the lack of a rigorous disproof left room for speculation [1]. Figures like Descartes and Euler, towering giants in the history of mathematics, deepened the intrigue by investigating the potential properties of these elusive numbers [1]. Euler, in particular, highlighted the challenge, noting, "Whether . . . there are any odd perfect numbers is a most difficult question". Their efforts revealed constraints—such as the necessity for an odd perfect number to have specific prime factorizations—but no concrete example emerged, leaving the question as a persistent challenge to mathematical rigor.

Today, the mystery of odd perfect numbers remains one of the oldest unsolved problems in number theory, a testament to the profound complexity hidden within simple definitions. Modern computational searches have pushed the boundaries, ruling out odd perfect numbers below staggeringly large thresholds, yet no proof confirms or denies their existence. The problem continues to captivate, not only for its historical significance but also for its ability to bridge elementary arithmetic with deep theoretical questions. As mathematicians wield advanced tools and novel approaches, the search for odd perfect numbers endures, embodying the timeless pursuit of truth in the face of uncertainty. This paper provides a rigorous proof by contradiction that no odd perfect numbers indivisible by 3 exist.

2. Background and Ancillary Results

In 1734, Leonhard Euler solved the celebrated Basel problem, determining the exact value of the Riemann zeta function at $s = 2$. This breakthrough not only demonstrated his extraordinary mathematical creativity but also forged deep connections between analysis, number theory, and the primes [2].

Proposition 1. *The Riemann zeta function evaluated at $s = 2$ satisfies:*

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \prod_{n=1}^{\infty} \frac{p_n^2}{p_n^2 - 1} = \frac{\pi^2}{6},$$

where:

- *p_n is the n-th prime number,*
- *n ranges over the natural numbers, and*
- *$\pi \gtrapprox 3.14159$ is the fundamental constant arising in diverse mathematical contexts, from geometry to number theory.*

Euler's proof ingeniously bridges the infinite series and an infinite product over primes, revealing the surprising appearance of π in the limit.

Definition 1. *In number theory, the p-adic order of a positive integer n, denoted $\nu_p(n)$, is the highest exponent of a prime number p that divides n. For example, if $n = 72 = 2^3 \cdot 3^2$, then $\nu_2(72) = 3$ and $\nu_3(72) = 2$.*

The divisor sum function, denoted $\sigma(n)$, is a fundamental arithmetic function that computes the sum of all positive divisors of a positive integer n, including 1 and n itself. For instance, the divisors of 12 are $1, 2, 3, 4, 6, 12$, yielding $\sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$. This function can be expressed multiplicity over the prime factorization of n, providing a powerful tool for analyzing perfect numbers.

Proposition 2. *For a positive integer $n > 1$ with prime factorization $n = \prod_{p \mid n} p^{\nu_p(n)}$ [3]:*

$$\sigma(n) = \prod_{p \mid n} \left(1 + p + p^2 + \cdots + p^{\nu_p(n)}\right) = n \cdot \prod_{p \mid n} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^{\nu_p(n)}}\right),$$

where $p \mid n$ indicates that p is a prime divisor of n.

The abundancy index, defined as $I(n) = \frac{\sigma(n)}{n}$, maps positive integers to rational numbers and quantifies how the divisor sum compares to the number itself. The following Proposition provides a precise formula for $I(n)$ based on the prime factorization.

Proposition 3. *Let $n = \prod_{i=1}^{j} p_i^{a_i}$ be the prime factorization of n, where $p_1 < \cdots < p_j$ are distinct primes and $a_1, \ldots, a_j$ are positive integers. Then [4]:*

$$I(n) = \prod_{i=1}^{j} \left(\sum_{k=0}^{a_i} \frac{1}{p_i^k}\right) = \prod_{i=1}^{j} \frac{p_i^{a_i+1} - 1}{p_i^{a_i}(p_i - 1)} = \left(\prod_{i=1}^{j} \frac{p_i}{p_i - 1}\right) \cdot \prod_{i=1}^{j} \left(1 - \frac{1}{p_i^{a_i+1}}\right).$$

We conclude that the abundancy index satisfies the upper bound

$$I(n) < \prod_{i=1}^{j} \frac{p_i}{p_i - 1},$$

where $p_1, \ldots, p_j$ are the distinct prime factors of n.

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Definition 2. *The radical of a positive integer n, denoted $\mathrm{rad}(n)$, is the largest square-free divisor of n, obtained as the product of distinct prime factors of n [5]. For example, if $n = 72 = 2^3 \cdot 3^2$, then rad(72) = 2 ·3 = 6.*

In our proof, we utilize the following propositions:

Proposition 4. *The inequality $1 + x \leq e^x$ holds (where $e^x = \exp(x)$) [6].*

Proposition 5. *A positive integer n is a perfect number if and only if $I(n) = 2$, meaning $\sigma(n) = 2n$.*

Proposition 6. *Any odd perfect number N must satisfy the following conditions [7–10]:*

- *Let $p_1, p_2, \ldots, p_j$ be the distinct odd prime factors of N.*
- *N has at least 10 distinct prime factors (i.e., $j \geq 10$).*
- *The reciprocals of its prime factors satisfy $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_j} < \ln 2$, where $\ln$ denotes the natural logarithm.*

These constraints provide critical insights into the potential structure of odd perfect numbers, guiding efforts to prove their non-existence.

By deriving a contradiction from the assumption that odd perfect numbers not divisible by 3 exist—using the properties established above—we provide a definitive proof of their non-existence.

3. Main Result

This is a main insight.

Lemma 1. *For a positive integer $n > 1$ with prime factorization $n = \prod_{p|n} p^{\nu_p(n)}$:*

$$I(n) = \prod_{p|n} \left(1 + \frac{I(p^{\nu_p(n)-1})}{p} \right),$$

where $I(n) = \frac{\sigma(n)}{n}$ is the abundancy index, $\nu_p(n)$ is the p-adic order of n, and $\nu_p(n) - 1 \geq 0$ is a non-negative integer for all primes p dividing n.

Proof. We express the function $I(n)$ in terms of the sum-of-divisors function $\sigma(n)$ and its prime factorization. First, recall that:

$$I(n) = \frac{\sigma(n)}{n}.$$

Using the multiplicative property of $\sigma(n)$, we write:

$$\sigma(n) = \prod_{p|n} \left(1 + p + p^2 + \cdots + p^{\nu_p(n)} \right),$$

where $\nu_p(n)$ is the p-adic valuation of n (Definition 1). Now, we manipulate the expression as follows:

$$I(n) = \frac{1}{n} \prod_{p|n} \left(1 + p + \cdots + p^{\nu_p(n)} \right).$$

Multiplying and dividing each term by $p^{\nu_p(n)-1}$, we obtain:

$$I(n) = \frac{1}{n} \prod_{p|n} p^{\nu_p(n)-1} \left(\frac{1 + p + \cdots + p^{\nu_p(n)}}{p^{\nu_p(n)-1}} \right).$$

By Definition 2, we have $\prod_{p|n} p^{\nu_p(n)-1} = \frac{n}{\mathrm{rad}(n)}$, where $\mathrm{rad}(n)$ is the radical of n. Substituting this in, we get:

$$I(n) = \frac{1}{n} \cdot \frac{n}{\mathrm{rad}(n)} \prod_{p|n} \left(1 + \frac{1}{p} + \cdots + \frac{1}{p^{\nu_p(n)-1}} + p\right).$$

Simplifying, this becomes:

$$I(n) = \frac{1}{\mathrm{rad}(n)} \prod_{p|n} \left(\sum_{k=0}^{\nu_p(n)-1} \frac{1}{p^k} + p\right).$$

By Proposition 2, the sum $\sum_{k=0}^{\nu_p(n)-1} \frac{1}{p^k}$ is recognized as $I(p^{\nu_p(n)-1})$, leading to:

$$I(n) = \frac{1}{\mathrm{rad}(n)} \prod_{p|n} \left(I(p^{\nu_p(n)-1}) + p\right).$$

Finally, applying Proposition 3, we rewrite this as:

$$I(n) = \prod_{p|n} \left(1 + \frac{I(p^{\nu_p(n)-1})}{p}\right),$$

which completes the proof. □

This is the main theorem.

Theorem 1. *If an odd perfect number N exists and is not divisible by 3, then N must satisfy contradictory bounds. Consequently, no such N exists.*

Proof. Assume for contradiction that N is an odd perfect number not divisible by 3. By definition, its abundancy index satisfies:

$$I(N) = \frac{\sigma(N)}{N} = 2,$$

where $\sigma(N)$ is the sum of divisors of N (Proposition 5). Since N is odd, its distinct prime factors $p_1, \ldots, p_{k-1}$ are all odd. Let $k-1$ be the number of distinct primes in N, with factorization:

$$N = \prod_{i=1}^{k-1} p_i^{e_i}, \quad \text{where } e_i = \nu_{p_i}(N) \geq 1.$$

The abundancy index decomposes multiplicity as:

$$I(N) = \prod_{i=1}^{k-1} I(p_i^{e_i}) = \prod_{i=1}^{k-1} \left(1 + \frac{1}{p_i} + \cdots + \frac{1}{p_i^{e_i}}\right) = 2.$$

Now consider $2N$. Its prime factorization includes the prime 2 with exponent 1:

$$2N = 2^1 \cdot \prod_{i=1}^{k-1} p_i^{e_i},$$

so $2N$ has $k = (k-1) + 1$ distinct prime factors. The abundancy index of $2N$ is:

$$I(2N) = \frac{\sigma(2N)}{2N}.$$

Since $\gcd(2, N) = 1$, the sum-of-divisors function σ is multiplicative:

$$\sigma(2N) = \sigma(2) \cdot \sigma(N) = 3 \cdot 2N = 6N.$$

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Thus:

$$I(2N) = \frac{6N}{2N} = 3.$$

Alternatively, using the multiplicity of I:

$$I(2N) = I(2) \cdot I(N) = \frac{3}{2} \cdot 2 = 3.$$

Following Lemma 1, define:

$$a_i = \frac{I(p_i^{\nu_{p_i}(2N)-1})}{p_i},$$

where:

- $p_1, \ldots, p_{k-1}$ are the distinct odd prime factors of N,
- $p_k = 2$,
- $k - 1 \geq 10$ (Proposition 6).

Since $\nu_{p_i}(2N) = \nu_{p_i}(N) = e_i$ for $i = 1, \ldots, k-1$, and $\nu_2(2N) = 1$, we have:

- For $i = 1, \ldots, k-1$:

$$a_i = \frac{I(p_i^{e_i-1})}{p_i}, \quad \text{where } I(p_i^{e_i-1}) = \sum_{j=0}^{e_i-1} \frac{1}{p_i^j}.$$

- For $i = k$ (the prime 2):

$$a_k = \frac{I(2^0)}{2} = \frac{1}{2}.$$

The abundancy index of $2N$ can be expressed as:

$$I(2N) = \prod_{i=1}^{k}(1+a_i) = \left(\prod_{i=1}^{k-1}(1+a_i)\right) \cdot \left(1+\frac{1}{2}\right) = I(N) \cdot \frac{3}{2} = 2 \cdot \frac{3}{2} = 3,$$

which is consistent with the earlier calculation. Expanding the product, we obtain:

$$\prod_{i=1}^{k}(1+a_i) = 1 + \sum_{m=1}^{k} A_m,$$

where:

$$A_m = \sum_{1 \leq i_1 < \cdots < i_m \leq k} a_{i_1} \cdots a_{i_m}$$

is the sum of all products of m distinct a_i's. Since $I(2N) = 3$, this gives:

$$1 + A_1 + A_2 + \cdots + A_k = 3 \quad \Longrightarrow \quad A_1 + A_2 + \cdots + A_k = 2.$$

For N, the expansion is:

$$I(N) = \prod_{i=1}^{k-1}(1+a_i) = 1 + \sum_{m=1}^{k-1} A'_m,$$

where:

$$A'_m = \sum_{1 \leq i_1 < \cdots < i_m \leq k-1} a_{i_1} \cdots a_{i_m}.$$

Since $I(N) = 2$, we have:

$$1 + A'_1 + A'_2 + \cdots + A'_{k-1} = 2.$$

Comparing with the expansion for $2N$, we deduce:

$$1 + A'_1 + A'_2 + \cdots + A'_{k-1} = A_1 + A_2 + \cdots + A_k.$$

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The following technical lemmas provide the necessary foundations for our proof.

Lemma 2. *The following inequality holds:*

$$2 \leq \exp(A_1'),$$

where $A_1' = \sum_{i=1}^{k-1} a_i$, *with* a_i *defined as in Lemma 1.*

Proof. By Proposition 4, we have $1 + a_i \leq \exp(a_i)$ for each i. Thus,

$$2 = \prod_{i=1}^{k-1}(1 + a_i) \leq \prod_{i=1}^{k-1} \exp(a_i) = \exp\left(\sum_{i=1}^{k-1} a_i\right) = \exp(A_1').$$

Since $1 + a_i \leq \exp(a_i)$, the inequality $2 \leq \exp(A_1')$ follows. □

Lemma 3. *If an odd perfect number N exists and is not divisible by 3, then the following inequality holds:*

$$\prod_{i=1}^{k} \frac{p_i}{p_i - 1} \geq \exp(A_1),$$

where $A_1 = \sum_{i=1}^{k} a_i$, *with* a_i *defined as in Lemma 1.*

Proof. From Proposition 3, we have:

$$\prod_{i=1}^{k} \frac{p_i}{p_i - 1} = 2 \prod_{i=1}^{k-1} \frac{p_i}{p_i - 1} > 2 \cdot I(2) = 4.$$

Taking square roots of both sides of the original inequality, we obtain the equivalent condition:

$$2 = \exp(\ln 2) \geq \exp\left(\frac{A_1}{2}\right).$$

This simplifies our goal to verifying this exponential inequality. Since $A_1 = A_1' + \frac{1}{2}$, we obtain:

$$\frac{A_1}{2} = \frac{A_1'}{2} + \frac{1}{4}.$$

For each i, Proposition 3 gives:

$$\frac{a_i}{2} = \frac{I(p_i^{v_{p_i}(N)-1})}{2p_i} < \frac{5}{8p_i},$$

because $I(p_i^{v_{p_i}(N)-1}) < \frac{5}{4}$ when $p_i \geq 5$. By Proposition 6, we bound the sum:

$$\frac{A_1'}{2} < \frac{5}{8} \sum_{i=1}^{k-1} \frac{1}{p_i} < \frac{5}{8} \ln 2.$$

Thus:

$$\frac{A_1}{2} < \frac{5}{8} \ln 2 + \frac{1}{4}.$$

To complete the proof, we verify:

$$\ln 2 \geq \frac{5}{8} \ln 2 + \frac{1}{4}.$$

Rearranging terms:

$$\frac{3}{8} \ln 2 \geq \frac{1}{4} \iff 3 \ln 2 \geq 2.$$

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The numerical value $3 \ln 2 \gtrapprox 2.0794$ indeed satisfies this inequality. □

We analyze the abundancy index $I(2N) = 3$ through several key steps:

$$3 = I(2N) = \left(\prod_{i=1}^{k} \frac{p_i}{p_i - 1}\right) \cdot \prod_{i=1}^{k} \left(1 - \frac{1}{p_i^{\nu_{p_i}(2N)+1}}\right) \tag{1}$$

$$\implies \left(\prod_{i=1}^{k} \frac{p_i^{\nu_{p_i}(2N)+1}}{p_i^{\nu_{p_i}(2N)+1} - 1}\right) \cdot 3 = \prod_{i=1}^{k} \frac{p_i}{p_i - 1}. \tag{2}$$

Since $\frac{p_i^2}{p_i^2-1} \geq \frac{p_i^{\nu_{p_i}(2N)+1}}{p_i^{\nu_{p_i}(2N)+1}-1}$ for all i, we have:

$$\prod_{i=1}^{k} \left(\frac{p_i^2}{p_i^2 - 1}\right) \geq \prod_{i=1}^{k} \left(\frac{p_i^{\nu_{p_i}(2N)+1}}{p_i^{\nu_{p_i}(2N)+1} - 1}\right) \tag{3}$$

Since N is odd and not divisible by 3, Proposition 1 gives:

$$\frac{\pi^2}{9} = \frac{3}{4} \cdot \frac{8}{9} \cdot \prod_{n=1}^{\infty} \left(\frac{p_n^2}{p_n^2 - 1}\right) > \prod_{i=1}^{k} \left(\frac{p_i^2}{p_i^2 - 1}\right) \tag{4}$$

Combining these results:

$$\begin{aligned} \frac{\pi^2}{3} &= \frac{\pi^2}{9} \cdot 3 > \prod_{i=1}^{k} \left(\frac{p_i^{\nu_{p_i}(2N)+1}}{p_i^{\nu_{p_i}(2N)+1} - 1}\right) \cdot 3 \\ &= \prod_{i=1}^{k} \frac{p_i}{p_i - 1} \geq \exp(A_1) \quad \text{(by Lemma 3)} \end{aligned} \tag{5}$$

From Lemma 2, we have $2 \leq \exp(A_1')$. Since $A_1 = A_1' + \frac{1}{2}$, this implies:

$$3.2974 \lessapprox 2 \cdot \exp\left(\frac{1}{2}\right) \leq \exp(A_1) < \frac{\pi^2}{3} \lessapprox 3.2899 \tag{6}$$

This yields the impossible inequality $3.2974 < 3.2899$. This contradiction completes the proof. □

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