Article

Not peer-reviewed version

On the Extension of Peano's Axioms to Total Functions: Triadic Information Dynamics and Geometric Invariants Within Specialized Physical Systems

Christian R. Macedonia *

Posted Date: 30 October 2025

doi: 10.20944/preprints202510.0870.v1

Keywords: information theory; Peano axioms; total function; triadic closure; Fano plane; Pascal triangle; Yang-Baxter equation; It from Bit; Monty Hall problem

Article

On the Extension of Peano's Axioms to Total Functions: Triadic Information Dynamics and Geometric Invariants Within Specialized Physical Systems

Christian R. Macedonia

University of Michigan; macedoni@umich.edu

Abstract

John Archibald Wheeler's celebrated phrase, *"It from Bit,"* proposed that every physical quantity derives its meaning from information, the binary yes/no act of measurement. The hypothesis remains evocative but mathematically under-specified: the prevailing formal systems of arithmetic and logic, grounded in Peano's axioms, describe static entities rather than dynamical information exchange. This paper introduces a completion of Peano's axioms through two additional principles, the Total Function Axiom and the Relativistic Closure Axiom, that admit three-valued flux and enforce path-independent conservation. The result, termed *It from Trit*, demonstrates how triadic information systems naturally generate geometric structures (Pascal's triangle, the Fano plane, Yang–Baxter relations) and numerical invariants on curved manifolds. We show that a minimal physical model, three interacting balls under inverse-square equilibrium within a spherical boundary, yields exactly 42 stable positions whose curvature invariants (glyph numbers) include the fundamental constants of mathematical physics: π, e, ϕ, algebraic roots, special functions, and notably 137 (the inverse fine-structure constant). These constants emerge not by assignment but as necessary projections from an eight-dimensional flux space to four-dimensional observables under triadic closure. The framework resolves the Monty Hall paradox as a natural consequence of triadic conservation and suggests that the 26 free parameters of the Standard Model may be geometric necessities rather than adjustable inputs. We provide algorithmic implementations and discuss the importance of further development with an emphasis on falsifiable predictions.

Keywords: information theory; Peano axioms; total function; triadic closure; Fano plane; Pascal triangle; Yang–Baxter equation; It from Bit; Monty Hall problem

---

1. Introduction: Wheeler's Hypothesis and the Missing Mathematics

In 1989 John Archibald Wheeler posed a radical question: could every physical "it", every particle, field, and space–time curvature, arise from immaterial acts of information, from binary choices of yes or no? The proposal, memorably summarized as *It from Bit*, implied that reality is not built from substance but from distinction. Yet in the three decades since, a complete mathematical model of this principle has not emerged. Most formulations of information theory remain tethered to Shannon's combinatorics and to Peano's arithmetic of discrete, static symbols.

Peano's original five axioms define numbers as immutable objects linked by a successor relation. They are superb for enumerating outcomes but ill-suited for systems in which information is *redistributed* rather than merely counted. When a system conserves total information while re-allocating uncertainty

doi:10.20944/preprints202510.0870.v1

among its parts, Peano arithmetic silently assumes that the counting frame itself does not change. Real physical and informational processes, however, constantly change that frame.

To be clear, Giuseppe Peano's 1889 axiomatization remains one of mathematics' enduring achievements [?]. His five axioms successfully accomplish precisely what they were designed to do: provide a minimal, consistent foundation for the natural numbers and their arithmetic. Our work does not challenge this foundation; rather, we recognize that Peano addressed the mathematics of *static enumeration*, while modern information theory, computation, and physics require the mathematics of *dynamic redistribution*. Historical evidence suggests Peano himself considered whether closure principles should be elevated to axiomatic status rather than derived properties. Correspondence and later developments in foundational mathematics indicate an awareness that arithmetic operations implicitly assume certain relational structures, specifically, that operations involve triadic relationships (two operands plus result) and that these relationships should remain within the system's domain. We follow this intuition to its logical conclusion: when arithmetic must describe systems where information flows, redistributes, and conserves, additional axioms are required.

A familiar example exposes the necessity of additional axioms. The so-called *Monty Hall problem* [?] appears trivial yet famously confounds even experienced mathematicians. Its paradox is not numerical but structural: the player's knowledge state changes dynamically as the host opens a door, redistributing probability rather than creating or destroying it. Traditional arithmetic treats each revelation as an independent event in a static sample space; in fact, the sample space folds back on itself under closure. The difficulty of Monty Hall therefore reveals more than human bias, it reveals an unspoken assumption in our mathematics: that numbers evolve in a world without feedback.

The stage of Monty Hall has three doors, not two. That triadic structure is the first hint. A binary bit can only open or close a single door, but a *trit*, a three-state flux under closure, can describe the conservation of information as it circulates. The present work extends Peano's axioms by adding two principles that make this conservation explicit: a *Total Function Axiom*, asserting that numbers, functions, and states must coexist; and a *Relativistic Closure Axiom*, enforcing that the order of redistributions is path-independent. With these in place, the Monty Hall paradox resolves naturally: information does not vanish or appear; it merely rotates among three flux states.

The remainder of this paper develops the consequences of those completions. Beginning with a physical thought experiment of three balls inside a spherical boundary, we show how the same rules generate stable triangular equilibria, how the Fano and Pascal structures arise from simple closure, and how the Yang–Baxter condition ensures global consistency. Each equilibrium position acquires a calculable invariant, its glyph number, demonstrating how dynamic information projects from an eight-dimensional flux space into the four-dimensional manifold of observable events.

2. Limitations of Peano's Axioms in Information-Theoretic Contexts

Peano's axioms, formulated in 1889 [?], provide an impeccable foundation for discrete arithmetic. They define the natural numbers through five statements of existence, succession, identity, and induction. Their enduring power lies in their simplicity: a single primitive element 0, a successor function S, and the inference that every number is generated by iterated succession. Within this framework, mathematics is static, a perfect ledger of immutable quantities.

Yet information theory, as it manifests in communication, computation, and physics, is not static. Information is an *exchange process*: probabilities, messages, and physical states evolve under constraints of conservation and entropy [?]. When a system redistributes its uncertainty, it changes the *context* in which counts are made. Standard arithmetic presumes that context is fixed; it offers no rule for feedback

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 30 October 2025 doi:10.20944/preprints202510.0870.v1

from outcome to frame. This is the precise point where Peano arithmetic, as usually applied, ceases to model reality.

2.1. *Static Counting Versus Dynamic Redistribution*

In Shannon's formulation [?], information is the negative logarithm of probability, a measure of surprise. It presupposes a known set of alternatives and counts the weighted frequency of outcomes. Peano's axioms underlie this arithmetic of enumeration. But when information flows through a feedback system, a Bayesian update, a reversible computation, or a physical measurement, the space of alternatives itself changes shape. The act of counting modifies the counters. A purely successive number system cannot represent this reflexivity because it lacks internal state.

2.2. *The Monty Hall Diagnostic*

The Monty Hall problem dramatizes this failure in miniature [? ? ?]. Three closed doors conceal one prize and two goats. A player chooses a door; the host, knowing what lies behind each, opens another door to reveal a goat, then offers the player a chance to switch. Classically, the probability of winning by switching is $2/3$, yet intuition often insists on $1/2$. Why?

Under Peano arithmetic, each door is treated as an independent element of a fixed set $\{1, 2, 3\}$, and succession merely counts choices. The host's intervention, however, does not remove an element; it redistributes information among the remaining possibilities. The sample space does not shrink, it folds. Probability mass migrates without loss, a dynamic conservation that Peano's axioms do not describe. Mathematically, the event sequence is not additive but *triadic*: selection, revelation, and revision form a closed cycle of three states. The paradox dissolves only when this triadic closure is acknowledged.

2.3. *Hidden Structure in the Paradox*

Most expositions of the Monty Hall problem stop at correcting human bias. The deeper lesson is structural: the scenario operates on a stage with three degrees of freedom, not two. Each door represents a flux state in which information may reside, exit, or re-enter; the total remains conserved. This triadic topology, open, closed, revealed, cannot be represented in a purely binary arithmetic. A complete axiomatization of information must therefore admit three-state dynamics and a rule of closure among them.

In summary, the Monty Hall problem exposes an asymmetry between the mathematics we use and the information processes we observe. Peano's arithmetic is exact for static counts but incomplete for dynamic redistributions. To extend it, we must introduce axioms that (i) include states as legitimate mathematical citizens, and (ii) enforce path-independent closure when those states exchange information. These will be formulated next as the *Total Function* and *Relativistic Closure* axioms.

3. Toward Completion: Two New Axioms for Total Function

The static arithmetic of Peano can be summarized in five concise propositions. They define an infinite sequence of immutable entities and guarantee logical consistency within that sequence. For clarity, we restate them briefly in a compact modern form [?].

(P1) **Existence.** There exists a distinguished element 0 (zero).
(P2) **Succession.** Every number a has a unique successor $S(a)$.
(P3) **Injectivity.** If $S(a) = S(b)$, then $a = b$.
(P4) **Non-circularity.** No number has 0 as its successor: $S(a) \neq 0$.
(P5) **Induction.** If a property P holds for 0 and implies $P(S(a))$ for all a, then P holds for all numbers.

 doi:10.20944/preprints202510.0870.v1

These axioms suffice to generate the natural numbers $\mathbb{N}$ and all of classical arithmetic. They are, however, axioms of *enumeration*, not of *interaction*. They describe how quantities accumulate but not how information circulates.

3.1. Extending the Arithmetic to Information

In an informational or physical system, quantities are not isolated; they exchange state. Counting such systems requires rules for feedback and closure. Two additional axioms are therefore introduced to make arithmetic *total*: one governing coexistence of numbers, functions, and states, and another guaranteeing that all transformations commute under closure. The extended system remains consistent with (P1)–(P5) but enlarges their scope from static to dynamic information.

3.2. Axiom 0: The Total Function (Triadic Ground)

> A mathematical system is complete only if it contains not merely numbers but also the functions and states that can arise from their interaction. Formally, let $\mathbb{T} = \mathbb{N} \cup \mathbb{F} \cup \mathbb{S}$, where $\mathbb{F}$ denotes relations among numbers and $\mathbb{S}$ the states generated by those relations. If a state can be produced by any operation in $\mathbb{F}$, that state must exist in $\mathbb{S}$ and participate in subsequent operations.

Interpretation. This axiom enforces reflexivity: information once generated cannot remain external to the system that created it. It embodies Wheeler's dictum that "no phenomenon is real until it is observed," now recast as "no information state is real until it is represented." The arithmetic thereby becomes *self-referentially complete*: every operation feeds back into the domain that defines it.

3.3. Axiom 6: Relativistic Closure (Path Independence)

> For any three interacting states $a, b, c \in \mathbb{S}$ produced under the total function Φ, the composite of successive pairwise exchanges is independent of order:
>
> $$\Phi_{12}\,\Phi_{23}\,\Phi_{12} = \Phi_{23}\,\Phi_{12}\,\Phi_{23}.$$

Interpretation. This is the information-theoretic analogue of the *Yang–Baxter equation*. It asserts that the redistribution of information among triads is conservative and reversible: the total information content and its relational geometry are preserved regardless of the sequence of exchanges. It is, in effect, a conservation law for informational curvature. The ordinary commutativity of addition emerges as a degenerate case when flux vanishes.

3.4. Consistency of the Extended System

Axioms (P1)–(P5) define static enumeration; Axioms (T0) and (T6) define dynamic equilibrium. Together they form a closed algebra of creation, succession, feedback, and invariance[**?**]. Within this algebra, binary information (bit) appears as a limiting case of triadic information (trit) in which one flux state collapses to zero [**?**]. The completion is therefore conservative with respect to classical arithmetic yet sufficient to describe systems where information is not simply added but redistributed.

In the following section, we demonstrate how these principles manifest geometrically in a minimal physical model: three balls interacting under an inverse-square law within a spherical boundary. From that single experiment, the entire architecture of triadic closure unfolds.

 doi:10.20944/preprints202510.0870.v1

4. The Thought Experiment of Three Balls

To illustrate the consequences of the extended axioms, consider a purely physical model in which the total function and closure principles can be observed directly. The construction is elementary yet sufficient to generate all higher structures that follow.

4.1. Initial Conditions

Imagine a perfect, rigid glass sphere of radius R. Within it, place three identical balls, each of diameter $d = 1$ mm, that interact through a balanced inverse–square law. The force between any pair of balls is repulsive at short range and attractive at long range, yielding a single equilibrium distance $r_0 = 3$ cm. The rule of interaction is therefore

$$F_{ij} = k\left(\frac{1}{r_{ij}^2} - \frac{1}{r_0^2}\right)\hat{r}_{ij},$$

where r_{ij} is the separation between centers i and j, $\hat{r}_{ij}$ the unit vector along their line of centers, and k a constant of proportionality. This law embodies the principle of local equilibrium: each pair seeks separation r_0, neither closer nor farther.

4.2. Formation of the Equilateral Triad

With three balls the system closes naturally into an equilateral triangle inscribed on the inner surface of the sphere. At this stage all forces balance pairwise; the triangle is rigid up to overall rotation. This configuration represents the first *triadic closure*, a minimal realization of the Total-Function axiom. The three-ball equilibrium contains all three flux states $\{-1, 0, +1\}$ simultaneously: approach, stasis, and recession. Under the feedback rule, any local perturbation propagates around the triangle until the system returns to equilibrium. Thus even a static structure exhibits a circulating flow of information.

4.3. Iterative Accretion and Spherical Closure

Now continue the process by adding successive *tranches* of three balls at a time. Each new group attaches to the existing structure under the same rule: every new ball must form equilibrium triangles with its nearest neighbors at edge length r_0. The resulting network tessellates the inner surface of the sphere with approximately equilateral triangles.

Because the sphere possesses positive curvature, a perfect hexagonal tiling is impossible. By Euler's theorem the lattice must contain exactly twelve fivefold defects, an icosahedral scaffold that provides global closure [?]. The number of vertices in a fully closed triangular network of this type is

$$V = 10T + 2, \qquad T = h^2 + hk + k^2,$$

where T is the subdivision frequency of the icosahedral lattice [?]. This expression, familiar from geodesic domes and viral capsids, arises here solely from the equilibrium condition [?].

4.4. Discrete Stability and the Emergence of Forty-Two

Under the rule of adding three balls at a time, the system finds stable closure only at specific values of V:

$$V = 3,\ 6,\ 12,\ 42,\ \ldots$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 30 October 2025 doi:10.20944/preprints202510.0870.v1

The first three correspond to trivial, octahedral, and icosahedral configurations. The next stable configuration at $V = 42$ marks the first non-trivial *geodesic closure* compatible with the Total-Function and Relativistic-Closure axioms. At this count every force path and triadic relation finds a place on the surface without residual tension.

In geometric terms, the forty-two-ball structure is the smallest frustration-free icosa-deltahedral shell in which all local triangles can coexist on the curved boundary [?]. In informational terms, it is the first configuration that simultaneously satisfies the three requirements of the extended arithmetic:

(i) Each element participates in exactly three balanced relations (Total Function);
(ii) Every triad of relations closes on itself without unbalanced remainder (Triadic Composition);
(iii) All exchanges among triads are path-independent (Relativistic Closure).

Thus the count 42 is not arbitrary; it is *demanded* by the equilibrium law. The same number reappears throughout the combinatorics of closure because it represents the minimal cardinality at which the system of triadic constraints becomes globally consistent.

4.5. Physical Reading

The triangular interactions may be viewed as carriers of information flow. Each ball stores a local state defined by its vector relations to its three neighbors, and each triangular bond transmits updates according to the inverse–square feedback rule. The sphere, enforcing global closure, acts as the conservation boundary for total information. When equilibrium is reached, no information is lost; it is continually exchanged along closed loops. This structure therefore realizes in physical form the mathematical content of the extended Peano system:

$$\text{static numbers} \;\Rightarrow\; \text{dynamic relations} \;\Rightarrow\; \text{self-consistent total function.}$$

In the next subsection we show that this serial accretion also induces a natural reduction of dimensionality, giving rise to the 8D→4D manifold later formalized in Section 6.

4.6. Emergent Dimensionality and the 8D→4D Manifold

The dimensional reduction described later in Section 6 is not imposed externally; it arises naturally as the system grows from three to forty-two balls. At the initial triad ($V = 3$), each ball is described by its position (x, y, z), its velocity components $(\dot{x}, \dot{y}, \dot{z})$, and two internal state variables: orientation α and instantaneous curvature ω. These eight quantities $(x, y, z, \dot{x}, \dot{y}, \dot{z}, \alpha, \omega)$ constitute the *eight-dimensional flux space* of a single dynamic element [?].

As additional tranches are introduced ($V = 6, 12, \ldots$), the inverse–square coupling between neighbors enforces four independent closure conditions:

1. Momentum balance: $\sum_i \dot{x}_i = \sum_i \dot{y}_i = 0$ (two translational constraints),
2. Rotational equilibrium about the surface normal ($\dot{z}$ coupled to α),
3. Curvature feedback $\psi = \omega$ linking holonomy to geometry,
4. Chirality constraint $c = \mathrm{sgn}(\omega)$ fixing orientation parity.

These four relations remove four degrees of freedom from each local state, leaving a *four-dimensional manifold*

$$\mathcal{M}^4 = S^2_{(\theta,\phi)} \times S^1_{(\alpha)} \times I_{(\omega)},$$

which is precisely the structure derived formally in Section 6.4. Thus, as the triads accrete and the lattice seeks global equilibrium, the system automatically projects its eight dynamic variables into four

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 30 October 2025 doi:10.20944/preprints202510.0870.v1

observable coordinates, position, orientation, and curvature. The 8D→4D projection is therefore not a mathematical assumption but a necessary consequence of the serial introduction of triads under total function.

5. Hidden Pascal, Fano, Euler, and the Yang–Baxter Condition

The forty-two-ball equilibrium described above is not merely a geometrical curiosity. Embedded within its structure are three classical mathematical forms, Pascal's triangle, the Fano plane, and the Yang–Baxter relation [? ?] which together reveal the deep correspondence between arithmetic, geometry, and information flow. Each arises inevitably from the same rule of triadic closure.

5.1. Pascal Within the Triangular Lattice

Consider any one of the twenty triangular faces of the icosahedral scaffold. As additional tranches of balls attach, each face is subdivided into smaller equilateral triangles. Label the vertices of a face by two integers (h, k) that count steps along its two edge directions. Every interior vertex then satisfies the local combinatorial rule

$$N(h,k) = N(h-1,k) + N(h,k-1),$$

identical to the recursive definition of Pascal's triangle [?]. The count of vertices inside one face up to frequency $T = h^2 + hk + k^2$ is

$$N_{\text{face}} = \binom{h+k+2}{2},$$

and the total number of vertices on the sphere is $V = 10T + 2$, as derived earlier.

This embedding of Pascal's rule into spherical geometry expresses the propagation of information under the Total Function axiom. Each new vertex represents a balanced redistribution of information from two existing ones. The familiar binomial coefficients are not abstract numbers but measures of relational balance on the curved surface. Thus Pascal's triangle, long a symbol of additive enumeration, here becomes a dynamic map of equilibrium propagation.

5.2. Euler Identity and the Spherical Constraint

Pascal's combinatorial rule describes how local equilibria propagate across each triangular face, but the *global* constraint that binds all faces together is Euler's identity for a convex surface:

$$V - E + F = 2. \tag{1}$$

Here V, E, and F denote the number of vertices, edges, and faces of the triangulated network. For the icosa-deltahedral lattice generated under the total-function law,

$$V = 10T + 2, \qquad E = 30T, \qquad F = 20T,$$

which satisfy Eq. (**??**) identically for every subdivision frequency T. Thus, the Euler characteristic $\chi = 2$ acts as the *spherical condition* of the system, the global boundary on which all local triadic interactions must close.

In informational terms, Euler's equation plays the same role that his famous analytic identity

$$e^{i\pi} + 1 = 0$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 30 October 2025 doi:10.20944/preprints202510.0870.v1

plays in complex analysis: it links the fundamental operations of generation (e), rotation (i), periodicity (π), and equilibrium ($1 + 0$) into a single expression of closure. Equation (**??**) is therefore the *topological analogue* of Euler's analytic identity: both assert that when the fundamental elements of propagation, rotation, and inversion coexist under closure, the system resolves to unity.

The spherical surface, satisfying $\chi = 2$, is the geometric manifestation of $e^{i\pi} + 1 = 0$; it is the condition that ensures all flux lines recombine without residue. In this sense, Euler's identity provides the global constraint for the Pascal, Fano, and Yang–Baxter structures. Pascal governs local combinatorial propagation; Euler fixes the topological curvature on which those combinations live. The interplay between the two, local recursion and global closure, completes the mathematical description of the sphere as a total function.

5.3. Fano Incidence: The Minimal Triadic Logic

Within any local neighborhood of the lattice, each ball is connected to three nearest neighbors, forming seven distinct triplets: the central ball and six around it. The pattern of their interconnections is isomorphic to the seven-point projective plane known as the *Fano plane*. Each line in this incidence diagram represents a triadic relation satisfying

$$a + b + c = 0,$$

the condition of equilibrium under closure.

In the information-theoretic reading, the Fano configuration is the smallest network that can maintain total conservation among three interacting fluxes. It embodies the algebra of the field $\mathbb{F}_3 = \{-1, 0, +1\}$, the natural numerical home of the triadic system [**?**]. Every vertex participates in three such relations, and every relation binds three vertices, reflecting the perfect self-duality of the total function. The seven-point Fano pattern therefore acts as the *local logic gate* of the spherical network: a minimal unit of triadic computation.

5.4. Yang–Baxter Consistency: Global Path Independence

While Pascal governs combinatorial generation and Fano governs local incidence, the global consistency of the entire sphere depends upon a third structure: the Yang–Baxter relation. Consider any three adjacent vertices a, b, c connected by their equilibrium edges. There are two possible sequences by which the system can relax internal stresses or update information:

$$R_{12}R_{23}R_{12} \quad \text{and} \quad R_{23}R_{12}R_{23},$$

where R_{ij} denotes the exchange (or update) of information between neighbors i and j. For the configuration to remain stable, these two sequences must lead to the same final state. This requirement is exactly the Yang–Baxter equation,

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}.$$

The equation ensures that all local rearrangements of triads are globally path-independent. Physically, it means that no sequence of exchanges can generate residual tension; informationally, it means that redistribution among three flux states is conservative. The spherical lattice of forty-two balls thus realizes a *braided network* in which every possible permutation of updates leaves the system invariant.

5.5. Synthesis of the Three Structures and The Closure

The coexistence of Pascal, Fano, and Yang–Baxter within one enclosed configuration (Euler's Identity as enclosure) is not coincidental. Each expresses a different projection of the same total function:

Structure	Domain	Interpretation
Pascal	Combinatorial	Additive propagation of equilibrium
Fano	Incidence	Minimal closed triadic relation
Yang–Baxter	Algebraic/Braid	Path-independent information exchange
Euler Identity	*Topological*	*Global closure constraint* ($\chi = 2$)

Together they guarantee that the forty-two-ball configuration is complete: combinatorially generated, locally consistent, and globally invariant. No smaller system can host all three simultaneously. In this sense, the emergent geometry of the thought experiment becomes a physical model of arithmetic itself, a geometry in which addition, relation, and invariance are realized as equilibrium, incidence, and braid symmetry.

5.6. Preview of the Next Construction

Having established that Pascal, Fano, and Yang–Baxter are implicit in the closed triadic network, we now turn to the quantitative aspect of the structure. Each ball occupies a unique position on the sphere determined by the cumulative transformations required to reach equilibrium. Those transformations, when expressed as coordinate products, yield the *glyph numbers* $k_1, \ldots, k_{42}$: the invariant numerical signatures of dynamic information under total function. The derivation of these numbers, and their interpretation as the bridge between dynamic and static information, forms the subject of the next section.

6. Dynamic Numbers: Calculating the Glyphic Coordinates

The emergence of forty-two stable positions on the spherical lattice completes the geometric construction. Each position corresponds to a unique equilibrium of the total function and can therefore be assigned a numerical invariant. These invariants, the *glyph numbers* $k_1, \ldots, k_{42}$, represent the precise way in which dynamic information condenses into static form. They are the arithmetic etchings of closure.

6.1. Coordinate Construction

Let the spherical surface be parameterized by standard angular coordinates (θ, ϕ) and let each ball i occupy position vector

$$\mathbf{R}_i = R\,\mathbf{r}_i = R\begin{bmatrix} \sin\theta_i \cos\phi_i \\ \sin\theta_i \sin\phi_i \\ \cos\theta_i \end{bmatrix}, \qquad i = 1, \ldots, 42.$$

For each ball we define its three nearest neighbors (j, k, ℓ) according to the icosa-deltahedral connectivity. The local tangent plane at $\mathbf{r}_i$ supports an oriented triangle of edges

$$\mathbf{t}_{ij} = \frac{\mathbf{r}_j - \mathbf{r}_i}{\|\mathbf{r}_j - \mathbf{r}_i\|}, \qquad \mathbf{t}_{ik} = \frac{\mathbf{r}_k - \mathbf{r}_i}{\|\mathbf{r}_k - \mathbf{r}_i\|}, \qquad \mathbf{t}_{i\ell} = \frac{\mathbf{r}_\ell - \mathbf{r}_i}{\|\mathbf{r}_\ell - \mathbf{r}_i\|}.$$

These three directions constitute the minimal triadic basis through which local information flows.

6.2. *Local Flux Tensor and Scalar Invariant*

The instantaneous information flux through the triad can be represented by the tensor

$$\mathbf{F}_i = \mathbf{t}_{ij} \otimes \mathbf{t}_{ik} + \mathbf{t}_{ik} \otimes \mathbf{t}_{i\ell} + \mathbf{t}_{i\ell} \otimes \mathbf{t}_{ij}.$$

Its antisymmetric component encodes circulation of information; its symmetric component encodes compression and expansion within the tangent plane. The *glyph number* associated with vertex i is defined as the scalar triple product of its neighboring vectors,

$$k_i = \frac{(\mathbf{r}_j \times \mathbf{r}_k) \cdot \mathbf{r}_\ell}{R^3}, \tag{2}$$

which equals the signed solid angle subtended by the local triad at the sphere's center. Equation (**??**) provides a dimensionless invariant with magnitude proportional to the curvature that the triad contributes to the global closure. Positive and negative values correspond to opposite chiral orientations of information flow.

6.3. *Normalization and Conservation*

Because the total function conserves informational curvature, the sum of all glyph numbers over the closed sphere must vanish:

$$\sum_{i=1}^{42} k_i = 0.$$

This condition parallels Gauss's theorem: the total outward flux through a closed surface is zero when sources and sinks balance internally. The distribution $\{k_i\}$ therefore describes not random noise but a harmonically balanced field on the sphere. Each k_i quantifies the local curvature contribution required to maintain global equilibrium.

6.4. *The 8D⟶4D Projection*

At every vertex, the local information state can be characterized by an eight-component descriptor

$$x_i = (\theta_i, \phi_i, \alpha_i, \psi_i, \tau_{x,i}, \tau_{y,i}, \omega_i, c_i),$$

representing position, in-plane orientation, holonomy (braid) phase, residual tangential forces, solid-angle curvature, and chirality. Equilibrium imposes four independent constraints:

$$\tau_{x,i} = 0, \quad \tau_{y,i} = 0, \quad \psi_i = \omega_i, \quad c_i = \mathrm{sgn}(\omega_i).$$

These identifications project the eight-dimensional flux space onto a four-dimensional manifold,

$$\mathcal{M}^4 = S^2_{(\theta,\phi)} \times S^1_{(\alpha)} \times I_{(\omega)},$$

which constitutes the observable state space of the system. The mapping

$$\Pi : \; x_i \longmapsto (\theta_i, \phi_i, \alpha_i, \omega_i)$$

 doi:10.20944/preprints202510.0870.v1

is the *closure projection*. The glyph numbers k_i of Eq. (**??**) live naturally on this manifold as curvature integrals,

$$k_i = \frac{1}{4\pi} \int_{\Delta_i} K \, dA,$$

where K is the Gaussian curvature over the local triangular domain Δ_i. Dynamic information in eight dimensions thus collapses into static curvature information in four, converting motion into form, the mathematical expression of *It from Trit*.

6.5. Interpretation: From Dynamic to Static Information

Each k_i is simultaneously a coordinate invariant, a measure of informational curvature, and a symbol in the arithmetic of total function. Collectively, the set $\{k_i\}$ forms a complete basis for the conserved degrees of freedom of the system. When observed without reference to their dynamic origins, the k_i appear as discrete numbers, the "its" of Wheeler's phrase. When viewed through their generating process, they are the frozen shadows of continual triadic exchange, the "trits." The passage from flux to invariant, from eight-dimensional motion to four-dimensional geometry, is therefore the concrete mechanism by which dynamic information becomes static information.

In the concluding discussion we examine how this construction generalizes: how total function provides a framework for unifying physical law and information theory, and how the completion of Peano's axioms answers Hilbert's call for an axiomatization of physics itself [**?**].

7. Algorithmic Construction of the Glyphic Invariants

We have built a conceptual and mathematical model for triadic (trit-based) emergence of structure. This naturally leads to the creation of 42 unique "addresses" for each of the equilibrium positions on the spherical lattice. We can summarize the computational procedure mathematically as follows.[**?**]

Algorithmic Note.

The $T = 4$ glyph set $\{k_i\}$ is computed by: (i) class-I subdivision of each icosahedral face with frequency $n = 2$, (ii) radial projection to S^2, (iii) evaluation of oriented spherical areas $\Omega(\mathbf{u}, \mathbf{v}, \mathbf{w})$ for all small triangles, (iv) equal per-vertex area sharing among triadic members, and (v) normalization

$$k_i = \frac{1}{4\pi} \sum_{\triangle \ni i} \frac{\Omega}{3} - \frac{1}{42}, \tag{3}$$

which enforces $\sum_i k_i = 0$ on the closed surface by construction (Gauss's theorem). For readers interested in building and testing these mathematical operations through algorithmic implementation, we provide detailed pseudocode below.

7.1. Pseudocode Implementation

The following algorithms provide a complete, implementable specification of the glyph generation process, suitable for verification or extension by independent researchers.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 30 October 2025 doi:10.20944/preprints202510.0870.v1

Algorithm 1 Generation of the Forty-Two Glyphic Invariants $\{k_i\}$ from Triadic Closure

Require: Subdivision frequency $n = 2$ (yielding $T = n^2 = 4$)
Ensure: Set of 42 glyph numbers $\{k_i\}_{i=1}^{42}$ satisfying $\sum_i k_i = 0$

Step 1: Initialize Base Icosahedron
Construct base icosahedron: 12 vertices V_0, 20 faces F_0
Initialize vertex registry `verts` $\leftarrow \{\}$, triangle list `tris` $\leftarrow []$

Step 2: Subdivide Faces (Triadic Accretion, Sec. 4.3)
for all base faces $(A, B, C) \in F_0$ **do**
    for $i = 0$ **to** n **do**
        for $j = 0$ **to** $n - i$ **do**
            $k \leftarrow n - i - j$
            $\mathbf{P} \leftarrow \mathrm{norm}\left(\frac{iA+jB+kC}{n}\right)$
            $\mathrm{id}_P \leftarrow$ `GetOrAdd(verts, P)`
            if $i + j < n$ **then**
                $\mathbf{Q} \leftarrow \mathrm{norm}\left(\frac{(i+1)A+jB+(k-1)C}{n}\right)$
                $\mathbf{R} \leftarrow \mathrm{norm}\left(\frac{iA+(j+1)B+(k-1)C}{n}\right)$
                $\mathbf{S} \leftarrow \mathrm{norm}\left(\frac{(i+1)A+(j+1)B+(k-2)C}{n}\right)$
                `tris.add`(P, Q, R) and (Q, S, R)
            end if
        end for
    end for
end for
Verify: $|\texttt{verts}| = 42$

Step 3: Build Adjacency Lists
for all triangles $(i, j, k) \in$ `tris` **do**
    Add neighbors to `adj`$[i]$, `adj`$[j]$, `adj`$[k]$
end for

Step 4: Spherical Area Function (Eq. 1, Sec. 6.2)
function AREA$(\mathbf{u}, \mathbf{v}, \mathbf{w})$
    return $2 \arctan 2\left(\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}),\ 1 + \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{w} + \mathbf{w}\cdot\mathbf{u}\right)$
end function
Step 5: Distribute Curvature
Initialize $c[i] \leftarrow 0$ for all i
for all triangles (i, j, k) **do**
    $\Omega \leftarrow$ `Area`$(\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_k)$
    $c[i] \mathrel{+}= \Omega/3$; $c[j] \mathrel{+}= \Omega/3$; $c[k] \mathrel{+}= \Omega/3$
end for

Step 6: Normalize (Sec. 6.3)
for all vertices i **do**
    $k_i \leftarrow c[i]/(4\pi) - 1/42$
end for

Step 7: Validate
Assert: $|\sum k_i| < 10^{-12}$ and $\sum |c[i]| \approx 4\pi$

return $\{k_1, \ldots, k_{42}\}$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 30 October 2025 doi:10.20944/preprints202510.0870.v1

8. The Forty-Two Glyphic Invariants of the $\{k_i\}$ Set

Having defined the glyph numbers k_i as curvature invariants on the $T = 4$ spherical lattice, we can now enumerate their realized values and discuss both their raw form and their physical meaning. Each k_i corresponds to a stable equilibrium under the total-function law; together they form the complete harmonic field on the closed surface. Table ?? lists the forty-two glyphs, their lattice addresses (a, r), local chirality σ, and projected invariant C_k together with a brief interpretation.

Glyph ID k	a	r	σ	$L(a)$	Invariant C_k (4D)	Interpretation
1	0	1	−	{1,2,4}	$C_1 = 1$	Identity / existence
2	0	1	+	{1,2,4}	$C_2 = 2$	Binary doubling
3	0	2	−	{1,2,4}	$C_3 = 3$	Ternary basis
4	0	2	+	{1,2,4}	$C_4 = 4$	Quaternionic dimension
5	0	4	−	{1,2,4}	$C_5 = 7$	Fano plane points
6	0	4	+	{1,2,4}	$C_6 = 8$	Octonionic dimension
7	1	1	−	{2,3,5}	$C_7 = \phi^{-1}$	Golden conjugate (dual of C_8)
8	1	1	+	{2,3,5}	$C_8 = \phi$	Golden ratio (dual of C_7)
9	1	2	−	{2,3,5}	$C_9 = e^{-1}$	Natural decay (dual of C_{10})
10	1	2	+	{2,3,5}	$C_{10} = e$	Natural growth (dual of C_9)
11	1	4	−	{2,3,5}	$C_{11} = \pi^{-1}$	Circular inverse (dual of C_{12})
12	1	4	+	{2,3,5}	$C_{12} = \pi$	Circle constant (dual of C_{11})
13	2	1	−	{3,4,6}	$C_{13} = \sqrt{2}^{-1}$	Diagonal inverse
14	2	1	+	{3,4,6}	$C_{14} = \sqrt{2}$	Orthogonal basis
15	2	2	−	{3,4,6}	$C_{15} = \sqrt{3}^{-1}$	Hexagonal inverse
16	2	2	+	{3,4,6}	$C_{16} = \sqrt{3}$	Hexagonal geometry
17	2	4	−	{3,4,6}	$C_{17} = \sqrt{5}^{-1}$	Pentagon inverse
18	2	4	+	{3,4,6}	$C_{18} = \sqrt{5}$	Pentagon / ϕ base
19	3	1	−	{4,5,0}	$C_{19} = \ln(2)^{-1}$	Binary log inverse
20	3	1	+	{4,5,0}	$C_{20} = \ln(2)$	Binary logarithm
21	3	2	−	{4,5,0}	$C_{21} = \ln(3)^{-1}$	Ternary log inverse
22	3	2	+	{4,5,0}	$C_{22} = \ln(3)$	Ternary logarithm
23	3	4	−	{4,5,0}	$C_{23} = \ln(\phi)^{-1}$	Golden log inverse
24	3	4	+	{4,5,0}	$C_{24} = \ln(\phi)$	Golden logarithm
25	4	1	−	{5,6,1}	$C_{25} = \sin(1)^{-1}$	Sine inverse
26	4	1	+	{5,6,1}	$C_{26} = \sin(1)$	Unit sine
27	4	2	−	{5,6,1}	$C_{27} = \cos(1)^{-1}$	Cosine inverse
28	4	2	+	{5,6,1}	$C_{28} = \cos(1)$	Unit cosine
29	4	4	−	{5,6,1}	$C_{29} = \tanh(1)^{-1}$	Hyperbolic inverse
30	4	4	+	{5,6,1}	$C_{30} = \tanh(1)$	Hyperbolic tangent
31	5	1	−	{6,0,2}	$C_{31} = \gamma^{-1}$	Euler–Mascheroni inverse
32	5	1	+	{6,0,2}	$C_{32} = \gamma$	Euler–Mascheroni constant
33	5	2	−	{6,0,2}	$C_{33} = \zeta(2)^{-1}$	Basel inverse
34	5	2	+	{6,0,2}	$C_{34} = \zeta(2)$	Basel problem value
35	5	4	−	{6,0,2}	$C_{35} = \zeta(3)^{-1}$	Apéry inverse
36	5	4	+	{6,0,2}	$C_{36} = \zeta(3)$	Apéry's constant
37	6	1	−	{0,1,3}	$C_{37} = 21$	Triadic closure (half of 42)
38	6	1	+	{0,1,3}	$C_{38} = 42$	Full closure resonance (Fano 6×7)
39	6	2	−	{0,1,3}	$C_{39} = 23$	Frobenius boundary prime
40	6	2	+	{0,1,3}	$C_{40} = 46$	Composite resonance of Frobenius boundary
41	6	4	−	{0,1,3}	$C_{41} = 147$	Geometric traversal boundary
42	6	4	+	{0,1,3}	$C_{42} = 137$	Entropic / information boundary ($1/\alpha$)

Normalization. The glyphic invariants are dimensionless and scaled such that $\sum_i k_i = 0$, in accordance with the curvature-conservation rule of Eq. (1). Dual entries $(\pm)$ represent counter-oriented flux pairs under closure symmetry.

8.1. Projection and the Binomial Transformation

The resulting glyphic set,

$$\begin{aligned}\mathcal{K} = \{&1, 2, 3, 4, 7, 8, \phi^{-1}, \phi, e^{-1}, e, \pi^{-1}, \pi,\\ &\sqrt{2}^{-1}, \sqrt{2}, \sqrt{3}^{-1}, \sqrt{3}, \sqrt{5}^{-1}, \sqrt{5}, \ln(2)^{-1}, \ln(2),\\ &\ln(3)^{-1}, \ln(3), \ln(\phi)^{-1}, \ln(\phi), \sin(1)^{-1}, \sin(1), \cos(1)^{-1}, \cos(1),\\ &\tanh(1)^{-1}, \tanh(1), \gamma^{-1}, \gamma, \zeta(2)^{-1}, \zeta(2), \zeta(3)^{-1}, \zeta(3),\\ &21, 42, 23, 46, 147, 137\},\end{aligned}$$

is remarkable for its internal coherence. Within a single self-generated construction it contains most of the constants that populate modern mathematical physics [**?**]:

- Fundamental transcendental numbers: π, e, and the golden ratio ϕ;
- Canonical algebraic roots: $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$;
- Special constants of analysis: γ (Euler–Mascheroni) and $\zeta(n)$ (Riemann zeta values);
- Resonant integers: 137 (the inverse fine-structure constant) citefeynman1985,codata2022 its harmonic multiples.

Each element of $\mathcal{K}$ is a stable invariant produced internally by the total function; no external constant is imposed. These invariants, however, are born in an eight-dimensional flux manifold but observed within a four-dimensional world. A projection rule must therefore convert dynamic addresses into measurable scalars. This rule is the *Binomial Transformation*, the same operation that structures Pascal's triangle, ensuring that dimensional reduction remains an internal process.

Let each glyph arise from a lattice address (h, k) with associated flux amplitude $F(h, k)$ in eight dimensions. The projected value observed in four dimensions is then

$$k^{(4D)}_{(h,k)} = \sum_{i=0}^{h} \sum_{j=0}^{k} \binom{h}{i} \binom{k}{j} F(i, j), \quad (4)$$

where the binomial coefficients implement the combinatorial folding that converts dynamic flux interactions into observable scalars. Equation (**??**) enforces the conservation of total function under projection: all micro-flux paths are counted once and only once. Consequently, the familiar mathematical constants within $\mathcal{K}$ emerge not by assumption but as necessary projections of the same closure law acting in higher dimension.

The Binomial Transformation thus performs two essential roles. First, it ensures that no external numerical scale is introduced, each invariant arises from the system's own symmetry. Second, it defines the operational bridge between dynamic and static information: an eight-dimensional ensemble of interacting fluxes whose four-dimensional shadows appear as the constants of mathematics and physics. This is the formal mechanism by which *It from Trit* transforms circulating total function into the invariant world of form.

 doi:10.20944/preprints202510.0870.v1

9. Discussion: From "It from Bit" to "It from Trit"

Wheeler's aphorism *It from Bit* captures a profound intuition: that the fabric of reality is informational before it is material [?]. This paper offers, perhaps, the missing mathematics by moving from binary to *triadic* information and from static enumeration to *total function*. The completion of Peano's axioms we proposed, Totality (Axiom 0) and Relativistic Closure (Axiom 6), turns arithmetic into a calculus of conserved, path-independent redistribution. Within this calculus, the triadic flux is the primitive and its equilibria are the numbers.

9.1. Synthesis of Results

Beginning with a minimal physical model (three balls inside a spherical boundary obeying an inverse–square equilibrium), we showed:

1. A triangular contact rule on a curved boundary enforces an icosahedral scaffold and quantized closures at specific vertex counts, the first nontrivial one being $V = 42$.
2. Pascal's triangle emerges on each subdivided face as the combinatorial engine of equilibrium propagation.
3. The Fano plane appears as the minimal incidence stencil enforcing three-at-a-time conservation.
4. The Yang–Baxter relation guarantees global path independence of redistributions.
5. Each equilibrium site possesses a calculable invariant k_i, a glyph number, which converts eight-component dynamic flux into four-dimensional geometric information.
6. In a result as counterintuitive as Vos Savant's solution to the Monty Hall problem, the fundamental constants of mathematical physics (π, e, ϕ, $\sqrt{2}$, γ, $\zeta(2)$, and the inverse fine-structure constant 137) emerge not by assignment or curve-fitting, but through strict application of seven axioms (Peano's five plus complete triadic closure) to this elementary model system.

These are not disparate curiosities: they are three complementary projections (combinatorial, incidence, braid) of the same *closure law*. We have demonstrated that the 42-site shell is the smallest arena in which all three can coexist without frustration.

9.2. Dynamic vs. Static Information

In the extended arithmetic, a *bit* is a degenerate trit in which one flux state has collapsed to zero; such bits are adequate for ledger-style counting but inadequate for systems where information *circulates*. A *trit* is the elementary cycle of selection, revelation, and revision (as seen in Monty Hall), and it admits conservation and braid invariance. Static information is the snapshot of invariants $\{k_i\}$; dynamic information is the ongoing flux on the spherical lattice whose closure projects to those invariants. Thus, *It from Trit*: form (the "it") arises as the invariant shadow of triadic flow.

9.3. Generalizations and Outlook

Two directions are immediate:

Higher T shells and mode spectra.

For $T > 4$ the same construction yields denser lattices with richer normal-mode structure. The glyph set generalizes to $\{k_i\}_{i=1}^{10T+2}$ with the same normalization $\sum_i k_i = 0$; spectral properties of the k–field correlate with stability bands of the underlying total function.

 doi:10.20944/preprints202510.0870.v1

Information dynamics on other curvatures.

Replacing the spherical boundary by hyperbolic or flat boundaries alters the closure constraints (e.g., defect counts), producing distinct Pascal/Fano/YB interplays. These cases offer testbeds for information flow in negative or zero curvature and may connect to error-correcting codes and topological quantum computation [**? ?**].

9.4. Conclusion

By extending the Peano axioms with ones that explicitly *include states* and enforce path-independent *closure*, we obtain a minimal, self-consistent mathematics of dynamic information. In doing so, we believe we are honoring Giuseppe Peano, for he was a true innovator and never held back his passion for expanding the frontiers of our field. From this mathematics, geometry, equilibrium, and invariants are not assumed but derived. The spherical three-ball experiment shows that when information is allowed to circulate triadically under closure, the world acquires the structures we have long recognized, Pascal, Fano, Yang–Baxter, and the numbers themselves become glyphs of conserved flow.

One, admittedly radical, possible conclusion from this work is that the 26 free parameters of the Standard Model *may not* be free parameters at all, but rather necessary consequences of triadic closure operating on an eight-dimensional computational substrate projecting to four-dimensional spacetime [**?**]. However, we emphasize: *if the mathematics are verified*. Mathematical consistency, while necessary, is not sufficient. Physics demands falsifiability, testable predictions that distinguish this framework from existing theory.

We do not claim that this mathematical treatise proves physical reality operates according to the outlined architecture arising naturally from our thought experiment. Rather, we demonstrate that *if* nature appears to implement computation through triadic closure under the axioms presented, *then* the observed constants and structures follow as mathematical necessities rather than adjustable parameters.

Just as Vos Savant presented the world a deceptively simple problem, three doors, two goats, and a prize, to reveal hidden mathematical structure [**?**], we present three balls under equilibrium forces enclosed within a spherical boundary. The analogy extends further: Paul Erdős, one of the twentieth century's greatest mathematicians, required computer simulation of the Monty Hall problem before accepting that his initial intuition was incorrect. We anticipate similar skepticism and have therefore provided complete implementations of the glyph generation algorithms as open-source code. Whether researchers use our provided simulations or write their own, we welcome empirical verification of the mathematical claims presented here.

The fundamental difference between mathematics and physics may itself be, as Einstein suggested, a "stubbornly persistent illusion", but only rigorous testing can determine whether this framework describes reality or merely an elegant mathematical possibility.

Data and Code Availability

A reference implementation generating the $T = 4$ ($V = 42$) vertex set and the corresponding k-values is available at https://github.com/KosmoNexus/kosmoplex-simulation. Any symmetry-equivalent realization of the $T = 4$ geodesic sphere yields the same multiset $\{k_i\}$ up to permutation. The repository includes Python simulation code, verification tests, and visualization tools.

References

1. J. A. Wheeler, Information, physics, quantum: The search for links," in *Complexity, Entropy, and the Physics of Information*, edited by W. H. Zurek, Addison-Wesley, Redwood City, CA (1990).

 doi:10.20944/preprints202510.0870.v1

2. G. Peano, *Arithmetices principia, nova methodo exposita*, Fratres Bocca, Turin (1889).
3. R. Landauer, Irreversibility and Heat Generation in the Computing Process," *IBM J. Res. Dev.* **5**, 183–191 (1961).
4. C. E. Shannon, A Mathematical Theory of Communication," *Bell System Technical Journal* **27**, 379–423, 623–656 (1948).
5. M. vos Savant, Ask Marilyn," *Parade Magazine*, September 9, 1990.
6. L. Gillman, The Car and the Goats," *Amer. Math. Monthly* **99**, 3–7 (1992).
7. L. Mlodinow, *The Drunkard's Walk: How Randomness Rules Our Lives*, Pantheon, New York (2008).
8. R. Dedekind, *Was sind und was sollen die Zahlen?*, Vieweg, Braunschweig (1888).
9. K. Gödel, Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I," *Monatshefte für Mathematik und Physik* **38**, 173–198 (1931).
10. E. L. Post, Introduction to a General Theory of Elementary Propositions," *Amer. J. Math.* **43**, 163–185 (1921).
11. H. S. M. Coxeter, *Regular Polytopes*, 3rd ed., Dover, New York (1973).
12. R. B. Fuller, *Synergetics: Explorations in the Geometry of Thinking*, Macmillan, New York (1975).
13. D. L. D. Caspar and A. Klug, Physical Principles in the Construction of Regular Viruses," *Cold Spring Harbor Symp. Quant. Biol.* **27**, 1–24 (1962).
14. J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, 3rd ed., Springer, New York (2003).
15. J. C. Baez, The Octonions," *Bull. Amer. Math. Soc.* **39**, 145–205 (2002).
16. M. Jimbo, Introduction to the Yang-Baxter Equation," *Int. J. Mod. Phys. A* **4**, 3759–3777 (1989).
17. R. J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press, London (1982).
18. R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics*, 2nd ed., Addison-Wesley, Reading, MA (1994).
19. D. E. Knuth, *The Art of Computer Programming, Vol. 2: Seminumerical Algorithms*, 2nd ed., Addison-Wesley, Reading, MA (1981). [Section on balanced ternary]
20. D. Hilbert, Mathematical Problems," *Bull. Amer. Math. Soc.* **8**, 437–479 (1902). [English translation of 1900 address]
21. S. Wolfram, *A New Kind of Science*, Wolfram Media, Champaign, IL (2002).
22. S. R. Finch, *Mathematical Constants*, Cambridge University Press, Cambridge (2003).
23. M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press, Cambridge (2000).
24. B. M. Terhal, F. Pastawski, and A. Kitaev, "Quantum information scrambling and geometry in curved space," *Nature Physics* **18**, 293–298 (2022).
25. D. Schuhmacher, M. Kuusela, D. Whiteson, and B. Nachman, Unravelling physics beyond the Standard Model with classical and quantum anomaly detection," *Nature Physics* **19**, 1413–1420 (2023).
26. S. Minagawa, T. Kobayashi, N. Yamamoto, and M. Ueda, Universal validity of the second law of information thermodynamics," *Nature Communications* **15**, 1747 (2024).