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Modular Deutsch Entropic Uncertainty Principle

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Abstract: Khosravi, Drnovšek and Moslehian [*Filomat, 2012*] derived Buzano inequality for Hilbert C*-modules. Using this inequality we derive Deutsch entropic uncertainty principle for Hilbert C*-modules over commutative unital C*-algebras.

Keywords: Buzano inequality; entropic uncertainty; Hilbert C*-module

MSC: 46L08; 42C15; 46L05

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1. Introduction

Let $\mathcal{H}$ be a finite dimensional Hilbert space. Given an orthonormal basis $\{\tau_j\}_{j=1}^n$ for $\mathcal{H}$, the **Shannon entropy** at a point $h \in \mathcal{H}_\tau$ is defined as

$$S_\tau(h) := -\sum_{j=1}^{n} \left|\langle h, \tau_j\rangle\right|^2 \log\left|\langle h, \tau_j\rangle\right|^2, \tag{1}$$

where $\mathcal{H}_\tau := \{h \in \mathcal{H} : \|h\| = 1, \langle h, \tau_j\rangle \neq 0, 1 \leq j \leq n\}$ [1]. In 1983, Deutsch derived following breakthrough entropic uncertainty principle for Shannon entropy [1].

Theorem 1 ([1] (**Deutsch Entropic Uncertainty Principle**)). *Let $\{\tau_j\}_{j=1}^n$, $\{\omega_k\}_{k=1}^n$ be two orthonormal bases for a finite dimensional Hilbert space $\mathcal{H}$. Then*

$$S_\tau(h) + S_\omega(h) \geq -2\log\left(\frac{1 + \max_{1\leq j,k\leq n}|\langle \tau_j, \omega_k\rangle|}{2}\right), \quad \forall h \in \mathcal{H}_\tau \cap \mathcal{H}_\omega. \tag{2}$$

The inequality (2) is recently derived for Banach spaces [2]. It is observed very recently that using Buzano inequality (see [3–5]) one can provide a simple proof of Theorem 1 (see Corollary 1 in [2]). As Hilbert C*-modules became more important in noncommutative geometry, we are mainly motivated from the following problem. What is the modular version of Theorem 1? Hilbert C*-modules are first introduced by Kaplansky [6] for modules over commutative C*-algebras and later developed for modules over arbitrary C*-algebras by Paschke [7] and Rieffel [8].

Definition 1 ([6–8]). *Let $\mathcal{A}$ be a unital C*-algebra. A left module $\mathcal{E}$ over $\mathcal{A}$ is said to be a (left) Hilbert C*-module if there exists a map $\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \to \mathcal{A}$ such that the following hold.*

(i) $\langle x, x\rangle \geq 0, \forall x \in \mathcal{E}$. *If* $x \in \mathcal{E}$ *satisfies* $\langle x, x\rangle = 0$, *then* $x = 0$.
(ii) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle, \forall x, y, z \in \mathcal{E}$.
(iii) $\langle ax, y\rangle = a\langle x, y\rangle, \forall x, y \in \mathcal{E}, \forall a \in \mathcal{A}$.
(iv) $\langle x, y\rangle = \langle y, x\rangle^*, \forall x, y \in \mathcal{E}$.
(v) $\mathcal{E}$ *is complete w.r.t. the norm* $\|x\| := \sqrt{\|\langle x, x\rangle\|}, \forall x \in \mathcal{E}$.

Our prime tool to derive modular Deutsch uncertainty is the following modular Buzano inequality by Khosravi, Drnovšek, and Moslehian [9].

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Theorem 2 ([9] (Modular Buzano Inequality)). *If $\mathcal{E}$ is a Hilbert C*-module over a unital C*-algebra $\mathcal{A}$, then*

$$\|\langle x,z\rangle\langle z,y\rangle\| \leq \frac{1}{2}(\|x\|\|y\| + \|\langle x,y\rangle\|), \quad \forall x,y,z \in \mathcal{E}, \langle z,z\rangle = 1.$$

In this paper we derive Theorem 1 for Hilbert C*-modules over commutative unital C*-algebras.

2. Modular Deutsch Entropic Uncertainty Principle

We begin by recalling the definition of frames for Hilbert C*-modules.

Definition 2 ([10]). *Let $\mathcal{E}$ be a Hilbert C*-module over a C*-algebra $\mathcal{A}$. A collection $\{\tau_j\}_{j=1}^\infty$ in $\mathcal{E}$ is said to be a (modular)* ***Parseval frame*** *for $\mathcal{E}$ if*

$$x = \sum_{j=1}^{\infty} \langle x, \tau_j\rangle \tau_j, \quad \forall x \in \mathcal{E}.$$

A collection $\{\tau_j\}_{j=1}^\infty$ in a Hilbert C*-module $\mathcal{E}$ over unital C*-algebra $\mathcal{A}$ with identity 1 is said to have **unit inner product** if

$$\langle \tau_j, \tau_j\rangle = 1, \quad \forall j \in \mathbb{N}.$$

In analogy with Equation (1), given a unit inner product Parseval frame $\{\tau_j\}_{j=1}^\infty$ for $\mathcal{E}$, we define **modular Shannon entropy** at a point $x \in \mathcal{E}_\tau$ is defined as

$$S_\tau(x) := -\sum_{j=1}^{\infty} \langle x, \tau_j\rangle\langle \tau_j, x\rangle \log(\langle x, \tau_j\rangle\langle \tau_j, x\rangle) \tag{3}$$

where $\mathcal{E}_\tau := \{x \in \mathcal{E} : \langle x,x\rangle = 1, \langle x, \tau_j\rangle \neq 0, \forall j \in \mathbb{N}\}$.

Theorem 3 (**Modular Deutsch Entropic Uncertainty Principle**). *Let $\mathcal{E}$ be a Hilbert C*-module over a commutative unital C*-algebra $\mathcal{A}$. Let $\{\tau_j\}_{j=1}^\infty$, $\{\omega_k\}_{k=1}^\infty$ be two Parseval frames for $\mathcal{E}$. Then*

$$S_\tau(x) + S_\omega(x) \geq -2\log\left(\frac{1 + \sup_{j,k\in\mathbb{N}} \|\langle \tau_j, \omega_k\rangle\|}{2}\right), \quad \forall x \in \mathcal{E}_\tau \cap \mathcal{E}_\omega.$$

Proof. Let $x \in \mathcal{E}_\tau \cap \mathcal{E}_\omega$. Using the Parseval frame property, the commutativity of C*-algebra, Theorem 2 and the result that 'function logarithm is operator monotone' [11], we get

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$$
\begin{aligned}
S_\tau(x)+S_\omega(x) &= -\sum_{j=1}^{\infty}\langle x,\tau_j\rangle\langle\tau_j,x\rangle\log(\langle x,\tau_j\rangle\langle\tau_j,x\rangle)-\sum_{k=1}^{\infty}\langle x,\omega_k\rangle\langle\omega_k,x\rangle\log(\langle x,\omega_k\rangle\langle\omega_k,x\rangle)\\
&= -\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\langle x,\tau_j\rangle\langle\tau_j,x\rangle\langle x,\omega_k\rangle\langle\omega_k,x\rangle\left[\log(\langle x,\tau_j\rangle\langle\tau_j,x\rangle)+\log(\langle x,\omega_k\rangle\langle\omega_k,x\rangle)\right]\\
&= -\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\langle x,\tau_j\rangle\langle\tau_j,x\rangle\langle x,\omega_k\rangle\langle\omega_k,x\rangle\log(\langle x,\tau_j\rangle\langle\tau_j,x\rangle\langle x,\omega_k\rangle\langle\omega_k,x\rangle)\\
&= -\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\langle x,\tau_j\rangle\langle\tau_j,x\rangle\langle x,\omega_k\rangle\langle\omega_k,x\rangle\log(\langle\tau_j,x\rangle\langle x,\omega_k\rangle\langle\omega_k,x\rangle\langle x,\tau_j\rangle)\\
&\geq -\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\langle x,\tau_j\rangle\langle\tau_j,x\rangle\langle x,\omega_k\rangle\langle\omega_k,x\rangle\log\left(\frac{[\|\tau_j\|\|\omega_k\|+\|\langle\tau_j,\omega_k\rangle\|][\|\omega_k\|\|\tau_j\|+\|\langle\omega_k,\tau_j\rangle\|]}{4}\right)\\
&= -\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\langle x,\tau_j\rangle\langle\tau_j,x\rangle\langle x,\omega_k\rangle\langle\omega_k,x\rangle\log\left(\frac{(\|\tau_j\|\|\omega_k\|+\|\langle\tau_j,\omega_k\rangle\|)^2}{4}\right)\\
&= -2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\langle x,\tau_j\rangle\langle\tau_j,x\rangle\langle x,\omega_k\rangle\langle\omega_k,x\rangle\log\left(\frac{\|\tau_j\|\|\omega_k\|+\|\langle\tau_j,\omega_k\rangle\|}{2}\right)\\
&\geq -2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\langle x,\tau_j\rangle\langle\tau_j,x\rangle\langle x,\omega_k\rangle\langle\omega_k,x\rangle\log\left(\frac{1+\|\langle\tau_j,\omega_k\rangle\|}{2}\right)\\
&\geq -2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\langle x,\tau_j\rangle\langle\tau_j,x\rangle\langle x,\omega_k\rangle\langle\omega_k,x\rangle\log\left(\frac{1+\sup_{j,k\in\mathbb{N}}\|\langle\tau_j,\omega_k\rangle\|}{2}\right)\\
&= -2\log\left(\frac{1+\sup_{j,k\in\mathbb{N}}\|\langle\tau_j,\omega_k\rangle\|}{2}\right)\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\langle x,\tau_j\rangle\langle\tau_j,x\rangle\langle x,\omega_k\rangle\langle\omega_k,x\rangle\\
&= -2\log\left(\frac{1+\sup_{j,k\in\mathbb{N}}\|\langle\tau_j,\omega_k\rangle\|}{2}\right)\langle x,x\rangle\langle x,x\rangle\\
&= -2\log\left(\frac{1+\sup_{j,k\in\mathbb{N}}\|\langle\tau_j,\omega_k\rangle\|}{2}\right).
\end{aligned}
$$

□

In 1988, Maassen and Uffink (motivated from the conjecture by Kraus made in 1987 [12]) improved Deutsch entropic uncertainty principle.

Theorem 4 ([13] **(Maassen-Uffink Entropic Uncertainty Principle)).** *Let $\{\tau_j\}_{j=1}^n$, $\{\omega_k\}_{k=1}^n$ be two orthonormal bases for a finite dimensional Hilbert space $\mathcal{H}$. Then*

$$S_\tau(h)+S_\omega(h)\geq -2\log\left(\max_{1\leq j,k\leq n}|\langle\tau_j,\omega_k\rangle|\right),\quad \forall h\in\mathcal{H}_\tau\cap\mathcal{H}_\omega.$$

In 2013, Ricaud and Torrésani [14] showed that orthonormal bases in Theorem 4 can be improved to Parseval frames.

Theorem 5 ([14] **(Ricaud-Torrésani Entropic Uncertainty Principle)).** *Let $\{\tau_j\}_{j=1}^n$, $\{\omega_k\}_{k=1}^m$ be two Parseval frames for a finite dimensional Hilbert space $\mathcal{H}$. Then*

$$S_\tau(h)+S_\omega(h)\geq -2\log\left(\max_{1\leq j\leq n,1\leq k\leq m}|\langle\tau_j,\omega_k\rangle|\right),\quad \forall h\in\mathcal{H}_\tau\cap\mathcal{H}_\omega.$$

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Proofs of Theorems 4 and 5 use Riesz-Thorin interpolation (RTI). To the best of author's knowledge, RTI does not exists for abstract Hilbert C*-modules. Therefore we end by formulating the following conjecture.

Conjecture 6 (**Modular Kraus Entropic Conjecture**). *Let $\mathcal{E}$ be a Hilbert C*-module over a commutative unital C*-algebra $\mathcal{A}$. Let $\{\tau_j\}_{j=1}^{\infty}$, $\{\omega_k\}_{k=1}^{\infty}$ be two Parseval frames for $\mathcal{E}$. Then*

$$S_\tau(x) + S_\omega(x) \geq -2\log\left(\sup_{j,k\in\mathbb{N}} \|\langle \tau_j, \omega_k\rangle\|\right), \quad \forall x \in \mathcal{E}_\tau \cap \mathcal{E}_\omega.$$

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