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Categorical Theory of Fundamental Dynamical Processes

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Abstract

With the aim of finding a generalization of the principle of general covariance applicable to certain models for a *fundamental dynamical*, a categorical approach to dynamics is developed. Given a category, the notion of pre-dynamics is introduced. The dynamics is then defined as a *finest pre-dynamics*. The *categorical recurrence principle*, which constitutes a categorical generalization of the principle of general covariance, is formulated. Natural consequences of the theory are the deterministic character of the fundamental dynamics and the evolving character of the physical dynamical law itself. One main problem considered is the way how time parameters can be defined in terms of the elements of the category and set theory, as a part of the implementation of the categorical recurrence principle. Aimed to address this question, a construction is presented showing how time parameters are defined from the elements of the theory recursively.

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1. Introduction

Emergent quantum mechanics encompasses several paradigms for fundamental physics whose main common assumption is the existence of a deeper deterministic and local description of physical systems than the one assumed in the standard quantum mechanical theory [1,8,16,18,22]. One of these approaches is *Hamilton-Randers theory*, where Hamiltonian dynamical systems of Randers type are postulated as describing the dynamics of a sub-quantum scale degrees of freedom [6–9,11,15]. One of the aims of Hamilton-Randers theory is to recover the quantum description of a physical system as a coarse-grained, effective description from a deeper causal, deterministic and local (in a generalized sense) physical description. Furthermore, the theory embraces enough structure to describe gravity as a classical, emergent interaction, intimately related with a process of natural, spontaneous collapse of the wave function [8–10,12,15].

Among the consequences of Hamilton-Randers theory there is one concerning an intrinsic limit in the knowledge acquirable by direct experimental methods. This is due to the *three-level ontological structure* assumed in the theory: 1. *Fundamental degrees of freedom*, 2. *Quantum degrees of freedom* and 3. *Classical degrees of freedom* and their associated descriptions and *physical scales*. The limits on the knowledge acquirable comes by the following mechanism. Information of the physical system or physical process is obtained by means of measurement processes that involve quantum and classical scale external systems. However, in order to obtain phenomenological information from the fundamental dynamics and processes, it is necessary to use as probes quantum systems, since the smallest probes available for a macroscopic observer are found in such a level. Quantum systems are individually uncontrollable by macroscopic means and furthermore, they exceed in several ways the scale of the degrees of freedom aimed to probe. These two conditions imply natural restrictions in the observability of the fundamental processes hypothesized by emergent quantum mechanics,

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limitations that are stronger than in the case of quantum physics. In particular, direct observation of the elementary degrees of freedom and processes is precluded.

Aware of such epistemological difficulties, a methodology is needed to tailor the theoretical research in Hamilton-Randers theory. One aspect of such methodology must pass by avoiding the concepts and notions directly motivated by local experience that do not have enough degree of generality. This method is along Einstein's methodology of suppressing particular points of view in the formulation of general physical statements, one of the strongest motivation of the principle of general covariance as an heuristic principle for the formulation of physical laws [5]. It is not suggested the substitution of the empirical method by pure and probably wild speculation, but the experimental limitations for the case under consideration implies the necessary use of abstract methods of analysis and construction. The case is similar to the property of confinement of quarks, real particles even if the theory precludes them from being detected as free particles, or to the physics of black holes, where the appearance of the event horizon makes un-accessible to obtain direct information extraction from the interior of the black hole, but does not devoids the concept the concept of the corresponding physical realization. It is suggested that part of the repertory of methods needed for the new theory correspond to theoretical methods, to be found in the sphere of maximal abstraction. Specific models are not investigated in this paper, but a methodology that restricts the possible specific models and more in particular, justify the methods of emergent quantum mechanics and possible alternative treatments is discussed.

The present work is motivated by the aim of finding a general framework for a fundamental dynamics and more specifically, to contextualized the theory of emergent quantum mechanics developed in [6–9,14,15] in a general framework. This contextualization should serve as justification for several fundamental notions in emergent quantum mechanics as it appears in [8,14]. Although it is possible to develop a notion of macroscopic coordinate system as an emergent notion [8], a natural notion of macroscopic coordinate system at the level of Hamilton-Randers systems is absent. This leads to the fundamental problem of the implementation of the general covariance principle. The general covariance principle is a way to implement the *impersonal viewpoint* in physical theories, an useful method to avoid unnecessary constructions in the theory, that can be misleading.

Motivated by this problem, the present paper sketches the construction of a meta-theory of dynamics in the framework of category theory [19]. Category theory offers a repertory of notions suitable to develop an abstract theory of fundamental dynamical systems, general enough to avoid unrequired logical restrictions. Essentially, it is that generality what makes the theory applicable to a fundamental dynamics. In mathematical physics, the categorical methods have found several interesting applications. For instance, in the context of dynamical systems, categorical methods have been employed in [17] in an attempt to formalize a systematic view of dynamical systems and as motivation for several developments in dynamic theory [3]. In quantum foundations, category theory has been also extensively used by several authors [2,4]. Closer to our motivation is the use the of categorical methods in causal set theory [21], although our theory takes a different route.

The main concepts of the theory developed in this work are the following. First, it is assumed that the notion of theory is carried out by a mathematical category, while particular models of the theory are associated to objects of the category and their endomorphisms. A dynamics is then associated to certain sequences of endomorphisms that satisfies a completeness condition in the form of *finest dynamics* condition. The concept establishes a correlation between the parameters used in the description and the sequence of transformations itself. The relations between objects are morphisms, while time parameters are constructed by means of a functor from the category to an algebraic category, whose objects are algebraic ones endowed with an structure of monoid.

The categorical approach to the fundamental dynamics leads to several general consequences on the nature of the fundamental dynamical systems. First of all, determinism in the dynamics arises from the associative property of composition of endomorphisms and from our notion of finest pre-dynamics. The form of determinism discussed adjusts very well to the requirements of Hamilton-Randers theory. Distinctive of our approach is the introduction of the *recursive principle*, a step towards the formalization of general covariance by means of category theory concepts. In view of the recursive principle, time parameters are realized as morphisms between objects of the category modulo allowed re-parameterizations. It is also shown that the categorical point of view on fundamental dynamics yields the possibility of a *dynamics of the category*. That is, the theory implies the evolution of the physical fundamental dynamical laws.

2. Categorical Approach to the Fundamental Dynamics

Fundamental dynamical systems refers to the hypothesised physical systems underlying quantum mechanical systems. Our categorical approach to a meta-theory of fundamental dynamical systems is build on the following notions. A fundamental theory is a methodology to construct models for the fundamental dynamics beneath the quantum mechanical scale systems. A candidate to fundamental theory is identified with a category Cat_{Fun}. A fundamental dynamical model is partially determined by an object $\mathcal{O}$ of Cat_{Fun}. The morphisms between different objects of Cat_{Fun} satisfy the property that allow for compositions: Given $f_{ij} : \mathcal{O}_i \to \mathcal{O}_j$ and $f_{kj} : \mathcal{O}_j \to \mathcal{O}_k$, the composition of morphisms $f_{kj} \circ f_{ji} : \mathcal{O}_i \to \mathcal{O}_k$ is defined when the codomain of f_{ij} is a subset of the domain of f_{kj}. Furthermore, each object $\mathcal{O}$ of Cat_{Fun} has associated a non-empty set of endomorphisms $End(\mathcal{O})$. The composition $f_j \circ \tilde{f}_j : \mathcal{O} \to \mathcal{O}$ of endomorphisms $f_j : \mathcal{O} \to \mathcal{O}$, $\tilde{f}_j : \mathcal{O} \to \mathcal{O}$ is always well defined. Also, $End(\mathcal{O})$ contains the identity morphism $Id_j : \mathcal{O}_j \to \mathcal{O}_j,\ a \mapsto a$. Therefore, $End(\mathcal{O})$ with the composition of morphisms is a monoid for each object $\mathcal{O}$.

The composition law of morphisms is assumed to be associative in the sense that the relations

$$f_{ij} \circ (f_{jk} \circ f_{kl}) = (f_{ij} \circ f_{jk}) \circ f_{kl}, \tag{2.1}$$

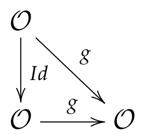
hold good whenever the compositions are defined. In particular, the composition of endomorphisms

$$f_j \circ (f_k \circ f_l) = (f_j \circ f_k) \circ f_l \tag{2.2}$$

holds always good, since the composition of endomorphisms is always defined.

The theory under consideration applies to dynamical systems interpreted within the framework of set theory. We restrict our theories to the case when the morphims are functions between sets. We still denote them as morphisms of the fundamental category. The restriction to functions has its role to implement determinism, although it is not logically necessary. On the other hand, one of the reasons to implement deeterminism in a mathematical description of physical systems is that it is the simplest alternative of generic type.

Within set theory and when the composition of the morphisms are defined, associativity holds. By assuming the associativity property for the composition of morphsims, $(End(\mathcal{O}), \circ)$ is a monoid for each object $\mathcal{O}$. Associativity ensures that, if there is an identity morphism $\mathcal{I}d : \mathcal{O} \to \mathcal{O}$ such that the diagrams



(2.3)

for each g, then it is unique. This is a nice property for our theory, since having two different neutral elements leads to difficulties related with cyclic dynamics, as a direct argument shows. Furthermore, associativity will be very useful for the construction of multiple composition of morphism.

It is not required that the composition of endomorphisms to be commutative. Neither it is required that the endomorphisms have inverse.

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Example 2.1. *Let us consider theories such that the spacetime is the dynamical object, as it is the case of the* $3+1$ *description of physical systems in general relativity. Such theories can be formalized by considering the category* Cat_{Fun} *as being the category* $\mathbf{Smo}_4$ *of smooth manifolds of dimension* 4 *and then considering foliations consistent with diffeomorphism invariance. However, when adopting* $\mathbf{Smo}_4$ *as the category for the description of fundamental processes, further assumptions of mathematical and physical nature are necessary:*

1. *Einstein equivalence principle, implying the existence of a spacetime Lorentzian metric on each object of* $\mathbf{Smo}_4$*,*
2. *General covariance principle, excluding the existence of other fundamental metric structures,*
3. *General covariant field equations,*
4. *A dynamical law for point particles generalizing the inertial law compatible with the above principles,*
5. *Phenomenological constraints limiting the possibilities for the field equations. These constraints regulate the specification of the subcategory of* $\mathbf{Smo}_4$ *compatible with the principles.*

Relaxing the amount of assumptions, implies a landscape theory.

In the construction of a given dynamical theory, it is necessary to introduce a parameter labeling the evolution. Hence we assume the existence of a *parameter space* or *time parameter*. The parameter variable is used to label the elements of sequences of morphisms. Such parameter spaces can be represented by subsets of the integer numbers or by subsets of the real numbers, for instance. However, other classes of parameter spaces can be used for this goal. As such, they lay first outside the category Cat_{Fun}, but it must be construable from set theory and Cat_{Fun}.

2.1. Construction of the Natural Numbers and Zero from Set Theory

Given a category, it is possible to construct the ordered set of natural numbers and the zero number $\mathbb{N}\cup\{0\}$ within the framework of set theory. We start considering a category and an object $\mathcal{O}$ of the category. Then we construct the following sets

$$\emptyset,\ \{\emptyset\},\ \{\emptyset,\{\emptyset\}\},\ \{\emptyset,\{\emptyset,\{\emptyset\}\}\},\ \ldots .$$

That, is the sequence of empty set, the subset of all the subsets of $\mathcal{O}$ consisting of the empty sets. These two mathematical objects are represented by symbols

$$\begin{aligned}\emptyset &\mapsto 0,\\ \{\emptyset\} &\mapsto 1.\end{aligned}$$

Then $0<1$ and 1 is the successor of 0. The next is to assign couples C_1 formed by the empty subset of $\mathcal{O}$ and the subset of the power set of $\mathcal{O}$ composed by one element, the empty set. Then one defines the above sequence recursively and uses the natural sequences to denote the corresponding elements of the sequence as follows,

$$\begin{aligned}&\emptyset \mapsto 0,\\ &\{\emptyset\} \mapsto 1,\\ &\{\emptyset,\{\emptyset\}\} \mapsto 2,\\ &\{\emptyset,\{\emptyset,\{\emptyset\}\}\} \mapsto 3,\\ &\ldots .\end{aligned}$$

Thus the sequence $\{\emptyset,\ \{\emptyset\},\ \{\emptyset,\{\emptyset\}\},\ \{\emptyset,\{\emptyset,\{\emptyset\}\}\},\ \ldots\}$ is identified with the natural numbers union with the zero number. Thus we call this sequence $\mathbb{N}\cup\{0\}$.

Being recursive, the above sequence is ordered. Thus, inducing the order $0<1<2<3<\ldots$.

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It is possible to define a sum operation in $\mathbb{N} \cup \{0\}$, just because it is recursive, if C_n, C_m are given, then $C_n + C_M := C_{n+m}$.

Proposition 2.2. $(\mathbb{N} \cup \{0\}, +)$ *is an ordered monoid.*

The requirement that the time parameter spaces should be endowed with an order relation is intrinsic in our theory, if not necessary. This requirement is a local condition on the parameter space. The assumption of partial order for the time parameter spaces serves to define the notion of *limits*. The above construction shows that category theory offers at least one natural parameter space, $\mathbb{N} \cup \{0\}$.

From $\mathcal{I}_N$, one can construct other parameter spaces as subsets of the extensions of the natural numbers. These parameter spaces are abstract, not linked to any specific physical system. There are at disposal as mathematical constructions for any theory based in Cat_{Fun}.

2.2. Notion of Fundamental Dynamics

Definition 2.3. *Let us consider an object $\mathcal{O}$ of a fundamental category Cat_{Fun} and a collection $\mathcal{S} \subset End(\mathcal{O})$ of endomorphisms of $\mathcal{O}$. Let us also consider an ordered parameter space. A sequence $\hat{\mathcal{S}}$ is a map that associates to each element of $\mathcal{S}$ a value of the time parameter space. A pre-dynamics defined on the model $\mathcal{O}$ with initial morphism $f_0 : \mathcal{O} \to \mathcal{O}$ is a couple of sequences $(Dyn^-(f_0), Dyn^+(f_0))$ of endomorphisms of $\mathcal{O}$ with initial element f_0.*

The *time parameter* serves to label the elements of sequences. The time parameter space can be realized as subsets of the integer $\mathbb{Z}$ or as subsets of the reals $\mathbb{R}$. In the theory, there is not an absolute rule for choosing the time parameter space, although the requirement that time parameter space being endowed with an order relation is adopted. Note that this is a local condition, not a global one, useful when constructing limits.

A typical sequence of endomorphisms is of the form $\hat{\mathcal{S}} \equiv \{f_{\alpha_0}, f_{\alpha_1}, f_{\alpha_2}, f_{\alpha_3}, ...\}$, where $\alpha_0 < \alpha_1 < \alpha_2 < ...$ are elements of the parameter space. Note that the association from morphisms to parameter values is not necessarily injective, neither is necessary to be surjective. If the sequence is injective, then the pre-dynamics is *deterministic*; otherwise, the pre-dynamics is *non-deterministic*. Associativity and the assumption that the theory is based upon a fundamental category within the set theory framework are fundamentally based to look for consistency of the models with determinism. Moreover, both sequences $(Dyn^-(f_0), Dyn^+(f_0))$ must be well-ordered.

In order that the elements of the sequences $(Dyn^-(f_0), Dyn^+(f_0))$ are ordered or partially ordered in our construction, the parameter space must be endowed with an order relation or a partial order relation. The elements of $\mathcal{S}$ acquire the order relation from the map $\hat{\mathcal{S}} : \mathcal{S} \to \mathcal{I}$ and the order in the parameter space $\mathcal{I}$. The parameter labeling the succession of endomorphisms is related with the notion of time parameters used in the dynamics. However, these two concepts do not fully coincide logically: The time parameter just needs to be *finer* than the parameter labelling the successions of $Dyn(f_0)$.

A *local cyclic pre-dynamics* on an object $\mathcal{O}$ is a pre-dynamics consisting of two morphisms sequences such that the composition of all the morphims of the second sequence β after the composition of all morphisms of thirst sequence λ is the identity, $\beta \circ \lambda = Id$. The associativity property of the composition of endomorphisms allows the possibility of well defined deterministic cyclic pre-dynamics. On the other hand, let us assume that for a given object $\mathcal{O}$ the composition of endomorphisms was not associative. Then the neutral element for the composition $Id : \mathcal{O} \to \mathcal{O}$ does not need to be unique, which implies that a possible inverse element of λ does not need to be unique neither. Thus the conditions of cycles will be of the form $\beta_1 \circ \lambda = Id_1$, $\beta_2 \circ \lambda = I_2$ and the second part of the local cycle dynamics is not well defined without further information, since there are two possibilities, β_1 and β_2 to complete the local cycle.

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Let us consider two pre-dynamics,

$$\begin{cases} Dyn(f_0) = (Dyn^-(f_0), Dyn^+(f_0)), \\ Dyn(f_\alpha) = (Dyn^-(f_\alpha), Dyn^+(f_\alpha)) \end{cases}$$

with initial endomorphisms f_0 and $f_\alpha \in Dyn(f_0)$ respectively and such that $Dyn^-(f_0) \cup Dyn^+(f_0) = Dyn^-(f_\alpha) \cup Dyn^+(f_\alpha))$. Then $Dyn(f_0)$ and $Dyn(f_\alpha)$ are equivalent. Thus, for example, if

$$Dyn^+(f_0) = \{f_0, f_1, f_2, f_3, ...\}, \quad Dyn^-(f_0) = \{f_0, f_{-1}, f_{-2}, f_{-3}, ...\}$$

and

$$\widetilde{Dyn}^+(f_k) = \{f_k, f_{k+1}, f_{k+2}, f_{k+3}, \cdots\}, \quad \widetilde{Dyn}^-(f_k) = \{f_k, f_{k-1}, f_{k-2}, f_{k-3}, ...\},$$

but the two sequences coincide as ordered sets, the pre-dynamics $(Dyn^-(f_0), Dyn^+(f_0))$ and $(\widetilde{Dyn}^-(f_k), \widetilde{Dyn}^+(f_k))$ are equivalent.

There is also a consistent condition for the definition of the morphisms. It can happen that a given endomorphism $f_j : \mathcal{O} \to \mathcal{O}$ of $Dyn^+(f_0)$ can be re-casted as a composition of two non-trivial morphisms $f_{j_2} \circ f_{j_1} = f_j$ with $codom(f_{j_2} \circ f_{j_1}) = codom(f_j)$ and such that $f_{j1}, f_{j2} \neq Id$. In this case, if otherwise the same, $Dyn^+(f_0) = \{f_0, f_1, f_2, f_3, ..., f_{j_1}, f_{j_2}, ..., \}$ *is finer* than $\widetilde{Dyn}^+(f_0) = \{f_0, f_1, f_2, f_3, ..., f_j, ..., \}$. Analogous considerations apply to the composition of elements $Dyn^-(f_0)$. This construction suggests a natural extension of the equivalence relation discussed above. The equivalence class containing $Dyn(f_0)$ is denoted by $[Dyn(f_0)]$.

Definition 2.4. *If given a morphism $f_\alpha \in Dyn(f_0)$, the condition $f_\alpha = f_\beta \circ g$ holds good for certain $f_\beta, g \in Dyn(f_0)$ only if $g = Id$, then $Dyn(f_0)$ is the finest pre-dynamics containing f_0.*

After the above preliminary considerations, we introduce our notion of *dynamics*,

Definition 2.5. *Given a fundamental category Cat_{Fun}, a dynamics $Dyn(\mathcal{O})$ on the object $\mathcal{O}$ is an equivalence class of pre-dynamics containing a finest pre-dynamics. A dynamics $Dyn(Cat_{Fun})$ in the category Cat_{Fun} is a collection consisting of one dynamics for each object of Cat_{Fun}.*

We have the following result:

Proposition 2.6. *Given an initial endomorphism f_0 of an object $\mathcal{O}$ of Cat_{Fun} for a pre-dynamics $(Dyn^-(f_0), Dyn^+(f_0))$, there is at most an unique dynamics $Dyn(\mathcal{O})$ with initial morphism f_0.*

Proof. Let us consider two pre-dynamics $Dyn(f_0)$ and $\widetilde{Dyn}(f_0)$ on $\mathcal{O}$ corresponding to the same dynamics $Dyn(\mathcal{O})$ with the same initial morphism f_0, but differing on the components $Dyn^+(f_0)$ and $\widetilde{Dyn}^+(f_0)$ such that $f_\alpha \in Dyn^+(f_0)$ and $f_\beta \in \widetilde{Dyn}^+(f_0)$, $f_\alpha \neq f_\beta$. This implies either that $f_\alpha = f_\beta \circ g$ or that $f_\beta = g' \circ f_\alpha$ for certain morphisms g and g'. This corresponds to the combined action of morphisms f_β, g or g', f_α. Thus one can construct a new pre-dynamics either by extending the original dynamics either by containing g or g'. However, this is impossible because being the pre-dynamics initially associated with the finest one, the original pre-dynamics $Dyn(f_0)$ and $\widetilde{Dyn}(f_0)$ are the finest sequences, except if $g = Id$. □

Note that the morphisms $g, g' : \mathcal{O} \to \mathcal{O}$ are not generally determined by the constructions.

It is not obvious that, given a pre-dynamics there is a dynamics as defined above.

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The relevance of the notion of dynamics with respect to the pre-dynamics consists of its implication of determinism. Therefore, this is the property to be required, either by proving the existence of a dynamics or assuming it.

Note that the notion of dynamics can be applied to both, the case when the parameters hold the property of the intermediate point, as the rational $\mathbb{Q}$ or real $\mathbb{R}$ fields, or for parameter spaces where it is not possible to define such intermediate.

3. Natural Monoids of Cummulant Endomorphisms

Let $\mathcal{I}_N$ be the parameter space as defined above and $\mathcal{J} \subset \mathcal{I}_N$ an arbitrary subset. Let us consider the *cummulants* of endomorphisms defined from pre-dynamics on an object $\mathcal{O}_j$, that is, endomorphisms of the form

$$\begin{cases} {}^{+}C(\mathcal{O}_j,\mathcal{J}) := \left\{\prod_{\alpha\in\mathcal{J}} f^j_\alpha,\ f^j_\alpha \in Dyn^{+}(\mathcal{O}_j),\ \mathcal{J} \subset \mathcal{I}_N\right\}, \\ {}^{-}C(\mathcal{O}_j,\mathcal{J}) := \left\{\prod_{\alpha\in\mathcal{J}} f^j_\alpha,\ f^j_\alpha \in Dyn^{-}(\mathcal{O}_j),\ \mathcal{J} \subset \mathcal{I}_N\right\}. \end{cases}$$

We can define a *convolution operation of cummulatives*,

$$\star :^{+} C(\mathcal{O}_j,\mathcal{I}_N) \times {}^{+}C(\mathcal{O}_j,\mathcal{I}_N) \to {}^{+}C(\mathcal{O}_j,\mathcal{I}_N),\ ({}^{+}C(\mathcal{O}_j,\mathcal{I}_1),^{+} C(\mathcal{O}_j,\mathcal{I}_2)) \mapsto {}^{+}C(\mathcal{O}_j,\mathcal{I}_1 \cup \mathcal{I}_2).$$

and similarly for ${}^{-}C(\mathcal{O}_j,\mathcal{I})$.

It is direct the following result,

Proposition 3.1. *$({}^{+}C(\mathcal{O}_j,\mathcal{I}_N),\star)$ and $(F\,{}^{-}C(\mathcal{O}_j,\mathcal{I}_N)),\star)$ are abelian monoids endowed with an order relation.*

Proof. The composition law is associative and it has a neutral element, when $\mathcal{J} = \emptyset$. By the definition, the composition law of cummulants is commutative.

Since $\mathcal{I}_N$ is ordered, we can state that $\mathcal{I}_1 < \mathcal{I}_2$ if the largest element of $mathcal I_1$ is smaller than the last element of $\mathcal{I}_2$. If they are equal, we check the previous one, etc... Then one has a criteria to decide when $\mathcal{I}_1 < \mathcal{I}_2$, ${}^{+}C(\mathcal{O}_j,\mathcal{I}_1) < {}^{+}C(\mathcal{O}_j,\mathcal{I}_2)$. □

The monoids $({}^{+}C(\mathcal{O}_j,\mathcal{I}_N),\star)$ and $({}^{-}C(\mathcal{O}_j,\mathcal{I}_N)),\star)$ are almost totally recursive constructions, since only depend upon the elements of the category and the parameter space $\mathcal{I}_N$.

4. Notion of Asymmetric Dynamics

Definition 4.1. *Given a dynamics determined by the finest pre-dynamics $(Dyn^{-}(f_0), Dyn^{+}(f_0))$, the inverted dynamics is the equivalence class of pre-dynamics equivalent to $(Dyn^{+}(f_0), Dyn^{-}(f_0))$.*

Let us consider the finest pre-dynamics $Dyn(f_0) = (Dyn^{+}(f_0), Dyn^{+}(f_0))$ such that for $Dyn^{+}(f_0) = \{f_0, f_1, f_2, ...\}$ $Dyn^{-}(f_0) = Dyn^{+}(f_0)$. This is a symmetric dynamics. It depends on the specific initial morphism f_0. In contrast, the following notion of asymmetric dynamics, does not depend on the election of the initial morphism,

Definition 4.2. *Given a dynamics determined by the finest pre-dynamics $(Dyn^{-}(f_0), Dyn^{+}(f_0))$, the inverted dynamics is the class containing the pre-dynamics $(Dyn^{+}(f_0), Dyn^{-}(f_0))$. A dynamics is asymmetric if there is an initial morphism f_0 such that $Dyn^{+}(f_0) \neq Dyn^{-}(f_0)$; it is strong asymmetric if it is asymmetric for any possible initial morphism f_0.*

Given a pre-dynamics $(Dyn^{-}(f_0), Dyn^{+}(f_0))$ constituted by a large number of morphisms, an heuristic argument shows that it must be generically asymmetric. Thus for each symmetric dynamics there are much more asymmetric dynamics with the same morphisms, but rearranged in different way than symmetric.

5. Recursive Principle

A fundamental theory should be complete in the sense of capable to describe the elements of the theory with elements of the theory. As a way towards the formalization of this idea, we introduce the following:

Recursive principle. *The elements of a dynamical theory of fundamental processes must be defined recursively in terms relative to the dynamical changes associated with the dynamical systems described by the theory.*

This principle should apply to all the elements of a dynamical theory: Objects, dynamical law and parameter space. It must also apply to the construction of the time parameters.

From the categorical point of view, a dynamics is described by collections of endomorphisms $\{f_j : \mathcal{O}_j \to \mathcal{O}_j\}$ for each object $\mathcal{O}_j$ of the fundmaental category Cat_{Fun}. Therefore, the principle is re-casted in the following form,

Recursive principle in categorical form. *The elements of a dynamical theory of fundamental processes must be defined from the objects and the morphisms of the fundamental category Cat_{Fun}.*

A category is determined by two type of fundamental elements: The objects of the category and the morphisms of the category. Therefore, according to this principle, the dynamical degrees of freedom, the dynamical law and the time parameter should be defined in terms of the objects and the morphisms of the category, within the framework of set theory.

6. Parameterized Dynamics from the Categorical Point of View

The indexes denoting the morphisms of the dynamics can be realized by subsets of the natural numbers $\mathbb{N} = \{1, 2, 3, ...\}$ together with zero. This parameter can be understood either as the domain set of labels for ordering the morphisms of the dynamics or as elements of an algebraic structure. There is a direct generalization of the construction, when we consider the indices as valuated in an ordered monoid $(mon, +, 0)$, since the associativity and the existence of a neutral element properties hold for composition of morphisms of the objects of Cat_{Fun} and the same must be required for the parameter space. The recursivity principle imposes that the monoids should be constructed from the elements of the category, modulo re-parameterizations.

Definition 6.1. *Let $(mon, +, 0)$ be an ordered monoid. A parameterized dynamics with starting point the endomorphism $f : \mathcal{O} \to \mathcal{O}$ is a functor*

$$\tilde{\mathcal{P}}_{mon} : Cat_{Fun} \times mon \to Cat_{Fun} \tag{6.1}$$

such that

1. *The action on an object is trivial, $(\mathcal{O}, t) \mapsto \mathcal{O}$.*
2. *The action on morphism is such that:*

$$\tilde{\mathcal{P}}_{mon}(f, \hat{\lambda}) \circ \tilde{\mathcal{P}}_{mon}(f, \hat{\beta}) = \tilde{\mathcal{P}}_{mon}(f, \hat{\lambda}') \circ \tilde{\mathcal{P}}_{mon}(f, \hat{\beta}'), \quad \forall \quad \lambda + \beta = \lambda' + \beta', \tag{6.2}$$

for the specific morphisms $\hat{\lambda} : mon \to mon$, $t \mapsto t + \lambda$, etc...

The ordered monoid $(mon, +, 0)$ is the parameter space.

The significance of the first condition is that measuring time does not affect the dynamics. This is a definition of *ideal clock*.

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The second condition states the compatibility of the functor $\tilde{\mathcal{P}}$ with the associativity law of composition of morphisms in Cat_{Fun}.

Let us consider another category $\mathcal{S}_{\mathcal{P}}$ that it will be identified with the category of time parameter spaces. From the previous discussion, $\mathcal{S}_{\mathcal{P}}$ is in first instance identified with the category of (partial) ordered monoids **OMon**. In this setting, the recursive principle is partially implemented by the application of *Proposition* 3.1, that defines a functor

$$\tilde{\mathcal{P}} : Cat_{Fun} \to \mathcal{S}_{\mathcal{P}}, \quad \mathcal{O}_j \mapsto {}^+C(\mathcal{O}_j, \mathcal{I})$$

where $\mathcal{S}_{\mathcal{P}}$ is a sub-category of **OMon**. and where the morphism $\tilde{\mathcal{P}}$ maps the morphism $f_{ji} : \mathcal{O}_i \to \mathcal{O}_j$ to the morphism $\tilde{\mathcal{P}}(f_{ji}) : {}^+C(\mathcal{O}_i, \mathcal{I}) \to {}^+C(\mathcal{O}_j, \mathcal{I})$ defined by the map on cummulants,

$$\prod_{\alpha \in J} f_i^\alpha \mapsto \prod_{\alpha \in J} f_j^\alpha.$$

The morphism $\tilde{\mathcal{P}}(f_{ji})$ does not depend upon the specific characteristics of f_{ji}, but only on the origin object and target object. In this sense, $\tilde{\mathcal{P}}(f_{ji})$ is a *constant morphism*.

The parameter space $\mathcal{I}$ is leaved un-specified, but it is required to be constructed from the elements of the category Cat_{Fun}, in particular, it must be constructible using the elements of each object $\mathcal{O}$. Moreover, it is assumed the same for the model based upon the object $\mathcal{O}_i$ and for the model bades upon $\mathcal{O}_j$.

The functor $\tilde{\mathcal{P}}$ assigns to each object $\mathcal{O}$ of Cat_{Fun} an object $\mathcal{I}$ of $\mathcal{S}_{\mathcal{P}}$ and to each morphism $f_{ij} : \mathcal{O}_i \to \mathcal{O}_j$ a morphism $\tilde{\mathcal{P}}(f_{ji}) : \tilde{\mathcal{P}}(\mathcal{O}_i) \to \tilde{\mathcal{P}}(\mathcal{O}_j)$ of the parameter category $\mathcal{S}_{\mathcal{P}}$. $\tilde{\mathcal{P}}$ is not surjective in the sense that time parameters that could be used in the mathematical description of a given dynamics are not in the codomain of $\tilde{\mathcal{P}}$, neither it is a *full functor*, since morphisms of an object *mon* in $\mathcal{S}_{\mathcal{P}}$ will not be taken into account by the morphisms of the given objects of Cat_{Fun}. In order to construct a full surjective parametrization functor we consider the category $\mathcal{S}_{\mathcal{P}}/Aut$ obtained by quotient each object $\mathcal{I}$ of $\mathcal{S}_{\mathcal{P}}$ by the set of order preserving automorphisms $Aut(\mathcal{I})$. The corresponding morphisms of $\mathcal{S}_{\mathcal{P}}/Aut$ are the morphisms between quotients $\mathcal{I}/Aut(\mathcal{I})$, that is, the quotient morphisms. From the construction of the category $\mathcal{S}_{\mathcal{P}}/Aut$ in this way we have the following result,

Proposition 6.2. *Let us consider a sub-category $\mathcal{S}_{\mathcal{P}}$ of* **OMon** *and the functor $\tilde{\mathcal{P}} : Cat_{Fun} \to \mathcal{S}_{\mathcal{P}}$. Then the quotient functor*

$$\mathcal{P} : Cat_{Fun} \to \mathcal{S}_{\mathcal{P}}/Aut, \tag{6.3}$$

is full.

Proposition 6.2 is a manifestation of the recursive principle, which is further fulfilled by assuming that $\mathcal{P}$ is surjective and by the characterization of evolution as given by sequences of endomorphisms of objects. According to this point of view, not only one can associate a clock to each possible fundamental system represented by objects $\mathcal{O}$ in Cat_{Fun}, as it is read from the assumption on the existence of the functor $\tilde{\mathcal{P}}$, but also any imaginable time parameter is realized in this way, up to re-parametrization, by means of the functor $\mathcal{P}$.

The characterization of the category of time parameters $\mathcal{S}_{\mathcal{P}}$ is intimately related with the assumptions concerning the dynamical laws and Cat_{Fun}, because the use of time parameterizations should not impose extra conditions on the dynamics. From the properties already discussed, it follows that each of the objects of $\mathcal{S}_{\mathcal{P}}$ that serves as time parameter space must be endowed at least with one algebraic binary relation, that furnishes each of them with the structure of monoid. Such binary

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relations constitute an integral part in the definition of dynamics, while the associative law in the objects of $\mathcal{S}_{\mathcal{P}}$ serves to be consistent with the associativity conditions for the morphisms,

$$\mathcal{I}_j \circ (\mathcal{I}_k \circ \mathcal{I}_l) = (\mathcal{I}_j \circ \mathcal{I}_k) \circ \mathcal{I}_l$$

without imposing conditions on the original category Cat_{Fun}.

In order to implement further the recursive principle, the parameters used to label the sequences of the dynamics need to be defined in terms of the elements of the category Cat_{Fun} or in terms of elements of categories constructed from Cat_{Fun}. One procedure to achieve this goal is the following. One can consider the convolution operators on morphisms. These operations define a category $\mathcal{F}$ whose objects are the monoids $(F^+(\mathcal{O}_j), \star)$ consisting on convolutions of endomorphisms given by the sequence of the dynamics and the morphisms are the induced morphisms between $(F^+(\mathcal{O}_j), \star)$ and $(F^+(\mathcal{O}_k), \star)$ induced from the morphims $f_{jk} : \mathcal{O}_j \to \mathcal{O}_k$ of Cat_{Fun}, modulo automorphisms of each monoid $(F^+(\mathcal{O}_j), \star)$. Since the use of re-parameterizations can be seen as a form of abstraction in the time labeling of the dynamics, the formal category of parameters is the quotient category $\mathcal{F}/Aut$, the quotient with respect to the automorphisms of each monoid $(F^+_j, \star)$.

The identification of the notion of time in the categorical theory of dynamics involves two steps. First, $\mathcal{S}_{\mathcal{P}}$ is identified with $\mathcal{F}$. Second, time parameters are restricted to be modelled as codomains of the functor $\mathcal{P} : Cat_{Fun} \to \mathcal{F}/Aut$. In this way, the categorical recursive principle is full filled.

7. Evolution of the Category

The notion of parameterized dynamics can be extended in natural way to the category itself.

Definition 7.1. *Given a functor $\Phi : Cat_{Fun} \to Cat_{Fun}$ and an partial ordered monoid mon, a generalized parameterized dynamics along Φ parameterized on mon is a functor*

$$\tilde{\mathcal{P}}_{mon} : Cat_{Fun} \times mon \to Cat_{Fun} \tag{7.1}$$

such that:

- *The action on objects of Cat_{Fun} is covariant in the sense of Φ, $\tilde{\mathcal{P}}_{mon}(\mathcal{O}, \lambda) = \Phi(\mathcal{O})$,*
- *The action on morphism is such that:*

$$\tilde{\mathcal{P}}_{mon}(f, \hat{\lambda}) \circ \tilde{\mathcal{P}}_{mon}(\Phi(f), \hat{\beta}) = \tilde{\mathcal{P}}_{mon}(f, \hat{\lambda}') \circ \tilde{\mathcal{P}}_{mon}(\Phi(f), \hat{\beta}'), \quad \forall \quad \lambda + \beta = \lambda' + \beta' \tag{7.2}$$

holds good.

The notion of generalized parameterized dynamics accommodates better to the view of general covariance, since also the law of evolution is exposed to evolution. The categorical point of view becomes in this context a natural framework, since the mere concept of dynamical model, identified with the notion of dynamics on a object of the category, needs to be transcended to consider the full changes in the category due to the evolution and measured by compatibility with Φ.

We observe in *Definition* (7.1) the germ of the notion of *evolution of the dynamical law*. Such evolution of the dynamical law is driven by the functor $\Phi : Cat_{Fun} \to Cat_{Fun}$.

8. Binary Composition Operation for Categories

Given a category Cat_{Fun}, a category of parameters $\mathcal{S}_{\mathcal{P}}$ and a parameterized dynamics $\mathcal{P}_{mon}$ on each of the objects mon of $\mathcal{S}_{\mathcal{P}}$, there is a binary operation $* : Cat_{Fun} \times Cat_{Fun} \to Cat_{Fun}$ such that the following diagram commutes,

$$\begin{array}{ccc} Cat_{Fun} \times Cat_{Fun} & & \\ \downarrow{\scriptstyle Id\times\mathcal{P}} & \searrow{\scriptstyle *} & \\ Cat_{Fun} \times \mathcal{S}_{\mathcal{P}}/Aut & \xrightarrow{\hat{\Phi}} & Cat_{Fun} \end{array} \tag{8.1}$$

The functor $\hat{\Phi}$ is defined in the following way: For each $\mathcal{O}$ of Cat_{Fun} and for each time parameter space I, an object of $\mathcal{S}_{\mathcal{P}}$, one has that $\hat{\Phi}(\mathcal{O}, I) = \mathcal{O}$, a definition that it is keep consistent under reparameterizations.

The action on morphisms of the product $Cat_{Fun} \times \mathcal{S}_{\mathcal{P}}$ is constructed in the following way. Let us consider $\tilde{\mathcal{P}}(O_i), \tilde{\mathcal{P}}(O_j)$ as objects in $\mathcal{S}_{\mathcal{P}}$. Therefore, we need to consider two generic objects I_α, I_β of $\mathcal{S}_{\mathcal{P}}$ and $\lambda_{\alpha\beta} : I_\alpha \to I_\beta$ a morphism. Then the action of the functor $\hat{\Phi}(f_{ij}, \lambda_{\alpha\beta})$ on product morphisms is

$$\tilde{P}_{\tilde{\mathcal{P}}(O_i)}(f_0, t) \to \tilde{P}_{\tilde{\mathcal{P}}(O_j)}(f_0, \lambda_{\alpha\beta}(t)).$$

By the relation (7.2) and since $\lambda_{\alpha\beta}$ is a morphism, $\hat{\Phi}(f_{ij}, \lambda_{\alpha\beta}) : \mathcal{O}_i \to \mathcal{O}_j$ is a morphism and it is independent of the election of $t \in I_\alpha$.

An analogous construction follows from the notion of generalized parameterized dynamics.

9. Conclusions

A categorical framework for fundamental dynamics has been developed. In the theory, the nature and role of the time parameters has been discussed. There are two type of time parameters. The first are the once used to choose the labels of the sequences of morphisms defining the dynamics. Such parameters are abstract mathematical parameters, not linked to any physical phenomena and in the model, build from the underlying set theory. The second type of parameters are linked with specific physical systems. It is discussed how time parameter must be naturally emerge from the principles of the theory and the elements of the category. This is in concordance with our recurrence principle.

An initial parameter space with an inherited order relation is need, in order to label the elements of the dynamics. Such parameter are defined by the first type discussed above. Besides such a parameter, the recurrence principle is totally fulfilled in the theory, since all the objects of the dynamics are re-written in terms of the elements of the dynamics.

The theory developed has a general character. It implies that also the laws of physics must evolve with time. Thus the categorical approach justifies a change dynamical law of physics. Second, the development of the theory implies deterministic laws for the fundamental dynamics.

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