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Default Risk with Imperfect Information under Regime-Switching Model

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Abstract: We propose a pricing formula for a defaultable zero-coupon bond with imperfect information under a regime-switching model using a structural form of credit risk modelling. While the value of the firm's equity is observed continuously, we assume that the total value of the firm is only observed at discrete times, such as the dates of the release of the firm's annual reports, or quarterly reports. This uncertainty about the true value of the firm results in credit spreads that do not approach zero as the debt approaches maturity, which is a problem with many structural models. The firm's value is typically defined as its equity and debt; however, we consider the asset-to-equity ratio, an accounting ratio used to examine a firm's financial well-being. A dependent regime-switching process models the components of this ratio, and the Markov chain represents the states of the economy. The main contribution is to study this problem when the dynamics of a firm's assets have different parameters with regime changes.

Keywords: risky debt; imperfect information; interest rate; regime switching

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1. Introduction

Structural bond pricing models value risky debt as a derivative on the firm's assets. Two famous papers first introduced this approach: [1] and [2] and later [3] and the approach has since then drawn considerable attention in understanding credit risk by specifying a firm's value process. One of the advantages of the structural bond pricing models is that it is possible to use price information for one class of securities, equity, to estimate the value of another, such as debt.

In the literature, it has been noted that one of the problems with structural models is that their credit spreads approach zero as the risky bond approaches its maturity. Alternatively, other approaches have added jumps to the model of the value of the firm's assets to add more uncertainty about whether or not there will be a default at maturity, such as [4–6] and so on for different types of jump processes. As [7] state one can assume that at least one of the firm's securities is traded and remain in a complete market setting. In such a framework, although a firm's assets are not traded, their value can be replicated".

Our approach is a simple way to keep the credit spreads from approaching zero at the bond's maturity. Our fundamental assumption is that the firm's total value V is not continuously observable; i.e., it is only directly observable on specific dates, in particular when the firm issues annual or possibly quarterly reports. On the other hand, the firm's equity value, E, is continually observed. Moreover, it turns out that it is more convenient to work with multiplicative variables. Therefore, we write $V = EH$, and thus define the ratio H as $H = V/E$. In industry, this is known as the *asset to equity ratio*, which is an accounting ratio often used to examine a firm's financial well-being. We then derive their stochastic differential equations, where the Brownian motion in E and H are correlated.

Another assumption is that the default happens only at bond maturity if the firm's value falls below a predetermined barrier. Our method is similar to the paper due to [8], which assumes noisy reports to determine the uncertainty from the imperfect information.

From there, we develop a simple approach to valuing risky debt that incorporates default risk in their valuation, and this can be extended nicely to price any derivative.

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We also consider a structural model in which the parameters in the presented dynamics change due to regime switching. Since [9], many researchers have used the idea of regime-switching (RS) in financial economics. One of their features is that the model dynamics can change over time according to the state of an underlying Markov chain, often interpreted as structural changes in economic conditions and different stages of business cycles. Default risk is influenced by business cycles and the macroeconomy and typically declines during economic expansion because this growth keeps the default rates low; the opposite is true in an economic recession. For this reason, it is reasonable to extend our idea to allow for regime-switching parameters, as there is a need to develop some credit risk models that can take into account changes in market regimes.

Although regime-switching models have been widely used in credit risk modelling, they mainly focus on reduced-form models; see for example, [10–13] among others. To our knowledge, some credit risk models with regime switching have recently been introduced to model the default risk using the structural model, making it a hot topic to consider. For example, in pricing defaultable bonds under this framework, [14] price risky bonds using the Esscher transform, where a Markov-modulated generalized jump-diffusion model with a Markov-switching compensator governs the firm value. [15] model the firm value and the default boundary by two dependent regime-switching jump diffusion processes, in which the Markov chain represents the states of an economy. Their numerical illustration suggests that the change of market regimes should be incorporated into the model for pricing credit derivatives.

This paper is arranged as follows. In Section 2, the model dynamics of the value of the firm and the Markov chain are introduced. Further, we study the moment-generating function of the logarithm of the value of the firm, $\ln V(T)$, conditional on $\mathcal{F}^*(t,t_k)$ and $\mathcal{F}^{\mathbf{X}}(t_k)$ and give a justification as to why we used the methodology (see Section 3.1). In Section 3, we price the risky debt with two assumptions about the interest rate, which will define which Markov chain is observed. Section 4 concludes the paper.

2. The Model Dynamics

We consider a continuous-time financial market with a finite time horizon where $T < \infty$, where we assume that $(\Omega, \mathcal{F}, \mathbb{Q})$ is a complete probability space and $\mathbb{Q}$ is a risk-neutral probability measure.

Following [16], we suppose that the continuous-time, finite-state, observable Markov chain $\{\mathbf{X}_t\}$ on the probability space takes values in a set of standard unit vectors $\mathcal{E} = \{e_1, e_2, \cdots, e_N\} \subset \mathbb{R}^N$, where for each $1 \le j \le N$, e_j is the vector with 1 in the j the entry and 0 elsewhere. The set $\mathcal{E}$ is called the canonical state space of $\mathbf{X}$.

Let $\mathbf{A} = (a_{ij})_{i,j=1,\cdots,N}$ be the transition rate matrix of $\mathbf{X}$ under the measure $\mathbb{Q}$. Here for $i \neq j$, a_{ij} is the (constant) transition intensity from state e_i to state e_j in a small interval of time, and satisfies $a_{ij} \geq 0$ for $i \neq j$ and $\sum_i^N a_{ij} = 0$, for each $i, j = 1, 2, \ldots, N$. The semi-martingale representation of $\mathbf{X}$ is

$$\mathbf{X}_t = X_0 + \int_t^T \mathbf{A}\mathbf{X}_s ds + \mathbf{M}_t, \quad t \in [0, T],$$

where $\mathbf{M}(t)$ is an $\mathbb{R}^N$- $(\mathbb{F}^{\mathbf{X}}, \mathbb{Q})$ martingale and for each t, $\mathbf{X}(t)$ is bounded and $\mathcal{F}_t$-measurable,

$$\mathbb{F}^{\mathbf{X}} := \left\{\mathcal{F}_t^{\mathbf{X}}, t \in \mathcal{T}\right\}, \quad \mathcal{T} = [0, T]. \tag{1}$$

Let $\mathbf{W}_t = (W_1, W_2)$ be two standard Brownian motions which are mutually independent and also independent of $\mathbf{X}_t$ on the space $(\Omega, \mathcal{F}, \mathbb{Q})$ and given a general column vector c, define a stochastic process c_s as $c^\top \mathbf{X}_s$. Recall that the fundamental assumption is that the firm's total value V is not always observable. It can, however, be reflected in the decomposition of the equity value E and H, which is defined by the relation $V = E\,H$ or $H = V/E$. In our case, E is observable and tradable, and H is a ratio that could be explained in a stochastic differential equation, where the Brownian motion in E and H are correlated. In particular, the dynamics of E, H, and V under the risk-neutral measure $\mathbb{Q}$ evolve

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according to the following stochastic differential equations, where their parameters depend on the Markov chain $\mathbf{X}$:

$$dE = r(t)Edt + \sigma_E(t)EdW_1 \tag{2}$$
$$dH = \mu_H(t)Hdt + \sigma_H(t)H\left(\rho_{12}(t)dW_1 + \sqrt{1-\rho_{12}^2(t)}dW_2\right) \tag{3}$$
$$dV = r(t)Vdt + \sigma_1(t)VdW_1 + \sigma_2(t)VdW_2, \tag{4}$$

where $\mu_H(t) = -\rho_{12}(t)\sigma_E(t)\sigma_H(t),\ \sigma_1(t) = \sigma_E(t) + \rho_{12}(t)\sigma_H(t)$ and $\sigma_2(t) = \sigma_H(t)\sqrt{1-\rho_{12}^2(t)}$.

For $\sigma_E(t)$, which is the equity's volatility, we assume that there is a constant $N \times 1$ vector, $\sigma_E = [\sigma_{E,1}, \sigma_{E,2}, \ldots, \sigma_{E,N}]^\top$ such that $\sigma_E(t) = \sigma_E^\top \mathbf{X}_t$, and similarly for the parameters $\sigma_H(t)$ and $\rho_{12}(t)$ and sometimes $r(t)$ (see Sections 3.1 and 3.2).

Let the σ-algebras be

$$\mathcal{F}_1(t) = \sigma\{W_1(s) : s \le t\}\ ,\ \mathcal{F}_2(t) = \sigma\{W_2(s) : s \le t\} \text{ and } \mathcal{F}^{\mathbf{X}}(t) = \sigma\{\mathbf{X}(s) : s \le t\},$$

where and $\mathcal{F}^{\mathbf{X}}$ are assumed to be the right-continuous, $\mathbb{Q}$-complete, natural filtration generated by $\mathbf{X}$. (Note that $\mathbf{X}$ is càdlàg, and so is Riemann integrable, which implies that, $\int_0^T \mathbf{X}_{s-}ds = \int_0^T \mathbf{X}_{s+}ds = \int_0^T \mathbf{X}_s ds$.) Moreover, we define an enlarged filtration

$$\mathcal{F}^*(t, t_k) = \mathcal{F}_1(t) \vee \mathcal{F}_2(t_k), \tag{5}$$

where the notation " $\vee$ " represents the minimal σ-field containing $\mathcal{F}_1(t)$ and $\mathcal{F}_2(t_k)$ and also we know that $\mathcal{F}_1(t_k) \subseteq \mathcal{F}_1(t)$ as $t_k \le t$. We assume that the filtration given above satisfies the usual conditions.

Conditioning On $\mathbf{X}(t_k)$

We derive some technical results regarding the moment-generating function of $\ln V(T)$ conditional on $\mathcal{F}^*(t, t_k)$ and $\mathcal{F}^{\mathbf{X}}(t_k)$. Throughout this section, we assume that the risk-free rate is regime-switching. Then, in the following section, we price risky debt and consider two cases. In the first case, we assume that the risk-free rate is constant, but the state of the Markov chain is only observed at date t_k. Then, in the following subsection, we assume the risk-free rate is regime-switching, and the Markov chain is observed at any date t.

As discussed in the introduction, we assume that the value of the firm's equity, E, is observed continuously over time, but the total value of the firm, V, can only be observed at dates $t_1, t_2, ..., t_N$. For the sake of generality, we also assume that the state of the Markov chain, $\mathbf{X}$, is not observed continuously. For convenience, we assume that $\mathbf{X}$ is observed at date t_k, where $t_k \le t < t_{k+1}$. The intuition behind this assumption is that $\mathbf{X}$ is hard to observe directly, and so it may take some time before the value of $\mathbf{X}$ can be determined, a technique known as *smoothing*. An interesting extension of this approach would be to assume that some states of $\mathbf{X}$ can be observed continuously (e.g., states that affect E, or the stochastic interest rate, r), but other states can only be determined with a time delay (e.g., states affecting V.) We leave this problem for future research. In Section 3.2, we will assume that $\mathbf{X}$ is observed at any date t, which is the most common assumption made in the literature.

For the calculations, we rewrite the processes for E, H in differential form using Itô's lemma as follows

$$d\ln(E_t) = \left(r(t) - \frac{1}{2}\sigma_E^2(t)\right)dt + \sigma_E(t)dW_1(t) \tag{6}$$

and

$$d\ln(H_t) = \left(\mu_H(t) - \frac{1}{2}\sigma_H^2(t)\right)dt + \sigma_H(t)\left(\rho(t)dW_1(t) + \sqrt{1-\rho^2(t)}dW_2(t)\right). \tag{7}$$

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Before proceeding to the main result, we will state some frequently used lemmas. The proofs of these results will be provided in the appendix.

Lemma 2.1. We assume that $t_k \le t < t_{k+1} \le T$. Given constant $N \times 1$ matrices $\mathbf{a} = [a_1, \ldots, a_N]^\top$, $\mathbf{b} = [b_1, \ldots, b_N]^\top$, and $\mathbf{c} = [c_1, \ldots, c_N]^\top$, and defining the processes $a(t) = \mathbf{a}^\top \mathbf{X}_t, b(t) = \mathbf{b}^\top \mathbf{X}_t$, and $c(t) = \mathbf{c}^\top \mathbf{X}_t$, we have,

$$\begin{aligned}
&\mathbb{E}\left[e^{\int_{t_k}^T c_s ds + \int_{t_k}^T a_s dW_1(s) + \int_{t_k}^T b_s dW_2(s)} \,\middle|\, \mathcal{F}^{\mathbf{X}}(T), \mathcal{F}^*(t, t_k)\right] \\
&\quad = \exp\left\{\int_{t_k}^T c_s ds + \int_{t_k}^t a_s dW_1(s)\right\} \exp\left\{\frac{1}{2}\int_t^T a_s^2 ds + \frac{1}{2}\int_{t_k}^T b_s^2 ds\right\}.
\end{aligned}$$

See Appendix A.1 for proof of Lemma 2.1.

Remark 2.2. The main expectation we want to evaluate under regime switching is

$$\mathbb{E}\left[\mathbb{E}\left[e^{u \ln(V_T)} \,\middle|\, \mathcal{F}^*(t, t_k), \mathcal{F}^{\mathbf{X}}(T)\right] \,\middle|\, \mathcal{F}^{\mathbf{X}}(t_k)\right], \tag{8}$$

which we now derive.

First, note that

$$\begin{aligned}
&\mathbb{E}\left[e^{u \ln(V_T)} \,\middle|\, \mathcal{F}^*(t, t_k), \mathcal{F}^{\mathbf{X}}(T)\right] \\
&= \mathbb{E}\left[e^{u \ln(E_T) + u \ln(H_T)} \,\middle|\, \mathcal{F}^*(t, t_k), \mathcal{F}^{\mathbf{X}}(T)\right] \\
&= \exp\left\{u \ln(E_t) + u \ln(H_{t_k}) + u \int_t^T \left(r(s) - \frac{1}{2}\sigma_E^2(s)\right) ds\right\} \\
&\quad \times \exp\left\{u \int_{t_k}^T \left(\mu_H(s) - \frac{1}{2}\sigma_H^2(s)\right) ds + u \int_{t_k}^t \sigma_H(s)\rho(s) dW_1(s)\right\} \\
&\quad \times \exp\left\{\frac{1}{2}u^2 \int_t^T (\sigma_E(s) + \sigma_H(s)\rho(s))^2 ds + \frac{1}{2}u^2 \int_{t_k}^T \sigma_H^2(s)\left(1 - \rho^2(s)\right) ds\right\} \\
&= \exp\{u \ln(E_t) + u \ln(H_{t_k})\} \\
&\quad \times \exp\left\{u \int_t^T \left(r(s) - \frac{1}{2}\sigma_E^2(s)\right) ds + u \int_t^T \left(\mu_H(s) - \frac{1}{2}\sigma_H^2(s)\right) ds\right\} \\
&\quad \times \exp\left\{\frac{1}{2}u^2 \int_t^T (\sigma_E(s) + \sigma_H(s)\rho(s))^2 ds + \frac{1}{2}u^2 \int_t^T \sigma_H^2(s)\left(1 - \rho^2(s)\right) ds\right\} \\
&\quad \times \exp\left\{u \int_{t_k}^t \left(\mu_H(s) - \frac{1}{2}\sigma_H^2(s)\right) ds + \frac{1}{2}u^2 \int_{t_k}^t \sigma_H^2(s)\left(1 - \rho^2(s)\right) ds\right\} \\
&\quad \times \exp\left\{u \int_{t_k}^t \sigma_H(s)\rho(s) dW_1(s)\right\}.
\end{aligned} \tag{9}$$

Here, the u^2 terms follow from Equations (6) and (7). Next, we need to discuss some properties of the integral $\int_{t_k}^t \sigma_H(s)\rho(s) dW_1(s)$. From Equation (6), note that

$$dW_1(t) = \frac{1}{\sigma_E(t)}\left(d \ln(E_t) - \left(r(t) - \frac{1}{2}\sigma_E^2(t)\right) dt\right) \tag{10}$$

which implies that

$$\begin{aligned}
\int_{t_k}^t \sigma_H(s)\rho(s) dW_1(s) &= \int_{t_k}^t \frac{\sigma_H(s)\rho(s)}{\sigma_E(s)}\left(d \ln(E_s) - \left(r(s) - \frac{1}{2}\sigma_E^2(s)\right) ds\right) \\
&= \int_{t_k}^t \frac{\sigma_H(s)\rho(s)}{\sigma_E(s)} d \ln(E_s) - \int_{t_k}^t \frac{\sigma_H(s)\rho(s)}{\sigma_E(s)}\left(r(s) - \frac{1}{2}\sigma_E^2(s)\right) ds.
\end{aligned} \tag{11}$$

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Let

$$\frac{\sigma_H(t)\rho(t)}{\sigma_E(t)} := \beta_{EH}(t), \tag{12}$$

which can be seen to be equal to $\frac{\sigma_H^i \rho^i}{\sigma_E^i} 1_{\{\mathbf{X}_t = \mathbf{e}_i\}} =: \beta^i 1_{\{\mathbf{X}_i = \mathbf{e}_i\}} = \beta_{EH}^\top \mathbf{X}_t$, where

$$\beta_{EH}^\top = \left(\beta^1, \ldots, \beta^K\right), \tag{13}$$

with

$$\beta^i := \frac{\sigma_H^i \rho^i}{\sigma_E^i}. \tag{14}$$

So,

$$u \int_{t_k}^t \sigma_H(s)\,\rho(s)\,dW_1(s) = u \int_{t_k}^t \beta_{EH}(s)\,d\ln(E_s) \;-\; u \int_{t_k}^t \beta_{EH}(s)\left(r(s) - \frac{1}{2}\sigma_E^2(s)\right)ds. \tag{15}$$

Remark 2.3. For the distribution of the integral, $\int_{t_k}^t \beta_{EH}(s)d\ln(E_s)$, we need to stress our assumption regarding when the state of the Markov chain, $\mathbf{X}$, is observed or not observed. If we assume that $\mathbf{X}$ is observed continuously, then at date t, this integral is non-random. However, for the sake of generality, in this section, we assume that E is observed throughout the interval $[t_k, t]$, but $\mathbf{X}$ is stochastic over this interval; that is, we want to find conditional expectations given $\mathcal{F}^*(t, t_k) \vee \mathcal{F}^X(t_k)$.

To understand the distribution of the integral $\int_{t_k}^t \beta_{EH}(s)d\ln(E_s)$, we can use integration by parts:

$$\begin{aligned}
u\int_{t_k}^t \beta_{EH}(s)d\ln(E_s) =& u\beta_{EH}(t)\ln(E_t) - u\beta_{EH}(t_k)\ln(E_{t_k}) - u\int_{t_k}^t \ln(E_s)d\beta_{EH}(s)\\
&= u\ln(E_t)\beta_{EH}^\top \mathbf{X}_t - u\ln(E_{t_k})\beta_{EH}^T \mathbf{X}_{t_k} - u\int_{t_k}^t \ln(E_s)d\beta_{EH}(s)\\
&= u\ln(E_t)\beta_{EH}^\top \int_{t_k}^t d\mathbf{X}_s + u\ln(E_t)\beta_{EH}^\top \mathbf{X}_{t_k} - u\ln(E_{t_k})\beta_{EH}^\top \mathbf{X}_{t_k}\\
&\qquad - u\int_{t_k}^t \ln(E_s)d\beta_{EH}(s)\\
&= u\ln\left(\frac{E_t}{E_{t_k}}\right)\beta_{EH}^\top \mathbf{X}_{t_k} + u\ln(E_t)\beta_{EH}^\top \int_{t_k}^t d\mathbf{X}_s - u\int_{t_k}^t \ln(E_s)d\beta_{EH}(s),
\end{aligned} \tag{16}$$

where we can define $\int_{t_k}^t \ln(E_s)d\beta_{EH}(s)$ as

$$u\int_{t_k}^t \ln(E_s)d\beta_{EH}(s) = u\int_{t_k}^t \ln(E_s)\beta_{EH}^\top d\mathbf{X}_s.$$

So, we have

$$\begin{aligned}
u\int_{t_k}^t \beta_{EH}(s)d\ln(E_s) =& u\ln\left(\frac{E_t}{E_{t_k}}\right)\beta_{EH}^\top \mathbf{X}_{t_k} + u\ln(E_t)\beta_{EH}^\top \int_{t_k}^t d\mathbf{X}_s - u\int_{t_k}^t \ln(E_s)\beta_{EH}^\top d\mathbf{X}_s\\
&= u\ln\left(\frac{E_t}{E_{t_k}}\right)\beta_{EH}^\top \mathbf{X}_{t_k} + u\int_{t_k}^t \ln\left(\frac{E_t}{E_s}\right)\beta_{EH}^\top d\mathbf{X}_s
\end{aligned} \tag{17}$$

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Substituting Equation (17) into Equation (15) and then adding it to Equation (9), we have shown that

$$\begin{aligned}\mathbb{E}\left[e^{u\ln(V_T)}\,\middle|\,\mathcal{F}^*(t,t_k),\mathcal{F}^{\mathbf{X}}(T)\right] &= \exp\left\{u\ln(E_t)+u\ln(H_{t_k})+u\ln\left(\frac{E_t}{E_{t_k}}\right)\beta_{EH}^T\mathbf{X}_{t_k}\right\}\\ &\times\exp\left\{u\int_t^T c_1(s)ds+u^2\int_t^T c_2(s)ds\right\}\\ &\times\exp\left\{u\int_{t_k}^t b_1(s)ds+u^2\int_{t_k}^t b_2(s)ds+u\int_{t_k}^t\ln\left(\frac{E_t}{E_s}\right)\beta_{EH}^T d\mathbf{X}_s\right\},\end{aligned} \tag{18}$$

where

$$\begin{aligned} c_1(s) &= r(s)-\frac{1}{2}\sigma_E^2(s)+\mu_H(s)-\frac{1}{2}\sigma_H^2(s),\\ c_2(s) &= \frac{1}{2}(\sigma_E(s)+\sigma_H(s)\rho(s))^2+\frac{1}{2}\sigma_H^2(s)\left(1-\rho^2(s)\right),\\ b_1(s) &= \left(\mu_H(s)-\frac{1}{2}\sigma_H^2(s)\right)-\beta_{EH}(s)\left(r(s)-\frac{1}{2}\sigma_E^2(s)\right),\\ b_2(s) &= \frac{1}{2}\sigma_H^2(s)\left(1-\rho^2(s)\right),\end{aligned}$$

and $\mu_H = -2\sigma_H\sigma_E$.

We define the process from $t_k \le s \le t$ by U_1 and from $t \le s \le T$ by U_2. We arrange these depending on the timeline. We find U_2 first, as it is easier to derive than U_1.

Theorem 2.4. Consider the process, for $t \le s \le T$,

$$U_2(t,T) = \exp\left\{\int_t^T \tilde{b}(s)ds\right\}, \tag{19}$$

where $\tilde{b}(s) = \tilde{\mathbf{b}}^\top\mathbf{X}_s$ with $\tilde{b}$ is equal to a constant $N\times 1$ vector. Then,

$$\mathbb{E}\left[U_2(t,T)\,\middle|\,\mathcal{F}^{\mathbf{X}}(t)\right] = U_2(t,t)\mathbf{1}^\top\exp\{(\text{diag}(\tilde{\mathbf{b}})+\mathbf{A})(T-t)\}\mathbf{X}_t. \tag{20}$$

See Appendix A.1 for proof.

The next Lemma will be used in proving Theorem 2.6.

Lemma 2.5. Let $H(t,T)$ be an $N\times 1$ vector-valued process such that

$$H(t,T) = \exp\left\{\mathbf{C}(T-t)+\int_t^T \mathbf{B}(s)ds\right\}H(t,t),$$

where $\mathbf{B}$ and $\mathbf{C}$ are constant $N\times N$ matrices. Then,

$$H(t,T) = H(t,t)+\mathbf{C}\int_t^T H(t,s)ds+\int_t^T \mathbf{B}(s)H(t,s)ds.$$

The converse holds because the integral equation has a unique solution.

See Appendix A.1 for proof.

Recall that $\mathbf{B}$ and $\mathbf{C}$ are two $N\times N$ matrices. The *Hadamard product* of $\mathbf{B}$ and $\mathbf{C}$, denoted by $\mathbf{B}\circ\mathbf{C}$, is the $N\times N$ matrix having $(i,j)^{th}$ element equal to $B_{ij}C_{ij}$.

The next three results are needed in subsection 3.1.

Theorem 2.6. Consider the process, for $t_k \leq s \leq t$,

$$U_1(t_k,t) = \exp\left\{\int_{t_k}^{t} \tilde{c}(s)ds + \int_{t_k}^{t} u \ln\left(\frac{E_t}{E_s}\right)\beta_{EH}^{\top} d\mathbf{X}_s\right\}$$
$$= \exp\left\{\int_{t_k}^{t} \tilde{c}(s)ds + \int_{t_k}^{t} u \ln\left(\frac{E_t}{E_s}\right)\mathbf{X}_{s-}^{\top}\mathbf{B}d\mathbf{X}_s\right\}, \quad (21)$$

where $\tilde{c}(s) = \tilde{\mathbf{c}}^{\top}\mathbf{X}_s$ with $\tilde{c}$ is equal to a constant $N \times 1$ vector and $\mathbf{B}$ is $N \times N$ matrix

$$\mathbf{B} = \begin{bmatrix} \beta_{EH}^{\top} \\ \vdots \\ \beta_{EH}^{\top} \end{bmatrix} = \begin{bmatrix} \beta^1 & \cdots & \beta^K \\ \vdots & \ddots & \vdots \\ \beta^1 & \cdots & \beta^K \end{bmatrix}.$$

Then,

$$\mathbb{E}\left[U_1(t_k,t)\mathbf{X}_t \,\middle|\, \mathcal{F}^{\mathbf{X}}(t_k)\right] = \exp\{(\text{diag}(\tilde{\mathbf{c}}) + \mathbf{A})(t - t_k)\}\exp\left\{\int_{t_k}^{t}(\mathbf{J}(s)\circ\mathbf{A} + \text{diag}(\mathbf{J}(s)\mathbf{A}))ds\right\}U_1(t_k,t_k)\mathbf{X}_{t_k}, \quad (22)$$

where $U_1(t_k,t_k) = 1$, and $\mathbf{J}(s)$ is the $N \times N$ matrix with $(i,j)^{\text{th}}$ element $e^{u \ln E_s(\beta_j - \beta_i)} - 1$.

Proof. With $t_k \leq s \leq t$, when $\mathbf{X}_{s-} = \mathbf{e}_i$ and $\mathbf{X}_s = \mathbf{e}_j$, $\mathbf{X}_{s-}^{\top}\mathbf{B}d\mathbf{X}_s$ can be seen as,

$$\mathbf{X}_{s-}^{\top}\mathbf{B}d\mathbf{X}_s = \mathbf{e}_i^{\top}\mathbf{B}(\mathbf{e}_j - \mathbf{e}_i) = \mathbf{e}_i^{\top}\begin{bmatrix} \beta^j \\ \vdots \\ \beta^j \end{bmatrix} - \mathbf{e}_i^{\top}\begin{bmatrix} \beta^i \\ \vdots \\ \beta^i \end{bmatrix} = \beta^j - \beta^i, \text{ where } \beta^i \neq 0.$$

We want to find $d(U_1(t_k,s)\mathbf{X}_s)$ but first, we will find $dU_1(t_k,s)$

$$dU_1(t_k,s) = U_1(t_k,s-)\tilde{c}(s)ds + U_1(t_k,s) - U_1(t_k,s-). \quad (23)$$

In this case,

$$\Delta U_1(t_k,s) = U_1(t_k,s) - U_1(t_k,s-) = U_1(t_k,s-)\left(e^{u \ln E_s(\beta_j - \beta_i)} - 1\right). \quad (24)$$

Let $\mathbf{J}(s)$ be the $N \times N$ matrix with $(i,j)^{\text{th}}$ element $e^{u \ln E_s(\beta_j - \beta_i)} - 1$. Note that the $(i,i)^{\text{th}}$ element of $\mathbf{J}(s)$ is zero for all $i = 1,\ldots,N$. However, we first note that for $\mathbf{e}_i^{\top}\mathbf{J}(\mathbf{s})\mathbf{e}_j = (\mathbf{J}(s))_{ij}$, the $(i,j)^{th}$ element of $\mathbf{J}(s)$ with $(\mathbf{J}(s))_{ii} = 0$. Then we see that $\Delta U_1(t_k,s)$ becomes

$$\Delta U_1(t_k,s) = U_1(t_k,s-)\mathbf{X}_{s-}^{\top}\mathbf{J}(s)\Delta\mathbf{X}_s. \quad (25)$$

Writing U^c for the continuous part of U_1, when $t_k \leq s \leq t$, we have,

$$d(U_1(t_k,s)\mathbf{X}_s) = U_1(t_k,s-)d\mathbf{X}_s + \mathbf{X}_{s-}dU_1^c(t_k,s) + \Delta U_1(t_k,s)\Delta\mathbf{X}_s. \quad (26)$$

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Next, $\Delta U_1(t_k,s)\Delta \mathbf{X}_s$ can be evaluated as

$$\begin{aligned}
\Delta U_1(t_k,s)\Delta \mathbf{X}_s &= \Big(U_1(t_k,s-)\mathbf{X}_{s-}^\top \mathbf{J}(s)\Delta \mathbf{X}_s\Big)\Delta \mathbf{X}_s \\
&= U_1(t_k,s-)\mathbf{e}_i^\top \mathbf{J}(s)(\mathbf{e}_j-\mathbf{e}_i)(\mathbf{e}_j-\mathbf{e}_i) \\
&= U_1(t_k,s-)\mathbf{e}_i^\top \mathbf{J}(s)\mathbf{e}_j(\mathbf{e}_j-\mathbf{e}_i) - U_1(t_k,s-)\mathbf{e}_i^\top \mathbf{J}(s)\mathbf{e}_i(\mathbf{e}_j-\mathbf{e}_i) \\
&= U_1(t_k,s-)\mathbf{e}_i^\top \mathbf{J}(s)\mathbf{e}_j(\mathbf{e}_j-\mathbf{e}_i) \\
&= U_1(t_k,s-)(\mathbf{J}(s))_{ij}(\mathbf{e}_j-\mathbf{e}_i) \\
&= U_1(t_k,s-)(\mathbf{J}(s))_{ij}\mathbf{e}_j - U_{s-}(\mathbf{J}(s))_{ij}\mathbf{e}_i \\
&= U_1(t_k,s-)\operatorname{diag}\Big(\mathbf{e}_i^\top \mathbf{J}(s)\Big)(\mathbf{e}_j-\mathbf{e}_i) - U_1(t_k,s-)\mathbf{e}_i\mathbf{e}_i^\top \mathbf{J}(s)(\mathbf{e}_j-\mathbf{e}_i) \\
&= U_1(t_k,s-)\operatorname{diag}\Big(\mathbf{X}_{s-}^\top \mathbf{J}(s)\Big)d\mathbf{X}_s - U_1(t_k,s-)\mathbf{X}_{s-}\mathbf{X}_{s-}^\top \mathbf{J}(s)d\mathbf{X}_s,
\end{aligned} \tag{27}$$

where $U_1(t_k,s-)\mathbf{e}_i^\top \mathbf{J}(s)\mathbf{e}_i(\mathbf{e}_j-\mathbf{e}_i)$ disappears as $(J(s))_{ii}=0$. Also, the seventh equality holds by the fact that $\operatorname{diag}(\mathbf{e}_i^\top \mathbf{J}(s)) = \operatorname{diag}(\mathbf{J}(s)_{i,\square}) = \begin{bmatrix} J(s)_{i,1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & J(s)_{i,K} \end{bmatrix}$.

Before deriving $d(U_1(t_k,s)\mathbf{X}_s)$, we should introduce the following lemma, which defines the compensator of these jump processes.

Lemma 2.7. 1. The process $\int_0^t U_1(t_k,s-)\operatorname{diag}(\mathbf{X}_{s-}^\top \mathbf{J}(s))d\mathbf{X}_s$ has compensator $\int_0^t U_1(t_k,s-)\mathbf{J}(s)\circ \mathbf{A}\mathbf{X}_{s-}ds$, where $\circ$ is the Hadamard product and
2. $\int_0^t U_1(t_k,s-)\mathbf{X}_{s-}\mathbf{X}_{s-}^\top \mathbf{J}(s)d\mathbf{X}_s$ has compensator $\int_{t_k}^t U_1(t_k,s-)\operatorname{diag}(\mathbf{J}(s)\mathbf{A})\mathbf{X}_{s-}ds$.

(See Section 3.4 of [17] for the proof of this result.)

Given the above compensators, we can now write,

$$\begin{aligned}
&d(U_1(t_k,s)\mathbf{X}_s) = U_1(t_k,s-)d\mathbf{X}_s + \mathbf{X}_{s-}dU_1^c(t_k,s) + \Delta U_1(t_k,s)\Delta \mathbf{X}_s \\
&= U_1(t_k,s-)d\mathbf{X}_s + \mathbf{X}_{s-}U_1(t_k,s-)\tilde{\mathbf{c}}^\top \mathbf{X}_{s-}ds + U_1(t_k,s-)\operatorname{diag}\Big(\mathbf{X}_{s-}^\top \mathbf{J}(s)\Big)d\mathbf{X}_s \\
&- U_1(t_k,s-)\mathbf{X}_{s-}\mathbf{X}_{s-}^\top \mathbf{J}(s)d\mathbf{X}_s \\
&= U_1(t_k,s-)d\mathbf{X}_s - U_1(t_k,s-)\mathbf{A}\mathbf{X}_s ds + U_1(t_k,s-)\operatorname{diag}(\tilde{\mathbf{c}})\mathbf{X}_{s-}ds \\
&+ U_1(t_k,s-)\operatorname{diag}\Big(\mathbf{X}_{s-}^\top \mathbf{J}(s)\Big)d\mathbf{X}_s - U_1(t_k,s-)\mathbf{J}(s)\circ \mathbf{A}\mathbf{X}_{s-}ds - U_1(t_k,s-)\mathbf{X}_{s-}\mathbf{X}_{s-}^\top \mathbf{J}(s)d\mathbf{X}_s \\
&+ U_1(t_k,s-)\operatorname{diag}(\mathbf{J}(s)\mathbf{A})\mathbf{X}_{s-}ds + U_1(t_k,s-)\mathbf{A}\mathbf{X}_s ds \\
&+ U_1(t_k,s-)\mathbf{J}(s)\circ \mathbf{A}\mathbf{X}_{s-}ds - U_1(t_k,s-)\operatorname{diag}(\mathbf{J}(s)\mathbf{A})\mathbf{X}_{s-}ds \\
&= U_1(t_k,s-)(d\mathbf{X}_s - \mathbf{A}\mathbf{X}_s ds) + U_1(t_k,s-)\operatorname{diag}(\tilde{\mathbf{c}})\mathbf{X}_{s-}ds \\
&+ U_1(t_k,s-)\Big(\operatorname{diag}\Big(\mathbf{X}_{s-}^\top \mathbf{J}(s)\Big)d\mathbf{X}_s - \mathbf{J}(s)\circ \mathbf{A}\mathbf{X}_{s-}ds\Big) \\
&- U_1(t_k,s-)\Big(\mathbf{X}_{s-}\mathbf{X}_{s-}^\top \mathbf{J}(s)d\mathbf{X}_s - \operatorname{diag}(\mathbf{J}(s)\mathbf{A})\mathbf{X}_{s-}ds\Big) + U_1(t_k,s-)\mathbf{A}\mathbf{X}_s ds \\
&+ U_1(t_k,s-)\mathbf{J}(s)\circ \mathbf{A}\mathbf{X}_{s-}ds - U_1(t_k,s-)\operatorname{diag}(\mathbf{J}(s)\mathbf{A})\mathbf{X}_{s-}ds \\
&= U_1(t_k,s-)dM_s^1 + U_1(t_k,s-)\operatorname{diag}(\tilde{\mathbf{c}})\mathbf{X}_{s-}ds + U_1(t_k,s-)dM_s^2 - U_1(t_k,s-)dM_s^3 \\
&+ U_1(t_k,s-)\mathbf{A}\mathbf{X}_s ds + U_1(t_k,s-)\mathbf{J}(s)\circ \mathbf{A}\mathbf{X}_{s-}ds - U_1(t_k,s-)\operatorname{diag}(\mathbf{J}(s)\mathbf{A})\mathbf{X}_{s-}ds,
\end{aligned} \tag{28}$$

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where M^1, M^2 and M^3 are martingales. In integral form, this can be written.

$$\begin{aligned} U_1(t_k,t)\mathbf{X}_t &= U_1(t_k,t_k)\mathbf{X}_{t_k} + \int_{t_k}^t U_1(t_k,s-)dM_s^1 + \int_{t_k}^t U_1(t_k,s-)\,\mathrm{diag}(\mathbf{c})\mathbf{X}_{s-}ds \\ &\quad + \int_{t_k}^t U_1(t_k,s-)dM_s^2 - \int_{t_k}^t U_1(t_k,s-)dM_s^3 + \int_{t_k}^t U_1(t_k,s-)\mathbf{A}\mathbf{X}_s ds \\ &\quad + \int_{t_k}^t U_1(t_k,s-)\mathbf{J}(s)\circ\mathbf{A}\mathbf{X}_{s-}ds - \int_{t_k}^t U_1(t_k,s-)\,\mathrm{diag}(\mathbf{J}(s)\mathbf{A})\mathbf{X}_{s-}ds. \end{aligned}$$

Taking the expected value, it follows that

$$\begin{aligned} \mathbb{E}\Big[U_1(t_k,t)\mathbf{X}_t \,\Big|\, \mathcal{F}^{\mathbf{X}}(t_k)\Big] &= U_1(t_k,t_k)\mathbf{X}_{t_k} + \mathrm{diag}(\mathbf{c})\int_{t_k}^t \mathbb{E}\Big[U_1(t_k,s)\mathbf{X}_s \,\Big|\, \mathcal{F}^{\mathbf{X}}(t_k)\Big]ds \\ &\quad + \mathbf{A}\int_{t_k}^t \mathbb{E}\Big[U_1(t_k,s)\mathbf{X}_s \,\Big|\, \mathcal{F}^{\mathbf{X}}(t_k)\Big]ds \\ &\quad + \int_{t_k}^t \mathbf{J}(s)\circ\mathbf{A}\mathbf{E}\Big[U_1(t_k,s)\mathbf{X}_s \,\Big|\, \mathcal{F}^{\mathbf{X}}(t_k)\Big]ds \\ &\quad - \int_{t_k}^t \mathrm{diag}(\mathbf{J}(s)\mathbf{A})\mathbb{E}\Big[U_1(t_k,s)\mathbf{X}_s \,\Big|\, \mathcal{F}^{\mathbf{X}}(t_k)\Big]ds. \end{aligned} \tag{29}$$

Using Lemma 2.5 with $H(t_k,t) = \mathbb{E}\Big[U_1(t_k,t)\mathbf{X}_t \,\Big|\, \mathcal{F}^{\mathbf{X}}(t_k)\Big]$ to Equation (29) give the desire result. □

Then Equation (18) can be written as

$$\mathbb{E}\Big[e^{u\ln(V_T)} \,\Big|\, \mathcal{F}^*(t,t_k), \mathcal{F}^{\mathbf{X}}(T)\Big] = \exp\Big\{u\ln(E_t) + u\ln(H_{t_k}) + u\ln\Big(\frac{E_t}{E_{t_k}}\Big)\beta_{EH}^T\mathbf{X}_{t_k}\Big\}U_1(t_k,t)U_2(t,T). \tag{30}$$

Now our main expectation in Equation (8) is presented in the following theorem.

Theorem 2.8. Using the tower property, we can prove the following result

$$\begin{aligned} &\mathbb{E}\Big[\mathbb{E}\Big[e^{u\ln(V_T)} \,\Big|\, \mathcal{F}^*(t,t_k), \mathcal{F}^{\mathbf{X}}(T)\Big] \,\Big|\, \mathcal{F}^{\mathbf{X}}(t_k)\Big] \\ = &\ \mathbf{1}^\top \exp\Big\{u\ln(E_t) + u\ln(H_{t_k}) + u\ln\Big(\frac{E_t}{E_{t_k}}\Big)\beta_{EH}^T\mathbf{X}_{t_k}\Big\} \\ &\times \exp\{(\mathrm{diag}(\tilde{\mathbf{b}}) + \mathbf{A})(T-t)\} \\ &\times \exp\Big\{(\mathrm{diag}(\tilde{\mathbf{c}}) + \mathbf{A})(t-t_k)\Big\} \\ &\times \exp\Big\{\int_{t_k}^t (\mathbf{J}(s)\circ\mathbf{A} + \mathrm{diag}(\mathbf{J}(s)\mathbf{A}))ds\Big\}\mathbf{X}_{t_k}. \end{aligned} \tag{31}$$

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Proof. We have,

$$
\begin{aligned}
&\mathbb{E}\left[\mathbb{E}\left[e^{u\ln(V_T)}\,\middle|\,\mathcal{F}^*(t,t_k),\mathcal{F}^{\mathbf{X}}(T)\right]\,\middle|\,\mathcal{F}^{\mathbf{X}}(t_k)\right]\\
&=\mathbb{E}\left[\exp\left\{u\ln(E_t)+u\ln(H_{t_k})+u\ln\left(\frac{E_t}{E_{t_k}}\right)\beta_{EH}^T\mathbf{X}_{t_k}\right\}U_1(t_k,t)U_2(t,T)\mathbf{X}_T\,\middle|\,\mathcal{F}^{\mathbf{X}}(t_k)\right]\\
&=\exp\left\{u\ln(E_t)+u\ln(H_{t_k})+u\ln\left(\frac{E_t}{E_{t_k}}\right)\beta_{EH}^T\mathbf{X}_{t_k}\right\}\mathbb{E}\left[U_1(t_k,t)U_2(t,T)\mathbf{X}_T\,\middle|\,\mathcal{F}^{\mathbf{X}}(t_k)\right]\\
&=\exp\left\{u\ln(E_t)+u\ln(H_{t_k})+u\ln\left(\frac{E_t}{E_{t_k}}\right)\beta_{EH}^T\mathbf{X}_{t_k}\right\}\mathbb{E}\left[\mathbb{E}\left[U_1(t_k,t)U_2(t,T)\mathbf{X}_T\,\middle|\,\mathcal{F}^{\mathbf{X}}(t)\right]\,\middle|\,\mathcal{F}^{\mathbf{X}}(t_k)\right]\\
&=\exp\left\{u\ln(E_t)+u\ln(H_{t_k})+u\ln\left(\frac{E_t}{E_{t_k}}\right)\beta_{EH}^T\mathbf{X}_{t_k}\right\}\mathbb{E}\left[U_1(t_k,t)\mathbb{E}\left[U_2(t,T)\mathbf{X}_T\,\middle|\,\mathcal{F}^{\mathbf{X}}(t)\right]\,\middle|\,\mathcal{F}^{\mathbf{X}}(t_k)\right]\\
&=\exp\left\{u\ln(E_t)+u\ln(H_{t_k})+u\ln\left(\frac{E_t}{E_{t_k}}\right)\beta_{EH}^T\mathbf{X}_{t_k}\right\}\exp\{(\mathrm{diag}(\tilde{\mathbf{b}})+\mathbf{A})(T-t)\}\\
&\qquad\times\mathbb{E}\left[U_1(t_k,t)U_2(t,t)\mathbf{X}_t\,\middle|\,\mathcal{F}^{\mathbf{X}}(t_k)\right]\\
&=\exp\left\{u\ln(E_t)+u\ln(H_{t_k})+u\ln\left(\frac{E_t}{E_{t_k}}\right)\beta_{EH}^T\mathbf{X}_{t_k}\right\}\\
&\qquad\times\exp\{(\mathrm{diag}(\tilde{\mathbf{b}})+\mathbf{A})(T-t)\}\mathbb{E}\left[U_1(t_k,t)\mathbf{X}_t\,\middle|\,\mathcal{F}^{\mathbf{X}}(t_k)\right]\\
&=\mathbf{1}^\top\exp\left\{u\ln(E_t)+u\ln(H_{t_k})+u\ln\left(\frac{E_t}{E_{t_k}}\right)\beta_{EH}^T\mathbf{X}_{t_k}\right\}\\
&\qquad\times\exp\{(\mathrm{diag}(\tilde{\mathbf{b}})+\mathbf{A})(T-t)\}\exp\left\{(\mathrm{diag}(\tilde{\mathbf{c}})+\mathbf{A})(t-t_k)\right\}\\
&\qquad\times\exp\left\{\int_{t_k}^{t}(\mathbf{J}(s)\circ\mathbf{A}+\mathrm{diag}(\mathbf{J}(s)\mathbf{A})ds\right\}\mathbf{X}_{t_k}.
\end{aligned}
$$

We use Theorem 2.4 and Theorem 2.6 as well as use power property in the third equality and in the sixth equality using the fact that $U_2(t,t)=1$. □

Now, in the next section, we want to price the risky debt under regime switching with imperfect information.

3. Risky debt under regime switching model

In this section, we present the price of a defaultable zero-coupon bond with regime-switching in the structural form of credit risk modelling. We model the firm's value, V, and its decomposition into H and E by dependent regime-switching processes, in which the Markov chain represents the states of the economy. Although regime-switching models have been widely used in credit risk modelling, it mainly focuses on the reduced-form models, and little research discusses the structural model with defaultable bonds.

The literature in this area is still maturing, and people continue to work on this problem. [14] extend [2] model as their dynamic of the firm value consists of a Markov-modulated generalized jump-diffusion model where the jumps component is described by a completely random measure, in which the jump sizes and jump times can be correlated. In addition, they allow for a Markov-switching compensator that switches over time as modelled by a continuous-time Markov chain according to the states of the economy.

For our calculations, we define the following σ-algebras

$$\mathcal{F}_1(t)=\sigma\{W_1(s):s\leq t\}\text{ and }\mathcal{F}_2(t)=\sigma\{W_2(s):s\leq t\}\text{ and }\mathcal{F}^{\mathbf{X}}(t)=\sigma\{\mathbf{X}(s):s\leq t\},$$

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where and $\mathcal{F}^{\mathbf{X}}$ is the right-continuous, $\mathbb{Q}$-complete, natural filtration generated by $\mathbf{X}$. Define an enlarged filtration $\mathcal{F}^*(t,t_k) = \mathcal{F}_1(t) \vee \mathcal{F}_2(t_k)$ where the notation " $\vee$ " represents the minimal σ-field containing $\mathcal{F}_1(t)$ and $\mathcal{F}_2(t)$. Also, we know that $\mathcal{F}_i(t_k) \subseteq \mathcal{F}_i(t)$, $i = 1,2$ and $\mathcal{F}^{\mathbf{X}}(t_k) \subseteq \mathcal{F}^{\mathbf{X}}(t)$, as $t_k \leq t$. We assume that these filtrations satisfy the usual conditions.

3.1. Constant Interest Rate

We now price risky debt in a regime-switching framework; i.e., the parameters for the processes, V, E, and H are regime-switching, according to the Markov chain, $\mathbf{X}$. Here, V represents the value of the firm, E is the market value of the firm's equity, and $V = HE$, as usual.

In this sub-section, we assume that the state of the Markov chain is only observed at certain dates. Therefore, we assume that the risk-free rate is constant; otherwise, the interest rate would not be continuously observable, in general. The technical results that we will mainly use are Lemma 2.1 and the moment generating function in Theorem 2.8.

Recall that state of the Markov chain, $\mathbf{X}$, is not observed continuously. For convenience, we assume that $\mathbf{X}$ is observed at date t_k, where $t_k \leq t < t_{k+1} \leq t_N$, So, as in Chapter 4, the price of a defaultable zero-coupon with face value \$1, but now conditional on $\mathbf{X}(t_k)$, is given by

$$\begin{aligned}
P_t(E,H,V) &= \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-t)}\left(1_{\{V(T)\geq D^*\}} + 1_{\{V(T)<D^*\}}.\frac{(1-\alpha)}{D}V(T)\right) \,\middle|\, \mathcal{F}^*(t,t_k), \mathcal{F}^{\mathbf{X}}(t_k)\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-t)}1_{\{V(T)\geq D^*\}} \,\middle|\, \mathcal{F}^*(t,t_k), \mathcal{F}^{\mathbf{X}}(t_k)\right] \\
&\quad + \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-t)}1_{\{V(T)<D^*\}}\frac{(1-\alpha)}{D}V(T) \,\middle|\, \mathcal{F}^*(t,t_k), \mathcal{F}^{\mathbf{X}}(t_k)\right] \\
&= \mathrm{e}^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[1_{\{V(T)\geq D^*\}} \,\middle|\, \mathcal{F}^*(t,t_k), \mathcal{F}^{\mathbf{X}}(t_k)\right] \\
&\quad + \frac{(1-\alpha)}{D}\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-t)}1_{\{V(T)<D^*\}}V(T) \,\middle|\, \mathcal{F}^*(t,t_k), \mathcal{F}^{\mathbf{X}}(t_k)\right],
\end{aligned} \tag{32}$$

where

- D^* is a constant default boundary such that a credit loss occurs if the value of the option writer's assets $V(T) < D^*$.
- D is the value of total liabilities given by D^* plus an additional liability as there is a possibility of a counter-party keeping operation even while $V(T) < D^*$, $D = D^* +$ additional liability.
- α is the deadweight cost related to the bankruptcy of the firm, expressed as a percentage of $V(T)$.

The entire claim is paid out when $V(T) \geq D^*$. However, if the default occurs, only a fraction $\frac{(1-\alpha)V(T)}{D}$ of the claim is paid out, where $\frac{V(T)}{D}$ is the ratio representing the value of the firm which are available to pay the claim.

For this expectation $\mathbb{E}^{\mathbb{Q}}\left[1_{\{V(T)\geq D^*\}} \,\middle|\, \mathcal{F}^*(t,t_k), \mathcal{F}^{\mathbf{X}}(t_k)\right]$ we need the characteristic function under the probability measure $\mathbb{Q}$ which is introduced in Theorem 2.8.
Now, for

$$\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-t)}1_{\{V(T)<D^*\}}V(T) \,\middle|\, \mathcal{F}^*(t,t_k), \mathcal{F}^{\mathbf{X}}(t_k)\right] \tag{33}$$

As mentioned in the introduction to this chapter, recall that V is not a discrete-time stochastic process. The firm's owners observe V continuously over time, they just don't allow outsiders to observe it, and so outsiders only observe it on specific dates. Thus, V is defined for all t. It's just that for pricing we have to condition our expectations on information at date t_k.

Define the following equivalent probability measure at time t

$$\frac{dQ^V}{dQ} = \frac{e^{-r(T-t)}V(T)}{V(t)}$$

or at time zero

$$\frac{dQ^V}{dQ} = \frac{e^{-r(T)}V(T)}{V(0)}.$$

Now, using the abstract Bayes' Theorem to evaluate the conditional expectation in Equation (33), we have

$$\mathbb{E}^V\left[1_{\{V(T)<D^*\}}\,\middle|\,\mathcal{F}^*(t,t_k),\mathcal{F}^{\mathbf{X}}(t_k)\right] = \frac{\mathbb{E}\left[\mathrm{e}^{-r(T-t)}1_{\{V(T)<D^*\}}V(T)\,\middle|\,\mathcal{F}^*(t,t_k),\mathcal{F}^{\mathbf{X}}(t_k)\right]}{\mathbb{E}\left[\mathrm{e}^{-r(T-t)}V(T)\,\middle|\,\mathcal{F}^*(t,t_k),\mathcal{F}^{\mathbf{X}}(t_k)\right]}, \tag{34}$$

and we see that

$$\begin{aligned}
&\mathbb{E}\left[\mathrm{e}^{-r(T-t)}1_{\{V(T)<D^*\}}V(T)\,\middle|\,\mathcal{F}^*(t,t_k),\mathcal{F}^{\mathbf{X}}(t_k)\right]\\
=\;&\mathbb{E}\left[\mathrm{e}^{-r(T-t)}V(T)\,\middle|\,\mathcal{F}^*(t,t_k),\mathcal{F}^{\mathbf{X}}(t_k)\right]\\
&\times\mathbb{Q}^V\left\{V(T)<D^*\,\middle|\,\mathcal{F}^*(t,t_k),\mathcal{F}^{\mathbf{X}}(t_k)\right\},
\end{aligned} \tag{35}$$

where the $V(t)$ terms have cancelled. Then Equation (35) becomes

$$\begin{aligned}
=&\mathbf{1}^\top\frac{E_t}{E_{t_k}}V_{t_k}\exp\left\{\ln\left(\frac{E_t}{E_{t_k}}\right)\beta_{EH}^T\mathbf{X}_{t_k}\right\}\exp\{(\mathrm{diag}(\tilde{\mathbf{b}})+\mathbf{A})(T-t)\}\exp\left\{(\mathrm{diag}(\tilde{\mathbf{c}})+\mathbf{A})(t-t_k)\right\}\\
&\times\exp\left\{\int_{t_k}^t(\mathbf{J}(s)\circ\mathbf{A}+\mathrm{diag}(\mathbf{J}(s)\mathbf{A})ds\right\}\mathbf{X}_{t_k}\mathbb{Q}^V\left\{V(T)<D^*\,\middle|\,\mathcal{F}^*(t,t_k),\mathcal{F}^{\mathbf{X}}(t_k)\right\},
\end{aligned} \tag{36}$$

where

- $\tilde{b}(s) = -\frac{1}{2}\sigma_E^2(s)+\mu_H(s)+\frac{1}{2}(\sigma_E(s)+\sigma_H(s)\rho(s))^2-\frac{1}{2}\sigma_H^2(s)\rho^2(s)$
- $\tilde{c}(s) = \left(\mu_H(s)-\frac{1}{2}\sigma_H^2(s)\right)-\beta_{EH}(s)\left(r-\frac{1}{2}\sigma_E^2(s)\right)+\frac{1}{2}\sigma_H^2(s)(1-\rho^2(s))$
- $J_{ij}(s) = \mathrm{e}^{\ln E_s(\beta_j-\beta_i)}-1$ as provided by Equation (24).

We have to evaluate $\mathbb{Q}^V\left(V(T)<D^*\,\middle|\,\mathcal{F}^*(t,t_k),\mathcal{F}^{\mathbf{X}}(t_k)\right)$. We need the characteristic function of V under the probability measure $\mathbb{Q}^V$. The moment generating function under this measure, which we denote by $\phi^V(u)$, is as follows:

$$\begin{aligned}
\phi^V(u) &= E^{\mathbb{Q}^V}\left[\mathrm{e}^{u\ln V(T)}\,\middle|\,\mathcal{F}^*(t,t_k),\mathcal{F}^{\mathbf{X}}(t_k)\right]\\
&= \frac{E^{\mathbb{Q}}\left[\mathrm{e}^{u\ln V(T)}\mathrm{e}^{-rT}V(T)\,\middle|\,\mathcal{F}^*(t,t_k),\mathcal{F}^{\mathbf{X}}(t_k)\right]}{E^{\mathbb{Q}}\left[\mathrm{e}^{-rT}V(T)\,\middle|\,\mathcal{F}^*(t,t_k),\mathcal{F}^{\mathbf{X}}(t_k)\right]},
\end{aligned} \tag{37}$$

where

$$
\begin{aligned}
& E^{\mathbb{Q}}\left[\mathrm{e}^{u \ln V(T)} \mathrm{e}^{-rT} V(T) \,\middle|\, \mathcal{F}^*(t,t_k), \mathcal{F}^{\mathbf{X}}(t_k)\right] \\
= \;& \mathbf{1}^\top \exp\left\{-rt + (u+1)\ln(E_t) + (u+1)\ln(H_{t_k}) + (u+1)\ln\left(\frac{E_t}{E_{t_k}}\right)\beta_{EH}^T \mathbf{X}_{t_k}\right\} \\
& \times \exp\{(\mathrm{diag}(\mathbf{b}') + \mathbf{A})(T-t)\} \exp\left\{(\mathrm{diag}(\mathbf{c}') + \mathbf{A})(t-t_k)\right\} \\
& \times \exp\left\{\int_{t_k}^{t} (\mathbf{J}^V(s) \circ \mathbf{A} + \mathrm{diag}(\mathbf{J}^V(s)\mathbf{A}) ds\right\} \mathbf{X}_{t_k}.
\end{aligned}
$$

We can evaluate by using Theorem 2.8 with the parameters defined in Equation (2.2)

- $b'(s) = (u+1)\left(-\frac{1}{2}\sigma_E^2(s) + \mu_H(s) - \frac{1}{2}\sigma_H^2(s)\right) + (u+1)^2 c_2(s)$,
- $c'(s) = (u+1)b_1(s) + (u+1)^2 b_2(s)$,
- $J_{ij}^V(s) = \mathrm{e}^{(u+1)\ln E_s(\beta_j - \beta_i)} - 1$,

and $E^{\mathbb{Q}}\left[\mathrm{e}^{-rT} V(T) \,\middle|\, \mathcal{F}^*(t,t_k), \mathcal{F}^{\mathbf{X}}(t_k)\right]$ following the same methodology with $u = 1$,

- $b'(s) = \left(-\frac{1}{2}\sigma_E^2(s) + \mu_H(s) - \frac{1}{2}\sigma_H^2(s)\right) + c_2(s)$,
- $c'(s) = b_1(s) + b_2(s)$,
- $J_{ij}^V(s) = \mathrm{e}^{\ln E_s(\beta_j - \beta_i)} - 1$.

The characteristic function is then just $\phi^V(iu)$, and once we evaluate the characteristic function, we can find the probability by using the inverse Fourier transform technique as discussed, for example, in [18], i.e.,

$$
P\{Y \geq a\} = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{e^{-iua}}{iu}\phi(u)\right) du. \tag{38}
$$

We will use this in Theorem 3.1 below.

Theorem 3.1. The price of risky debt under regime-switching with constant interest rate can be represented as the following

$$
\begin{aligned}
P_t(E,H,V) =& \mathrm{e}^{-r(T-t)} \mathbb{Q}\left(V(T) \geq D^* \,\middle|\, \mathcal{F}^*(t,t_k), \mathcal{F}^{\mathbf{X}}(t_k)\right) \\
& + \frac{E_t}{E_{t_k}} V_{t_k} \frac{(1-\alpha)}{D} \mathbf{1}^\top \exp\left\{-r(T-t) + \ln\left(\frac{E_t}{E_{t_k}}\right)\beta_{EH}^T \mathbf{X}_{t_k}\right\} \exp\{(\mathrm{diag}(\tilde{\mathbf{b}}) + \mathbf{A})(T-t)\} \\
& \times \exp\left\{(\mathrm{diag}(\tilde{\mathbf{c}}) + \mathbf{A})(t-t_k)\right\} \exp\left\{\int_{t_k}^{t} (\mathbf{J}(s) \circ \mathbf{A} + \mathrm{diag}(\mathbf{J}(s)\mathbf{A}) ds\right\} \mathbf{X}_{t_k} \\
& \times \mathbb{Q}^V\left(V(T) < D^* \,\middle|\, \mathcal{F}^*(t,t_k), \mathcal{F}^{\mathbf{X}}(t_k)\right),
\end{aligned} \tag{39}
$$

where $\tilde{\mathbf{b}}$, $\tilde{\mathbf{c}}$ and $\mathbf{J}$ are defined as in Equation (36) and these two probabilities are represented by

$$
\mathbb{Q}\left(\ln(V(T)) \geq \ln(D^*) \,\middle|\, \mathcal{F}^*(t,t_k), \mathcal{F}^{\mathbf{X}}(t_k)\right) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{e^{-iv\ln(D^*)}}{iv}\phi(iv)\right) dv
$$

$$
\mathbb{Q}^V\left(\ln(V(T)) < \ln(D^*) \,\middle|\, \mathcal{F}^*(t,t_k), \mathcal{F}^{\mathbf{X}}(t_k)\right) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{e^{-iv\ln(D^*)}}{iv}\phi^V(iv)\right) dv,
$$

and $\phi(iv)$ defined as in Theorem 2.8 and $\phi^V(iv)$ as in Equation (37).

3.2. Regime Switching Interest Rate

We now assume that the interest rate is regime-switching, that is, $r(t) = \mathbf{r}^\top \mathbf{X}(t)$ for a constant $N \times 1$ matrix, $\mathbf{r} = [r_1, \ldots, r_N]^\top$. So, we assume that we can observe when there is a regime-switch at any date. Loosely speaking, because E and r are observed continuously, we assume that $\mathbf{X}$ is, as well.

The price of a credit-risky bond is given by

$$\begin{aligned} P_t(E_t, H_{t_k}, V) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds} \left(1_{\{V(T) \geq D^*\}} + 1_{\{V(T) < D^*\}} \cdot \frac{(1-\alpha)}{D} V(T) \right) \middle| \mathcal{F}^*(t, t_k),\ \mathcal{F}^{\mathbf{X}}(t) \right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds} 1_{\{V(T) \geq D^*\}} \middle| \mathcal{F}^*(t, t_k), \mathcal{F}^{\mathbf{X}}(t) \right] \\ &\quad + \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds} 1_{\{V(T) < D^*\}} \cdot \frac{(1-\alpha)}{D} V(T) \middle| \mathcal{F}^*(t, t_k),\ \mathcal{F}^{\mathbf{X}}(t) \right] \end{aligned} \tag{40}$$

For solving $\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds} 1_{\{V(T) \geq D^*\}} \middle| \mathcal{F}^*(t, t_k), \mathcal{F}^{\mathbf{X}}(t) \right]$, define a new measure as:

$$\frac{dQ^B}{d\mathbb{Q}} = \frac{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds} \middle| \mathcal{F}^{\mathbf{X}}(t) \right]}{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r(s)ds} \right]}.$$

Let $B(t, T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds} \middle| \mathcal{F}^{\mathbf{X}}(t) \right]$ represent the price, as of date t, of a risk-free bond that pays \$1 at date T. This implies that

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds} 1_{\{V(T) \geq D^*\}} \middle| \mathcal{F}^*(t, t_k), \mathcal{F}^{\mathbf{X}}(t) \right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds} \middle| \mathcal{F}^{\mathbf{X}}(t) \right] \mathbb{E}^{Q^B}\left[1_{\{V(T) \geq D^*\}} \middle| \mathcal{F}^*(t, t_k), \mathcal{F}^{\mathbf{X}}(t) \right] \\ &= B(t, T)\ \mathbb{E}^{Q^B}\left[1_{\{V(T) \geq D^*\}} \middle| \mathcal{F}^*(t, t_k), \mathcal{F}^{\mathbf{X}}(t) \right] \\ &= B(t, T)\ Q^B\left(V(T) \geq D^* \middle| \mathcal{F}^*(t, t_k), \mathcal{F}^{\mathbf{X}}(t) \right). \end{aligned} \tag{41}$$

Here, $B(t, T)$ can be represented as in [19] on page 284. We have

$$B(t, T) = \mathbb{E}\left[e^{-\int_t^T r(s)ds} \middle| \mathcal{F}_X(t) \right] = \mathbf{1}^{\mathrm{T}} \exp\{(-\operatorname{diag}[\mathbf{r}] + \mathbf{A})(T - t)\} \mathbf{X}_t. \tag{42}$$

To evaluate $Q^B\left(V(T) \geq D^* \middle| \mathcal{F}^*(t, t_k), \mathcal{F}^{\mathbf{X}}(t) \right)$, we need the characteristic function of V under the probability measure Q^B, which also called T-forward measure. We have

$$\begin{aligned} \phi^{Q^B}(iu) &= \mathbb{E}^{\mathbb{Q}^B}\left[e^{iu \ln V(T)} \middle| \mathcal{F}^*(t, t_k),\ \mathcal{F}^{\mathbf{X}}(t) \right] \\ &= \frac{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(t)dt} e^{iu \ln V(T)} \middle| \mathcal{F}^*(t, t_k),\ \mathcal{F}^{\mathbf{X}}(t) \right]}{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(t)dt} \middle| \mathcal{F}^{\mathbf{X}}(t) \right]} \\ &= \frac{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(t)dt} e^{iu \ln V(T)} \middle| \mathcal{F}^*(t, t_k),\ \mathcal{F}^{\mathbf{X}}(t) \right]}{B(t, T)}, \end{aligned} \tag{43}$$

where

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$$
\begin{aligned}
& \mathbb{E}\left[\mathrm{e}^{-\int_t^T r(s)ds} e^{iu\ln(V_T)} \,\middle|\, \mathcal{F}^*(t,t_k),\ \mathcal{F}^{\mathbf{X}}(T)\right] \\
= \ & \exp\left\{ iu\ln(E_t) + iu\ln(H_{t_k}) + iu\ln\left(\frac{E_t}{E_{t_k}}\right)\beta_{EH}^T \mathbf{X}_{t_k} \right\} \\
& \times\ \exp\left\{ (iu-1)\int_t^T c_0(s)ds + iu\int_t^T c_1(s)ds + (iu)^2\int_t^T c_2(s)ds \right\} \\
& \times\ \exp\left\{ iu\int_{t_k}^t b_1(s)ds + (iu)^2\int_{t_k}^t b_2(s)ds + iu\int_{t_k}^t \ln\left(\frac{E_t}{E_s}\right)\beta_{EH}^T d\mathbf{X}_s \right\},
\end{aligned} \tag{44}
$$

and

$$
\begin{aligned}
c_0(s) &= r(s) \\
c_1(s) &= -\frac{1}{2}\sigma_E^2(s) + \mu_H(s) - \frac{1}{2}\sigma_H^2(s) \\
c_2(s) &= \frac{1}{2}(\sigma_E(s) + \sigma_H(s)\rho(s))^2 + \frac{1}{2}\sigma_H^2(s)\left(1-\rho^2(s)\right) \\
b_1(s) &= \left(\mu_H(s) - \frac{1}{2}\sigma_H^2(s)\right) - \beta_{EH}(s)\left(r(s) - \frac{1}{2}\sigma_E^2(s)\right) \\
b_2(s) &= \frac{1}{2}\sigma_H^2(s)\left(1-\rho^2(s)\right).
\end{aligned}
$$

Using Theorem 2.4, we have,

$$
\begin{aligned}
& \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r(t)dt} \mathrm{e}^{iu\ln V(T)} \,\middle|\, \mathcal{F}^*(t,t_k),\ \mathcal{F}^{\mathbf{X}}(t)\right] \\
= \ & \exp\left\{ iu\ln(E_t) + iu\ln(H_{t_k}) + iu\ln\left(\frac{E_t}{E_{t_k}}\right)\beta_{EH}^\top \mathbf{X}_{t_k} \right\} \\
& \times\ \exp\left\{ iu\int_{t_k}^t b_1(s)ds + (iu)^2\int_{t_k}^t b_2(s)ds + iu\int_{t_k}^t \ln\left(\frac{E_t}{E_s}\right)\mathbf{X}_{s-}^\top \mathbf{B} d\mathbf{X}_s \right\} \\
& \times\ \mathbf{1}^{\mathrm{T}} \exp\left\{ \left(\mathrm{diag}(b') + \mathbf{A}\right)(T-t) \right\} \mathbf{X}_t,
\end{aligned} \tag{45}
$$

where

- $\beta_{EH}^\top d\mathbf{X}_s = \mathbf{X}_{s-}^\top \mathbf{B} d\mathbf{X}_s$
- $b' = (iu-1)c_0(s) + (iu)c_1(s) + (iu)^2 c_2(s)$.

Combining Equations (45) and (42) gives Equation (43), the characteristic function under Q^B:

$$
Q^B\left(\ln(V(T)) \geq \ln(D^*) \,\middle|\, \mathcal{F}^*(t,t_k), \mathcal{F}^{\mathbf{X}}(t)\right) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left(\frac{e^{-iu\ln(D^*)}}{iu}\phi^{Q^B}(u)\right).
$$

Now, for the second term in Equation (40), we have

$$
\begin{aligned}
& \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r(t)dt} 1_{\{V(T)<D^*\}} \cdot \frac{(1-\alpha)}{D} V(T) \,\middle|\, \mathcal{F}^*(t,t_k),\ \mathcal{F}^{\mathbf{X}}(t)\right] \\
& = \frac{(1-\alpha)}{D}\,\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r(t)dt} V(T) \,\middle|\, \mathcal{F}^*(t,t_k),\ \mathcal{F}^{\mathbf{X}}(t)\right] \\
& \times\ \mathbb{E}^{\mathbb{Q}^{V^r}}\left[1_{\{V(T)<D^*\}} \,\middle|\, \mathcal{F}^*(t,t_k),\ \mathcal{F}^{\mathbf{X}}(t)\right],
\end{aligned} \tag{46}
$$

where we used Abstract Bayes' Theorem and the change of measure

$$
\frac{d\mathbb{Q}^{V^r}}{d\mathbb{Q}} = \frac{e^{-\int_0^T r dt} V(T)}{V(0)}.
$$

We have to evaluate $\mathbb{Q}^{V^r}\left(V(T) < D^* \,\middle|\, \mathcal{F}^*(t,t_k),\ \mathcal{F}^{\mathbf{X}}(t)\right)$. We need the characteristic function of V under the probability measure $\mathbb{Q}^{V^r}$. The moment generating function under this measure, which we denote by $\phi^{V^r}(u)$, is as follows:

$$\begin{aligned}\phi^{V^r}(u) &= \mathbb{E}^{\mathbb{Q}^{V^r}}\left[\mathrm{e}^{u\ln V(T)} \,\middle|\, \mathcal{F}^*(t,t_k), \mathcal{F}^{\mathbf{X}}(t)\right] \\ &= \frac{\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{u\ln V(T)} e^{-\int_0^T r dt} V(T) \,\middle|\, \mathcal{F}^*(t,t_k), \mathcal{F}^{\mathbf{X}}(t)\right]}{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r dt} V(T) \,\middle|\, \mathcal{F}^*(t,t_k), \mathcal{F}^{\mathbf{X}}(t)\right]}.\end{aligned} \tag{47}$$

Conditional on the given filtration above, the $e^{-\int_0^t r dt}$ terms cancel out.

For $\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{u\ln V(T)} e^{-\int_0^T r dt} V(T) \,\middle|\, \mathcal{F}^*(t,t_k), \mathcal{F}^{\mathbf{X}}(t)\right]$, from Equation (18) and conditional on $\mathcal{F}^*(t,t_k), \mathcal{F}^{\mathbf{X}}(t)$, we have

$$\begin{aligned}&= \exp\left\{(u+1)\ln(E_t) + (u+1)\ln(H_{t_k}) + (u+1)\ln\left(\frac{E_t}{E_{t_k}}\right)\beta_{EH}^T \mathbf{X}_{t_k}\right\} \\ &\quad\times \exp\left\{(u+1)\int_{t_k}^t b_1(s)ds + ((u+1))^2\int_{t_k}^t b_2(s)ds + (u+1)\int_{t_k}^t \ln\left(\frac{E_t}{E_s}\right)\mathbf{X}_{s-}^\top \mathbf{B} d\mathbf{X}_s\right\} \\ &\quad\times \mathbf{1}^{\mathrm{T}} \exp\left\{\left(\mathrm{diag}(b'') + \mathbf{A}\right)(T-t)\right\}\mathbf{X}_t,\end{aligned}$$

where

- $b'' = ur(s) + (u+1)\left(-\frac{1}{2}\sigma_E^2(s) + \mu_H(s) - \frac{1}{2}\sigma_H^2(s)\right) + (u+1)^2 c_2(s)$
- c_2, b_1 and b_2 is defined by Equation (18),

and $\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r dt} V(T) \,\middle|\, \mathcal{F}^*(t,t_k), \mathcal{F}^{\mathbf{X}}(t)\right]$ is the same as above with $u+1=0$. Then, the characteristic function is just $\phi^{V^r}(iu)$. Once we evaluate the characteristic function, we can find the probability by using Equation (38), as we discuss in Theorem 3.2 below.

Theorem 3.2. The price of risky debt under regime-switching with regime-switching interest rate can be represented as the following

$$\begin{aligned}&P_t(E,H,V) \\ =\ & B(t,T)Q^B\left(V(T) \geq D^* \,\middle|\, \mathcal{F}^*(t,t_k),\ \mathcal{F}^{\mathbf{X}}(t)\right) \\ &+ \frac{E_t}{E_{t_k}} V_{t_k} \frac{(1-\alpha)}{D} \exp\left\{\ln\left(\frac{E_t}{E_{t_k}}\right)\beta_{EH}^T \mathbf{X}_{t_k}\right\} \\ &\times\ \exp\left\{\int_{t_k}^t b_1(s)ds + \int_{t_k}^t b_2(s)ds + \int_{t_k}^t \ln\left(\frac{E_t}{E_s}\right)\mathbf{X}_{s-}^\top \mathbf{B} d\mathbf{X}_s\right\} \\ &\times\ \mathbf{1}^{\mathrm{T}} \exp\left\{\left(\mathrm{diag}(\hat{b}) + \mathbf{A}\right)(T-t)\right\}\mathbf{X}_t \\ &\times\ \mathbb{Q}^{V^r}\left(V(T) < D^* \,\middle|\, \mathcal{F}^*(t,t_k), \mathcal{F}^{\mathbf{X}}(t)\right),\end{aligned} \tag{48}$$

where $\hat{b}$ is the same as b'' with $u = 1$. Also, $B(t,T)$, $\mathbb{Q}^B\left(V(T) \geq D^* \,\middle|\, \mathcal{F}^*(t,t_k), \mathcal{F}^{\mathbf{X}}(t)\right)$, and $\mathbb{Q}^{V^r}\left(V(T) < D^* \,\middle|\, \mathcal{F}^*(t,t_k), \mathcal{F}^{\mathbf{X}}(t)\right)$ given by Equations (42), (43) and (47), respectively.

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4. Conclusion

In this paper, we consider the problem of pricing risky debt under the regime-switching model with imperfect information. In particular, we provide a technical representation of risky debt under regime-switching with a constant interest rate, see Theorem 3.1, and risky debt under regime-switching with a regime-switching interest rate, see Theorem 3.2.

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Appendix A

Appendix A.1 Proofs of Section 2

In this appendix, we provide the proofs of the results of this paper.
Proof of Lemma 2.1: Note that

$$\begin{aligned}
&\mathbb{E}\left[\exp\left\{\int_{t_k}^{T} c_s ds + \int_{t_k}^{T} a_s dW_1(s) + \int_{t_k}^{T} b_s dW_2(s)\right\} \,\middle|\, \mathcal{F}^{\mathbf{X}}(T), \mathcal{F}^*(t,t_k)\right] \\
&= \exp\left\{\int_{t_k}^{T} c_s ds + \int_{t_k}^{t} a_s dW_1(s)\right\}\mathbb{E}\left[\exp\left\{\int_{t}^{T} a_s dW_1(s) + \int_{t_k}^{T} b_s dW_2(s)\right\} \,\middle|\, \mathcal{F}^{\mathbf{X}}(T), \mathcal{F}^*(t,t_k)\right] \\
&= \exp\left\{\int_{t_k}^{T} c_s ds + \int_{t_k}^{t} a_s dW_1(s)\right\} \exp\left\{\frac{1}{2}\int_{t}^{T} a_s^2 ds + \frac{1}{2}\int_{t_k}^{T} b_s^2 ds\right\}.
\end{aligned}$$

The second equality follows from the fact that, conditional on $\mathcal{F}^{\mathbf{X}}(T)$, $\int_t^T a_s dW_1(s)$ and $\int_t^T b_s dW_2(s)$, are conditionally normally distributed with mean zero, and by the Itô isometry, their variances are given by $\int_t^T a_s^2 ds$ and $\int_t^T b_s^2 ds$ respectively. This proves the Lemma.

□

Proof of Theorem 2.4:

To prove this theorem, we start by applying the product rule to $\exp\left\{\int_t^T \tilde{b}(s)ds\right\}\mathbf{X}_T$ which will allow us to find its expected value $\mathbb{E}\left[U_2(t,T) \,\middle|\, \mathcal{F}^{\mathbf{X}}(t)\right]$.

$$\begin{aligned}
d\left(\exp\left\{\int_t^T \tilde{b}(s)ds\right\}\mathbf{X}_T\right) &= \exp\left\{\int_t^T \tilde{b}(s)ds\right\}d\mathbf{X}_t + \exp\left\{\int_t^T \tilde{b}(s)ds\right\}\tilde{b}(t)\mathbf{X}_{t_} \\
&= U_2(t,T)(\mathbf{A}\mathbf{X}_t dt + M_t) + U_2(t,T)\tilde{b}(t)\mathbf{X}_{t_}.
\end{aligned}$$

Using Remark 2.5 from [20], integrating over t and then taking the expectation, (note that M is martingale), we find that

$$\mathbb{E}\left[\exp\left\{\int_t^T \tilde{b}(s)ds\right\}\mathbf{X}_T\right] = U_2(t,t)\mathbf{X}_t + \int_t^T (\mathrm{diag}(\tilde{b}) + \mathbf{A})\mathbb{E}\left[\exp\left\{\int_t^T \tilde{b}(s)ds\right\}\mathbf{X}_s\right]ds,$$

knowing that the differential has a solution of an exponential function, we obtain the result as desired.

□

***Proof of Lemma* 2.5**: By the definition of the exponential of a matrix, it is not hard to see that

$$\begin{aligned}\frac{d}{dT}H(t,T) &= \frac{d}{dT}\exp\left\{\mathbf{C}(T-t)+\int_t^T \mathbf{B}(s)ds\right\}H(t,t)\\ &= \exp\left\{\mathbf{C}(T-t)+\int_t^T \mathbf{B}(s)ds\right\}(\mathbf{C}+\mathbf{B}(T))H(t,t)\\ &= (\mathbf{C}+\mathbf{B}(T))\exp\left\{\mathbf{C}(T-t)+\int_t^T \mathbf{B}(s)ds\right\}H(t,t).\end{aligned}$$

Using the third equality and integration gives

$$\begin{aligned}H(t,T) &= H(t,t)+\int_t^T(\mathbf{C}+\mathbf{B}(s))\exp\left\{\mathbf{C}(s-t)+\int_t^s \mathbf{B}(v)dv\right\}H(t,t)\,ds\\ &= H(t,t)+\int_t^T(\mathbf{C}+\mathbf{B}(s))H(t,s)\,ds,\end{aligned}$$

which proves the lemma.

□

References

1. Black, F.; Scholes, M. The Pricing of Options and Corporate Liabilities. *Journal of Political Economy* **1973**, *81*, 637–654.
2. Merton, R.C. On the pricing of corporate debt: The risk structure of interest rates. *The Journal of Finance* **1974**, *29*, 449–470.
3. Cox, J.C.; Ross, S.A. The valuation of options for alternative stochastic processes. *Journal of Financial Economics* **1976**, *3*, 145–166.
4. Zhou, C. The term structure of credit spreads with jump risk. *Journal of Banking & Finance* **2001**, *25*, 2015–2040.
5. Hilberink, B.; Rogers, L.C. Optimal capital structure and endogenous default. *Finance and Stochastics* **2002**, *6*, 237–263.
6. Ma, Y.; Xu, W. Structural credit risk modelling with Hawkes jump-diffusion processes. *Journal of Computational and Applied Mathematics* **2016**, *303*, 69–80.
7. Ericsson, J.; Reneby, J. A note on contingent claims pricing with non-traded assets. *SSE/EFI Working Paper Series in Economics and Finance* **2002**, *314*.
8. Duffie, D.; Lando, D. Term structures of credit spreads with incomplete accounting information. *Econometrica* **2001**, *69*, 633–664.
9. Hamilton, J.D. A new approach to the economic analysis of nonstationary time series and the business cycle. *Econometrica: Journal of the Econometric Society* **1989**, pp. 357–384.
10. Dionne, G.; Gauthier, G.; Hammami, K.; Maurice, M.; Simonato, J.G. A reduced form model of default spreads with Markov-switching macroeconomic factors. *Journal of Banking & Finance* **2011**, *35*, 1984–2000.
11. Liang, X.; Wang, G. On a reduced form credit risk model with common shock and regime switching. *Insurance: Mathematics and Economics* **2012**, *51*, 567–575.
12. Liang, X.; Wang, G.; Li, H. Pricing credit default swaps with bilateral counterparty risk in a reduced form model with Markov regime switching. *Applied Mathematics and Computation* **2014**, *230*, 290–302.
13. Dong, Y.; Yuen, K.C.; Wang, G.; Wu, C. A reduced-form model for correlated defaults with regime-switching shot noise intensities. *Methodology and Computing in Applied Probability* **2016**, *18*, 459–486.
14. Lau, J.W.; Siu, T.K. Pricing risky debts under a Markov-modudated Merton model with completely random measures. *Computational Economics* **2008**, *31*, 255–288.
15. Xu, C.; Dong, Y.; Wang, G. The pricing of defaultable bonds under a regime-switching jump-diffusion model with stochastic default barrier. *Communications in Statistics-Theory and Methods* **2019**, *48*, 2185–2205.
16. Buffington, J.; Elliott, R.J. American options with regime switching. *International Journal of Theoretical and Applied Finance* **2002**, *5*, 497–514.
17. Zarban, A.A. Pricing of Exchange and Vulnerable Exchange option with Regime Switching. Phd thesis, UNSW Sydney, 2023.

 doi:10.20944/preprints202406.0833.v1

18. Heston, S.L. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *The Review of Financial Studies* **1993**, *6*, 327–343.
19. Elliott, R.J.; Kopp, P.E. *Mathematics of Financial Markets*; Springer Finance, 2005.
20. Colwell, D.B. Hitting times, number of jumps, and occupation times for continuous-time finite state Markov chains. *Statistics & Probability Letters* **2023**, *195*, 109786.