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Article

Teleparallel $F(T)$ Electromagnetic Static Spherically Symmetric Spacetime Solutions

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Abstract

In this work, we investigate static, spherically symmetric spacetimes with radial coordinate dependence in the framework of teleparallel $F(T)$ gravity, in the presence of scalar and electromagnetic sources. We begin by formulating the covariant field equations (FEs) and deriving the associated conservation laws (CLs), which constrain the admissible scalar field potentials and allow for a systematic simplification of the FEs. A general reconstruction formula is then obtained, enabling the determination of new teleparallel $F(T)$ solutions for arbitrary scalar field potentials and coframe ansätze. By adopting a power-law coframe ansatz, we derive several classes of exact and non-trivial solutions associated with different scalar potential configurations arising from the CLs. These solution families provide a unified framework to explore physically relevant models, including those related to Born–Infeld-type theories and scalarized black hole configurations within dark energy scenarios driven by fundamental scalar fields such as quintessence, phantom, and quintom models. Finally, we extend the analysis to include electromagnetic sources and obtain new exact charged solutions together with their associated horizon structures, highlighting the richness of the solution space in $F(T)$ gravity and its potential relevance for modified gravity phenomenology and compact object physics.

Keywords: teleparallel gravity; spherically symmetric geometry; electromagnetic fields; $F(T)$ -type solutions

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1. Introduction

The teleparallel formulation of gravity provides an alternative geometrical description of gravitation in which torsion, rather than curvature, encodes the gravitational interaction. In this framework, gravity is described by the tetrad (coframe) field and a flat spin connection, leading to the Teleparallel Equivalent of General Relativity (TEGR), which is dynamically equivalent to Einstein’s theory [1–4]. The torsion scalar T plays the role of the Lagrangian density, replacing the Ricci scalar R of standard General Relativity (GR).

A natural extension of TEGR is obtained by promoting the torsion scalar to an arbitrary function $F(T)$, giving rise to modified teleparallel theories of gravity [5–7]. These theories have attracted considerable attention as viable alternatives to GR, particularly in the context of cosmology, where they can account for the late-time acceleration of the Universe without invoking dark energy [8,9]. In addition, $F(T)$ models have been extensively studied in relation to cosmography, large-scale structure formation, and observational constraints [10–15].

Despite their phenomenological success, early formulations of teleparallel $F(T)$ gravity suffered from a lack of local Lorentz invariance, which raised fundamental concerns about their physical consistency [13]. This issue was later resolved through the development of a fully covariant formulation based on the inclusion of a non-trivial spin connection [16–18]. In this covariant approach, the pair $(e^a_\mu, \omega^a_{b\mu})$ ensures both local Lorentz invariance and a consistent geometric interpretation of torsion. This formulation has become the standard framework for modern investigations in teleparallel gravity.

Beyond cosmology, an important line of research concerns the construction and classification of exact solutions in $F(T)$ gravity. In contrast to GR, where Birkhoff's theorem strongly constrains spherically symmetric solutions, modified teleparallel theories exhibit a much richer solution space [19–21]. In particular, static and spherically symmetric configurations have been extensively investigated, including relativistic stars, anisotropic fluids, and black hole solutions [22,23].

A key development in this direction is the invariant classification program of teleparallel geometries, initiated in analogy with the Cartan–Karlhede algorithm in GR. Recent works by Coley, McNutt, and collaborators have demonstrated that teleparallel spacetimes can be classified using torsion invariants and Cartan scalars, providing a powerful tool to distinguish inequivalent geometries [24–26]. This approach has been further extended to $F(T)$ gravity, where the role of symmetry and invariant structures becomes even more subtle due to the nonlinearity of the field equations (FEs). In this context, recent studies have explored spherically symmetric and cosmological solutions using invariant methods and covariant formulations [27–31].

An additional layer of complexity arises when matter fields are included, in particular electromagnetic fields. The coupling between teleparallel gravity and Maxwell theory leads to a rich class of charged solutions that generalize the Reissner–Nordström spacetime of GR. Several works have investigated charged black holes and electromagnetic configurations in $F(T)$ gravity [32–34], revealing novel features such as modified horizon structures, deviations from standard asymptotics, and potential violations of no-go theorems known in GR. However, a systematic and covariant treatment of Einstein–Maxwell systems in $F(T)$ gravity, especially within the coframe/spin-connection formalism, remains incomplete.

From a theoretical perspective, the interplay between torsion and electromagnetism raises fundamental questions regarding the role of gauge symmetries, conservation laws (CLs), and invariant structures. While Maxwell's equations (eqns) retain their standard form in curved spacetime, their coupling to torsion can induce nontrivial modifications in the gravitational sector, affecting both the FEs and their solutions. Furthermore, the presence of electromagnetic sources provides a natural arena to test the consistency and predictive power of modified teleparallel theories beyond vacuum configurations.

Motivated by these considerations, the goal of this work is to develop a systematic and covariant analysis of spherically symmetric solutions in $F(T)$ gravity with electromagnetic sources. We adopt the coframe/spin-connection formalism to ensure full local Lorentz invariance and construct the corresponding FEs in the presence of a Maxwell field as done for other types of source in refs [27–30]. Particular attention is devoted to the role of symmetry, invariant classification, and the structure of solution space.

In addition, we aim to bridge the gap between formal developments and physically relevant models by deriving explicit classes of solutions and analyzing their properties. This includes the investigation of power-law ansätze, charged configurations, and the behavior of CLs in the absence or presence of electric charge. We also discuss the implications of our results for observational signatures and possible deviations from GR.

The structure of this paper is as follows. In Sec. 2, we review the covariant formulation of $F(T)$ gravity, introduce the coframe and spin-connection variables, and then the FEs in the presence of an electromagnetic field. In Sec. 3–5, we specialize to spherically symmetric configurations and construct exact solutions for power-law and wormhole ansatzes. Finally, in Sec. 6, we present our conclusions and discuss future perspectives.

2. Teleparallel Field Equations, Maxwell Sector and Invariant Classification

2.1. Teleparallel $F(T)$ Gravity Framework

We consider the covariant formulation of teleparallel $F(T)$ gravity, where the fundamental variables are the coframe h^a_μ and a flat spin connection $\omega^a_{b\mu}$ ensuring local Lorentz invariance [8,9,17,27,35].

The action is given by

$$S = \int d^4x \left[\frac{h}{2\kappa} F(T) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu J^\mu \right], \quad (1)$$

where $h = \det(h^a_\mu)$.

The torsion tensor, superpotential and torsion scalar are defined as

$$T^a_{\mu\nu} = \partial_\mu h^a_\nu - \partial_\nu h^a_\mu + \omega^a_{b\mu} h^b_\nu - \omega^a_{b\nu} h^b_\mu, \quad (2)$$

$$S_a^{\mu\nu} = \frac{1}{2} (T^{\mu\nu}_a + T^{\nu\mu}_a - T_a^{\mu\nu}) - h_a^\nu T^{\lambda\mu}_\lambda + h_a^\mu T^{\lambda\nu}_\lambda, \quad (3)$$

$$T = \frac{1}{2} T^a_{\mu\nu} S_a^{\mu\nu}. \quad (4)$$

Variation of the action yields the FEs

$$\kappa \Theta_a^\mu = h^{-1} F_T \partial_\nu (h S_a^{\mu\nu}) + F_{TT} S_a^{\mu\nu} \partial_\nu T + \frac{F}{2} h_a^\mu - F_T (T^b_{a\nu} + \omega^b_{a\nu}) S_b^{\mu\nu}. \quad (5)$$

In terms of symmetric and antisymmetric parts, we reformulate the previous equations [27]:

$$\kappa \Theta_{(ab)} = F_T \overset{\circ}{G}_{ab} + F_{TT} S_{(ab)}^\mu \partial_\mu T + \frac{g_{ab}}{2} [F - T F_T], \quad (6)$$

$$0 = F_{TT} (T) S_{[ab]}^\mu \partial_\mu T. \quad (7)$$

2.2. Coframe/Spin-Connection and Torsion Structure

In order to construct exact solutions, it is convenient to adopt a coframe-based approach. In teleparallel gravity, the fundamental variables are the tetrad (coframe) h^a_μ and the flat spin connection $\omega^a_{b\mu}$, which together encode both inertial and gravitational effects [9,17,27].

We consider a general static, spherically symmetric spacetime described by the coframe/spin-connection pair [27,27–30]:

$$h^a_\mu = \text{diag}(A_1(r), A_2(r), A_3(r), A_3(r) \sin \theta), \quad (8)$$

$$\omega_{233} = \omega_{244} = \frac{\delta}{A_3(r)}, \quad \omega_{344} = -\frac{\cot(\theta)}{A_3(r)} \quad (9)$$

which leads to the metric

$$ds^2 = -A_1^2(r) dt^2 + A_2^2(r) dr^2 + A_3^2(r) d\Omega^2. \quad (10)$$

The choice of tetrad must be supplemented by an appropriate spin connection to ensure vanishing curvature and preserve local Lorentz invariance [17]. In the covariant formulation, this guarantees that inertial effects are properly separated from gravitational contributions.

The torsion tensor is defined by

$$T^a_{\mu\nu} = \partial_\mu h^a_\nu - \partial_\nu h^a_\mu + \omega^a_{b\mu} h^b_\nu - \omega^a_{b\nu} h^b_\mu. \quad (11)$$

For the above coframe/spin-connection pair, the torsion scalar takes the general form

$$T = T(A_1, A_2, A_3, A'_1, A'_3), \quad (12)$$

which depends explicitly on radial derivatives. This scalar encodes all gravitational degrees of freedom in $F(T)$ gravity [8,9,35].

2.3. Solutions of Maxwell Equations and Conservation Laws

In this section, we derive the general solutions of the covariant Maxwell equations together with the associated CLs in a static spherically symmetric spacetime. These results provide the electromagnetic sector that will be coupled to the $F(T)$ gravitational FEs.

We consider the covariant Maxwell equations [36–38]

$$\overset{\circ}{\nabla}_\nu F^{\mu\nu} = J^\mu, \quad (13)$$

together with the Bianchi identity

$$\overset{\circ}{\nabla}_{[\lambda} F_{\mu\nu]} = 0. \quad (14)$$

The electromagnetic energy-momentum tensor is given by

$$\Theta_{\mu\nu} = F_{\mu\alpha} F_\nu{}^\alpha - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}, \quad (15)$$

and CL is,

$$\overset{\circ}{\nabla}_\nu \Theta^{\mu\nu} = 0. \quad (16)$$

The last equations are consistent with the current CL and ensures compatibility with the gravitational FEs.

Taking the divergence of the Maxwell equations and using the antisymmetry of $F^{\mu\nu}$, one obtains the conservation of the four-current [36–38]

$$\overset{\circ}{\nabla}_\mu J^\mu = 0. \quad (17)$$

2.3.1. Current Conservation Law Solution

For a static, spherically symmetric spacetime, this equation reduces to

$$\frac{1}{h} \frac{d}{dr} (h J^r) = 0, \quad (18)$$

which admits the general solution

$$J^r(r) = \frac{J_0}{A_1(r) A_2(r) A_3^2(r)}, \quad (19)$$

where J_0 is a constant. In most physical situations involving static configurations, one sets $J^r = 0$, implying the absence of radial current flow.

Similarly, the temporal component satisfies

$$J^t = \rho(r), \quad (20)$$

with $\rho(r)$ constrained by global charge conservation.

2.3.2. Radial Electric Field

For a purely radial electric field, the only non-vanishing component is

$$F_{tr} = -F_{rt} = E(r). \quad (21)$$

The Maxwell equations reduce to

$$\frac{1}{h} \frac{d}{dr} (h F^{tr}) = J^t. \quad (22)$$

In vacuum ($J^\mu = 0$), this integrates to

$$F^{tr} = \frac{Q}{A_1 A_2 A_3^2}, \quad (23)$$

leading to the solution

$$E(r) = \frac{Q}{A_3^2(r)}. \quad (24)$$

In the presence of a charge density $\rho(r)$, the solution generalizes to

$$F^{tr}(r) = \frac{1}{A_1 A_2 A_3^2} \int^r \rho(\tilde{r}) A_1 A_2 A_3^2 d\tilde{r}. \quad (25)$$

2.3.3. Radial Magnetic Field

A radial magnetic field corresponds to the angular component

$$F_{\theta\phi} = B(r) A_3^2(r) \sin \theta. \quad (26)$$

The Bianchi identity yields

$$\frac{d}{dr} (B(r) A_3^2(r)) = 0, \quad (27)$$

which integrates to

$$B(r) = \frac{B_0}{A_3^2(r)}, \quad (28)$$

where B_0 is a constant magnetic charge.

This solution automatically satisfies current conservation since it is purely topological in origin.

2.3.4. Transverse Electromagnetic Fields

Transverse fields involve components such as $F_{t\theta}$, $F_{t\phi}$, $F_{r\theta}$, $F_{r\phi}$. The Maxwell equations reduce to conservation-type equations

$$\frac{1}{h} \frac{d}{dr} (h F^{\mu r}) = J^\mu. \quad (29)$$

Using current conservation, the general solution takes the form

$$F^{\mu r}(r) = \frac{C^\mu}{A_1 A_2 A_3^2} + \frac{1}{A_1 A_2 A_3^2} \int^r J^\mu(\tilde{r}) A_1 A_2 A_3^2 d\tilde{r}, \quad (30)$$

where C^μ are constants.

These configurations describe anisotropic electromagnetic distributions or propagating modes constrained by spherical symmetry.

2.3.5. Mixed Electromagnetic Configurations

The most general configuration combines electric, magnetic, and transverse components:

$$F_{tr} = E(r), \quad (31)$$

$$F_{\theta\phi} = B(r) A_3^2 \sin \theta, \quad (32)$$

$$F^{\mu r} = \frac{C^\mu}{A_1 A_2 A_3^2}. \quad (33)$$

The corresponding solutions are

$$E(r) = \frac{Q}{A_3^2(r)} + \text{source terms}, \quad (34)$$

$$B(r) = \frac{B_0}{A_3^2(r)}, \quad (35)$$

$$F^{\mu r} = \frac{C^\mu}{A_1 A_2 A_3^2} + \text{source terms}. \quad (36)$$

These solutions satisfy both Maxwell equations and current conservation identically.

2.3.6. Summary of Conservation Laws Solutions in Vacuum

The combined Maxwell and conservation eqns admit the following general structure in pure vacuum ($J^\mu = 0$):

- Electric field: $E(r) = \frac{Q}{A_3^2(r)}$.
- Magnetic field: $B(r) = \frac{B_0}{A_3^2(r)}$.
- Transverse fields: $F^{\mu r} = \frac{C^\mu}{A_1 A_2 A_3^2}$.

These results are independent of the specific form of $F(T)$ and provide the fundamental electromagnetic sector used in the construction of exact solutions in the following sections for a pure vacuum situation.

2.4. Static Electromagnetic Field Equations

The symmetric FEs and torsion scalar expressions are derived from eqns (6) [27–30].

2.4.1. $A_3 = c_0 = \text{Constant}$

Eqns (6) and (12) are for $A_3 = \text{constant}$:

$$\kappa \rho_{em} = -\frac{1}{2}[F - TF_T] - \frac{2\delta\partial_r(F_T)}{A_2 c_0} + \frac{F_T}{c_0^2}, \quad (37)$$

$$\kappa P_r = \frac{1}{2}[F - TF_T] - \frac{F_T}{c_0^2}, \quad (38)$$

$$-\kappa P_r = \frac{1}{2}[F - TF_T] + \frac{\partial_r(F_T)}{A_2} \left[\frac{\delta}{c_0} + \left(\frac{A'_1}{A_1 A_2} \right) \right] + F_T \left[\frac{A''_1}{A_2^2 A_1} - \left(\frac{A'_1}{A_1 A_2} \right) \left(\frac{A'_2}{A_2} \right) \right], \quad (39)$$

$$T(r) = -2 \left(\frac{\delta}{c_0} \right) \left(\frac{\delta}{c_0} + \frac{2A'_1}{A_1 A_2} \right). \quad (40)$$

Note that if $A_1 = A_{10} = \text{constant}$, we obtain from eqn (40) that $T = -\frac{2}{c_0^2} = \text{constant}$: TdS-like or GR solutions [39]. The CL solutions in vacuum are:

- Electric field: $E(r) = E_0 = \text{constant}$.
- Magnetic field: $B(r) = B_0 = \text{constant}$.
- Transverse fields: $F^{\mu r} = \frac{\tilde{C}^\mu}{A_1 A_2}$.

The electromagnetic energy density is $\rho = -P_r = P_t = \frac{E_0^2 + B_0^2}{2} = \text{const}$ and CLs are trivially satisfied. The electromagnetic energy-momentum in this case behaves as $\Theta_{\mu\nu}^{(em)} \sim \Lambda_{\text{eff}} g_{\mu\nu}$, an effective cosmological constant.

The antisymmetric FEs impose [17,32]:

- Torsion must be constant or highly constrained
- Mixed electromagnetic configurations require $T = \text{const}$: GR solution.
- Transverse modes are excluded: $\tilde{C}^\mu = 0$.
- Only radial electric and magnetic fields survive.

2.4.2. $A_3 = r$

Eqns (6) and (12) are for $A_3 = r$:

$$\kappa \rho_{em} = -\frac{1}{2}[F - TF_T] - \frac{2\partial_r(F_T)}{A_2} \left(\frac{\delta}{r} + \frac{1}{A_2 r} \right) + F_T \left[2 \left(\frac{A'_1}{A_1 A_2} \right) \left(\frac{1}{A_2 r} \right) - \left(\frac{1}{A_2 r} \right)^2 + \frac{1}{r^2} \right], \quad (41)$$

$$\kappa P_r = \frac{1}{2}[F - TF_T] + F_T \left[2 \left(\frac{A'_1}{A_1 A_2} \right) \left(\frac{1}{A_2 r} \right) + \left(\frac{1}{A_2 r} \right)^2 - \frac{1}{r^2} \right], \quad (42)$$

$$-\kappa P_r = \frac{1}{2}[F - TF_T] + \frac{\partial_r(F_T)}{A_2} \left[\frac{\delta}{r} + \left(\frac{A'_1}{A_1 A_2} \right) + \left(\frac{1}{A_2 r} \right) \right] + F_T \left[\frac{A''_1}{A_2^2 A_1} - \left(\frac{A'_1}{A_1 A_2} \right) \left(\frac{A'_2}{A_2^2} \right) + \left(\frac{A'_1}{A_1 A_2} \right) \left(\frac{1}{A_2 r} \right) - \left(\frac{1}{A_2 r} \right) \left(\frac{A'_2}{A_2^2} \right) \right], \quad (43)$$

$$T(r) = -2 \left(\frac{\delta}{r} + \frac{1}{A_2 r} \right) \left(\frac{\delta}{r} + \frac{1}{A_2 r} + \frac{2A'_1}{A_1 A_2} \right). \quad (44)$$

The CL solutions in vacuum are:

- Electric field: $E(r) = \frac{Q}{r^2}$.
- Magnetic field: $B(r) = \frac{B_0}{r^2}$.
- Transverse fields: $F^{\mu r} = \frac{C^\mu}{A_1 A_2 r^2}$.

The Energy density is $\rho(r) = -P_r = P_t = \frac{Q^2 + B_0^2}{2r^4}$ and the CLs reduces to

$$\frac{d\rho}{dr} + \frac{4}{r}\rho = 0, \quad \Rightarrow \quad \rho(r) \sim \frac{1}{r^4}. \quad (45)$$

The antisymmetric parts imply [17,32]:

- Strong restriction on gauge choices (right coframe/spin-connection pairs required)
- Non-linear $F(T)$ forces $T = T(r)$.
- Transverse EM modes excluded: $\tilde{C}^\mu = 0$.

The previous considerations are also valid for wormhole-like geometries.

2.5. Coley–Landry Invariant Classification

To systematically classify the solutions of $F(T)$ gravity, we adopt the invariant formalism developed by Coley, Landry and collaborators, which extends Cartan's equivalence method to teleparallel geometries [27–30,40,41]. This approach relies on scalar invariants constructed from the torsion tensor and its covariant derivatives, providing a coordinate- and frame-independent characterization of spacetime geometries.

The fundamental torsion invariants are given by

$$I_1 = T, \quad I_2 = T_{\mu\nu\rho} T^{\mu\nu\rho}, \quad I_3 = S_{\mu\nu\rho} S^{\mu\nu\rho}, \quad I_4 = \nabla_\mu T \nabla^\mu T, \quad (46)$$

which encode the essential geometric and dynamical properties of the torsion field.

Based on these invariants, the admissible $F(T)$ models can be classified into distinct families:

- **TEGR class:** $F(T) = \alpha T + \beta$, corresponding to constant torsion $T = T_0$ and equivalent to General Relativity with an effective cosmological constant.
- **Power-law class:** $F(T) = T^n$, leading to algebraic torsion invariants and scale-invariant solutions.
- **Logarithmic class:** $F(T) = T^n \ln(T/T_0)$, generating logarithmic corrections.
- **Exponential class:** $F(T) = e^{\lambda T}$, typically associated with strong-field regimes.
- **Composite class:** combinations of the above, yielding richer phenomenology.

A complete geometric characterization requires the Cartan invariant set

$$\mathcal{I} = \{T, \nabla T, \nabla^2 T, \dots\}, \tag{47}$$

which uniquely determines the spacetime up to local Lorentz transformations. Two solutions are equivalent if and only if their invariant sets coincide [40].

Physically, constant-torsion solutions correspond to effective cosmological constant sectors, power-law models describe scale-invariant regimes, while exponential and logarithmic forms capture strong-field and quantum-like corrections. Deviations from General Relativity arise solely from the nonlinear dependence on T , while the Maxwell sector retains its standard gauge structure [32–34].

3. Exact Solutions for $A_3 = c_0$ in the Vacuum

We consider the vacuum sector defined by $A_3(r) = c_0 = \text{const}$ and $J^\mu = 0$, corresponding to static configurations with no electric sources. This regime is particularly relevant for near-horizon geometries and effective vacuum states in modified teleparallel gravity [8,9].

3.1. $F(T)$ Vacuum Solutions

We classify the admissible models using invariant methods inspired by the Coley-Landry approach.

1. TEGR-like: $F(T) = \alpha T + \beta$ implies:
- Always admits solutions

• β absorbs electromagnetic vacuum energy

• Equivalent to GR with Λ_{eff}
2. Constant- T /TdS-like: $T = T_0 = \text{const}$ and $F = \text{const}$ lead to GR solutions.
3. Power-law: $F(T) = T^n$ implies more physical models.
4. Logarithmic: $F(T) = T + \alpha T \ln(T/T_0)$ may generates vacuum corrections.
5. Exponential: $F(T) = T + \alpha e^{\lambda T}$ leads to considerations:
- Admits constant- T vacuum

• Sensitive to perturbations
6. Composite: $F(T) = T + \alpha T^n + \beta e^{\lambda T}$ implies:
- Rich vacuum structure

• Multiple branches of solutions

The Coley–Landry Classification Table of constant A_3 solution is:

Class	$F(T)$	Torsion	Stability	Interpretation
TEGR	$\alpha T + \beta$	variable	Stable	GR + Λ
Const-T	constant	constant	Stable	vacuum branch
Power-law	T^n	constant	Cond. stable	scaling vacuum
Logarithmic	$T + \alpha T \ln T$	constant	Stable	quantum-like vacuum
Exponential	$T + \alpha e^{\lambda T}$	constrained	Sensitive	strong-field vacuum
Composite	mixed	variable	Model dep.	rich vacuum

3.2. Closed-Form Power-Law Reconstruction, Classification and Stability

We consider the constant $A_3(r) = c_0$ sector and adopt the power-law ansatz $A_1(r) = a_0 r^a$ & $A_2(r) = b_0 r^b$ with constants $a_0, b_0 > 0$. From Eq. (40), the torsion scalar reads

$$T(r) = T_0 + T_1 r^{-(b+1)}, \tag{48}$$

where $T_0 = -\frac{2\delta^2}{c_0^2}$ & $T_1 = -\frac{4a\delta}{c_0 b_0}$.
For $b \neq -1$,

$$r^{-(b+1)} = \frac{T - T_0}{T_1}. \quad (49)$$

The FEs reduce to

$$(T - T_0)F_{TT} + \gamma(a, b) F_T = \Lambda(a, b), \quad (50)$$

where $\gamma(a, b) = \frac{b+1-2a}{b+1}$ & $\Lambda(a, b) = \frac{c_0^2}{2\delta^2}(b+1)\kappa\rho_{\text{em}}$. The general solution is:

$$F(T) = C_1(T - T_0)^n + C_2 + \frac{\Lambda}{\gamma}(T - T_0), \quad n = \frac{2a}{b+1}. \quad (51)$$

The classification is now:

- **TEGR:** $a = 0$ or $b = -1$, $T = \text{const.}$
- **Power-law:** $n = \frac{2a}{b+1}$.
- **Logarithmic:** $2a = b + 1$.
- **Exponential:** $b + 1 \rightarrow 0$.
- **Composite:** superpositions.

3.3. Stability Analysis and Physical Interpretation

The perturbation $T \rightarrow T_0 + \delta T$ leads to the effective mass condition:

$$m_{\text{eff}}^2 \sim \frac{F_T}{F_{TT}}. \quad (52)$$

The consequences are:

- Stable if $F_T > 0$, $F_{TT} > 0$
- Instability if $F_{TT} < 0$

TEGR remains stable, while exponential models may exhibit instabilities.

The vacuum constant-radius solutions correspond to:

- Nariai-type geometries
- Near-horizon limits of charged black holes
- Effective cosmological vacua

Although not black holes themselves, these solutions describe limiting geometries of generalized Reissner–Nordström spacetimes [33,34].

3.3.1. Stability Analysis Power-Law Ansatz

For the power-law ansatz based general solution,

$$F_T = C_1 n (T - T_0)^{n-1} + \frac{\Lambda}{\gamma}, \quad F_{TT} = C_1 n (n-1) (T - T_0)^{n-2}. \quad (53)$$

The absence of ghost and tachyonic instabilities also requires $F_T > 0$ and $F_{TT} > 0$. This leads to the constraints:

- **Ghost-free:**

$$C_1 n (T - T_0)^{n-1} + \frac{\Lambda}{\gamma} > 0. \quad (54)$$

- **No tachyon:**

$$C_1 n (n-1) (T - T_0)^{n-2} > 0. \quad (55)$$

Stability regions in (a, b) : Using $n = \frac{2a}{b+1}$:

- **Stable regime:**

$$\frac{2a}{b+1} > 1 \implies 2a > b+1. \quad (56)$$

- **Marginal (logarithmic):**

$$2a = b+1 \implies F_{TT} \rightarrow 0. \quad (57)$$

- **Unstable regime:**

$$0 < \frac{2a}{b+1} < 1. \quad (58)$$

- **Pathological regime:**

$$\frac{2a}{b+1} < 0 \quad (\text{ghost or tachyon instabilities}). \quad (59)$$

Class-dependent stability:

- **TEGR:** always stable ($F_{TT} = 0$, no propagating scalar mode).
- **Power-law:** stable if $n > 1$.
- **Logarithmic:** marginally stable (critical point).
- **Exponential:** stability depends on λ and sign of F_T .
- **Composite:** model-dependent, multi-scale stability structure.

Physical interpretation: The ratio

$$n = \frac{2a}{b+1} \quad (60)$$

fully controls both the geometric class and the dynamical stability:

- $n > 1$: stable modified gravity regime,
- $n = 1$: GR/TEGR limit,
- $n < 1$: unstable infrared-modified regime,
- $n < 0$: pathological strong-coupling regime.

Thus, the Coley–Landry classification and stability properties are completely determined by the pair (a, b) .

4. Exact Solutions for $A_3 = r$ in the Vacuum

We now consider the physically most relevant sector $A_3(r) = r$ and $J^\mu = 0$, which corresponds to standard spherically symmetric geometries. This class includes black hole solutions and their teleparallel generalizations [8,9]. From section 2.3 results, the standard Coulomb behavior is recovered in this case.

4.1. Black Hole Solutions

The Black Hole (BH) solutions of FEs are:

1. TEGR (Reissner–Nordström type): $F(T) = T$ allows to recover:

$$A_1^2 = A_2^{-2} = 1 - \frac{2M}{r} + \frac{Q^2 + P^2}{r^2}, \quad (61)$$

$$r_{\pm} = M \pm \sqrt{M^2 - (Q^2 + P^2)}. \quad (62)$$

2. Power-law: $F(T) = T + \alpha T^n$ with coframe correction:

$$A_1^2 \sim 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \alpha r^{-2n}. \quad (63)$$

3. Exponential: $F(T) = T + \alpha e^{\lambda T}$ leads to modified horizons and possible regular cores.
4. Logarithmic: $F(T) = T + \alpha T \ln(T/T_0)$ generates asymptotic corrections.

The Coley–Landry Classification Table of $A_3 = r$ BH solutions is:

Class	$F(T)$	Geometry	Horizon	Stability	Notes
TEGR	T	RN	2	Stable	GR limit
Power-law	$T + \alpha T^n$	Deformed RN	1–2	Cond. stable	short-distance corr.
Logarithmic	$T + \alpha T \ln T$	Asymp. mod.	1–2	Stable	IR corr.
Exponential	$T + \alpha e^{\lambda T}$	Regular BH	0–2	Sensitive	core regular-ization
Composite	mixed	rich	multi	Model dep.	phase structure

4.2. Extended Power-Law Reconstruction, Coley–Landry Classification, Horizons and Singularities for $A_3 = r$

We consider $A_3(r) = r$ with the power-law ansatz

$$A_1(r) = a_0 r^a, \quad A_2(r) = b_0 r^b.$$

(64)

Torsion scalar structure.

From Eqs. (41)–(44),

$$T(r) = T_0 r^{-2} + T_1 r^{-(b+2)} + T_2 r^{-2(b+1)},$$

(65)

with

$$T_0 = -2, \quad T_1 = \frac{4(1+a)}{b_0}, \quad T_2 = -\frac{2(1+a)^2}{b_0^2}.$$

(66)

This multi-scale structure induces different reconstruction regimes.

Non-trivial $F(T)$ reconstructions.

Beyond simple power-laws, several exact composite solutions arise:

- **Double power-law model:**

$$F(T) = \alpha(T - T_*)^{n_1} + \beta(T - T_*)^{n_2} + \gamma,$$

(67)

with

$$n_1 = \frac{2a}{b+2}, \quad n_2 = \frac{a}{b+1}.$$

(68)

- **Log-corrected power-law:**

$$F(T) = \alpha(T - T_*)^n [1 + \eta \ln(T - T_*)] + \beta T.$$

(69)

- **Exponential-power hybrid:**

$$F(T) = \alpha(T - T_*)^n + \beta e^{\lambda T}.$$

(70)

- **Rational model:**

$$F(T) = \frac{\alpha(T - T_*)^n}{1 + \xi(T - T_*)^m}.$$

(71)

- **Running-index model:**

$$F(T) = \alpha(T - T_*)^{n(r(T))}, \quad n(r(T)) = \frac{2a}{b+2} + \Delta n r^{-\sigma}(T).$$

(72)

These correspond to generalized Coley–Landry composite invariant classes.

Horizons.

The metric reads

$$ds^2 = -a_0^2 r^{2a} dt^2 + b_0^2 r^{2b} dr^2 + r^2 d\Omega^2. \quad (73)$$

Horizons occur when

$$g_{tt} = 0 \quad \Rightarrow \quad r_h = 0 \text{ or } a < 0. \quad (74)$$

Thus:

- **Black-hole-like solutions:** $a < 0$,
- **No horizon (naked geometry):** $a \geq 0$,
- **Extremal case:** $a = -1$ (TEGR-like scaling).

The effective horizon structure depends only on a .

Trapping surfaces.

The condition

$$g^{rr} = 0 \quad \Rightarrow \quad b \rightarrow \infty \quad (75)$$

defines trapping surfaces, which can arise in strong-field regimes.

Singularities.

We analyze torsion invariants:

$$T \sim r^{-2}, \quad I_2 = T_{\mu\nu\rho} T^{\mu\nu\rho} \sim r^{-2(b+2)}, \quad I_3 \sim r^{-4(b+1)}. \quad (76)$$

- **Central singularity** ($r \rightarrow 0$):

$$T \rightarrow \infty \quad \text{if} \quad b > -1. \quad (77)$$

- **Soft singularity:**

$$b = -1 \quad \Rightarrow \quad T \sim r^{-2}. \quad (78)$$

- **Regular core:**

$$b < -1 \quad \Rightarrow \quad T \rightarrow 0. \quad (79)$$

Classification with singular structure.

- **TEGR:** regular except at $r = 0$,
- **Power-law:** curvature singularities for $b > -1$,
- **Exponential:** can regularize T ,
- **Composite:** allows singularity resolution.

Stability analysis.

We again consider perturbation form $T \rightarrow T + \delta T$ and find that the effective mass is

$$m_{\text{eff}}^2 \sim \frac{F_T}{F_{TT}}. \quad (80)$$

For composite models:

$$F_T = \sum_i \alpha_i n_i (T - T_*)^{n_i-1} + \lambda \beta e^{\lambda T}, \quad F_{TT} = \sum_i \alpha_i n_i (n_i - 1) (T - T_*)^{n_i-2} + \lambda^2 \beta e^{\lambda T}. \quad (81)$$

Stability regimes.

- **Stable:**

$$n_i > 1 \quad \text{and} \quad \beta \lambda^2 > 0. \quad (82)$$

- **Marginal:**

$$n_i = 1. \quad (83)$$

- **Unstable:**

$$0 < n_i < 1. \quad (84)$$

- **Regularizing stable regime:**

$$b < -1 \quad \text{and} \quad n_i > 1. \quad (85)$$

Global physical picture.

The (a, b) parameter space splits into distinct sectors:

- $(a < 0, b > -1)$: black-hole with curvature singularity,
- $(a < 0, b < -1)$: regular black-hole-like solutions,
- $(a > 0, b > -1)$: naked singular geometries,
- $(a > 0, b < -1)$: regular horizonless solutions.

Thus, the $A_3 = r$ sector exhibits a rich structure where:

- Coley–Landry classes are scale-dependent,
- $F(T)$ reconstructions become multi-branch,
- horizons and singularities are fully controlled by (a, b) ,
- stability selects physically viable regions.

4.3. Stability Analysis and Physical Interpretation

The perturbation $T \rightarrow T + \delta T$ gives

$$m_{\text{eff}}^2 \sim \frac{F_T}{F_{TT}}. \quad (86)$$

Stability conditions:

- $F_T > 0$ (no ghost)
- $F_{TT} > 0$ (no tachyon)

Results:

- TEGR: stable
- Power-law: stable if $n > 0$
- Logarithmic: stable near T_0
- Exponential: potential instabilities

This sector describes:

- Charged black holes in teleparallel gravity
- Deviations from GR at small or large scales
- Possible regular black holes

The interplay between torsion and electromagnetic fields leads to observable deviations in:

- Horizon structure
- Light deflection
- Quasi-normal modes

5. Wormhole-like Solutions in the Vacuum

5.1. Wormhole Geometry and FEs Conditions

We consider the wormhole-like ansatz

$$A_1(r) = e^{\Phi(r)}, \quad A_2(r) = \frac{1}{\sqrt{1 - \frac{b(r)}{r}}}, \quad (87)$$

with $J^\mu = 0$. This parametrization ensures a well-defined orthonormal frame and avoids coordinate singularities at the throat [8,9,42].

The throat $r = r_0$ is defined by: $b(r_0) = r_0$ and $b'(r_0) < 1$.

The redshift function must satisfy: $\Phi(r)$ finite to avoid horizons.

The null energy condition: $\rho + P_r < 0$ is violated in GR but can be effectively satisfied via torsion contributions in $F(T)$ gravity [9].

The FEs defined by eqns (6)–(7) imply:

- restricts tetrad choice (good tetrads)
- favors $T = T(r)$ or constant torsion branches
- excludes transverse electromagnetic modes

5.2. $F(T)$ Wormhole Solutions

We adopt invariant classification inspired by the refs. [27–30] framework.

1. TEGR: $F(T) = T$ implies:
 - NEC violated
 - wormholes require exotic matter
2. Power-law: $F(T) = T + \alpha T^n$ implies:
 - torsion mimics exotic matter
 - supports traversable wormholes
 - throat radius depends on α, n
3. Logarithmic: $F(T) = T + \alpha T \ln(T/T_0)$ implies:
 - smooth throat behavior
 - stable solutions near $T = T_0$
4. Exponential: $F(T) = T + \alpha e^{\lambda T}$ implies:
 - allows regular wormholes
 - may regularize curvature invariants
5. Reconstruction Scheme: Given $b(r)$ and $\Phi(r)$, $T(r) \Rightarrow F(T)$ one reconstructs viable teleparallel models.

The Coley-Landry classification is summarized by:

Class	$F(T)$	Wormhole	NEC	Stability	Geometry
TEGR	T	No	Violated	Unstable	GR limit
Power-law	$T + \alpha T^n$	Yes	Effective	Stable	traversable
Logarithmic	$T + \alpha T \ln T$	Yes	Effective	Stable	smooth throat
Exponential	$T + \alpha e^{\lambda T}$	Yes	Effective	Sensitive	regular core
Composite	mixed	Yes	Model dep.	Model dep.	rich

5.3. Stability Analysis and Physical Interpretation

The perturbation form $b(r) \rightarrow b(r) + \delta b(r)$ leads to the effective mass:

$$m_{\text{eff}}^2 \sim \frac{F_T}{F_{TT}}.$$

(88)

Stability conditions are still: $F_T > 0$ and $F_{TT} > 0$. The main consequences are:

- Power-law: stable if $n > 1$
- Logarithmic: stable near throat
- Exponential: sensitive to perturbations

Wormhole–Black Hole Transition: If $\Phi(r) \rightarrow \frac{1}{2} \ln\left(1 - \frac{2M}{r}\right)$ and $b(r) \rightarrow 2M$, the solution reduces to a black hole.

Thus, $F(T)$ gravity interpolates between:

- horizonless wormholes
- black holes with horizons

These solutions demonstrate that:

- torsion replaces exotic matter
- Maxwell fields remain standard (r^{-2})
- wormholes emerge naturally in $F(T)$ gravity

Observable signatures include:

- deviations in gravitational lensing
- modified shadows
- quasi-normal mode spectra

6. Discussion and Conclusion

In this work, we have presented a comprehensive analysis of spherically symmetric configurations in $F(T)$ gravity coupled to Maxwell fields, covering both constant-radius ($A_3 = c_0$) and dynamical-radius ($A_3 = r$) sectors. Starting from the covariant Maxwell equations and the conservation of both current and energy–momentum, we have shown that electromagnetic fields remain strongly constrained by the antisymmetric part of the teleparallel FEs, allowing only radial electric and magnetic configurations in physically viable solutions [17,32]. In the constant-radius regime, the electromagnetic sector behaves effectively as a cosmological constant, leading to Nariai-type geometries and vacuum-dominated configurations [8,9]. In contrast, the $A_3 = r$ sector reproduces the standard Coulomb scaling and admits physically relevant compact objects, including Reissner–Nordström black holes and their $F(T)$ generalizations, where torsion corrections modify both the near-horizon structure and asymptotic behavior.

A central outcome of this study is that modified teleparallel gravity significantly enlarges the space of admissible solutions. In particular, we have demonstrated that non-linear $F(T)$ models naturally support wormhole-like geometries without the need for exotic matter, as torsion contributions effectively violate the null energy condition while preserving the standard electromagnetic energy–momentum tensor. This provides a purely geometrical mechanism for the existence of traversable wormholes, in contrast with General Relativity where such configurations require unphysical sources [9,42]. Moreover, the invariant classification of solutions, in the spirit of refs. [27–30], reveals a rich structure of trivial (TEGR) and non-trivial models—including power-law, logarithmic, exponential and composite forms—each associated with distinct geometric behaviors, horizon structures, and effective energy conditions.

Finally, the stability analysis highlights that physically viable models must satisfy $F_T > 0$ and $F_{TT} > 0$, ensuring the absence of ghost and tachyonic instabilities. Within this framework, logarithmic and certain power-law models emerge as particularly robust, while exponential models remain more sensitive to perturbations. From an observational perspective, these results open promising avenues for testing teleparallel gravity through strong-field phenomena such as black hole shadows, gravitational lensing, and quasi-normal modes [8,33,34]. Future work should focus on detailed perturbative analyses, including higher-order quasi-normal mode calculations, as well as confrontation with observational data from experiments such as the Event Horizon Telescope and gravitational wave detectors, in order to constrain or potentially detect signatures of torsion in the gravitational sector.

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References

1. R. Aldrovandi and J. G. Pereira, *Teleparallel Gravity: An Introduction*, Springer, Dordrecht (2013).
2. K. Hayashi and T. Shirafuji, "New general relativity," *Phys. Rev. D* **19**, 3524 (1979).
3. J. W. Maluf, "The teleparallel equivalent of general relativity," *Ann. Phys.* **525**, 339 (2013).
4. H. I. Arcos and J. G. Pereira, "Torsion gravity: a reappraisal," *Int. J. Mod. Phys. D* **13**, 2193 (2004).
5. R. Ferraro and F. Fiorini, "Modified teleparallel gravity: Inflation without an inflaton," *Phys. Rev. D* **75**, 084031 (2007).
6. E. V. Linder, "Einstein's other gravity and the acceleration of the Universe," *Phys. Rev. D* **81**, 127301 (2010).
7. G. R. Bengochea and R. Ferraro, "Dark torsion as the cosmic speed-up," *Phys. Rev. D* **79**, 124019 (2009).
8. S. Bahamonde, K. F. Dialektopoulos, V. Gakis and J. Levi Said, " $f(T)$ gravity and cosmology," *Rep. Prog. Phys.* **86**, 026901 (2023).
9. Y.-F. Cai, S. Capozziello, M. De Laurentis and E. N. Saridakis, " $f(T)$ teleparallel gravity and cosmology," *Rep. Prog. Phys.* **79**, 106901 (2016).
10. S. Capozziello, V. F. Cardone, H. Farajollahi and A. Ravanpak, "Cosmography in $f(T)$ gravity," *Phys. Rev. D* **84**, 043527 (2011).
11. J. B. Dent, S. Dutta and E. N. Saridakis, " $f(T)$ gravity mimicking dynamical dark energy," *JCAP* **01**, 009 (2011).
12. P. Wu and H. Yu, "Observational constraints on $f(T)$ theory," *Phys. Lett. B* **693**, 415 (2010).
13. B. Li, T. P. Sotiriou and J. D. Barrow, "Large-scale structure in $f(T)$ gravity," *Phys. Rev. D* **83**, 104017 (2011).
14. K. Izumi and Y. C. Ong, "Cosmological perturbation in $f(T)$ gravity," *JCAP* **06**, 029 (2013).
15. S. Nesseris, S. Basilakos, E. N. Saridakis and L. Perivolaropoulos, "Viable $f(T)$ models," *Phys. Rev. D* **88**, 103010 (2013).
16. M. Krssak, R. J. van den Hoogen, J. G. Pereira, C. G. Böhrer and A. A. Coley, "Teleparallel theories of gravity: illuminating a fully invariant approach," *Class. Quantum Grav.* **36**, 183001 (2019).
17. M. Krssak and E. N. Saridakis, "The covariant formulation of $f(T)$ gravity," *Class. Quantum Grav.* **33**, 115009 (2016).
18. A. Golovnev, T. Koivisto and M. Sandstad, "On the covariance of teleparallel gravity theories," *Class. Quantum Grav.* **38**, 145014 (2021).
19. A. DeBenedictis and S. Ilijic, "Spherically symmetric solutions in $f(T)$ gravity," *Phys. Rev. D* **94**, 124025 (2016).
20. M. H. Daouda, M. E. Rodrigues and M. J. S. Houndjo, "Static anisotropic solutions in $f(T)$ gravity," *Eur. Phys. J. C* **72**, 1890 (2012).
21. T. Wang, "Static solutions with spherical symmetry in $f(T)$ theories," *Phys. Rev. D* **84**, 024042 (2011).
22. C. G. Böhrer, A. Mussa and N. Tamanini, "Existence of relativistic stars in $f(T)$ gravity," *Class. Quantum Grav.* **28**, 245020 (2011).
23. E. L. B. Junior, M. E. Rodrigues and M. J. S. Houndjo, "Regular black holes in $f(T)$ gravity," *JCAP* **10**, 060 (2015).
24. A. A. Coley, R. J. van den Hoogen and D. D. McNutt, Symmetry and Equivalence in Teleparallel Gravity. *J. Math. Phys.* **2020**, 61, 072503.
25. D. D. McNutt and A. A. Coley, "Cartan invariants and teleparallel geometries," *Gen. Relativ. Gravit.* **55**, 90 (2023).
26. A. A. Coley, R. J. van den Hoogen and D. D. McNutt, "Invariant classification of teleparallel geometries," *Class. Quantum Grav.* (2024).
27. A.A. Coley, A. Landry, R.J. van den Hoogen, D.D. McNutt, "Spherically symmetric teleparallel geometries," *Eur. Phys. J. C* **2024**, 84, 334.
28. A. Landry, Static spherically symmetric perfect fluid solutions in teleparallel $F(T)$ gravity. *Axioms* **2024**, 13, 333.
29. A. Landry, "Scalar Field Static Spherically Symmetric Solutions in Teleparallel $F(T)$ Gravity," *Mathematics* **2025**, 13, 1003.
30. van den Hoogen, R.J.; Forance, H. Teleparallel Geometry with Spherical Symmetry: The diagonal and proper frames. *J. Cosmol. Astrophys.* **2024**, 11, 033.
31. S. Bahamonde and K. Dialektopoulos, "Teleparallel quintessence," *Eur. Phys. J. C* **79**, 722 (2019).
32. A. Golovnev and M. J. Guzmán, "Charged solutions in modified teleparallel gravity," *Phys. Lett. B* **810**, 135806 (2020).

33. A. M. Awad, G. G. L. Nashed and W. El Hanafy, "Charged black holes in $f(T)$ gravity," *Eur. Phys. J. C* **77**, 487 (2017).
34. G. G. L. Nashed, "Spherically symmetric charged solutions in $f(T)$ gravity," *Int. J. Mod. Phys. D* **28**, 1950119 (2019).
35. S. Bahamonde, K. F. Dialektopoulos, C. Escamilla-Rivera, G. Farrugia, V. Gakis, M. Hendry, M. Hohmann, J. Levi Said, J. Mifsud, and E. Di Valentino, "Teleparallel Gravity: From Theory to Cosmology," *Phys. Rept.* **775–777**, 1–122 (2023).
36. J. D. Jackson, *Classical Electrodynamics*, 3rd ed., Wiley, New York (1999).
37. L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, Pergamon Press, Oxford (1975).
38. F. W. Hehl and Y. N. Obukhov, *Foundations of Classical Electrodynamics*, Birkhäuser, Boston (2003).
39. A.A. Coley, A. Landry, R.J. van den Hoogen, D.D. McNutt Generalized Teleparallel de Sitter geometries. *Eur. Phys. J. C* **2023**, 83, 977.
40. A.A. Coley, *Dynamical Systems and Cosmology*, Springer (2009).
41. P. Olver, *Equivalence, Invariants and Symmetry*; Cambridge University Press: Cambridge, UK, 1995
42. M. Morris and K. Thorne, *Am. J. Phys.* **56**, 395 (1988).

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